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Equations of motion and laws of conservation in the discrete elasticity

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IN THE PAPER [1] the foundations of the mechanics of discretized bodies (i.e. the bodies obtained by the process of discretization [5]) were formulated. If such bodies are elastic we arrive at the equations of discrete elasticity, analised in [2]. In the present note, the equation of motion and the laws of conservation are analysed in the general case and in some special cases of elastic discretized bodies.

W pracy [1] sformułowano podstawy mechaniki ciał dyskretyzowanych (otrzymanych w procesie dyskretyzacji [5]). Gdy ciało dyskretyzowane jest sprężyste, to opisujące je równania nazywamy równaniami dyskretnej teorii sprężystości [2]. W tej pracy wyprowadzono równania ruchu i prawa zachowania dla przypadku ogólnego i niektórych szczególnych przypadków dyskretyzowanych ciał sprężystych.

В работе [1] сформулированы основы механики дискретизированных тел, получаемых в процессе дискретизации [5]. В случае, когда дискретизированное тело является упругим, описывающие его уравнения будем называть уравнениями дискретной теории упругости [2]. В данной работе выведены уравнения движения и законы сохранения, как для общего случая, так и для некоторых частных случаев дискретизированных упругих тел.

Notations

The indices $\Lambda, \Phi, \ldots$ run over the sequence $\mathrm{I}, \mathrm{II}, \ldots, m$, the indices $a, b, \ldots$ take the values $1, 2, \ldots, n$ and the indices $k, l, \ldots$ run over the sequence $1, 2, 3$. The summation convention holds for all kinds of indices.

1. General form of the conservation laws

THE SUBJECT of our considerations is the elastic discretized body (the elastic discrete medium) defined in [1] as the body obtained in the process of discretization [5]. Such body is a pair $(D, \mathscr{E})$, where each $d \in D$ is the holonomic dynamic system with n degrees of freedom (a particle of the discretized body), each $E \in \mathscr{E}$ is a given subset of D (a discrete element), and $\mathscr{E}$ is a covering of D, and, moreover, to each $E \in \mathscr{E}$ we assign the potential ε_E which determines the internal forces among particles of the subset $E \subset D$. To simplify the considerations, we assume in what follows that the global difference structure can be prescribed on $(D, \mathscr{E})$ [1]. Using this structure, we are able to write the following equations of motion [1]

$$
\begin{aligned}
&\bar{\Delta}_\Lambda T_a{}^\Lambda(d, \tau) + t_a(d, \tau) + f_a(d, \tau) = r_a(d, \ldots), \\
(1.1) \qquad &r_a(d, \ldots \stackrel{\mathrm{df}}{=} \frac{d}{d\tau} \frac{\partial T(d, \ldots)}{\partial \dot{q}^a(d, \tau)} - \frac{\partial T(d, \ldots)}{\partial q^a(d, \tau)}, \quad d \in D,
\end{aligned}
$$

and the constitutive equations [1]

$$T_a{}^A(d, \tau) = \frac{\partial \varepsilon(d, \ldots)}{\partial \Delta_A q^a(d, \tau)}, \quad t_a(d, \tau) = -\frac{\partial \varepsilon(d, \ldots)}{\partial q^a(d, \tau)}, \quad d \in D, \tag{1.2}$$

of the elastic discretized body. We assume that

$$\begin{aligned} \varepsilon(d, \ldots) &= \varepsilon\big(d, q^a(d,\tau), \Delta_A q^a(d, \tau)\big) = \varepsilon_E \text{ when } d \in D_*, \\ &\qquad \varepsilon(d, \ldots) = 0 \text{ when } d \sim \in D^*, \\ T(d, \ldots) &= T\big(d, q^a(d, \tau), \dot{q}^a(d, \tau)\big), \end{aligned} \tag{1.3}$$

where $\varepsilon(d, \ldots) = \varepsilon_E$ is the elastic potential at the discrete element $E = E_d$, $d \in D_*$, and $T(d, \ldots)$ is the kinetic energy of the particle $d \in D$, and where $q^a(d, \tau)$, $a = 1, 2, \ldots, n$, are independent generalized coordinates of the particle $d \in D$. Moreover, the functions $T_a{}^A(d, \tau)$, $t_a(d, \tau)$ represent the forces among the particles of the discrete element E_d, and the functions $f_a(d, \tau)$ are generalized external forces acting at the particle $d \in D$. The Eqs. (1.1) and (1.2) were introduced in [1], where also some examples of the discretized bodies were given. The Eqs. (1.1) and (1.2) are said to be the equations of discrete elasticity [2, 4]. Some special problems of discrete elasticity have been studied in [2, 4, 6, 7]; in this note, the equations of motion and the conservation laws of the discrete elasticity will be analysed.

