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Reduction of the number of independent variables and optimization in swirling fluid flow

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IN THE APPROACH utilized for this investigation, transformations obtained by the application of continuous one-parameter group theory are applied to the fundamental system of non-dimensional equations for conservation of mass, conservation of momentum, and conservation of energy. The number of independent variables is reduced from three first to two then to one using the absolute invariants determined for each transformation group. The resulting system of nonlinear ordinary differential equations is then solved numerically by Hamming's modified predictor-corrector technique. The one-dimensional solutions are transformed back to the three-dimensional space. A Mayer type optimization problem is solved using the one-dimensional representations for conservation of energy and the swirl parameter to optimize the one-dimensional representation for pressure.

W sposobie wykorzystanym w tym badaniu zastosowano transformacje otrzymane przez wprowadzenie ciągłej jednoparametrowej teorii grup do podstawowego układu jednowymiarowych równań zachowania masy, pędu i energii. Liczba zmiennych niezależnych została zredukowana z trzech do jednej używając bezwzględnych niezmienników określonych dla każdej grupy transformacji. Końcowy układ nieliniowych zwyczajnych równań różniczkowych rozwiązano numerycznie stosując zmodyfikowaną metodę Hamminga prób i błędów. Jednowymiarowe rozwiązania zostały przekształcone z powrotem do przestrzeni trójwymiarowej. Zagadnienie optymizacyjne typu Mayera rozwiązano używając reprezentacji jednowymiarowej dla zachowania energii i parametru wirowości w celu zoptymalizowania jednowymiarowej reprezentacji dla ciśnienia.

Используется преобразование, основанное на применении теории непрерывных одно-параметрических групп к исследованию основной системы безразмерных уравнений сохранения массы, количества движения и энергии. При помощи абсолютных инвариантов, определенных для каждой из групп преобразований, число независимых переменных сокращено с трех до двух, а затем до одного. Полученная система нелинейных обыкновенных дифференциальных уравнений решена численным путем, с применением техники предиктор-корректор, модифицированной Геммингом. Одномерные решения преобразованны обратно в трехмерное пространство. Решена оптимизационная задача типа Мейера. С целью оптимизации одномерного представления давления используются одномерные представления принципа сохранения энергии и параметра вихря.

Notations

p pressure,
r, θ, z physical coordinates, independent variables,
u, v, w velocity components,
ϱ density,
y_j dummy dependent variable,
h Bernoulli constant,
λ_j Lagrange multipliers,
K augmented function,

ϕ_j functional forms of optimal constraint equations,
π_1 swirl aspect ratio $= r^2/z^2$,
π_2 tangential energy fraction $= v^2/(h/\varrho)$,
S swirl parameter $= \pi_1, \pi_2$,
z_0 axial swirl length,
v_0 tangential velocity at the outer swirl radius,
R dimensionless independent variable $= r/z_0$,
Z dimensionless independent variable $= z/z_0$,
U dimensionless radial velocity component $= u/v_0$,
V dimensionless tangential velocity component $= v/v_0$,
W dimensionless axial velocity component $= w/v_0$,
P dimensionless static pressure $= p/(\varrho v_0^2)$,
H dimensionless Bernoulli constant $= h/(\varrho v_0^2)$,
t_i constants,
a group parameter,
A_1 one-parameter group,
Φ differential form of k-th order in m independent variables,
z^i arguments of the k-th order differential form,
η_i independent variables, absolute invariants of A_1,
f_i absolute invariants of A_1,
ξ group constant $= t_2/t_1$,
α group constant $= t_4/t_2$,
F_1 two-dimensional representation for the radial velocity component,
F_2 two-dimensional representation for the tangential velocity component,
F_3 two-dimensional representation for the axial velocity component,
F_4 two-dimensional representation for static pressure,
k constant,
γ_i constants,
c group parameter,
B_1 one-parameter group,
X independent variable, absolute invariant of B_1,
σ group constant $= \gamma_1/k$,
g_i absolute invariants of B_1,
G_1 one-dimensional representation for the radial velocity component,
G_2 one-dimensional representation for the tangential velocity component,
G_3 one-dimensional representation for the axial velocity component,
G_4 one-dimensional representation for static pressure,
L^2 constant $= G_1^2+G_3^2 = G^2$,
$'$ denotes differentiation with respect to the independent variable.

1. Introduction

GENERALLY, swirling flow may be categorized as any flow in which the tangential velocity component has a finite magnitude such that the fluid has a macroscopic rotary motion. Typical flow examples occur in Hilsch tubes, vortex amplifiers, tornados, and rotating fluid machinery.

A large number of studies have been directed toward various aspects of swirling flow. The system of fundamental equations describing swirling flow in the general case includes three-dimensional, non-linear partial differential equations which are not amenable to solution by standard analytical techniques. Therefore, investigators of swirling type flows were compelled to make various simplifying assumptions. All velocity components were assumed to be functions of only one independent variable [5, 10, 13, 14, 15, 25, 27, 35, 47] or particular velocity components were assumed to have only certain special forms [12, 24, 26].

2. Mathematical model([1])

For this model, inviscid-incompressible flow is assumed. The natural choice of coordinates is a cylindrical polar coordinate system, since the fluid has a macroscopic rotary motion. The equations governing swirling flow are based on the fundamental laws of conservation of mass, conservation of momentum, and conservation of energy. The model equations presented here are all in non-dimensional form.

The conservation of mass law is given by the continuity equation.

Conservation of mass (continuity equation)

$$\frac{\partial U}{\partial R}+\frac{U}{R}+\frac{1}{R}\frac{\partial V}{\partial \theta}+\frac{\partial W}{\partial Z}=0. \tag{2.1}$$

The conservation of momentum law is given in scalar form by the three Euler equations.

$$U\frac{\partial U}{\partial R}+\frac{V}{R}\frac{\partial U}{\partial \theta}+W\frac{\partial U}{\partial Z}-\frac{V^2}{R}=-\frac{\partial P}{\partial R}, \quad R\text{ — momentum}; \tag{2.2}$$

$$U\frac{\partial V}{\partial R}+\frac{V}{R}\frac{\partial V}{\partial \theta}+W\frac{\partial V}{\partial Z}+\frac{UV}{R}=-\frac{1}{R}\frac{\partial P}{\partial \theta}, \quad \theta\text{ — momentum}; \tag{2.3}$$

$$U\frac{\partial W}{\partial R}+\frac{V}{R}\frac{\partial W}{\partial \theta}+W\frac{\partial W}{\partial Z}=-\frac{\partial P}{\partial Z}, \quad Z\text{ — momentum}. \tag{2.4}$$

The conservation of energy law is given by Bernoulli's equation for this case.

Conservation of energy (Bernoulli equation)

$$H=P+\frac{1}{2}(U^2+V^2+W^2). \tag{2.5}$$

The boundary conditions are all specified as constants at $R=R_0$.

At $R=R_0$,

$$U=U_0, \quad V=V_0=1, \quad W=W_0, \quad \text{and} \quad P=P_0. \tag{2.6}$$

The product of the swirl aspect ratio (R^2/Z^2) and the tangential energy fraction (V^2/H) is defined to be the swirl parameter for this investigation.

Swirl parameter

$$S=\left(\frac{RV}{Z}\right)^2\frac{1}{H}. \tag{2.7}$$

3. Three to two reduction in the number of independent variables

Continuous one-parameter group theory is now applied to the system of equations ain Sec. 2 to reduce the three independent variable (R, θ, Z) to two new independent vari-

([1]) Mathematical fundamentals are given in the Appendix.

bles (η_1, η_2). The general approach followed here for applying group theory to achieve a reduction in the number of independent variables for systems of partial differential equations was developed in two classic papers by A. D. MICHAL [31] and A. J. A. MORGAN [33]. The new independent variables (η_1, η_2) are functionally independent absolute invariants of a subgroup of the one-parameter continuous group of transformations, A_1. There is no unique method for specifying such groups and the group A_1 is chosen as shown:

$$
(3.1)\qquad A_1:\quad \begin{aligned} &\overline{R} = e^{at_1}R, \quad \overline{U} = e^{at_4}U, \quad \overline{P} = e^{2at_4}P,\\ &\overline{\theta} = \theta + at_2, \quad \overline{V} = e^{at_4}V,\\ &\overline{Z} = e^{at_1}Z, \quad \overline{W} = e^{at_4}W, \end{aligned}
$$

where t_1, t_2, and t_4 are constants and a is the group parameter.

It can easily be shown that the partial differential equations in Sec. 2 are conformally invariant under the group A_1. A differential form Φ is conformally invariant under a one-parameter group A_1, if, under the transformations of the group, it satisfies the relation

$$
(3.2)\qquad \Phi(\overline{Z}^1, \ldots, \overline{Z}^p) = M(Z^1, \ldots, Z^p;\ a)\, \Phi(Z^1, \ldots, Z^p),
$$

where Φ is exactly the same function of the Z's as it is of the $\overline{Z}$'s and M is some function of the Z's and the parameter a. When the condition for conformal invariance is satisfied by each differential form in a system of partial differential equations, a reduction in the number of independent variables is possible for a system of partial differential equations.

