Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 2, pp. 259–268, Warszawa 1973

On the surface behaviour of gradient-sensitive liquids

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A NON-LINEAR theory of the gradient-sensitive elastic liquids is proposed. Expressions for the constitutive equation and the equations of motion are derived from the concept of gradient-of-density dependent elastic potential. It is shown that the theory is able to supply a proper solution of the problem of pressure difference across a thin spherical liquid shell, as well as the classical theory of capillarity.

Zaproponowano nieliniową teorię sprężystych cieczy gradientowych. Bazując na koncepcji zależnego od gradientów gęstości potencjału otrzymano równanie konstytutywne i równania ruchu. Wykazano, że teoriaèjest w stanie dać prawidłową i zgodną z klasyczną teorią napięcia powierzchniowego odpowiedź na pytanie o różnicy ciśnień po dwu stronach cienkiej sferycznej płynnej powłoki.

Предложена нелинейная теория градиентно-чувствительной упругой жидкости. На основе понятия зависимого от градиентов плотности упругого потенциала выведены уравнения движения и уравнение состояния. Показано, что теория способна обеспечить правильный и согласующийся с классической теорией поверхностного натяжения ответ на вопрос о разности давлений по две стороны тонкой жидкой оболочки.

1. Introduction

DURING the recent decade, many attempts have been made to generalize the concepts of surface tension. In most of the cases considered, surfaces or surface layers were regarded as two-dimensional media or thin strata obeying constitutive relations independent of the bulk properties of the media. These constitutive relations were postulated or deduced using concepts other than those of continuum mechanics.

To this problem, different approach is also possible. A phenomenological theory of surface tension based on the concepts of the bulk gradient-sensitive properties of liquids was proposed by KORTEWEG [1] as early as 1901 (see also [2]). In recent years, these ideas have been developed by HART [3, 4, 5] and MINDLIN [6]. According to Mindlin's linear approach, in the case of liquids an elastic potential can be expressed as a quadratic function of the dilatation and its first and second spatial gradients. Applying Mindlin's theory to the free-surface problem yields results showing the existence of the dilatation gradient in the vicinity of the free surface. This behaviour seems to be in agreement with that observed in experiments.

On the other hand, using Mindlin's equations to analyze the problem of a Mindlin liquid enclosing a gas bubble and in turn enclosed by a reservoir of gas, it can easily be seen that the difference of pressures between the individual regions of gas is zero. This result stands in direct contradiction to both the classical theory of capillarity and the physical reality.

In the present paper, we shall attempt to show that a proper solution of the problem can be obtained within the framework of the non-linear theory using the concept of elastic potential.

Note that Korteweg's theory gives the proper solution of the problem under consideration, but his constitutive equations were not derived from energetic considerations but were rather assumed ad hoc. It is also not clear whether or not they are compatible with the existence of the elastic potential.

2. Kinematics

Let us consider any scalar density field w,

$$w = w(\varrho, \nabla\varrho, \nabla\nabla\varrho), \tag{2.1}$$

where ϱ denotes the mass density and ∇ is the spatial gradient operator. If V is any material region, then

$$\overline{\int_V w\varrho\, dV}^{\boldsymbol{\cdot}} = \int_V \dot{w}\varrho\, dV = \int_V \left(\frac{\partial w}{\partial \varrho}\dot{\varrho} + \frac{\partial w}{\partial(\nabla\varrho)}\cdot\overline{\nabla\varrho}^{\boldsymbol{\cdot}} + \frac{\partial w}{\partial(\nabla\nabla\varrho)} : \overline{\nabla\nabla\varrho}^{\boldsymbol{\cdot}}\right) dV, \tag{2.2}$$

where a dot over a symbol denotes the material time derivative, and dots between the symbols denote scalar multiplication.

We denote now by X^K $(K = 1, 2, 3)$ the Lagrangian coordinates, and by $\mathbf{G}^K$ — their reciprocal base-vectors. We have for a tensor field $\mathbf{R}$ of any rank

$$\operatorname{Grad}\mathbf{R} \equiv \mathbf{G}^K \frac{\partial \mathbf{R}}{\partial X^K} \tag{2.3}$$

and for the deformation gradient $\mathbf{F}$

$$\mathbf{F} = \mathbf{G}^K \mathbf{g}_k \frac{\partial x^k}{\partial X^K}, \tag{2.4}$$

where $x^k, \mathbf{g}^k$ $(k = i, 2, 3)$ are correspondingly the spatial coordinates and their base vectors. Recalling the well-known identities

$$\dot{\mathbf{F}} = \mathbf{F}\cdot(\nabla\mathbf{v}), \quad \dot{\varrho} + \varrho\nabla\cdot\mathbf{v} = 0,$$