Let us denote by z^k, $k = 1, 2, 3$, the inertial Cartesian orthogonal coordinates in the physical space. The infinitesimal translations and rotations of the physical space are given by the transformation formulas

$$z^k \to z^k + \epsilon^k + \epsilon^{kl} z_l,$$

where ϵ^k, $\epsilon^{kl} = -\epsilon^{lk}$, are arbitrary infinitesimal constants. Let us assume that the variations of the dynamic variables $q^a(d, \tau)$ due to the translations and rotations of the physical space are given by

$$q^a(d, \tau) \to q^a(d, \tau) + \epsilon^k C_k^a + \epsilon^{kl} C_{kl}^{ab} q_b(d, \tau), \quad q_b \equiv q^b, \tag{1.4}$$

where C_k^a, $C_{kl}^{ab} = -C_{lk}^{ab}$ are constants and $C_{kl}^{ab} = 0$ for $a \neq b$. Assuming that the elastic potential $\varepsilon(d, \ldots)$ and the kinetic energy $T(d, \ldots)$ are invariant under arbitrary translations and rotations of the physical space, we arrive at

$$\begin{gathered} C_k^a \frac{\partial \varepsilon}{\partial q^a} = 0, \qquad C_k^a \frac{\partial T}{\partial q^a} = 0, \\ C_{kl}^{ab}\left(\frac{\partial \varepsilon}{\partial q^a} q^b + \frac{\partial \varepsilon}{\partial \Delta_A q^a} \Delta_A q_b\right) = 0, \quad C_{kl}^{ab}\left(\frac{\partial T}{\partial q^a} q^b + \frac{\partial T}{\partial \dot{q}^a} \dot{q}^b\right) = 0, \end{gathered} \tag{1.5}$$

for each $d \in D$. We also assume that the functions $\varepsilon(d, \ldots)$, $T(d, \ldots)$, $d \in D$, are invariant under an arbitrary "translation" of the time coordinate $\tau \to \tau + \epsilon$, where ϵ is an arbitrary infinitesimal constant. It follows that

$$\frac{\partial \varepsilon}{\partial \tau} = 0, \quad \frac{\partial T}{\partial \tau} = 0 \quad \text{or} \quad \dot{\varepsilon} = \frac{\partial \varepsilon}{\partial q^a} \dot{q}^a + \frac{\partial \varepsilon}{\partial \Delta_A q^a} \Delta_A \dot{q}^a, \tag{1.6}$$

$$\dot{T} = \frac{\partial T}{\partial q^a} \dot{q}^a + \frac{\partial T}{\partial \dot{q}^a} \ddot{q}^a.$$

The Eqs. (1.5) and (1.6) are the sufficient conditions for existence of the conservation laws in discrete elasticity. By virtue of (1.5), (1.6) and rewriting the Eqs. (1.1), (1.2) in the form

$$\frac{\partial \varepsilon}{\partial q^a} = \bar{\Delta}_A T_a{}^A + f_a - \frac{d}{d\tau}\frac{\partial T}{\partial \dot{q}^a} + \frac{\partial T}{\partial q^a},$$

we obtain

$$
\begin{aligned}
&\frac{d}{d\tau}\left(C_k^a \frac{\partial T}{\partial \dot{q}^a}\right) = C_k^a \bar{\Delta}_A T_a{}^A + C_k^a f_a,\\
(1.7)\qquad &\frac{d}{d\tau}\left(C_{kl}^{ab}\frac{\partial T}{\partial \dot{q}^a} q_b\right) = C_{kl}^{ab}(\bar{\Delta}_A T_a{}^A q_b + T_a{}^A \Delta_A q_b) + C_{kl}^{ab} f_a q_b,\\
&\frac{d}{d\tau}\left(\varepsilon + \frac{\partial T}{\partial \dot{q}^a}\dot{q}^a - T\right) = \bar{\Delta}_A T_a{}^A \dot{q}^a + T_a{}^A \Delta_A \dot{q}^a + f_a \dot{q}^a, \quad d \in D.
\end{aligned}
$$