There is no well defined method for selecting absolute invariants and the absolute invariants for the group A_1 are chosen as shown:

$$
(3.3)\qquad \begin{aligned} &\eta_1 = e^{-\theta}R^{\xi}, \quad \eta_2 = e^{-\theta}Z^{\xi}, \quad f_1 = Ue^{-\alpha\theta}, \quad f_2 = Ve^{-\alpha\theta}, \quad f_3 = We^{-\alpha\theta},\\ &f_4 = Pe^{-2\alpha\theta}, \quad \text{where} \quad \xi \equiv t_2/t_1 \quad \text{and} \quad \alpha \equiv t_4/t_2. \end{aligned}
$$

Any absolute invariant of a group is expressible in terms of the functionally independent absolute invariants. Therefore,

$$
(3.4)\qquad f_j(U, V, W, P, R, \theta, Z) = F_j(\eta_1, \eta_2)
$$

$j = 1, 2, 3, 4$. Using Eqs. (3.3) and (3.4), the original dependent variables become functions of the new independent variables:

$$
(3.5)\qquad U = e^{\alpha\theta}F_1(\eta_1, \eta_2), \quad V = e^{\alpha\theta}F_2(\eta_1, \eta_2), \quad W = e^{\alpha\theta}F_3(\eta_1, \eta_2), \quad \text{and} \quad P = e^{2\alpha\theta}F_4(\eta_1, \eta_2).
$$

Application of Eqs. (3.3) and (3.5) to the system of equations in Sec. 2 yields a three to two reduction in the number of independent variables from (R, θ, Z) to (η_1, η_2). The equations resulting from the transformations are:

$$
(3.6)\qquad \xi\eta_1\frac{\partial F_1}{\partial \eta_1} + F_1 + \alpha F_2 - \eta_1\frac{\partial F_2}{\partial \eta_1} - \eta_2\frac{\partial F_2}{\partial \eta_2} + \xi\left(\frac{\eta_1}{\eta_2}\right)^{1/\xi}\eta_2\frac{\partial F_3}{\partial \eta_2} = 0,
$$

$$
(3.7)\qquad (\xi F_1 - F_2)\eta_1\frac{\partial F_1}{\partial \eta_1} + \left(\xi F_3(\eta_1/\eta_2)^{1/\xi} - F_2\right)\eta_2\frac{\partial F_1}{\partial \eta_2} \cdot - F_2^2 + \alpha F_1F_2 = \xi\eta_1\frac{\partial F_4}{\partial \eta_1},
$$

$$(3.8)\quad (\xi F_1 - F_2)\eta_1 \frac{\partial F_2}{\partial \eta_1} + \left(\xi F_3(\eta_1/\eta_2)^{1/\xi} - F_2\right)\eta_2 \frac{\partial F_2}{\partial \eta_2} + F_1 F_2 + \alpha F_2^2 = \eta_1 \frac{\partial F_4}{\partial \eta_1} + \eta_2 \frac{\partial F_4}{\partial \eta_2} - 2\alpha F_4,$$

$$(3.9)\quad (\xi F_1 - F_2)\eta_1 \frac{\partial F_3}{\partial \eta_1} + \left(\xi F_3(\eta_1/\eta_2)^{1/\xi} - F_2\right)\eta_2 \frac{\partial F_3}{\partial \eta_2} + \alpha F_2 F_3 = \xi \left(\frac{\eta_1}{\eta_2}\right)^{1/\xi} \eta_2 \frac{\partial F_4}{\partial \eta_2},$$

$$(3.10)\quad H = e^{2\alpha\theta}\left(F_4 + \frac{1}{2}(F_1^2 + F_2^2 + F_3^2)\right),$$

$$(3.11)\quad S = \left(\frac{\eta_1}{\eta_2}\right)^{2/\xi} e^{2\alpha\theta} F_2^2 / H.$$

The following table relates the new equations in two independent variables to the original equations in three independent variables.

Equation name	(R, θ, Z) space	(η_1, η_2) Space
Conservation of mass	(2.1)	(3.6)
R — momentum	(2.2)	(3.7)
θ — momentum	(2.3)	(3.8)
Z — momentum	(2.4)	(3.9)
Conservation of energy	(2.5)	(3.10)
Swirl parameter	(2.7)	(3.11)

4. Two to one reduction in the number of independent variables

Continuous one-parameter group theory is now applied to the system of equations in Sec. 3 to reduce the two independent variables (η_1, η_2) to one new independent variable (X). The new independent variable (X) is a functionally independent absolute invariant of a subgroup of the one-parameter continuous group of transformations, B_1. Again, noting that there is no unique way of specifying such groups, the group B_1 is chosen as shown:

$$(4.1)\quad B_1:\quad \begin{aligned} \bar{\eta}_1 &= c^k \eta_1, & \bar{F}_2 &= c^{\gamma_1} F_2, \\ \bar{\eta}_2 &= c^k \eta_2, & \bar{F}_3 &= c^{\gamma_1} F_3, \\ \bar{F}_1 &= c^{\gamma_1} F_1, & \bar{F}_4 &= c^{2\gamma_1} F_4, \end{aligned}$$

where k and γ_1 are constants and c is the group parameter.

It can easily be shown that the partial differential equations in Sec. 3 are conformally invariant under the group B_1. Therefore, utilizing the group B_1, a further reduction in the number of independent variables is possible for the system of partial differential equations in Sec. 3.

Since there is no well defined method for selecting absolute invariants, the absolute invariants for the group B_1 are chosen as shown:

$$(4.2)\quad X = \eta_1/\eta_2, \quad g_1 = F_1 \eta_1^{-\sigma}, \quad g_2 = F_2 \eta_1^{-\sigma}, \quad g_3 = F_3 \eta_1^{-\sigma}, \quad g_4 = F_4 \eta_1^{-2\sigma},$$

where $\sigma \equiv \gamma_1 / k$.

Note that the choice for X is suggested by the appearance of this ratio in the system of equations in Sec. 3.

Any absolute invariant of a group is expressible in terms of the functionally independent absolute invariants. Therefore,

$$g\,(F_1, F_2, F_3, F_4, \eta_1, \eta_2) = G_j(X), \quad j = 1, 2, 3, 4. \tag{4.3}$$

Using Eqs. (4.2) and (4.3), the original dependent variables become functions of the new independent variable:

$$F_i = \eta_1^{\varkappa\sigma} G_i(X), \quad i = 1, 2, 3, 4, \quad \varkappa = \begin{cases} 1, & i \neq 4, \\ 2, & i = 4. \end{cases} \tag{4.4}$$

Application of Eqs. (4.2) and (4.4) to the system of equations in Sec. 3 yields a two to one reduction in the number of independent variables from (η_1, η_2) to (X). The equations resulting from the transformations are:

$$(1+\xi\sigma)G_1 + (\alpha-\sigma)G_2 + \xi X G_1' - \xi X^{1+1/\xi} G_3' = 0, \tag{4.5}$$

$$\xi X G_1 G_1' + \xi\sigma G_1^2 - \sigma G_1 G_2 - \xi X^{1+1/\xi} G_3 G_1' - G_2^2 + \alpha G_1 G_2 = -\xi X G_4' - 2\xi\sigma G_4, \tag{4.6}$$

$$\xi X G_1 G_2' + (\xi\sigma+1) G_1 G_2 + (\alpha-\sigma) G_2^2 - \xi X^{1+1/\xi} G_3 G_2' = 2(\sigma-\alpha) G_4, \tag{4.7}$$

$$\xi X G_1 G_3' + \xi\sigma G_1 G_3 + (\alpha-\sigma) G_2 G_3 - \xi X^{1+1/\xi} G_3 G_3' = \xi X^{1+1/\xi} G_4', \tag{4.8}$$

$$G_4 = \frac{H}{e^{2\alpha\theta}\eta_1^{2\sigma}} - \frac{1}{2}(G_1^2 + G_2^2 + G_3^2), \tag{4.9}$$

$$S = X^{2/\xi} e^{2\alpha\theta} \eta_1^{2\sigma} G_2^2 / H. \tag{4.10}$$

To obtain a complete reduction to one independent variable in Eqs. (4.9) and (4.10), the constants α and σ must vanish. Let $\alpha = \sigma = 0$, Eqs. (4.5), (4.6), (4.7), (4.8), (4.9) and (4.10) then become:

$$G_1 + \xi X G_1' - \xi X^{1+1/\xi} G_3' = 0, \tag{4.11}$$

$$\xi X G_1 G_1' - \xi X^{1+1/\xi} G_3 G_1' - G_2^2 = -\xi X G_4', \tag{4.12}$$

$$\xi X G_1 G_2' + G_1 G_2 - \xi X^{1+1/\xi} G_3 G_2' = 0, \tag{4.13}$$

$$\xi X G_1 G_3' - \xi X^{1+1/\xi} G_3 G_3' = \xi X^{1+1/\xi} G_4', \tag{4.14}$$

$$G_4 = H - \frac{1}{2}(G_1^2 + G_2^2 + G_3^2), \tag{4.15}$$

$$S = X^{2/\xi} G_2^2 / H. \tag{4.16}$$

Equations (4.11) through (4.16) comprise the new system of equations for the fluid flow model in (X) space. Note that the original system of partial differential equations has been transformed into a system of ordinary differential equations.

The following table relates the new system of equations in one independent variable to both systems of equations in two and three independent variables.