$$\overline{\operatorname{Grad}\varrho}^{\boldsymbol{\cdot}} = \overline{G^K \frac{\partial\varrho}{\partial X^K}}^{\boldsymbol{\cdot}} = \mathbf{G}^K \frac{\partial\dot{\varrho}}{\partial X^K} = \operatorname{Grad}\dot{\varrho}, \tag{2.5}$$

$$\nabla\varrho = \mathbf{F}^{-1}\cdot\operatorname{Grad}\varrho,$$

where $\mathbf{v}$ denotes the velocity vector field, we can easily obtain:

$$\begin{aligned}\overline{\nabla\varrho}^{\boldsymbol{\cdot}} &= \overline{\mathbf{F}^{-1}}^{\boldsymbol{\cdot}}\cdot\operatorname{Grad}\varrho + \mathbf{F}^{-1}\cdot\operatorname{Grad}\dot{\varrho} = \overline{\mathbf{F}^{-1}}^{\boldsymbol{\cdot}}\cdot\mathbf{F}\cdot\nabla\varrho + \mathbf{F}^{-1}\cdot\mathbf{F}\cdot\nabla\dot{\varrho} \\ &= -\mathbf{F}^{-1}\cdot\dot{\mathbf{F}}\cdot\nabla\varrho + \nabla\dot{\varrho} = -\mathbf{F}^{-1}\cdot\mathbf{F}\cdot\nabla\mathbf{v}\cdot\nabla\varrho - \nabla(\varrho\nabla\cdot\mathbf{v}) \\ &= -\nabla\mathbf{v}\cdot\nabla\varrho - \nabla\varrho(\nabla\cdot\mathbf{v}) - \varrho\nabla(\nabla\cdot\mathbf{v})\end{aligned} \tag{2.6}$$

and

$$\overline{\nabla\nabla\varrho}^{\cdot} = \overline{\mathbf{F}^{-1}\cdot \mathrm{Grad}\nabla\varrho}^{\cdot} = \dot{\overline{\mathbf{F}^{-1}}}\cdot \mathbf{F}\cdot \nabla\nabla\varrho+\mathbf{F}\mathbf{F}^{-1}\nabla\dot{\overline{(\nabla\varrho)}} = -\nabla\mathbf{v}\cdot\nabla\nabla\varrho+\nabla\dot{\overline{(\nabla\varrho)}}$$

$$(2.7)\qquad = -\nabla\mathbf{v}\cdot\nabla\nabla\varrho-\nabla\nabla\mathbf{v}\cdot\nabla\varrho-\nabla\nabla\varrho\cdot(\nabla\mathbf{v})^T-\nabla\nabla\varrho(\nabla\cdot\mathbf{v})$$

$$-\nabla(\nabla\cdot\mathbf{v})\nabla\varrho-\varrho\nabla\nabla(\nabla\cdot\mathbf{v})-\nabla\varrho(\nabla\cdot\mathbf{v}).$$

Taking into account the fact that $\nabla\nabla\varrho$ and $\partial w/\partial(\nabla\nabla\varrho)$ are symmetric tensor fields, we obtain:

$$\frac{\partial w}{\partial\varrho}\dot{\varrho} = \frac{\partial w}{\partial\varrho}\mathbf{1}:\nabla\mathbf{v},$$

$$(2.8)\qquad \frac{\partial w}{\partial(\nabla\varrho)}\cdot\dot{\overline{\nabla\varrho}} = -\left[\frac{\partial w}{\partial(\nabla\varrho)}\nabla\varrho+\left(\frac{\partial w}{\partial(\nabla\varrho)}\cdot\nabla\varrho\right)\mathbf{1}\right]:\nabla\mathbf{v}-\varrho\frac{\partial w}{\partial(\nabla\varrho)}\mathbf{1}\,\vdots\,\nabla\nabla\mathbf{v},$$

$$\frac{\partial w}{\partial(\nabla\nabla\varrho)}:\dot{\overline{\nabla\nabla\varrho}} = -\left[2\left(\frac{\partial w}{\partial(\nabla\nabla\varrho)}\cdot\nabla\nabla\varrho\right)+\left(\frac{\partial w}{\partial(\nabla\nabla\varrho)}:\nabla\nabla\varrho\right)\mathbf{1}\right]:\nabla\mathbf{v}$$

$$-\left[\frac{\partial w}{\partial(\nabla\nabla\varrho)}\nabla\varrho+2\left(\nabla\varrho\cdot\frac{\partial w}{\partial(\nabla\nabla\varrho)}\right)\mathbf{1}\right]\vdots\,\nabla\nabla\mathbf{v}-\varrho\frac{\partial w}{\partial(\nabla\nabla\varrho)}\mathbf{1}::\nabla\nabla\nabla\mathbf{v}.$$

