Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 2, pp. 165–177, Warszawa 1973

# Stochastic plastic analysis

M. A. MUSPRATT (MONTREAL)

THE OBJECT of this paper is to demonstrate the versatility of non-linear mathematical programming for solution of lower bound plastostatic analysis of a concrete plate and shell. In addition, this numerical technique is coupled with Monte Carlo simulation to permit convolved probability distributions of collapse loads to be obtained from component exogenous frequency densities of properties of plate or shell material, loading, geometry, etc.

Celem pracy jest wykazanie przydatności nieliniowego, matematycznego programowania do oceny dolnego ograniczenia w plastycznostatycznej analizie płyt i powłok żelbetowych. Ta numeryczna technika jest sprzężona z metodą Monte Carlo, aby zapewnić warunkowe rozkłady prawdopodobieństwa obciążeń niszczących, które mogą być otrzymane ze składowej zmian gęstości własności płyt i powłok materiału, obciążenia, geometrii itp.

Целью данной работы является доказательство пригодности нелинейного математического программирования для построения нижних оценок в статическом предельном анализе бетонных пластин и оболочек. Данная вычислительная техника сопряжена дополнительно с методом Монте-Карло с целью нахождения условных распределений вероятностей разрущающих нагрузок на основе свойств материала пластины или оболочки, нагрузки, геометрии и пр.

## Notation

### Slab

$i, j$ $0.0, 0.33, 0.66, 1.0$ = finite difference node indices,
$m_x, m_y$ bending moments normal to $x, y$ axes,
$m_{xy}$ torsional moment,
$q$ distributed load,
$x, y$ rectangular Cartesian coordinates,
$\delta_s$ finite difference grid size $= 0.33$;

### Shell

$B$ unit width,
$C^2$ $N_\theta L/RM_0$,
$F_x, F_\theta$ areas of axial, circumferential steel,
$2H$ wall thickness,
$L$ axial length,
$M_0$ $R_c BH^2$,
$M$ axial bending moment,
$m$ $M/M_0$,
$N_0$ $2R_c BH$,
$N_x, N_\theta$ axial, circumferential forces,
$n_x, n_\theta$ $N_x/N_0$, $N_\theta/N_0$,
$P_\infty$ $\gamma R^2/2\mu N_0$ = ultimate radial pressure as $L \to \infty$,
$P_0$ $P_\infty(1-e^{-t})$ = radial pressure at base ($x = L$),

$R$ shell radius,
$R_c, R_t$ compressive, tensile strength of concrete,
$t$ $2\mu kL/R$,
$U_x, U_\theta$ $F_x/2BH$, $F_\theta/2BH$ = axial, circumferential percentages of reinforcement,
$V$ reinforcement volume,
$x$ axial ordinate, origin at top of shell (free end),
$z$ $x/L = 0.0, 0.2, 0.4, \ldots, 1.0$ for finite difference nodes,
$\alpha$ $R_t/R_c$,
$\beta$ $\sigma_0/R_c$,
$\gamma$ density of silo contents,
$\delta$ $0.2$ = finite difference grid size,
$k$ internal friction coefficient of silo contents,
$\theta$ circumferential ordinate,
$\sigma_0$ yield strength of reinforcing steel,
$\mu$ friction coefficient of silo contents against wall.

## 1. Introduction

BECAUSE of the brittle nature of concrete and the wide variances of its properties, exact specification is impossible. Some of the more important causes of uncertainty with reinforced concrete are now listed:

- error and variation in mix, compaction and curing of the concrete;
  variation of tensile strength, concrete rigidity with cracking, and dilatation of the concrete for plastic strains;
- errors in layout of reinforcement patterns, and variations in surface conditions and hence bond capacity of individual bars;
- effect of discrete reinforcement bars, and curtailment and workhardenability;
  degenerative temporal changes such as creep and shrinkage of the concrete, relaxation and corrosion of the reinforcing steel and fatigue effects.

  In addition, variations peculiar to reinforced concrete slabs and shells are:
- sensitivity to changes in geometrical and spatial configuration such as effective span and depth, support conditions, geometry of middle surface, and loading distribution and intensity.