Equation name	(R, θ, Z) Space	(η_1, η_2) Space	(X) Space
Conservation of mass	(2.1)	(3.6)	(4.11)
R — momentum	(2.2)	(3.7)	(4.12)
θ — momentum	(2.3)	(3.8)	(4.13)
Z — momentum	(2.4)	(3.9)	(4.14)
conservation of energy	(2.5)	(3.10)	(4.15)
swirl parameter	(2.7)	(3.11)	(4.16)

5. One-dimensional solutions of the ordinary differential equations

The exponential term $1+1/\xi$ appears in each of the Eqs. (4.11) through (4.14). These equations are greatly simplified by choosing $\xi = -1$.

For $\xi = -1$, the system of ordinary differential equations consisting of Eqs. (4.11), (4.12), (4.13) and (4.14) simplifies to the following system of ordinary differential equations:

$$-XG_1'+G_1+G_3' = 0, \tag{5.1}$$

$$(G_3-XG_1)G_1'-G_2^2 = HG_4', \tag{5.2}$$

$$(G_3-XG_1)G_2'+G_1G_2 = 0, \tag{5.3}$$

$$(G_3-XG_1)G_3' = -G_4'. \tag{5.4}$$

Since this system of ordinary differential equations is non-linear, a numerical analysis approach is required to determine its solutions.

Hamming's modified predictor-corrector numerical method is utilized to obtain solutions for the system consisting of Eqs. (5.1), (5.2), (5.3) and (5.4). The method employs the following numerical calculations:

$$\text{predictor:} \quad P_{j+1} = G_{j-3}+\frac{4h}{3}(2G_j'-G_{j-1}'+2G_{j-2}'); \tag{5.5}$$

$$\text{modifier:} \quad M_{j+1} = P_{j+1}-\frac{112}{121}(P_j-C_j); \tag{5.6}$$

$$\text{corrector:} \quad C_{j+1} = \frac{1}{8}[9G_j-G_{j-2}+3h(M_{j+1}'+2G_j'-G_{j-1}')]; \tag{5.7}$$

$$\text{final value:} \quad G_{j+1} = C_{j+1}+\frac{9}{121}(P_{j+1}-C_{j+1}). \tag{5.8}$$

If the results are known at the equidistant points $X_{j-3}, X_{j-2}, X_{j-1}$ and X_j, the results at point $X_{j+1} = X_j+h$ (where h is the step size) can be computed using the numerical Eqs. (5.5), (5.6), (5.7) and (5.8). Hamming's modified predictor-corrector method is not self starting; that is, the functional values at a single previous point are not enough to get the functional values ahead. To obtain the starting values, a special Runge-Kutta procedure followed by one iteration step is added to the predictor-corrector method. For an extensive discussion on Hamming's modified predictor-corrector method, the reader is referred to CARNAHAN [8] or WILF [38].

The IBM scientific subroutine DHCPG, a double precision arithmetic routine utilizing Hamming's method, was used in this investigation to solve the system of non-linear ordinary differential equations. The numerical computations were performed by an IBM 360 computer at General Motors Institute, Flint, Michigan, and the numerical results are displayed in Table 1.

The boundary conditions (2.6) are constants and transform to X space unaltered. The values for the boundary conditions are selected as shown:

$$U_0 \to F_{1*} \to G_{1*} = 1.0, \qquad V_0 \to F_{2*} \to G_{2*} = 1.0,$$

(5.9)
$$W_0 \to F_{3*} \to G_{3*} = 1.0, \quad \text{and} \quad P_0 \to F_{4*} \to G_{4*} = 0.5.$$

In X space, $G_{j*} \equiv G_j(X_*)$, where $j = 1, 2, 3, 4$.

The boundary point X_* is the left limit of the X interval $(X_* \leqslant X \leqslant X_r)$. Using the result obtained from substituting Eq. (3.3) into Eq. (4.2), X_* is defined to transform as indicated by the following equation:

(5.10)
$$X_* \equiv \frac{\eta_{1*}}{\eta_2} = \frac{e^{-\theta} R_0^{-1}}{e^{-\theta} Z^{-1}} = \frac{Z}{R_0}.$$

For the selected boundary conditions, the numerical solutions to Eqs. (5.1), (5.2), (5.3) and (5.4) are plotted for the interval $0.1 \leqslant X \leqslant 1.0$ in Fig. 1. The output from the computer program using the DHCPG subroutine was stored as the input to a subroutine which controls a Calcomp plotter from which the plots were made.

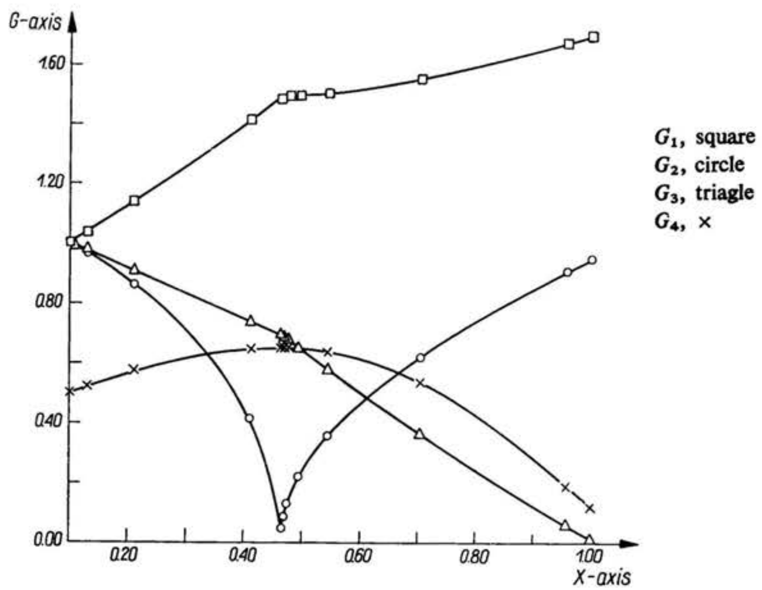

FIG. 1. One-dimensional functions.

At the point $X = 0.46531$ in Fig. 1, the solution functions G_1, G_2 and G_3 all have a discontinuous derivative. G_4 (the one-dimensional representation for pressure) reaches its maximum value and G_2 (the one-dimensional representation for the tangential veloctiy component) reaches its minimum value. Therefore, the point $X = 0.46531$ is designated as the "tangential quasi-stagnation point" for this flow.

The solutions G_1, G_2, G_3 and G_4 were determined by simultaneously solving the one-dimensional representations for conservation of momentum and conservation of mass. To verify these results, the one-dimensional representation for conservation of energy, Eq. (4.15), is used to determine the Bernoulli constant H at each point X.

Rearranging Eq. (4.15), the equation for H is:

$$H = G_4 + \frac{1}{2} \int_{i=1}^{3} G_i^2. \tag{5.11}$$

The value of H is fixed by the selected boundary conditions and for this case $H = 2$. Data values for H were computed at each point X from Eq. (5.11) using the solutions G_1, G_2, G_3 and G_4. From the H data values listed in Table 1, the maximum per cent deviation of H from the value $H = 2$ is determined to be 0.28%. This maximum per cent deviation occurs at one point only. Most of the per cent deviations for H are in the range from 0.00% to 0.09% and hence Eq. (4.15) is satisfied.

Polynomial expressions for the solutions G_1, G_2, G_3 and G_0 are determined using a Gaussian least squares method. The polynomial expressions for the interval $0.1 \leqslant X \leqslant 0.46531$ are:

$$G_1 = 0.88531 + 1.104\,X + 0.44657\,X^2, \tag{5.12}$$

$$G_2 = 1.4366 - 9.5497X + 82.897\,X^2 - 383.95\,X^3 + 829.38\,X^4 - 691.74\,X^5, \tag{5.13}$$

$$G_3 = 1.0909 - 0.91970\,X + 0.13199\,X^2, \tag{5.14}$$

$$G_4 = 0.40740 + 1.0314\,X - 1.0941\,X^2. \tag{5.15}$$

The polynomial expressions for the interval $0.46531 \leqslant X \leqslant 1.0$ are:

$$G_1 = 1.5338 - 0.29709\,X + 0.43202\,X^2, \tag{5.16}$$

$$G_2 = -36.184 + 241.04\,X - 636.39\,X^2 + 838.23\,X^3 - 546.94\,X^4 + 141.25\,X^5, \tag{5.17}$$

$$G_3 = 1.4350 - 1.7466\,X + 0.30862\,X^2, \tag{5.18}$$

$$G_4 = 0.31091 + 1.5504\,X - 1.7556\,X^2. \tag{5.19}$$

6. Three-dimensional pressure and velocity curves

Application of Eqs. (3.3), (3.5), (4.2) and (4.4) to this case results in the following transformation equations:

$$G_1 X = \eta_1/\eta_2 = Z/R, \tag{6.1}$$

$$G_1(X) = F_1(\eta_1, \eta_2) = U(R, \theta, Z), \tag{6.2}$$

$$G_2(X) = F_2(\eta_1, \eta_2) = V(R, \theta, Z), \tag{6.3}$$

$$G_3(X) = F_3(\eta_1, \eta_2) = W(R, \theta, Z), \tag{6.4}$$

$$G_4(X) = F_4(\eta_1, \eta_2) = P(R, \theta, Z). \tag{6.5}$$

The three-dimensional equations for pressure and velocity components are obtained by substituting the transformation relations (6.1) through (6.5) into Eqs. (5.12) through (5.19). For the interval $0.1 \leqslant Z/R \leqslant 0.46531$, the polynomial expressions become:

(6.6) $$U = 0.88531+1.1040(Z/R)+0.44657/(Z/R)^2,$$

(6.7) $$V = 1.4366-9.5497(Z/R)+82.897(Z/R)^2-383.95(Z/R)^3+829.38(Z/R)^4-\\ -691.74(Z/R)^5,$$

(6.8) $$W = 1.0909-0,91970(Z/R)+0.13199(Z/R)^2,$$

(6.9) $$P = 0.40740+1.0314(Z/R)-1.0941(Z/R)^2.$$

For the interval $0.46531 \leqslant Z/R \leqslant 1.0$, the polynomial expressions become:

(6.10) $$U = 1.5338-0.29709(Z/R)+0.43202(Z/R)^2,$$

(6.11) $$V = -36.184+241.04(Z/R)-636.39(Z/R)^2+838.23(Z/R)^3-546.94(Z/R)^4\\ +141.25(Z/R)^5,$$

(6.12) $$W = 1.4350-1.7466(Z/R)+0.30862(Z/R)^2,$$

(6.13) $$P = 0.31091+1.5504(Z/R)-1.7556(Z/R)^2.$$

The absence of the θ coordinate in the transformed equations for U, V, W and P signifies that for this case an axially symmetric flow is obtained.