Thus we have

$$(2.9)\qquad \int_V \overline{\varrho w\,dV}^{\cdot} = \int_V (\mathbf{A}:\nabla\mathbf{v}+\mathbf{B}\,\vdots\,\nabla\nabla\mathbf{v}+\mathbf{C}::\nabla\nabla\nabla\mathbf{v})\,dV,$$

where

$$\mathbf{A} = -\left(\varrho^2\frac{\partial w}{\partial\varrho}+\varrho\frac{\partial w}{\partial(\nabla\varrho)}\cdot\nabla\varrho+\varrho\frac{\partial w}{\partial(\nabla\nabla\varrho)}:\nabla\nabla\varrho\right)\mathbf{1}-\varrho\frac{\partial w}{\partial(\nabla\varrho)}\nabla\varrho-2\varrho\frac{\partial w}{\partial(\nabla\nabla\varrho)}\cdot\nabla\nabla\varrho,$$

$$(2.10)\qquad \mathbf{B} = -\left(\varrho^2\frac{\partial w}{\partial(\nabla\varrho)}+2\varrho\nabla\varrho\cdot\frac{\partial w}{\partial(\nabla\nabla\varrho)}\right)\mathbf{1}-\varrho\frac{\partial w}{\partial(\nabla\nabla\varrho)}\nabla\varrho,$$

$$\mathbf{C} = -\varrho^2\frac{\partial w}{\partial(\nabla\nabla\varrho)}\mathbf{1}.$$

Thus by (2.9) we are able to express the material time derivative of the integral of ϱw over the region V in terms of velocity gradients.

3. Dynamics

Let us assume that at every point in the liquid we can define tensors $\mathbf{T}$, $\mathbf{Q}$ and $\mathbf{R}$ such that the following energy balance is valid for every material subregion and every velocity field

$$(3.1)\qquad \overline{\int w\varrho\,dV}^{\cdot} = \int \varrho\mathbf{f}\cdot\mathbf{v}\,dV+\int_{\partial V}\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{v}\,dS+\int_{\partial V}\mathbf{n}\cdot\mathbf{Q}\cdot\nabla\mathbf{v}\,dS$$

$$+\int_{\partial V}\mathbf{n}\cdot\mathbf{R}\;\nabla\nabla\mathbf{v}\,dS-\int_V \overline{\frac{\varrho\mathbf{v}\cdot\mathbf{v}}{2}}^{\cdot}\,dV,$$

where w denotes the internal energy density, $\mathbf{f}$ the density of mass forces and $\mathbf{n}$ the outward unit normal vector. Using (2.9) and the Gauss-Ostrogradskij formula, we immediately obtain:

$$\int_V [\mathbf{A}:\nabla\mathbf{v}+\mathbf{B}\vdots\nabla\nabla\mathbf{v}+\mathbf{C}::\nabla\nabla\nabla\mathbf{v}-\nabla\cdot(\mathbf{T}\cdot\mathbf{v}+\mathbf{Q}:\nabla\mathbf{v}+\mathbf{R}\vdots\nabla\nabla\mathbf{v})-\varrho\mathbf{f}\cdot\mathbf{v}-\varrho\dot{\mathbf{v}}\cdot\mathbf{v}]dV = 0. \tag{3.2}$$

But (3.2) has to be valid for every subregion V and every velocity field $\mathbf{v}$; therefore, we can write:

$$\begin{gathered}\nabla\cdot\mathbf{T}+\varrho\mathbf{f}-\varrho\dot{\mathbf{v}} = 0, \quad \mathbf{T}+\nabla\cdot\mathbf{Q}-\mathbf{A} = 0,\\ \mathbf{Q}+\nabla\cdot\mathbf{R}-\mathbf{B}-\mathbf{B}^* = 0, \quad \mathbf{R}-\mathbf{C}-\mathbf{C}^* = 0,\end{gathered} \tag{3.3}$$

where $\mathbf{B}^*$ and $\mathbf{C}^*$ are some undefined tensor fields. $\mathbf{B}^*$ is antisymmetric over the first two indices, and $\mathbf{C}^*$ is antisymmetric over at least one pair of the first three of its indices. We can easily see, that if the Eqs. (3.3) hold, then (3.2) becomes an identity for every $\mathbf{B}^*$ and $\mathbf{C}^*$ with the above symmetry properties. Bearing in mind that $\nabla\nabla\nabla:\mathbf{C}^*$ and $\nabla\nabla:\mathbf{B}^*$ are equal to zero, we can rewrite (3.3) as follows:

$$\begin{gathered}\nabla\nabla\nabla\vdots\mathbf{C}-\nabla\nabla:\mathbf{B}+\nabla\cdot\mathbf{A}+\varrho\mathbf{f} = \varrho\dot{\mathbf{v}},\\ \mathbf{T} = \nabla\nabla:(\mathbf{C}+\mathbf{C}^*)-\nabla\cdot(\mathbf{B}+\mathbf{B}^*)+\mathbf{A},\\ \mathbf{Q} = -\nabla\cdot(\mathbf{C}+\mathbf{C}^*)+\mathbf{B}+\mathbf{B}^*, \quad \mathbf{R} = \mathbf{C}+\mathbf{C}^*.\end{gathered} \tag{3.4}$$