  Other important stochastic factors are:
- errors in calculations, errors in instruments indicating material properties, etc.,
  serial correlation of degenerative temporal changes of materials due to temperature and humidity fluctuations,
- various spatial and temporal cross-correlations, e.g. between concrete strength and elastic properties.

It can be seen that normal simplifying assumptions such as elasticity, isotropy, homogeneity, or rigid-plasticity, etc., are not consistent with reality unless qualified and quantified by probability spectra.

It should be noted that elementary probability theory has been included in the recommendations for an international code of practice for reinforced concrete by the European Committee for concrete, Ref. [1].

SAWYER, Ref. [2], had calculated the probability of collapse of a typical reinforced concrete framed structure designed by contemporary methods and found it was of the order of $10^{-7}$. No similar work has been attempted for concrete slabs or shells.

Advantages of the non-deterministic design are quantification and uniformity of reliability often with cost optimisation and without attendant information loss as occurs if a single discrete value, e.g. the mean, is used to represent a frequency spectrum. However, the required acumen in system definition with strict limitations on simplifying assumptions and increase in volume of numerical or analytical work makes this approach unattractive to many engineers.

## 2. Mathematical programming

In plastic analysis of structures, mathematical programming is increasingly displacing variational calculus as a means of establishing bounds for the collapse loads because of greater versatility in solution of novel problems, Ref. [3 & 4].

Mathematical programming techniques may be summarized as below:

| Programming method | Objective | Constraints |
|---|---|---|
| Linear programming. Simplex | Linear | Linear |
| Quadratic programming. Simplex | Quadratic | Linear |
| Fractional programming. Simplex | Linear Fractional | Linear |
| Integer programming — cutting plane | Linear Integer | Linear Integer |
| Non-linear programming — gradient methods | Non-linear | Non-linear |
| — search methods | Non-linear | Non-linear |
| Geometric programming | Polynomial | Polynomial |
| Dynamic programming | Non-linear | Non-linear |
| Stochastic programming | Non-linear | Non-linear |
| S.U.M.T. | Non-linear | Non-linear |

Most of the above techniques have been recently used for structural optimisation, Ref. [5], while in addition, duality and its significance in establishing upper and lower bounds are also discussed in this reference.

Non-linear programming obviates simplifying assumptions such as piece-wise linearisation of yield hypersurfaces, or consideration of sandwich plate instead of solid plate construction and, as will be subsequently shown, facilitates inclusion of general stochastic effects.

However, general non-linear programming introduces uncertainty about globality of any extremum obtained. There is no completely general method of finding a global optimum of any arbitrary non-linear function, subject to non-linear constraints.

## Stochastic programming

The general solution of the stochastic non-linear programming problem without invariants can easily be handled, at the expense of increased computational effort, by coupling Monte Carlo methods with non-linear programming techniques. The heuristic used involves discretisation of component probability distributions and optimisation of various combinations of the discrete values according to some strategy. Further details of this method are given in Ref. [3, 4, 5, 6].

The non-linear mathematical programming technique, with which Monte Carlo simulation is to be subsequently combined, is described in Ref. [6]. It is the sequential unconstrained minimization technique (S.U.M.T.) and it is possibly the most sophisticated of the non-linear techniques at present to hand. S.U.M.T. has been used successfully by SCHMIT, Ref. [7], for the synthesis of stiffened shells and by HODGE, Ref. [8], for the plastic analysis of shells.

## 3. Slab analysis

### 3.1. Deterministic problem

The example demonstrated is a stochastic lower-bound plastic analysis of a simply-supported square slab, subject to a uniform load.

LANCE *et al.*, Ref. [9], have obtained a lower bound analysis of a uniformly loaded steel plate with an aspect ratio of unity and assuming Tresca's yield criterion, while WOLFENSBERGER, Ref. [10], has solved the same problem for a concrete slab. Both of these solutions were obtained using linearised, deterministic systems and linear programming, whereas the numerical results presented below were obtained from a non-linear, stochastic system.