The U, V, W and P curves are plotted as functions of R using Z as a parameter. For Z as a parameter, the plot intervals are: $1.0Z \leqslant R \leqslant 2.1491\,Z$ and $2.1491\,Z \leqslant R \leqslant 10.0\,Z$. Figures 2, 3, 4 and 5 display the curves as plotted by the Calcomp plotter for pressure, radial velocity, tangential velocity, and axial velocity, respectively. For the parametric

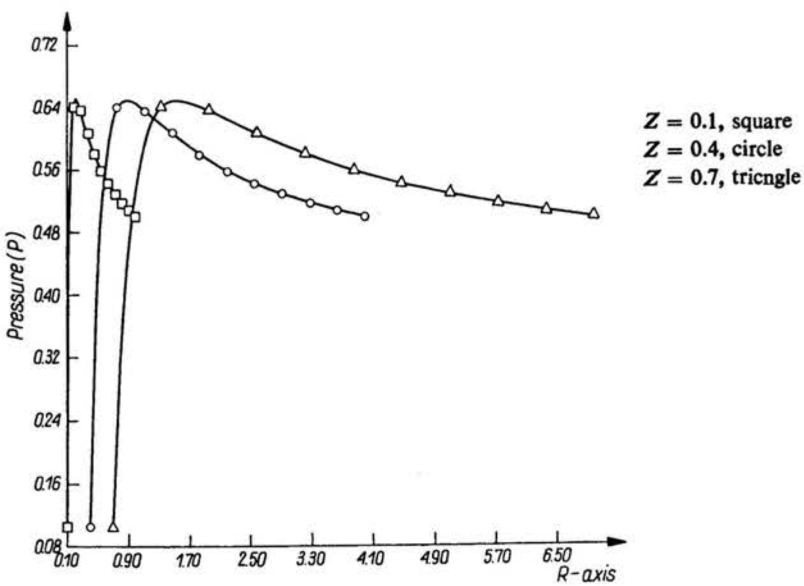

FIG. 2. Pressure curves.

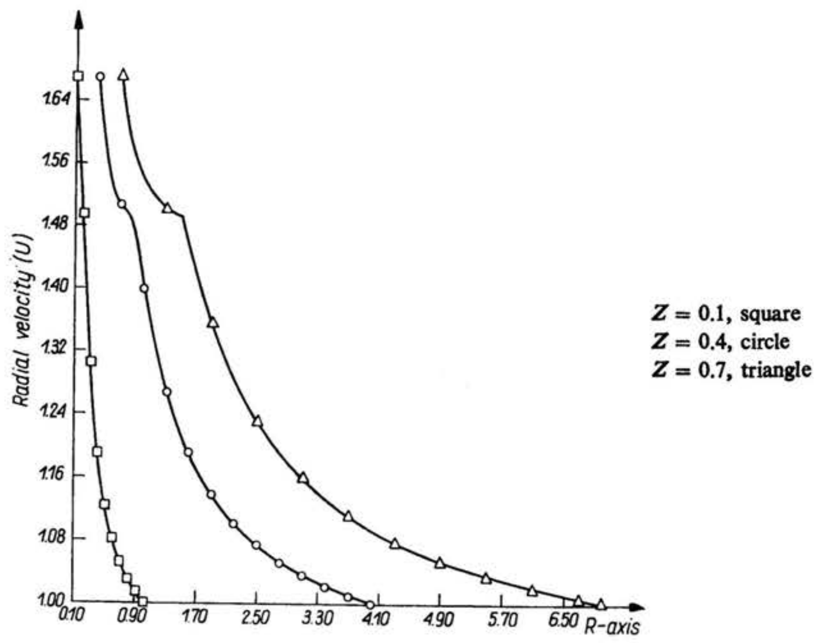

FIG. 3. Radial velocity curves.

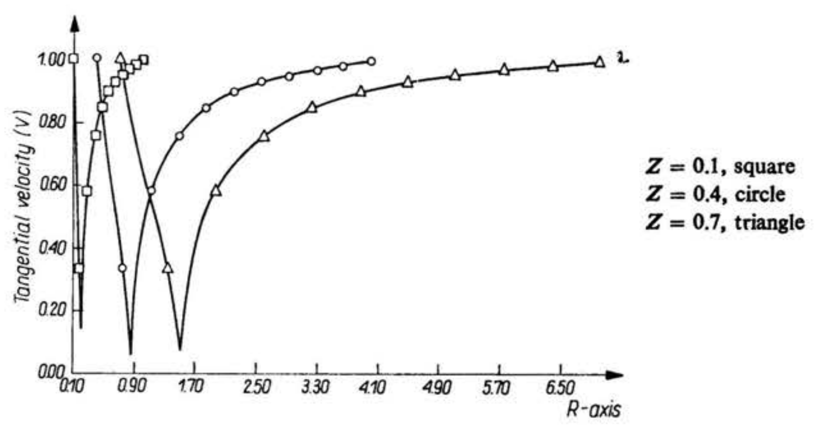

FIG. 4. Tangential velocity curves.

values $Z = 0.1$, $Z = 0.4$ and $Z = 0.7$, the corresponding tangential quasi-stagnation point locations are at $R = 0.21491$, $R = 0.85964$ and $R = 1.5044$. In each set of curves except the set for axial velocity, the location of the tangential quasi-stagnation point is pronounced by an abrupt slope change. Figure 6 displays the real space functions U, V, W and P on a single diagram for the parametric value $Z = 0.7$.

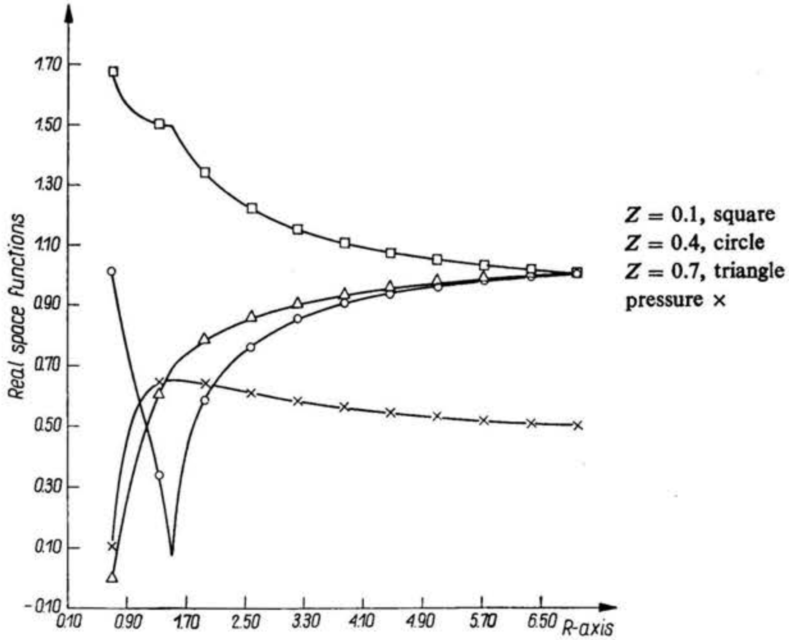

FIG. 5. Real space functions.

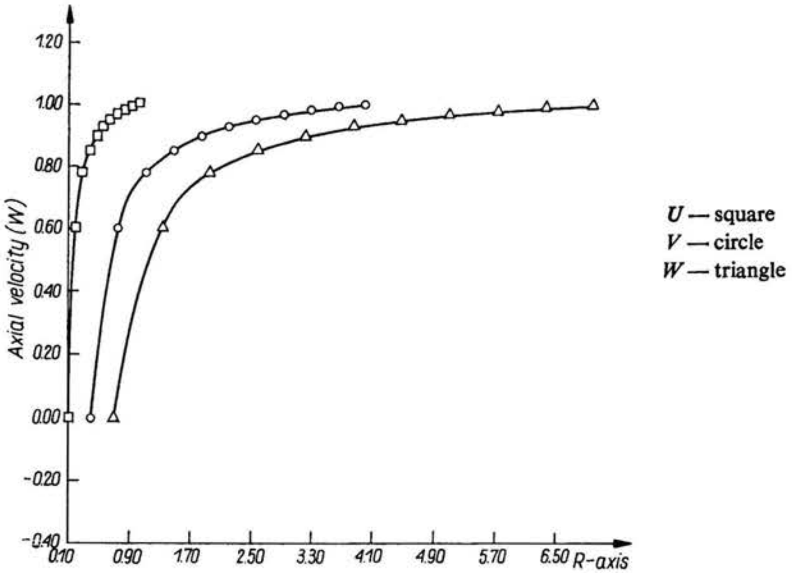

FIG. 6. Axial velocity curves.