Boundary conditions can be specified by prescribing some generalized tractions $\mathbf{t}$, $\boldsymbol{\varkappa}$, $\boldsymbol{\pi}$ for every point on ∂V, where

$$\mathbf{t} \equiv \mathbf{n}\cdot\mathbf{T}, \quad \boldsymbol{\varkappa} \equiv \mathbf{n}\cdot\mathbf{Q}, \quad \boldsymbol{\pi} \equiv \mathbf{n}\cdot\mathbf{R}. \tag{3.5}$$

The set of Eqs. (3.4) and boundary conditions (3.5) is not complete because $\mathbf{C}^*$ and $\mathbf{B}^*$ still remain undefined. Moreover the physical meaning of the generalized tractions is also unclear. Let us note however that $(3.4)_1$ contains neither $\mathbf{C}^*$ nor $\mathbf{B}^*$. This enables us to employ the theory for our particular problem provided that we have chosen the proper geometrical form for the shell and need no information about the stress field inside the material (where the stress tensor can be defined in several different ways). Thus we have to confine ourselves to considerations of the pressure difference across the shell between the two regions of gas, which is a well-defined concept.

4. Spherical shell

So long as we wish only to show that the above considerations can give the proper solution to our particular "rainbow bubble problem", we are free to choose any simple expression for the energy density which is compatible with symmetry properties of the material and frame-indifference principle. For the isotropic liquid, we have

$$\begin{gathered}w = w[\varrho;\ \nabla\varrho\cdot\nabla\varrho;\ \mathbf{1}:\nabla\nabla\varrho;\ \mathbf{1}:(\nabla\nabla\varrho\cdot\nabla\nabla\varrho);\\ \mathbf{1}:(\nabla\nabla\varrho\cdot\nabla\nabla\varrho\cdot\nabla\nabla\varrho);\ \nabla\varrho\cdot\nabla\nabla\varrho\cdot\nabla\varrho;\ \nabla\varrho\cdot\nabla\nabla\varrho\cdot\nabla\nabla\varrho\cdot\nabla\varrho]\end{gathered} \tag{4.1}$$

(see e.g. [7]).

Let us assume the following expression for w:

$$w = f(\varrho-\varrho_0)+\frac{\alpha}{2}\frac{\nabla\varrho\cdot\nabla\varrho}{\varrho}+\frac{\beta}{2}\frac{\nabla\nabla\varrho:\nabla\nabla\varrho}{\varrho}+\gamma\frac{\mathbf{1}:\nabla\nabla\varrho}{\varrho}, \tag{4.2}$$

where f is some scalar function and $\alpha \geqslant 0$, $\beta > 0$, and γ are constants. In view of (4.2), we obtain:

$$\begin{aligned} \frac{\partial w}{\partial \varrho} &= \frac{\partial f}{\partial \varrho}-\frac{\alpha}{2\varrho^2}\nabla\varrho\cdot\nabla\varrho-\frac{\beta}{2\varrho^2}\nabla\nabla\varrho:\nabla\nabla\varrho-\frac{\gamma}{\varrho^2}\mathbf{1}:\nabla\nabla\varrho,\\ \frac{\partial w}{\partial(\nabla\varrho)} &= \frac{\alpha}{\varrho}\nabla\varrho,\\ \frac{\partial w}{\partial(\nabla\nabla\varrho)} &= \frac{\beta}{\varrho}\nabla\nabla\varrho+\frac{\gamma}{\varrho}\mathbf{1} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \mathbf{A} &= -\left\{\left(\varrho^2\frac{\partial f}{\partial\varrho}-\frac{\alpha}{2}\nabla\varrho\cdot\nabla\varrho-\frac{\beta}{2}\nabla\nabla\varrho:\nabla\nabla\varrho+\alpha\nabla\varrho\cdot\nabla\varrho+\beta\nabla\nabla\varrho:\nabla\nabla\varrho+\gamma\nabla^2\varrho\right)\mathbf{1}\right.\\ &\qquad \left.+\alpha\nabla\varrho\nabla\varrho+2\gamma\nabla\nabla\varrho+\beta\nabla\nabla\varrho\cdot\nabla\nabla\varrho\right\},\\ \mathbf{B} &= -\{(\alpha\varrho\nabla\varrho+2\beta\nabla\varrho\cdot\nabla\nabla\varrho+2\gamma\nabla\varrho)\mathbf{1}+\beta\nabla\nabla\varrho\nabla\varrho+\gamma\mathbf{1}\nabla\varrho\},\\ \mathbf{C} &= -(\beta\varrho\nabla\nabla\varrho+\gamma\mathbf{1})\mathbf{1}. \end{aligned} \tag{4.4}$$