If the particle d is the scleronomic holonomic dynamic system, then the kinetic energy $T(d, \ldots)$ is the homogeneous quadratic form of the generalized valocities and we can write:

$$(1.8)\qquad 2T = \frac{\partial T}{\partial \dot{q}^a}\dot{q}^a.$$

Let us denote $D_0 \overset{\mathrm{df}}{=} \bigcap_{A=I}^{m} (D_A \cap D_{-A})$ and assume that $D_0 \neq \phi$. It can be verified that for arbitrary real-valued functions $\varphi^A : D \to R$, $\zeta : D \to R$, the following indentities hold

$$(1.9)\qquad \zeta \bar{\Delta}_A \varphi^A + \varphi^A \Delta_A \zeta = \Delta_A(\zeta \bar{\varphi}^A), \quad \bar{\Delta}_A \varphi^A = \Delta_A \bar{\varphi}^A,$$

for each $d \in D_0$; in (1.9) we have denoted $\bar{\varphi}^A(d) \overset{\mathrm{df}}{=} \varphi^A(f_{-A}d)$. By virtue of (1.8) and (1.9), the Eqs. (1.7) can be transformed to the form

$$
\begin{aligned}
&\frac{d}{d\tau}\left(C_k^a \frac{\partial T}{\partial \dot{q}^a}\right) = C_k^a \Delta_A \bar{T}_a{}^A + C_k^a f_a,\\
(1.10)\qquad &\frac{d}{d\tau}\left(C_{kl}^{ab}\frac{\partial T}{\partial \dot{q}^a} q_b\right) = C_{kl}^{ab}\Delta_A(\bar{T}_a{}^A q_b) + C_{kl}^{ab} f_a q_b,\\
&\frac{d}{d\tau}(T+\varepsilon) = \Delta_A(\bar{T}_a{}^A \dot{q}^a) + f_a \dot{q}^a, \quad d \in D_0.
\end{aligned}
$$

The expressions in parenthesis on the left-hand sides of the Eqs. $(1.10)_{1,2}$ are the momentum and the moment of momentum of the particle $d \in D_0$, respectively. The expression in parenthesis on the left-hand sides of $(1.10)_3$ is the sum of the kinetic energy of the particle d and internal energy of the discrete element E_d. The formulas (1.10) represent the local form of the conservation laws in discrete elasticity. To obtain the global form of these laws, we have to introduce some auxiliary concepts. Let us denote by K an arbitrary subset of D_0, and let us define the subset ΔK, putting

$$d \in \Delta K : [(d \in K) \wedge (\bigvee_A f_{-A} d \sim \in K)] \vee [(d \sim \in K) \wedge (\bigvee_A f_{-A} d \in K)].$$

The subset ΔK is said to be the Δ-boundary of K, and depends on the permissible difference structure on $(D, \mathscr{E})$. Let $\varphi^A: D_0 \to R$, $A = \mathrm{I}, \mathrm{II}, \ldots, m$, be arbitrary real-valued functions. We can verify that the following identities hold

$$\sum_K \Delta_A \varphi^A = \sum_{\Delta K} \varphi^A N_A,$$

for each $K \subseteq D_0$, where we have denoted

$$N_A = N_A(d) = \begin{cases} 1 \text{ when } (d \sim \in K) \wedge (f_{-A} d \in K), \\ -1 \text{ when } (d \in K) \wedge (f_{-A} d \sim \in K), \\ 0 \text{ in other cases.} \end{cases}$$