The heuristic used for obtaining the field of stress resultants $(m_x, m_y, m_{xy})$ and the collapse load is firstly to replace the continuum with a quantised form, i.e. the differential

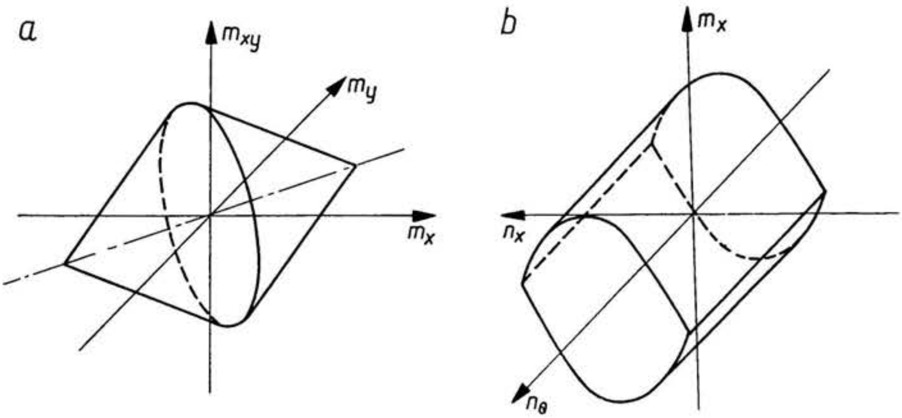


FIG. 1. Yield surfaces for reinforced concrete structures.

equations of equilibrium for the domain are replaced by finite differences. A statically admissible moment field is obtained if these finite difference equations are solved, subject to yield constraints on the moments. The octal symmetry of the slab is invoked and the finite different mesh used is as shown in Fig. 2. Even for the ten points shown, the initial programming matrix is substantial, involving 31 variables and 76 constraints. Of these

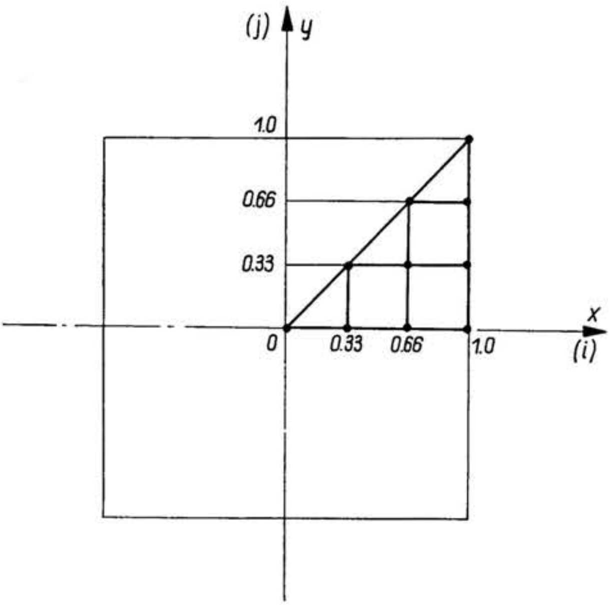


FIG. 2. Finite-difference mesh for octant of square slab.

constraints, 20 are non-linear, and additional first and second variations[1] occur resulting in tedious compilation of input matrices.

However, the law of diminishing returns prevail if an excessively fine finite different mesh is utilised[2], i.e. rounding and overflow machine errors become proportionately larger; and so accuracy may actually decrease or oscillate if an excessively fine mesh is used.

i. *Equilibrium equation.* TIMOSHENKO *et al*, Ref. [11], give the biharmonic equation for slab equilibrium

$$\frac{\partial^2 m_x}{\partial x^2}+\frac{\partial^2 m_y}{\partial y^2}+\frac{2\partial^2 M_{xy}}{\partial x\,\partial y}=q.$$

(1) These derivatives are necessary input for the S.U.M.T. routine but search methods of optimization could be considered to obviate gradient estimations.

(2) Computer store in the Burroughs B5500 computer used was quite large, e.g. $1000\times1000$ matrices could be handled for linear programming problems.