Let us consider now the static equilibrium conditions for a spherical shell. We assume that our shell is thin — i.e.,

$$R_2-R_1 \ll \frac{1}{2}(R_1+R_2), \tag{4.5}$$

where R_1, R_2 are correspondingly the inner and outer radii of the shell. We assume also that the mean radius of the shell is large when compared with some characteristic length $\sqrt{\beta/\alpha}$,

$$\frac{1}{2}(R_1+R_2) \gg \sqrt{\frac{\beta}{\alpha}}. \tag{4.6}$$

Finally, we assume (as in Sec. 1) that the media inside and outside of the spherical shell are classical and exert pressures on both surfaces uniformly in the radial direction only.

The spherical symmetry of the problem under consideration requires that at least one solution (if any solution exists) be expressible as

$$\varrho = \varrho(r), \tag{4.7}$$

where r is the radial coordinate in the spherical coordinate system.

According to the boundary conditions, we have for $r = R_1$; $r = R_2$

$$\mathbf{n}\cdot\mathbf{R} = \mathbf{0}. \tag{4.8}$$

In view of (4.7), we have in the spherical coordinate system:

$$\nabla\varrho = \frac{\partial\varrho}{\partial r}\mathbf{e}^r, \tag{4.9}$$
$$\nabla\nabla\varrho = \frac{\partial^2\varrho}{\partial r^2}(\mathbf{e}^r\mathbf{e}^r)+\frac{1}{r}\frac{\partial\varrho}{\partial r}(\mathbf{1}-\mathbf{e}^r\mathbf{e}^r).$$

We can rewrite (4.8) as

$$\mathbf{n}\cdot\left\{\beta\varrho\left[\frac{\partial^2\varrho}{\partial r^2}(\mathbf{e}^r\mathbf{e}^r)+\frac{1}{r}\frac{\partial\varrho}{\partial r}(\mathbf{1}-\mathbf{e}^r\mathbf{e}^r)\right]+\varrho\gamma\mathbf{1}\right\}\mathbf{1}+\mathbf{n}\cdot\mathbf{C}^* = 0. \tag{4.10}$$

Bearing in mind that for spherical surfaces $\mathbf{n} = \mathbf{e}^r$ or $\mathbf{n} = -\mathbf{e}^r$ and multiplying (4.10) by $(\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r)$, we obtain:

$$\beta\varrho\frac{\partial^2\varrho}{\partial r^2}+\varrho\gamma+(\mathbf{n}\cdot\mathbf{C}^*)\vdots(\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r) = 0. \tag{4.11}$$

But according to the symmetry properties of $\mathbf{C}^*$, we have

$$(\mathbf{n}\cdot\mathbf{c}^*)\vdots(\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r) = 0; \tag{4.12}$$

hence,

$$\left.\frac{\partial^2\varrho}{\partial r^2}\right|_{r=R_1;\ r=R_2} = -\frac{\gamma}{\beta}. \tag{4.13}$$

Bearing in mind that on both surfaces only radial forces are applied, we have

$$\mathbf{t}\equiv\mathbf{n}\cdot\mathbf{T} = -p_{(i)}\mathbf{n}, \quad i = 1, 2, \tag{4.14}$$

where

$$p_{(i)} = -\mathbf{nn}:\mathbf{T}|_{r=R_{(i)}}. \tag{4.15}$$

It can be shown (see Appendix) that, at least for our particular problem, $\mathbf{B}^*$ and $\mathbf{C}^*$ do not contribute to p and

$$p = -\mathbf{nn}:(\nabla\nabla:\mathbf{C}-\nabla\cdot\mathbf{B}+\mathbf{A}). \tag{4.16}$$