Using the formulas given above, we arrive at the global form of the conservation laws in discrete elasticity:

$$\begin{aligned} &\Big[\sum_K C_k^a \frac{\partial T}{\partial \dot{q}^a}\Big]_{\tau_0}^{\tau_1} = \sum_{\Delta K} \int_{\tau_0}^{\tau_1} C_k^a T_a^{(\mathrm{N})} d\tau + \sum_K \int_{\tau_0}^{\tau_1} C_k^a f_a d\tau, \\ (1.11)\qquad &\Big[\sum_K C_{kl}^{ab} \frac{\partial T}{\partial \dot{q}^a} q_b\Big]_{\tau_0}^{\tau_1} = \sum_{\Delta K} \int_{\tau_0}^{\tau_1} C_{kl}^{ab} T_a^{(\mathrm{N})} q_b d\tau + \sum_K \int_{\tau_0}^{\tau_1} C_{kl}^{ab} f_a q_b d\tau, \\ &\Big[\sum_K (T+\varepsilon)\Big]_{\tau_0}^{\tau_1} = \sum_{\Delta K} \int_{\tau_0}^{\tau_1} T^{(\mathrm{N})} \dot{q}^a d\tau + \sum_K \int_{\tau_0}^{\tau_1} f_a \dot{q}^a d\tau, \end{aligned}$$

where

$$T_a^{(\mathrm{N})} = \overline{T}_a{}^A N_A.$$

The equations of motion and the laws of conservation can be obtained from the variational approach. We shall see that the action functional has to be assumed in the form:

$$\mathscr{W}(K) = \int_{\tau_0}^{\tau_1} \sum_K (T-\varepsilon) d\tau. \tag{1.12}$$

The total variation $\delta\mathscr{W}(K)$ of the action functional is equal to

$$\begin{aligned} (1.13)\qquad \delta\mathscr{W}(K) = \sum_K \int_{\tau_0}^{\tau_1} \Big(&\frac{\partial(T-\varepsilon)}{\partial q^a} \delta_0 q^a + \frac{\partial(T-\varepsilon)}{\partial \Delta_A q^a} \delta_0 \Delta_A q^a \\ &+ \frac{\partial(T-\varepsilon)}{\partial \dot{q}^a} \delta_0 \dot{q}^a + \frac{d(T-\varepsilon)}{d\tau} \delta\tau \Big) d\tau = \sum_K \int^{\tau_1} (\overline{\Delta}_A T_a{}^A - r_a) \delta_0 q^a d\tau \\ &+ \sum_K \Big[\frac{\partial T}{\partial \dot{q}^a} \delta_0 q^a + (T-\varepsilon)\delta\tau\Big]_{\tau_0}^{\tau_1} + \sum_\tau \int^{\tau_1} T_a^{(\mathrm{N})} \delta_0 q^a d\tau, \end{aligned}$$

where $\delta_0 q^a$ are variations of the dynamic variables $q^a(d, \tau)$, due to a change in the functional form of the functions $q^a(d, \tau)$, and $\delta\tau$ is the variation of the time coordinate. If the external forces are absent, then, from the principle of stationary action, we obtain:

$$\sum_K \int_{\tau_0}^{\tau_1} (\bar{\Delta}_\Lambda T_a{}^\Lambda - r_a)\delta_0 q^a d\tau = 0 \tag{1.14}$$

for an arbitrary subset $K \subseteq D_0$. It follows that

$$\bar{\Delta}_\Lambda T_a{}^\Lambda - r_a = 0.$$

If the external forces acting at the particles are present, then the right-hand sides of the equations given above are not equal to zero; denoting them by $-f_a$, we obtain:

$$\bar{\Delta}_\Lambda T_a{}^\Lambda - r_a = -f_a.$$

Thus we have derived the equations of motion (1.1) which were obtained in [1] in a different manner. If these equations are satisfied, we may rewrite the variation (1.11) of the action functional in the form:

$$\delta \mathscr{W}(K) = -\sum_K \int_{\tau_0}^{\tau_1} f_a \delta_0 q^a d\tau + \sum_K \left[\frac{\partial T}{\partial \dot{q}^a}\delta_0 q^a + (T-\varepsilon)\delta\tau\right]_{\tau_0}^{\tau_1} + \sum_{\Delta K} \int_{\tau_0}^{\tau_1} T_a^{(N)}\delta_0 q^a d\tau. \tag{1.15}$$