The simplest procedure for expressing this elliptic partial differential equation in finite difference form is to use trifurcation:

$$\frac{\partial^2 m_x}{\partial x^2} = -P_x = (M_x^{i+1} - 2M_x^i + M_x^{i-1})/\delta_s^2,$$

$$\frac{\partial^2 m_y}{\partial y^2} = -P_y = (M_y^{j+1} - 2M_y^j + M_y^{j-1})/\partial_s^2,$$

$$\frac{2\partial^2 m_{xy}}{\partial x \partial y} = -P_{xy} = (-M_{xy}^{i-1,j+1} + M_{xy}^{i+1,j+1} + M_{xy}^{i-1,j-1} - M_{xy}^{i+1,j-1})/2\delta_s^2,$$

with the additional constraint

$$P_x + P_y + P_{xy} = q.$$

ii. *Boundary conditions.* For the contour of the slab octant, we have:

At the free edge $M_x^{1\cdot 0,j} = 0.0$;

Along the diagonal $M_x^{i,j} = M_y^{i,j}$ for $i = j$;

Along the $X$-axis $M_{xy}^{i,0\cdot 0} = 0.0$.

iii. *Yield criteria.* The yield criteria for a concrete slab is shown in Fig. 1. The equation of the yield surface was given by NIELSEN, Ref. [12],

$$\frac{1}{2}(M_x + M_y) \pm \left[\frac{1}{4}(M_x - M_y)^2 + M_{xy}^2\right]^{\frac{1}{2}} \leqslant \pm M_p.$$

For the purpose of illustration, the numeric value $M_p = +1.0$ is assumed in the constitutive equation. No negative steel is provided.

### 3.2. Stochastic aspect

The stochastic form of the problem is assembled by imbedding the deterministic initial matrix with random variates conforming to prespecified probability density functions.

In the following, *a priori* $RN1, \ldots, RN5$ are arbitrarily taken, and the replacement statements used in the system definition of the analysis problem are also given

$$M_x \to M_x(1 + RN1),$$

where $RN1$ = normal scatter of $M_x$, range = $\pm 0.05$, variance = 0.0083. The range used is assumed to include $\pm 2\times$ (standard deviation)

$$M_y \to M_y(1 + RN2),$$

where $RN2$ = negative lognormal scatter of the flexural bending moment normal to the $Y$-axis, range = $\begin{matrix} +0.0 \\ -0.1 \end{matrix}$

$$M_{xy} \to M_{xy}(1 + RN3),$$

where $RN3$ = uniform scatter of the torsional moment, range $\pm 0.01$,

$$q \to q(1 + RN4),$$

where $RN4$ = normal scatter of the essentially-uniform load range = $\pm 0.02$, variance = = 0.004,

$$M_p \rightarrow M_p(1+RN5),$$

where $RN5$ = lognormal scatter of the plastic capacity of the slab, range = $\begin{matrix} +0.2 \\ -0.0 \end{matrix}$.

The solution in standard canonical form becomes the following:

1. Objective function = $-q$;

2. Constraints:

(a) Equality

equilibrium equations

$$(M_x^{i+1}-2M_x^i+M_x^{i-1})(1+RN1)/\delta_s^2 = -P_x,$$

$$(M_y^{j+1}-2m_y^j+m_y^{j-1})(1+RN2)/\delta_s^2 = -P_y,$$

$$(M_{xy}^{i-1,j-1}-M_{xy}^{i-1,j+1}+M_{xy}^{i+1,j+1}-M_{xy}^{i+1,j-1})(1+RN3)/2\delta_s^2 = -P_{xy},$$

$$P_x+P_y+P_{xy} = q(1+RN4),$$

boundary conditions

$$M_x^{1.0,j}(1+RN1) = 0.0,$$

$$M_y^{i,j}(1+RN2) = M_x^{i,j}(1+RN1), \quad i=j,$$

$$M_{xy}^{i,0.0}(1+RN3) = 0.0.$$

(b) Inequality

yield criteria

$$0.0 \leqslant \frac{1}{2}[M_x(1+RN1)+M_y(1+RN2)] \pm \left\{\frac{1}{4}[M_x(1+RN1)-M_y(1+RN2)]^2+[M_{xy}(1+RN3)]^2\right\}^{\frac{1}{2}} \leqslant (1+RN5).$$

The solution method is to pulsate the feasible set input to the S.U.M.T. routine by perturbations generated in the levels of the stochastic parameters ($M_x$, $M_y$, $M_{xy}$, $q$, $M_p$), according to the prespecified probability density functions. This is effectively a Monte Carlo sampling technique coupled with non-linear mathematical programming.