By differentiation of $(4.4)_2$ and $(4.4)_3$, we obtain:

$$\nabla\nabla:\mathbf{C} = -\{\nabla\nabla:(\beta\nabla\nabla\varrho+\varrho\gamma\mathbf{1})\mathbf{1}\}, \tag{4.17}$$
$$\nabla\cdot\mathbf{B} = -\{[\nabla\cdot(\alpha\varrho\nabla\varrho+2\beta\nabla\varrho\cdot\nabla\nabla\varrho+2\gamma\nabla\varrho]\mathbf{1}$$
$$\beta\nabla\cdot(\nabla\nabla\varrho)\cdot\nabla\varrho+\beta\nabla\nabla\varrho\cdot\nabla\nabla\varrho+\gamma\nabla\nabla\varrho\}$$

and

$$\nabla\nabla:\mathbf{C}-\nabla\cdot\mathbf{B}+\mathbf{A} = -P\mathbf{1}-\alpha\nabla\varrho\nabla\varrho+\beta\nabla(\nabla^2\varrho)\nabla\varrho+\gamma\nabla\nabla\varrho, \tag{4.18}$$

where

$$P = \varrho^2\frac{\partial f}{\partial\varrho}-\alpha\varrho\nabla^2\varrho+\gamma\nabla^2\varrho+\beta\varrho\nabla^2\nabla^2\varrho. \tag{4.19}$$

Bearing in mind

$$\nabla(\nabla\cdot\nabla\varrho) = \left(\frac{\partial^3\varrho}{\partial r^3}+\frac{2}{r}\frac{\partial^2\varrho}{\partial r}-\frac{2}{r^2}\frac{\partial\varrho}{\partial r}\right)\mathbf{e}^r, \tag{4.20}$$

and making use of (4.9), we obtain:

$$\nabla\nabla:\mathbf{C}-\nabla\cdot\mathbf{B}+\mathbf{A} = -[P\mathbf{1}+M(\mathbf{e}^r\mathbf{e}^r)+N(\mathbf{1}-\mathbf{e}^r\mathbf{e}^r)], \tag{4.21}$$

where

$$M \equiv \alpha\left(\frac{\partial\varrho}{\partial r}\right)^2-\beta\frac{\partial\varrho}{\partial r}\left(\frac{\partial^3\varrho}{\partial r^3}+\frac{2}{r}\frac{\partial^2\varrho}{\partial r^2}-\frac{2}{r^2}\frac{\partial\varrho}{\partial r}\right)-\gamma\frac{\partial^2\varrho}{\partial r^2},$$
$$N \equiv -\gamma\frac{1}{r}\frac{\partial\varrho}{\partial r}. \tag{4.22}$$

Now $(3.4)_1$ can be rewritten as

$$\nabla P+\nabla M\cdot(\mathbf{e}^r\mathbf{e}^r)+M\nabla\cdot(\mathbf{e}^r\mathbf{e}^r)+\nabla N\cdot(\mathbf{1}-\mathbf{e}^r\mathbf{e}^r)-N\nabla\cdot(\mathbf{e}^r\mathbf{e}^r) = 0. \tag{4.23}$$

But

$$\nabla N\cdot(\mathbf{1}-\mathbf{e}^r\mathbf{e}^r) = 0 \tag{4.24}$$

and multiplying (4.23) by $\mathbf{e}^r$, we obtain:

$$\frac{\partial(P+M)}{\partial r} = -(M-N)\,[\nabla\cdot(\mathbf{e}^r\mathbf{e}^r)\cdot\mathbf{e}^r]. \tag{4.25}$$

For the term in brackets we have:

$$\nabla\cdot(\mathbf{e}^r\mathbf{e}^r) = (\nabla\cdot\mathbf{e}^r)\mathbf{e}^r+\mathbf{e}^r\cdot\nabla\mathbf{e}^r,$$
$$\nabla\mathbf{e}^r = \mathbf{e}^\varphi\frac{\partial\mathbf{e}^r}{\partial\varphi}+\mathbf{e}^\theta\frac{\partial\mathbf{e}^r}{\partial\theta}, \tag{4.26}$$
$$\mathbf{e}^r\cdot\nabla\mathbf{e}^r = 0, \qquad \nabla\cdot\mathbf{e}^r = \frac{2}{r}.$$

According to (4.16) and (4.21),

$$P+M = p \tag{4.27}$$

for both surfaces and

$$\int\limits_{R_1}^{R_2}\frac{\partial(P+M)}{\partial r}dr = p_2-p_1; \tag{4.28}$$

thus

$$p_1-p_2 = 2\int\limits_{R_1}^{R_2}\frac{M-N}{r}dr, \tag{4.29}$$

or

$$p_1-p_2 = \int\limits_{R_1}^{R_2}\frac{1}{r}\left[\left(\alpha+\frac{2\beta}{r}\right)\left(\frac{\partial\varrho}{\partial r}\right)^2-\beta\frac{\partial\varrho}{\partial r}\frac{\partial^3\varrho}{\partial r^3}-\frac{2\beta}{r}\frac{\partial\varrho}{\partial r}\frac{\partial^2\varrho}{\partial r^2}+\frac{2\beta}{r^2}\left(\frac{\partial\varrho}{\partial r}\right)^2-\gamma\frac{\partial^2\varrho}{\partial r^2}+\frac{\gamma}{r}\frac{\partial\varrho}{\partial r}\right]dr. \tag{4.30}$$