7. Optimal solutions in X space

The methematical optimization model employs the one-dimensional representations for the conservation of energy Eq. (4.15) and the swirl parameter Eq. (4.16). For the case $\xi = -1$, these equations become:

$$(7.1) \qquad G_4 = H - \frac{1}{2}(G_2^2 + G^2), \quad \text{where} \quad G^2 \equiv G_1^2 + G_3^2;$$

$$(7.2) \qquad S = G_2^2/(HX^2).$$

The optimization problem is posed to discover that function G that will optimize G_4, subject to the restriction imposed by the swirl parameter S.

Let

$$(7.3) \qquad \phi_1 = G_4 - H + \frac{1}{2}(G_2^2 + G^2) = 0 \quad \text{and} \quad \phi_2 = G_2^2 - SHX^2 = 0.$$

The augmented function for this case is written as:

$$(7.4) \qquad K = \lambda_1\phi_1 + \lambda_2\phi_2 = \lambda_1 G_4 - \lambda_1 H + \frac{\lambda_1}{2}(G_2^2 + G) + \lambda_2 G_2^2 - \lambda_2 SHX^2.$$

This optimization problem is formulated as a classical Mayer type optimization problem.

Following the proposition given in reference [32, p. 33] for solving problems not involving derivatives of the dependent variables, a change of dependent variables is introduced.

Let

$$(7.5) \qquad \alpha' = G_4, \quad \beta' = G_2 \quad \text{and} \quad \gamma' = G.$$

Substituting Eq. (7.5) into Eq. (7.4) results in the following expression for the augmented function:

$$(7.6) \qquad K = \lambda_1\alpha' - \lambda_1 H + \frac{\lambda_1}{2}(\beta'^2 + \gamma'^2) + \lambda_2\beta'^2 - \lambda_2 SHX^2.$$

Similarly, the constraint equations become:

$$(7.7) \qquad \phi_1 = \alpha' - H + \frac{1}{2}(\beta'^2 + \gamma'^2) = 0 \quad \text{and} \quad \phi_2 = \beta'^2 - SHX^2 = 0.$$

A single degree of freedom exists for this problem as there are three dependent variables (α, β, γ) and two constraint Eqs. (7.7). Hence one optimum requirement can be imposed on α. The following end conditions are specified for this problem: $X_i, X_f, \alpha_f, \gamma_i$, and α_f. Now, the optimization problem is specifically formulated as follows: In the class of functions $\alpha(X)$, $\beta(X)$ and $\gamma(X)$, which are consistent with the constraint Eqs. (7.7) and the specified end conditions, find that special set which minimizes the difference $\Delta\alpha = \alpha_f - \alpha_i$. Note that for α_f specified, minimizing the difference $\Delta\alpha$ corresponds to maximizing α_i.

There is one Euler-Lagrange equation for each dependent variable [1]

$$(7.8) \qquad \frac{d}{dx}\left(\frac{\partial K}{\partial y_j'}\right) - \frac{\partial K}{\partial y_j} = 0, \quad j = 1, 2, 3.$$

Application of Eqs. (7.8) to the augmented function K Eq. (7.6) leads to the following results:

$$\lambda_1' = 0 \quad \text{or} \quad \lambda_1 = \text{constant}, \tag{7.9}$$

$$(\lambda_1+2\lambda_2)\beta''+(\lambda_1'+2\lambda_2')\beta' = 0, \tag{7.10}$$

$$\lambda_1\gamma''+\lambda_1'\gamma' = 0. \tag{7.11}$$

Substitution of Eq. (7.9) into Eq. (7.11) demonstrates that,

$$\gamma' = L/\lambda_1 \tag{7.12}$$

where L is an integration constant. Since λ_1 is a constant, an obvious solution to Eq. (7.10) is obtained by choosing λ_2 to be a constant. Let

$$\lambda_2 = \text{constant} = -\lambda_1/2. \tag{7.13}$$

The value of $\lambda_1 = -1$ is determined by application of the transversality condition [Eq. (7.14)] from the calculus of variations,

$$d\alpha+\left(K-\sum_{j=1}^{n}\frac{\partial K}{\partial y_j'}y_j'\right)dX+\sum_{j=1}^{n}\frac{\partial K}{\partial y_j'}dy_j\Bigg]_i^f = 0. \tag{7.14}$$

It is easily demonstrated that the Legendre-Clebsch necessary condition [Eq. (7.15)] is negative for this case which means that the optimum obtained for this case is a maximum,

$$\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 K}{\partial y_k'\partial y_j'}\,\delta y_k'\delta y_j' < 0. \tag{7.15}$$

The optimum expression for G_4 is obtained by substituting Eqs. (7.5), (7.7), (7.12) and $\lambda_1 = -1$ into the first constraint Eq. (7.7),

$$G_4 = H-\frac{1}{2}(SHX^2+L^2). \tag{7.16}$$

Equation (7.16) represents the optimum G_4 when the sum $G^2 = G_1^2+G_3^2+ = L^2 =$ a constant and G_2 is governed by the swirl parameter S as given by Eq. (7.2). G_4 attains its maximum value at $X = X_i$.

Since G_4 is the one-dimensional representation for pressure, it is restricted to positive values. This restriction on G_4 restricts the swirl parameter as shown:

$$S < 2-L^2/H. \tag{7.17}$$

From Table 1, the values $H = 2.0000$ and $L^2 = 2.6951$ are selected, since they occur at the point, where the function G_4 is a maximum as determined by the solution of the system of ordinary differential equations in Sec. 5. For the selected values of H and L^2, Eq. (7.17) requires that $S < 0.6525$.

Computing the maximum G_4 at $X = X_i = 0.1$ using Eq. (7.16) with the previously mentioned values for H and L^2, the computed maximum G_4 for $S = 0.25$ is found to be within 0.02% of the maximum value given in Table 1. For $S = 0.65$, the computed maximum value of G_4 is found to be within 0.63% of the maximum value given in Table 1. Therefore, the maximum G_4 value determined by Eq. (7.16) compares favorably with the maximum G_4 value determined by the solution of the system of ordinary differential equations in Sec. 5.

Figures 7 and 8 illustrate the G_2 and G_4 curves generated from Eqs. (7.2) and (7.16). The maximum value of G_4 occurs at the left end of the X interval, where G_2 is at a minimum.

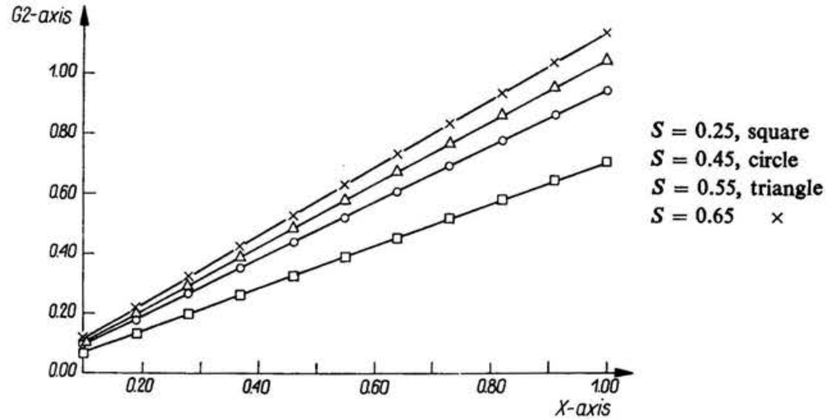

FIG. 7. G_2 curves, S parameter.

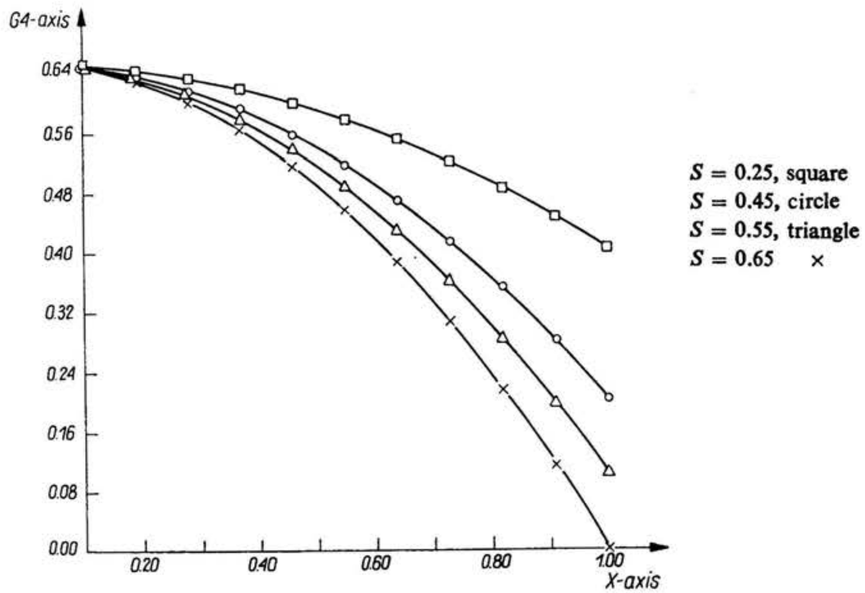

FIG. 8. G_4 curves, S parameter.