More discussion of this approach is given in Ref. [5], together with methods of estimating simulation accuracy, the use of statistics of extremes to simplify details in distribution tails, and also the use of variance reducing techniques and experimental design techniques to limit the number of discrete simulations required to detail the convolved probability spectrum of the dependent variable. In addition, methods of estimating reliability are indicated, together with convenient methods for including temporal and/or spatial cross correlations between variates.

A convolved histogram of relative frequencies for yield load is shown in Fig. 3 for a sampling size of 100.

Closed form analytical verification of the given stochastic slab solution is at present impossible, and the testing of enough slabs to permit empirical estimation of probability density functions of collapse loads is an economic impossibility. Attempts have been made

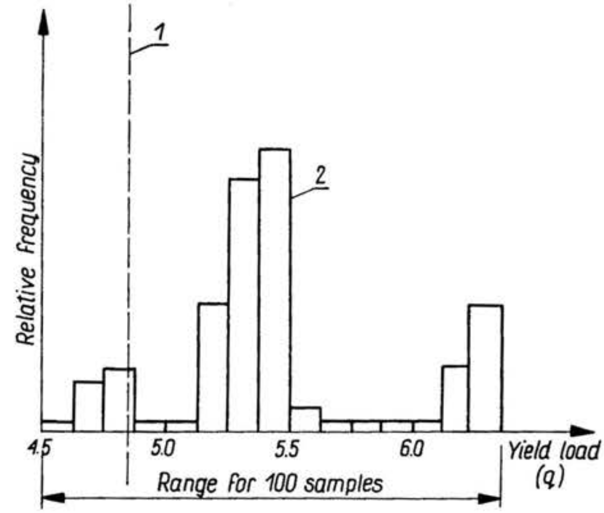


FIG. 3. Probability distribution of the lower bound plastic collapse load for a square simply-supported slab.
*1*–deterministic yield load for linearised system, *2*–histogram for 100 samples.

with slabs to verify individual experimental solutions using the deterministic mathematical programming approach, Ref. [4], and to verify programming solutions, Ref. [5], by reference to standard deterministic analytical solutions, Ref. [13].

Using affine convolution and optimization procedures, stochastic optimal design and upper bound analysis may be performed on structural systems by applying mathematical programming and Monte Carlo simulation to the mechanical ensembles outlined below.

## Related topics

*Optimal design.* Optimal design for minimum reinforcement volume may also be attempted using mathematical programming, Ref. [14], i.e.,

$$\iint (A_1+A_2)\,dx\,dy$$

is to be minimized when $A_1, A_2$ = volumes of orthogonal sets of reinforcement per unit area.

Analytical solutions to this problem have been obtained by ROZVANY, Ref. [15], and MORLEY, Ref. [16], by using the following two-dimensional subset of the yield hypersurface,

$$f(M_x, M_y) \subset f(M_x, M_y, M_{xy}),$$

and planforms of simply geometry.

The neglect of torsional moments is a necessity for mathematical tractability.

WOLFENSBERGER, Ref. [10], has treated a linearised version of the complete problem numerically.

Prestressed slabs may be handled in a similar way, using yield criteria after MORLEY, Ref. [17], and equilibrium equations given in Ref. [11]. It should be noted, however, that these yield criteria become invalid for semi-monocoque structures, where large in-plane forces together with bending and twisting movements exist — in fact, the presence of large in-plane forces in shells or prestressed slabs presents substantial difficulties both to yield line thoery, Ref. [18], and to limit analysis using mathematical programming, Ref. [19].

Ribbed shell or waffle slab constructions may also be investigated using the limit analysis theories and mathematical programming to determine stress and velocity fields — however the geometrical orthotropy presents extra problems, Ref. [20].