Let us assume now that the action functional $\mathscr{W}(K)$ is invariant under a group of infinitesimal transformations $z^k \to z^k + \epsilon^k + \epsilon^{kl} z_l$, $\tau \to \tau + \epsilon$, of the space-time. This means that the relation $\delta\mathscr{W}(K) = 0$ holds when $\delta q^a = \delta_0 q^a + \dot{q}^a \delta\tau = C_k^a \epsilon^k + C_{kl}^{ab} q_b \epsilon^{kl}$, $\delta\tau = \epsilon$, where ϵ^k, $\epsilon^{kl} = -\epsilon^{lk}$, ϵ are arbitrary constants. Substituting into $(1.15)_1$ the right-hand sides of $\delta_0 q^a = C_k^a \epsilon^k + C_{kl}^{ab} q_b \epsilon^{kl} - \dot{q}^a \epsilon$ and $\delta\tau = \epsilon$, we obtain $\delta\mathscr{W}(K) = 0$ for arbitrary ϵ^k, $\epsilon^{kl} = -\epsilon^{lk}$, ϵ. It follows that

$$\sum_K \int_{\tau_0}^{\tau_1} \frac{\partial(T-\varepsilon)}{\partial q^a} C_k^a d\tau = 0,$$

$$\sum_K \int_{\tau_0}^{\tau_1} \left(\frac{\partial(T-\varepsilon)}{\partial q^a} q_b + \frac{\partial(T-\varepsilon)}{\partial \Delta_\Lambda q^a}\Delta_\Lambda q_b + \frac{\partial(T-\varepsilon)}{\partial \dot{q}^a}\dot{q}_b\right) C_{(kl)}^{ab} d\tau = 0, \tag{1.16}$$

$$\sum_K \int_{\tau_0}^{\tau_1} \left(\frac{d(T-\varepsilon)}{d\tau} - \frac{\partial(T-\varepsilon)}{\partial q^a}\dot{q}^a - \frac{\partial(T-\varepsilon)}{\partial \Delta_\Lambda q^a}\Delta_\Lambda \dot{q}^a - \frac{\partial(T-\varepsilon)}{\partial \dot{q}^a}\ddot{q}^a\right) d\tau = 0$$

$$\text{or} \sum_K \int_{\tau_0}^{\tau_1} \frac{\partial(T-\varepsilon)}{\partial\tau} d\tau = 0.$$

The Eqs. (1.16) are said to be the strong conservation laws in discrete elasticity. They are the necessary and sufficient conditions for the weak conservation laws [3], given by the Eqs. (1.11). The conservation laws (1.11) can be obtained either from (1.16) and (1.1) or by substituting into (1.15) the right-hand sides of the expressions $\delta_0 q^a = C_k^a \epsilon^k + C_{kl}^{ab} q_b \epsilon^{kl} - \dot{q}^a \epsilon$, $\delta\tau = \epsilon$, and making use of the equality $\delta\mathscr{W}(K) = 0$, which holds for arbitrary ϵ^k, $\epsilon^{kl} = -\epsilon^{lk}$, ϵ.

2. Conservation laws of the discrete oriented media

The discretized body in which each particle is a set of $p+1$ free material points, $p > 0$, is said to be the discrete oriented (or multipolar) medium. The motion of an arbitrary particle can be given in the form:

$$z^k = \psi^k(d, \tau), \quad z^k = \psi^k(d, \tau) + d^k_{\mathfrak{a}}(d, \tau), \quad \mathfrak{a} = 1, 2, \ldots, p, \tag{2.1}$$

where z^k stand for orthogonal Cartesian coordinates in physical space and vectors with components $d^k_{\mathfrak{a}}(d, \tau)$ are called directors. Moreover, we assume that the discrete oriented medium is elastic and the elastic potential in a given coordinate system [1] has the form $\varepsilon = \varepsilon\big(d, \psi^k(d, \tau), \Delta_\Lambda \psi^k(d, \tau), d^k_{\mathfrak{a}}(d, \tau)\big)$. The indices $\mathfrak{a}, \mathfrak{b}$ in this Section have the range $1, 2, \ldots, p$, and we put $q^a(d, \tau) = \delta^a_k \psi^k(d, \tau) + \sum\limits_{\mathfrak{a}=l}^{p} \delta^a_{k+3\mathfrak{a}} d^k_{\mathfrak{a}}(d, \tau)$. The formulas (1.4) have the form:

$$\psi^k(d, \tau) \to \psi^k(d, \tau) + \in^k + \in^{kl} \psi_\Lambda(d, \tau), \quad d^k_{\mathfrak{a}}(d, \tau) \to d^k_{\mathfrak{a}}(d, \tau) + \in^{kl} d_{\mathfrak{a}l}(d, \tau), \tag{2.2}$$