8. Three-dimensional optimal pressure and velocity curves

The three-dimensional equations for optimum pressure and tangential velocity are obtained by substituting the transformation relations (6.1), (6.3) and (6.5) into Eqs. (7.2) and (7.16):

$$P = H-\frac{1}{2}\left(SH(Z/R)^2+L^2\right), \tag{8.1}$$

(8.2) $$V = [SH]^{1/2} Z/R.$$

The optimum P and V curves are plotted as functions of R using Z as a parameter and $S = 0.65$. For Z as a parameter, the plot interval becomes: $Z \leqslant R \leqslant 10Z$. The maximum value of P occurs at the right end of the R interval, where V is a minimum. This same result is obtained in Sec. 6 for the transformed equations of the solutions to the system of ordinary differential equations. Figures 9 and 10 display the curves for optimum pressure and tangential velocity, respectively. For the parameteric values $Z = 0.1$, $Z = 0.4$ and $Z = 0.7$, the corresponding maximum pressure locations are $R = 1$, $R = 4$, and $R = 7$.

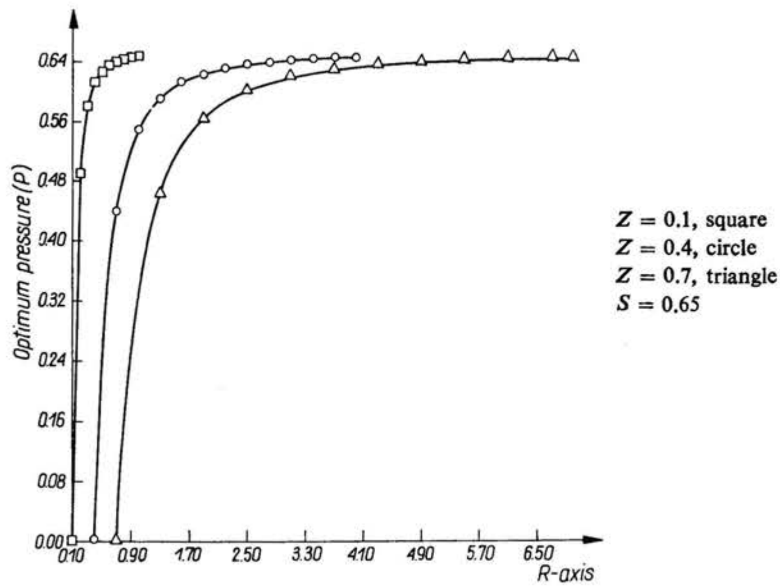

FIG. 9. Optimum pressure curves.

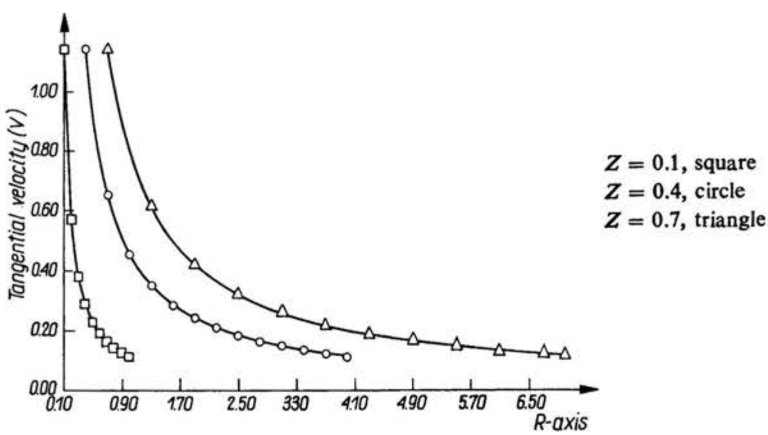

FIG. 10. Velocity curves for optimum pressure.

Table 1 presents the numerical data for the solutions to the system of non-linear ordinary differential equations given in Sec. 5. In addition to values for G_1, G_2, G_3 and G_4, values for H, S, L^2 and π_2 are listed. The values of these functions at each value of X are displayed in the order shown in the following 2×5 matrix:

$$X \text{ value}\begin{bmatrix} G_1 & G_2 & G_3 & G_4 & H \\ S & L^2 & \pi_2 & - & - \end{bmatrix}$$

Table 1. Numerical data for solutions to the system of ordinary differential equations

No.	X	Y(1)	Y(2)	Y(3)	Y(4)	Y(5)
1	0.10000E 00	0.10000E 01	0.10000E 01	0.10000E 01	0.50000E 00	0.20000E 01
		0.50000E 02	0.20000E 01	0.50000E 00		
2	0.10500E 00	0.10060E 01	0.99441E 00	0.99560E 00	0.50394E 00	0.20000E 01
		0.44846E 02	0.20033E 01	0.49443E 00		
3	0.11000E 00	0.10120E 01	0.98876E 00	0.99120E 00	0.50783E 00	0.20000E 01
		0.40399E 02	0.20067E 01	0.48882E 00		
4	0.11500E 00	0.10181E 01	0.98304E 00	0.98681E 00	0.51167E 00	0.20000E 01
		0.36536E 02	0.20103E 01	0.48318E 00		
5	0.12000E 00	0.10242E 01	0.97725E 00	0.98242E 00	0.51547E 00	0.20000E 01
		0.33160E 02	0.20140E 01	0.47751E 00		
6	0.12500E 00	0.10302E 01	0.97139E 00	0.97803E 00	0.51922E 00	0.20000E 01
		0.30195E 02	0.20179E 01	0.47180E 00		
7	0.13000E 00	0.10364E 01	0.96547E 00	0.97364E 00	0.52292E 00	0.20000E 01
		0.27578E 02	0.20220E 01	0.46606E 00		
8	0.14000E 00	0.10487E 01	0.95339E 00	0.96487E 00	0.53019E 00	0.20000E 01
		0.23188E 02	0.20307E 01	0.45448E 00		
9	0.15000E 00	0.10610E 01	0.94102E 00	0.95612E 00	0.53725E 00	0.20000E 01
		0.19678E 02	0.20400E 01	0.44276E 00		
10	0.16000E 00	0.10735E 01	0.92833E 00	0.94738E 00	0.54413E 00	0.20000E 01
		0.16832E 02	0.20499E 01	0.43090E 00		
11	0.17000E 00	0.10861E 01	0.91532E 00	0.93865E 00	0.55080E 00	0.20000E 01
		0.14495E 02	0.20606E 01	0.41890E 00		

Table 1. (Cont'D.)

No.	X	Y(1)	Y(2)	Y(3)	Y(4)	Y(5)
12	0.19000E 00	0.11114E 01	0.88824E 00	0.92124E 00	0.56355E 00	0.20000E 01
		0.10928E 02	0.20839E 01	0.39449E 00		
13	0.21000E 00	0.11371E 01	0.85965E 00	0.90390E 00	0.57548E 00	0.20000E 01
		0.83788E 01	0.21100E 01	0.36951E 00		
14	0.23000E 00	0.11031E 01	0.82939E 00	0.88662E 00	0.58657E 00	0.20000E 01
		0.65019E 01	0.21389E 01	0.34395E 00		
15	0.25000E 00	0.11895E 01	0.79725E 00	0.86942E 00	0.59682E 00	0.20000E 01
		0.50849E 01	0.21707E 01	0.31781E 00		
16	0.29000E 00	0.12431E 01	0.72629E 00	0.83527E 00	0.61471E 00	0.20000E 01
		0.31362E 01	0.22431E 01	0.26375E 00		
17	0.33000E 00	0.12981E 01	0.64385E 00	0.80148E 00	0.62905E 00	0.20000E 01
		0.19033E 01	0.23273E 01	0.20727E 00		
18	0.37000E 00	0.13542E 01	0.54460E 00	0.76809E 00	0.63975E 00	0.20000E 01
		0.10832E 01	0.24238E 01	0.14830E 00		
19	0.41000E 00	0.14116E 01	0.41622E 00	0.73517E 00	0.64674E 00	0.19999E 01
		0.51533E 00	0.25331E 01	0.86626E-01		
20	0.45000E 00	0.14725E 01	0.19415E 00	0.70384E 00	0.64992E 00	0.20007E 01
		0.93040E-01	0.26638E 01	0.18841E-01		
21	0.45500E 00	0.14824E 01	0.14255E 00	0.70096E 00	0.65003E 00	0.20046E 01
		0.48961E-01	0.26888E 01	0.10136E-01		
22	0.46000E 00	0.14885E 01	0.93135E-01	0.69633E 00	0.65012E 00	0.20047E 01
		0.20448E-01	0.27006E 01	0.43268E-02		
23	0.46500E 00	0.14934E 01	0.53592E-01	0.69115E 00	0.65014E 00	0.20056E 01
		0.66230E-02	0.27080E 01	0.14321E-02		
24	0.46531E 00	0.14900E 01	0.47948E-01	0.68910E 00	0.65013E 00	0.19988E 01
		0.53122E-02	0.26951E 01	0.11502E-02		
25	0.46562E 00	0.14900E 01	0.50620E-01	0.68864E 00	0.65013E 00	0.19986E 01
		0.59134E-02	0.26944E 01	0.12821E-02		

Table 1. (cont'D.)