*Upper bound analysis.* Upper bound analysis by mathematical programming has been developed by LANCE, Ref. [21] (design techniques are invariably lower bound). The problem is to minimize the collapse load for all kinematically admissible velocity fields such that the specific power of dissipation is everywhere non-negative and the loading and continuity conditions are satisfied. The disadvantage of this approach is that a yield mechanism must be assumed. Thus as incorrect choice of the position of yield hinges results in an optimistic (unsafe) estimation of the collapse load. Johansen's yield-line theory for plane continua suffers from the same deficiency, although, as with the programming approach, a concomitant minimization of the collapse load for an assumed yield mechanism, defined in terms of pattern parameters may be attempted, Ref. [22]. BIRON *et al.*, Ref. [23], have obtained satisfactory upper bounds for the collapse loads of shells.

## 4. Silo analysis

### 4.1. Deterministic problem

An example is now given of the stochastic lower-bound analysis of a circular cylindrical silo.

i. *Equilibrium equations.* SAWCZUK *et al.*, Ref. [24], give specific dimensionless equilibrium equations for a cylindrical shell. Typical particular values for various parameters were assumed for the purpose of illustration: $C^2 = 10.0$, $t = 1.0$, $\mu L/R = 1.0$. The equilibrium equations must be expressed in finite-difference form to permit machine manipulation, Ref. [4].

ii. *Boundary conditions.* For a rigid-plastic material, forced boundary conditions have no significance, Ref. [10]. If the silo is assumed encastre at the base, and free at the top edge, no natural boundary conditions are required at a fixed support, but two are required at a free support, viz. zero moment and shear. In addition, axial force is zero at the top edge.

iii. *Yield criteria.* Four yield criteria for reinforced concrete cylindrical shells are given by SAWCZUK *et al.*, Ref. [25]. A representative yield surface is shown in Fig. 1, with typical specific parameter values $\alpha = 0.0$, $U_x = U_\theta = 0.1$, $\beta = 15.0$.

Constitutive equations may be further simplified if fourth order terms are neglected.

#### 4.2. Stochastic aspect

For the purpose of illustration, the following replacement:
statements were used:

$$p_\infty \leftarrow p_\infty(1+RN1), \quad m \leftarrow m(1+RN2),$$

$$n_x \leftarrow n_x(1+RN3), \quad n_\theta \leftarrow n_\theta(1+RN4).$$

In addition, the free upper edge may be restrained by a cover, and so a partially pinned condition may be achieved;

$$m^{0.0} - m^{0.2}/2\delta = 0.0 - RN5.$$

$RN1, RN2, \ldots, RN5$ are stochastic effects as defined in the slab analysis.

The solution in standard canonical form becomes the following:

1. Objective function $= -P_\infty$;

2. Constraints

(a) equality:

two sets of six equilibrium equations

$$[(1+RN2)(m^{z-1} - 2M^z + m^{z+1})/\delta^2] + 10n_\theta(1+RN4) = 10.P_\infty(1+RN1)1 - e^{-z}),$$

$$[(1+RN3)(n_x^{z+1} - n_x^{z-1})/2\delta] + P_\infty(1+RN1)(1-e^{-z}) = 0.0;$$

three boundary conditions

$$m^{0.0}(1+RN2) = 0.0,$$

$$\frac{(1+RN2)(m^{0.2} - m^{0.0})}{2\delta} = 0.0 + RN5,$$

$$n^{0.0}(1+RN3) = 0.0;$$

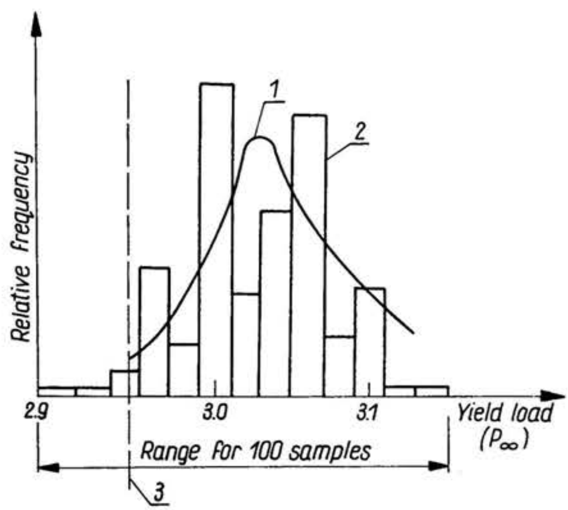


FIG. 4. Probability distribution of the lower bound plastic collapse load for a circular cylindrical silo. *1*–frequency = $6.1e^{-206}(P_\infty - 3.0)^2$, *2*–histogram for 100 samples, *3*–deterministic yield load for linearised system.