If we now make use of the assumptions (4.6) and (4.7), we can write:

$$p_1 - p_2 \approx \frac{2}{R_{\text{mean}}} \int_{R_1}^{R_2} \left[\alpha \left(\frac{d\varrho}{dr} \right)^2 - \beta \frac{d\varrho}{dr} \frac{d^3\varrho}{dr^3} - \gamma \frac{d^2\varrho}{dr^2} \right] dr, \tag{4.31}$$

where

$$R_{\text{mean}} = \frac{1}{2}(R_1 + R_2). \tag{4.32}$$

After some rearrangement, we obtain:

$$p_1 - p_2 \approx \frac{2}{R_{\text{mean}}} \int_{R_1}^{R_2} \left[\alpha \left(\frac{d\varrho}{dr} \right)^2 + \beta \left(\frac{d^2\varrho}{dr^2} \right)^2 \right] dr - \beta \frac{d\varrho}{dr} \frac{d^2\varrho}{dr^2} \Bigg|_{R_1}^{R_2} - \gamma \frac{d\varrho}{dr} \Bigg|_{R_1}^{R_2}, \tag{4.33}$$

but in view of (4.13), the last two terms cancel each other.

Note also that invie w of (4.13) and our assumptions: $\alpha \geqslant 0, \beta > 0$, the integral cannot be equal to zero. Finally, we obtain:

$$p_1 - p_2 \approx \frac{2}{R_{\text{mean}}} \int_{R_1}^{R_2} \left[\alpha \left(\frac{d\varrho}{dr} \right)^2 + \beta \left(\frac{d^2\varrho}{dr^2} \right)^2 \right] dr > 0. \tag{4.34}$$

This result seems to correspond, at least qualitatively, to physical reality.

Appendix

We shall prove that the quantities $\mathbf{C}^*$ and $\mathbf{B}^*$ do not make any contribution to p. We have for p on both surfaces:

$$-p = \mathbf{nn}:\mathbf{T} = (\mathbf{e}^r\mathbf{e}^r):\mathbf{A} - (\mathbf{e}^r\mathbf{e}^r):\left(\mathbf{e}^r \cdot \frac{\partial \mathbf{Q}}{\partial r} + \mathbf{e}^\varphi \cdot \frac{\partial \mathbf{Q}}{\partial \varphi} + \mathbf{e}^\theta \cdot \frac{\partial \mathbf{Q}}{\partial \theta} \right). \tag{A.1}$$

But on each surface we also have:

$$\mathbf{e}^r \cdot \mathbf{Q} = \mathbf{0} \tag{A.2}$$

and

$$\frac{\partial^{k+l}(\mathbf{e}^r \cdot \mathbf{Q})}{\partial \varphi^k \partial \theta^l} = \mathbf{0}. \tag{A.3}$$

Making use of (A.3) with $(k, l) = (0,1); (0,2); (1,0); (2,0)$, and calculating the derivatives of the reciprocal base vectors, we obtain:

$$-p = (\mathbf{e}^r\mathbf{e}^r):\mathbf{A} - (\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r):\left[\frac{\partial \mathbf{Q}}{\partial r} - \frac{1}{2r} \left(\frac{1}{\sin^2\theta} \frac{\partial^2 \mathbf{Q}}{\partial \varphi^2} + \frac{\partial^2 \mathbf{Q}}{\partial \theta^2} - \cot\theta \frac{\partial \mathbf{Q}}{\partial \theta} \right) \right]. \tag{A.4}$$

Since the partial derivatives of the antisymmetric tensor field in Euclidean space are also antisymmetric (or zero), we can rewrite (A.4) as

$$-p = a_1 - a_2, \tag{A.5}$$

where

$$(A.6)\qquad -a_1 = (\mathbf{e}^r\mathbf{e}^r):\mathbf{A}-(\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r)\vdots\left[\frac{\partial\mathbf{B}}{\partial r}-\frac{1}{2r}\left(\frac{1}{\sin^2\theta}\frac{\partial^2\mathbf{B}}{\partial\varphi^2}+\frac{\partial\mathbf{B}}{\partial\theta^2}+\cot\theta\frac{\partial\mathbf{B}}{\partial\theta}\right)\right],$$