26	0.46594E 00	0.14899E 01	0.52570E-01	0.68811E 00	0.65012E 00	0.19982E 01
		0.63709E-02	0.26933 E01	0.13831E-02		
27	0.46625E 00	0.14901E 01	0.57925E-01	0.68776E 00	0.65012E 00	0.19986E 01
		0.77227E-02	0.26935E 01	0.16788E-02		
28	0.46656E 00	0.14902E 01	0.61595E-01	0.68732E 00	0.65012E 00	0.19986E 01
		0.87206E-02	0.26931E 01	0.18983E-02		
29	0.46687E 00	0.14903E 01	0.65260E-01	0.68690E 00	0.65011E 00	0.19986E 01
		0.97758E-02	0.26928E 01	0.21308E-02		
30	0.46750E 00	0.14900E 01	0.77124E-01	0.68582E 00	0.65010E 00	0.19983E 01
		0.13619E-01	0.26904E 01	0.29766E-02		
31	0.46812E 00	0.14901E 01	0.83619E-01	0.68492E 00	0.65009E 00	0.19983E 01
		0.15967E-01	0.26894E 01	0.34991E-02		
32	0.46875E 00	0.14901E 01	0.89673E-01	0.68402E 00	0.65008E 00	0.19983E 01
		0.18314E-01	0.26884E 01	0.40240E-02		
33	0.46937E 00	0.14902E 01	0.94929E-01	0.68314E 00	0.65006E 00	0.19983E 01
		0.20469E-01	0.26875E 01	0.45096E-02		
34	0.47062E 00	0.14905E 01	0.10404E 00	0.68140E 00	0.65003E 00	0.19984E 01
		0.24455E-01	0.26859E 01	0.54165E-02		
35	0.47187E 00	0.14907E 01	0.11286E 00	0.67963E 00	0.64999E 00	0.19984E 01
		0.28624E-01	0.26841E 01	0.63736E-02		
36	0.47312E 00	0.14909E 01	0.12145E 00	0.67784E 00	0.64995E 00	0.19984E 01
		0.32976E-01	0.26821E 01	0.73816E-02		
37	0.47437E 00	0.14910E 01	0.12948E 00	0.67605E 00	0.64990E 00	0.19984E 01
		0.37282E-01	0.26802E 01	0.83895E-02		
38	0.47687E 00	0.14916E 01	0.14251E 00	0.67258E 00	0.64977E 00	0.19985E 01
		0.44686E-01	0.26771E 01	0.10162E-01		
39	0.47937E 00	0.14919E 01	0.15575E 00	0.66903E 00	0.64962E 00	0.19985E 01
		0.52823E-01	0.26735E 01	0.12139E-01		
40	0.48187E 00	0.14923E 01	0.16780E 00	0.66549E 00	0.64945E 00	0.19985E 01
		0.60678E-01	0.26699E 01	0.14090E-01		

Table 1. (cont'D.)

No.	X	Y(1)	Y(2)	Y(3)	Y(4)	Y(5)
41	0.48437E 00	0.14927E 01 0.68489E-01	0.17920E 00 0.26664E 01	0.66195E 00 0.16069E-01	0.64924E 00	0.19985E 01
42	0.48937E 00	0.14933E 01 0.84357E-01	0.20092E 00 0.26586E 01	0.65476E 00 0.20202E-01	0.64875E 00	0.19982E 01
43	0.49437E 00	0.14941E 01 0.99020E-01	0.21991E 00 0.26518E 01	0.64769E 00 0.24201E-01	0.64816E 00	0.19982E 01
44	0.49937E 00	0.14949E 01 0.11299E 00	0.23728E 00 0.26452E 01	0.64064E 00 0.28176E-01	0.64747E 00	0.19982E 01
45	0.50437E 00	0.14958E 01 0.12634E 00	0.25342E 00 0.26389E 01	0.63361E 00 0.32140E-01	0.64667E 00	0.19982E 01
46	0.51437E 00	0.14976E 01 0.15183E 00	0.28332E 00 0.26266E 01	0.61953E 00 0.40173E-01	0.64476E 00	0.19982E 01
47	0.52437E 00	0.14995E 01 0.17506E 00	0.31013E 00 0.26153E 01	0.60556E 00 0.48136E-01	0.64244E 00	0.19982E 01
48	0.53437E 00	0.15016E 01 0.19659E 00	0.33492E 00 0.26047E 01	0.59163E 00 0.56138E-01	0.63971E 00	0.19982E 01
49	0.54437E 00	0.15037E 01 0.21643E 00	0.35799E 00 0.25950E 01	0.57777E 00 0.64136E-01	0.63658E 00	0.19982E 01
50	0.56437E 00	0.15085E 01 0.25112E 00	0.39979E 00 0.25784E 01	0.55028E 00 0.79987E-01	0.62913E 00	0.19982E 01
51	0.58437E 00	0.15136E 01 0.28126E 00	0.43809E 00 0.25643E 01	0.52296E 00 0.96049E-01	0.62008E 00	0.19982E 01
52	0.60437E 00	0.15190E 01 0.30706E 00	0.47341E 00 0.25533E 01	0.49589E 00 0.11216E 00	0.60948E 00	0.19982E 01
53	0.62437E 00	0.15249E 01 0.32923E 00	0.50643E 00 0.25452E 01	0.46905E 00 0.12835E 00	0.59733E 00	0.19982E 01

Table 1. (Cont'D.)

54	0.66437E 00	0.15376E 01	0.56710E 00	0.41602E 00	0.56848E 00	0.19980E 01
		0.36467E 00	0.25373E 01	0.16097E 00		
55	0.70437E 00	0.15519E 01	0.62235E 00	0.36401E 00	0.53379E 00	0.19979E 01
		0.39075E 00	0.25409E 01	0.19387E 00		
56	0.74437E 00	0.15676E 01	0.67350E 00	0.31300E 00	0.49343E 00	0.19979E 01
		0.40975E 00	0.25553E 01	0.22704E 00		
57	0.78437E 00	0.15846E 01	0.72151E 00	0.26296E 00	0.44756E 00	0.19979E 01
		0.42351E 00	0.25801E 01	0.26056E 00		
58	0.86437E 00	0.16220E 01	0.81063E 00	0.16556E 00	0.33974E 00	0.19974E 01
		0.44032E 00	0.26582E 01	0.32899E 00		
59	0.94437E 00	0.16639E 01	0.89268E 00	0.72077E-01	0.21199E 00	0.19973E 01
		0.44737E 00	0.27737E 01	0.39898E 00		
60	0.95133E 00	0.16680E 01	0.89992E 00	0.63766E-01	0.20016E 00	0.19982E 01
		0.44782E 00	0.27863E 01	0.40529E 00		
61	0.95828E 00	0.16720E 01	0.90697E 00	0.55612E-01	0.18774E 00	0.19983E 01
		0.44826E 00	0.27986E 01	0.41164E 00		
62	0.96523E 00	0.16755E 01	0.91292E 00	0.48681E-01	0.17672E 00	0.19982E 01
		0.44767E 00	0.28095E 01	0.41708E 00		
63	0.97219E 00	0.16793E 01	0.91964E 00	0.40860E-01	0.16425E 00	0.19981E 01
		0.44784E 00	0.28219E 01	0.42328E 00		
64	0.98609E 00	0.16868E 01	0.93182E 00	0.26939E-01	0.14256E 00	0.19995E 01
		0.44640E 00	0.28459E 01	0.43407E 00		
65	0.10000E 01	0.16947E 01	0.94480E 00	0.11529E-01	0.11680E 00	0.19992E 01
		0.44649E 00	0.28723E 01	0.44649E 00		
66	0.10000E 01	0.16950E 01	0.94522E 00	0.11034E-01	0.11599E 00	0.19993E 01
		0.44689E 00	0.28731E 01	0.44689E 00		
67	0.10000E 01	0.16951E 01	0.94534E 00	0.10897E-01	0.11576E 00	0.19993E 01
		0.44700E 00	0.28734E 01	0.44700E 00		

Appendix

The idea of the association of ordinary and partial differential equations with the group theory and algebra is not a new one and was originated a long time ago. The present century has demonstrated some concrete approaches in this direction. Thus L. E. DICKSON (1924) [A. 3] showed how some differential equations could be integrated with the aid of group theory. G. BIRKHOFF [A. 1] in 1949 suggested that the reduction of independent variables in systems of partial differential equations could be attacked by algebraic methods. Perhaps the most significant achievement obtained by the use of this technique was the first solution of the boundary layer equations proposed by L. PRANDTL (1904) and used by BLASIUS (1908) [A. 2].Recently, some continuations to this technique were done by MICHAL [A. 5], MORGAN [A. 6], KRZYWOBLOCKI [4. A] and others. In the eastern part of the world, significant results in that respect were obtained by the Soviet and Polish schools of mathematics. The method of reduction of the number of dimensions by a transformation from one space into another does not appear amenable to the standard methods of the theory of functions, since the Jacobian determinant associated with such a transformation vanishes, whereas the algebraic methods do not break down under these conditions. The main problem in question is the following one: "Given a differential system in the space A of n dimensions, transform this system into the space B of $(n-r)$ dimensions $(1 \leqslant r)$ such that a solution of the system in B determines a solution of it in A. The transformations involved in these operations are not to be one-to-one". The resulting theorems involve the use of the theory of Banach spaces and some elements of the theory of continuous transformation groups.