(b) inequality:
six of each of the following yield criterions

$$2n_x(1+RN3)[n_x(1+RN3)-1] \leqslant m(1+RN2) \leqslant -2n_x(1+RN3)[n_x(1+RN3)-1];$$
$$6.0\big(n_x(1+RN3)+1\big)-1 \leqslant n_\theta(1+RN4) \leqslant 1.5.$$

The solution method is as for the slab. A histogram of collapse loads ($q$) is shown in Fig. 4 for a sample size 100. A non-linear regression model has been fitted to the results to facilitate subsequent description, Ref. [4, 6].

#### 4.3. Related topics

*Optimal design.* The problem of optimal lower bound design may be handled as follows: The objective is, for example, to minimize the reinforcement volume for a given concrete thickness, i.e.

$$V = 2\pi R \int_{0.0}^{1.0} (F_x+F_\theta)dz$$

is to be minimized (where $F_x$, $F_\theta$ are interpreted as reinforcement volumes per unit surface area of the silo). This extremization procedure is constrained by the equilibrium equations and boundary conditions, and is for a given applied load. The integral above is easily expressed in quadrature form, to permit programming solution, Ref. [26].

The major difficulty in the optimal design of silos is that both force and moment stress resultants act in the $x$ — direction: so $F_x$ is not easily expressed in terms of $n_x$ and $m$, the actual relation depending on whether tensile or compressive failure arises, Ref. [27, 28]. In general,

$$F_x = F_x(m, n_x), \qquad F_\theta \alpha n_\theta,$$
$$V = 2\pi R \int_{0.0}^{1.0} (F_x(|m|, n_x)+c_1 n_\theta)dz,$$

where $c_1$ = constant.

It seems that iterative back-substitution is required in this direct design procedure, and the advantages over iterated analysis are less definite. The absolute values of moment required may be handled by means of an extra dummy negative variable, e.g. $m = \hat{m}^+ - m^-$. Thus, because the moment signs may be positive or negative, the actual number of moment variables is doubled, although half of these will be zero in the final solution.

An alternative approach is to neglect axial forces initially — thus to assume $F_x \alpha m/d$ (approx.), where steel percentages are small, Ref. [27], and $d$ = effective depth of the reinforcing. Thus

$$V = 2\pi R c_2 \int_{0.0}^{1.0} \left(\frac{|m|}{d}+n_\theta\right) dz;$$

here $c_2$ = constant.

The problem is thus solved to a first order approximation using simplified equilibrium equations and boundary conditions and subsequently checked for inclusion of $n_x$. The approach would only be valid for short shells (e.g. $c^2 < 25$ for $t = 1.0$, Ref. [24]), in which axial forces are relatively small.

The inclusion of stochastic variables into the design procedure may again be accomplished by Monte Carlo techniques and probability spectra of the stress resultants may be found.

## 5. Conclusion

Systematic application of stochastic programming algorithms to structural ensembles should provide a very fruitful area for future research and solutions to hitherto intractable problems.

## Acknowledgements

The author is indebted to the Civil Engineering School, Monash University, for interest in this project, and to Miss. M. Wynn for help in preparation of the manuscript.