$$(A.7)\qquad a_2 = (\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r)\vdots\left\{\frac{\partial}{\partial r}\left(\mathbf{e}^r\cdot\frac{\partial\mathbf{R}}{\partial r}+\mathbf{e}^\varphi\cdot\frac{\partial\mathbf{R}}{\partial\varphi}+\mathbf{e}^\theta\cdot\frac{\partial\mathbf{R}}{\partial\theta}\right)\right.$$
$$-\frac{1}{2r}\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\left(\mathbf{e}^r\cdot\frac{\partial\mathbf{R}}{\partial r}+\mathbf{e}^\varphi\cdot\frac{\partial\mathbf{R}}{\partial\varphi}+\mathbf{e}^\theta\cdot\frac{\partial\mathbf{R}}{\partial\theta}\right)+\frac{\partial^2}{\partial\theta^2}\left(\mathbf{e}^r\cdot\frac{\partial\mathbf{R}}{\partial r}+\mathbf{e}^\varphi\cdot\frac{\partial\mathbf{R}}{\partial\varphi}+\mathbf{e}^\theta\cdot\frac{\partial\mathbf{R}}{\partial\theta}\right)\right.$$
$$\left.\left.+\cot\theta\frac{\partial}{\partial\theta}\left(\mathbf{e}^r\cdot\frac{\partial\mathbf{R}}{\partial r}+\mathbf{e}^\varphi\cdot\frac{\partial\mathbf{R}}{\partial\varphi}+\mathbf{e}^\theta\cdot\frac{\partial\mathbf{R}}{\partial\theta}\right)\right]\right\}.$$

Let us bear in mind now, that the pressure is uniform over the whole surface. Thus we are able to confine ourselves to determining p for any chosen point on the surface. In the interest of simplicity, we choose some point on the circle $\theta = \pi/2$.

Performing the differentiation and evaluating the derivatives at $\theta = \pi/2$, we obtain for a_2

$$(A.8)\qquad a_2 = (\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r)::\left(\frac{\partial^2\mathbf{R}}{\partial r^2}+\frac{1}{r}\frac{\partial\mathbf{R}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\mathbf{R}}{\partial\varphi^2}+\frac{1}{r^2}\frac{\partial^2\mathbf{R}}{\partial\theta^2}-\frac{1}{2r}\frac{\partial^3\mathbf{R}}{\partial r\partial\theta^2}\right.$$
$$\left.-\frac{1}{2r}\frac{\partial^3\mathbf{R}}{\partial r\,\partial\varphi^2}\right)-\frac{1}{2r}(\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r)\left(\mathbf{e}^\theta\cdot\frac{\partial\mathbf{R}}{\partial\theta}-\mathbf{e}^\theta\cdot\frac{\partial^3\mathbf{R}}{\partial\theta^3}\right.$$
$$\left.-\mathbf{e}^\varphi\cdot\frac{\partial^3\mathbf{R}}{\partial\varphi^3}+\mathbf{e}^\theta\cdot\frac{\partial^3\mathbf{R}}{\partial\theta\partial\varphi^2}+\mathbf{e}^\varphi\cdot\frac{\partial^3\mathbf{R}}{\partial\theta^2\partial\varphi}\right).$$

But according to the boundary conditions, we have:

$$(A.9)\qquad \mathbf{e}^r\cdot\mathbf{R} = \mathbf{0}.$$

and

$$(A.10)\qquad \frac{\partial^{i+j}(\mathbf{e}^r\cdot R)}{\partial\theta^i\partial\varphi^j} = \mathbf{0}.$$

Making use of (A.10) with $(i,j) = (0.1); (0.2); (0.4); (1.0); (2.0); (4.0); (2.2)$, we obtain

$$(A.11)\qquad a_2 = (\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r\mathbf{e}^r)::\left(\frac{\partial^2\mathbf{R}}{\partial r}+\frac{1}{r}\frac{\partial\mathbf{R}}{\partial r}+\frac{2}{r^2}\frac{\partial^2\mathbf{R}}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial^2\mathbf{R}}{\partial\varphi^2}-\frac{1}{2r}\frac{\partial^3\mathbf{R}}{\partial r\partial\theta}\right.$$
$$\left.-\frac{1}{2r}\frac{\partial^3\mathbf{R}}{\partial r\,\partial\varphi^2}-\frac{1}{8r^2}\frac{\partial^4\mathbf{R}}{\partial\theta^4}+\frac{1}{4r^2}\frac{\partial^4\mathbf{R}}{\partial\theta^2\partial\varphi^2}\cdots\frac{1}{8r^2}\frac{\partial^4\mathbf{R}}{\partial\varphi^4}\right).$$

It is evident now that a_2 is independent of $\mathbf{C}^*$. Thus p does not depend on either $\mathbf{B}^*$ or $\mathbf{C}^*$.

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POLISH ACADEMY OF SCIENCES.

Received March 6, 1972.