Consider an elementary example. Given a Laplace equation:

(A.1) $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad u=(x, y), \quad L[u]=0,$$

and a group of transformations:

(A.2) $$\bar{x}=f_1(x, y; a), \quad \bar{y}=f_2(x, y; a), \quad \bar{u}=f_3(u; a),$$

a being a parameter, one then obtains

(A.3) $$\frac{\partial \bar{u}}{\partial \bar{y}}=F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}; a\right)\frac{\partial u}{\partial y}.$$

A continuation of this procedure and the introduction of new variables

(A.4) $$\eta=x^2+y^2, \quad u=v(\eta),$$

leads to the form:

(A.5) $$L[u]=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=4\frac{dv}{d\eta}+4\eta\frac{d^2 v}{d\eta^2}=0,$$

and

(A.6) $$v=C_1\ln\eta+C_2, \quad u=C_1\ln(x^2+y^2)+C_2.$$

The group of transformations chosen (or sometimes guessed by the intuition) should be selected so that a solution satisfies the boundary conditions proposed in the original physical space and has the appropriate physical meaning. Technically, this may imply that some correspondence with test experiments should confirm the analytical results. This item may be considered a disadvantage of this technique which is otherwise very interesting and often very powerful. Note that the famous Prandtl-Blasius [A.2] transformation involved the function $\eta \approx yx^{-1/2}$.

References

1. Naum I. AKHIEZER, *The calculus of variations*, Blaisdell Publishing Co., New York 1962.
2. P. S. ALEXANDROFF, *Einführung in die Gruppentheorie*, Deutscher Verlag der Wissenschaften, Berlin 1954.
3. G. K. BATCHELOR, *On steady laminar flow with closed streamlines at large Reynolds number*, J. Fluid Mech., **12**, Part 2, 177–190, 1956.
4. T. B. BENJAMIN, *Significance of the vortex breakdown phenomenon*, Transactions of the ASME, 518–524, June 1965.
5. R. T. BICHARA and P. A. ORNER, *Analysis and modelling of the vortex amplifier*, ASME Publication, 69-FLCS-20, 1969.
6. Garrett BIRKHOFF, *Hydrodynamics*, Second Edition, Princeton University Press, Princeton 1960.
7. J. M. BOOKSTON and D. D. MCKEACHIE, *Notes on the numerical solution of differential equations*, General Motors Corporation Research Laboratories Report GMR 161, July 1958.
8. Brice Luther H. A. CARNAHAN and James O. WILKES, Applied numerical methods, John Wiley and Sons, Inc., New York 1969.
9. CHUEN-YEN CHOW, *Swirling flow in tubes of non-uniform cross-sections*, J. Fluid Mech., **38**, Part 4, 843–854, October 1969.
10. Robert G. DEISSLER and Morris PERLMUTTER, *An analysis of the energy separation in laminar and turbulent compressible vortex flows*, Proc. Heat Transfer and Fluid Mech. Inst., 1958.
11. L. E. DICKSON, *Differential equations from the group standpoint*, Annals of Mathematics, Second Series, **25**, 4, 287–378, June 1924.
12. P. Donaldson COLEMAN and Roger D. SULLIVAN, *Behaviour of solutions of the Navier-Stokes equations for a complete class of three-dimensional viscous vortices*, 1960 Proc. Heat Transfer and Fluid Mech. Inst. Stanford University Press, pp. 16–30.
13. J. L. DUSSOURD and G. P. SCHIVLEY, *An analytical and experimental study of a vortex pump*, ASME Publication, 70-FE-32, 1970.
14. E. R. G. ECKERT and J. P. HARTNETT, *Experimental study of the velocity and temperature distribution in a high velocity vortex type flow*, University of Minnesota Heat Transfer Laboratory Technical Report No. 6, September 1955.
15. Hans Albert EINSTEIN and HUON LI, *Steady vortex flow in a real fluid*, Heat Transfer and Fluid Mech. Proc., Stanford University Press, 1951.
16. I. S. GARTSHORE, *Recent work in swirling incompressible flow*, Nat. Research Council of Canada Aeronautical Report LR-343, June 1962.
17. Donald GREENSPAN, *Numerical studies of prototype cavity flow problems*, The Computer J., **12**, 1, 89–94, February 1969.
18. Donald GREENSPAN, *Numerical studies of viscous, incompressible flow for arbitrary Reynold number*, University of Wisconsin Computer Sciences Technical Report No. 11, January 1968.
19. Donald GREENSPAN, *Numerical studies of two-dimensional, steady state Navier-Stokes equations for arbitrary Reynolds number*, University of Wisconsin Computer Sciences Technical Report No. 9, October 1967.
20. H. P. GREENSPAN, *The theory of rotating fluids*, Cambridge University Press, London 1968.
21. G. G. HALL, *Applied group theory*, American Elsevier Publishing Co., New York 1967.
22. R. HILSCH, *The use of the expansion of gases in a centrifugal field as cooling process*, The Review of Scientific Instruments, **18**, 2, 108–113, February 1947.

23. M. KAWAGUTI, *Numerical solutions of the Navier-Stokes equations for the flow in a two-dimensional cavity*, J. Phys. Soc. of Japan, **16**, 2307–2318, April 1961.
24. R. C. KOLF, *Vortex flow from horizontal thin-plate orifices*, Ph. D. Thesis, University of Wisconsin, Madison, Wisconsin 1956.
25. C. C. K. KWOK, *A theoretical study of vortex flow in a thin cylindrical chamber*, Bendix Technical J., **1**, 4, 1969.
26. Hans J. LUGT and Ernst W. SCHWIDERSKI, *Birth and decay of vortices*, Physics of Fluids, **9**, 5, 851–859, May 1966.
27. Leslie M. MACK, *The compressible, viscous, heat-conducting vortex*, Cal. Tech. Jet Propulsion Laboratory Progress Report No. 20–382, 1959.
28. Paul MAKER and Endre A. MAYER, *Control characteristics of vortex valves*, Proc. of the Fluid Amplification Symposium, **2**, U.S. Army Material Command, 61–83, May 1964.
29. Endre A. MAYER, *Large-signal vortex valve analysis*, Advances in Fluidics, Proc. of the 1967 ASME Fluidics Symposium, ASME Publication, 233–249, May 1967.
30. G. L. MELLOR, P. J. CHAPPLE and V. K. STOKES, *On the flow between a rotating and stationary disk*, J. Fluid Mech., **31**, Part 1, 95–112, January 1968.
31. A. D. MICHAL, *Differential invariants and invariant partial differential equations in normed linear spaces*, Proc. of the National Academy of Sciences, **37**, 9, 623–627, September 1952.
32. Angelo MIELE, *Theory of optimum aerodynamic shapes*, Academic Press, New York 1965.
33. A. J. A. MORGAN, *The reduction by one of the number of independent variables in some systems of partial differential equations*, Quart. J. of Math., Oxford Second Series, **3**, 12, 250–259, Dec. 1952.
34. Carl E. PEARSON, *A computational method for viscous flow problems*, J. Fluid Mech., **21**, Part 4, 611–622, 1965.
35. Desmond C. PENGELLEY, *Flow in a viscous vortex*, J. Appl. Phys., **28**, 1, 86–92, January 1957.
36. Kent E. SPRINGER and F. M. PATTERSON, *Experimental investigation of critical submergence for vortexing in a vertical cylindrical tank*, ASME Publication, 69-FE-49, 1969.
37. T. P. TORDA and Urmila AGARWAL, *Numerical investigation of unsteady laminar incompressible boundary layer flows with cylindrical symmetry*, ASME Publication, 69-FE-47, 1969.
38. Herbert S. WILF and Anthony RALSTON, *Mathematical methods for digital computers*, John Wiley and Sons, Inc., New York 1960.

List of references to Appendix

A.1. G. BIRKHOFF, *Hydrodynamics*, 1st Edition, Princeton University Press, 1950.
A.2. H. BLASIUS, *Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Z. Math. u. Phys., 56, 1 1908.
A.3. L. E. DICKSON, *Differential equations from the group standpoint*, Annals of Mathematics, Second Series, **25**, 4, 287–378, June 1924.
A.4. M. Z. KRZYWOBLOCKI, *On the generalized theory of the laminar, two-dimensional boundary layer along a flat plate in continuum and slip flow regimes*, Part II, On the Fundamental Existence Theorems for Partial Differential Systems of the First Order, Bulletin de la Société Math. de Greece, **29**, Parts 1, 2, 3, 43–73, April 1954.
A.5. A. D. MICHAL, *Differential invariants and invariant partial differential equations in normal linear spaces*, Proc. Nat. Acad. Sci., **37**, 9, 623–627, September 1952.
A.6. A. J . A. MORGAN, *The reduction by one of the number of independent variables in some systems of partial differential equations*, Quart. J. Math., Oxford Second Series, **3**, 12, 250–259, December 1952.

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