## References

1. Comité Europeen du Beton, *Recommendations for an international code of practice for reinforced concrete* [English translation], 156 p., A.C.I. and C. and C.A., London.
2. H. SAWYER, *Status and potentiality of non-linear design of concrete frames*, Proc. Sym. Flex. Mech. of Reinforced Concrete, Sp. Pub. SP-12, A.C.I., 7–28, Detroit 1965.
3. M. MUSPRATT, *Shakedown of steel plates*, J. Appl. Mech. (ASME) **38**, 4, 1088–1090, 1971.
4. M. MUSPRATT, *Behaviour of footing slabs*, Build. Sci., **4**, 3, 108–118, 1969.
5. M. MUSPRATT, *Behaviour of concrete slabs*, Ph.D. Thesis, Dept. of Civil Eng. Monash University, December 1969.
6. M. MUSPRATT, *Stochastic plastic analysis of a shell*, Int. J. Num. Methods in Eng., **4**, 3, 315–333, 1972.
7. L. SCHMIT, W. MORROW and T. KICHER, *A structure synthesis capability of integrally stiffened cylindrical shells*, AIAA/ASME 9th Struct. Conf., Palm Springs, Calif., April 1968.
8. P. HODGE and A. BIRON, *Limit analysis of rotationally symmetric shells through nonlinear programming*, DOMIIT Rept. No. 1–31, Tll. Inst. Tech., January 1967.
9. R. KOOPMAN and R. LANCE, *On linear programming and plastic limit analysis*, J. Mech. Phys. Sol., **13**, 77–87, 1965.
10. R. WOLFENSBERGER, *Ultimate load and optimal design of slabs* [in German], Inst. Struct. Mechs., E.T.H., Zurich, June 1964.
11. S. TIMOSHENKO and S. WOINOWSKY-KRIEGER, *Theory of plates and shells*, p. 580, McGraw-Hill, New York 1959.
12. M. NIELSEN, *Limit analysis of reinforced concrete slabs*, Acta Polytechnica Scandinavica, Civ. Eng. and Build. Const., N. 26, 167p., 1964.
13. A. SAWCZUK and T. JAEGER, *Grenztragfähigkeitstheorie der Platten*, 522p., Springer Verlag, Berlin 1963.
14. H. CHAN, *Mathematical programming in optimal design*, J. Mech. Phys. Sol., **4**, 9, 885–895, September 1968.
15. G. ROZVANY, *Rational approach to plate design*, A.C.I., **63**, 10, October 1965.
16. C. MORLEY, *The minimum reinforcement of concrete slabs*, Int. J. Mech. Sci., **8**, 305–319, 1966.

17. C. MORLEY, *On the yield criterion of an orthogonally reinforced slab element*, J. Mech. Phys. Sol., **14**, 33–47, 1966.
18. K. NIELSEN, *Limit analysis of reinforced concrete shells*, in "Non-classical Shell Problems" ed. by W. OLSZAK and A. SAWCZUK, 1185 p., North-Holland Pub. Co., Amsterdam 1964.
19. R. LANCE, C. LEE, *The yield point load of a conical shell*, Int. J. Mech. Sci., **11**, January 1969.
20. V. NIEMIROVSKII and N. RABOTNOV, *Limit analysis of ribbed plates and shells*, in "Non-classical Shell Problems", North-Holland Pub. Co., Amsterdam 1964.
21. R. LANCE, *On automatic construction of velocity fields in plastic limit analysis*, Acta Mechanica, **3**, 22–33, 1966.
22. R. JOHNSON, *Structural concrete*, 271p., McGraw-Hill, London 1967.
23. A. BIRON and P. HODGE, *Non-linear programming method for limit analysis of rotationally symmetric shells*, Int. J. Non-linear Mech., **3**, 2, 201–213, June 1968.
24. A. SAWCZUK and J. KÖNIG, *Limit analysis of a reinforced concrete silo* [in Polish], Arch. Inż. Ląd., **8**, 161–183, 1962.
25. A. SAWCZUK and W. OLSZAK, *A method of limit analysis of reinforced concrete tanks*, Simpl. Shell Calc. Methods, 416–437, North-Holland Pub. Co., Amsterdam 1962.
26. M. MUSPRATT, *Optimal design of axisymmetric slabs*, Israel J. of Tech., **9**, 6, 559–564, 1971.
27. American Concrete Institute, *Building code requirements for reinforced concrete*, ACI — 318–63, June 1963.
28. American Concrete Institute, *Ultimate strength design of reinforced concrete columns*, Special Publication SP–7, 1964.

RESEARCH ASSOCIATE
DEPARTMENT OF CIVIL ENGINEERING, McGILL UNIVERSITY
MONTREAL, P.Q., CANADA

*Received October* 30, 1971.