where $\psi_l(d, \tau) \equiv \psi^l(d, \tau)$, $d_{\mathfrak{a}l}(d, \tau) \equiv d^l_{\mathfrak{a}}(d, \tau)$. It follows that

$$C^a_k = \delta^a_k, \quad C^{ab}_{kl} = \delta^a_k \delta^b_l + \sum_{\mathfrak{a}=l}^{p} \delta^a_{k+3\mathfrak{a}} \delta^b_{l+3\mathfrak{a}}, \quad a, b = 1, 2, \ldots, 3p+3. \tag{2.3}$$

The conditions (1.5) yield:

$$\begin{gathered} \frac{\partial \varepsilon}{\partial \psi^k} = 0, \quad \frac{\partial T}{\partial \psi^k} = 0, \\ \frac{\partial \varepsilon}{\partial \Delta_\Lambda \psi^{[k}} \Delta_\Lambda \psi^{l]} + \frac{\partial \varepsilon}{\partial d^{[k}_{\mathfrak{a}}} d_{\mathfrak{a}l]} = 0, \quad \frac{\partial T}{\partial d^{[k}_{\mathfrak{a}}} d^{\mathfrak{a}l]} + \frac{\partial T}{\partial \dot{\psi}^{[k}} \dot{\psi}_{l]} + \frac{\partial T}{\partial \dot{d}^{[k}_{\mathfrak{a}}} \dot{d}_{\mathfrak{a}l]} = 0, \end{gathered} \tag{2.4}$$

and will be satisfied if the kinetic energy is independent of ψ^k, $d^k_{\mathfrak{a}}$ and is the quadratic function of the velocities $|\dot{\psi}^k|$, $|\dot{d}^k_{\mathfrak{a}}|$, and if the elastic potential is assumed in the form:

$$\varepsilon = \varepsilon(d, \gamma_{\Lambda\Phi}, \gamma_{\mathfrak{a}\Lambda}, \gamma_{\mathfrak{a}\mathfrak{b}}), \tag{2.5}$$

where

$$\begin{aligned} \gamma_{\Lambda\Phi} &= \frac{1}{2} \Delta_\Lambda \psi^k(d, \tau) \Delta_\Phi \psi^l(d, \tau) \delta_{kl}, \\ \gamma_{\mathfrak{a}\Lambda} &= d^k_{\mathfrak{a}}(d, \tau) \Delta_\Lambda \psi^l(d, \tau) \delta_{kl}, \\ \gamma_{\mathfrak{a}\mathfrak{b}} &= \frac{1}{2} d^k_{\mathfrak{a}}(d, \tau) d^l_{\mathfrak{b}}(d, \tau) \delta_{kl}, \end{aligned} \tag{2.6}$$

are said to be stresses in the discrete element E_d [1]. The constitutive equations (1.2) can now be represented in the form:

$$\begin{aligned} T_a{}^\Lambda &= \delta^k_a T_k{}^\Lambda + \sum_{\mathfrak{a}} \delta^k_{a-3\mathfrak{a}} T^{\mathfrak{a}\Lambda}_k, \quad t_a = \delta^k_a t_k + \sum_{\mathfrak{a}} \delta^k_{a-3\mathfrak{a}} t^{\mathfrak{a}}_k; \\ T_k{}^\Lambda &= \frac{\partial \varepsilon}{\partial \gamma_{\Lambda\Phi}} \Delta_\Phi \psi_k + \frac{\partial \varepsilon}{\partial \gamma_{\mathfrak{a}\Lambda}} d_{\mathfrak{a}k}, \quad t_k = -\frac{\partial \varepsilon}{\partial \psi^k} = 0, \\ T^{\mathfrak{a}\Lambda}_k &= \frac{\partial \varepsilon}{\partial \Delta_\Lambda d^k_{\mathfrak{a}}} = 0, \quad t^{\mathfrak{a}}_k = -\frac{\partial \varepsilon}{\partial \gamma_{\mathfrak{a}\Lambda}} \Delta_\Lambda \psi_k - \frac{\partial \varepsilon}{\partial \gamma_{\mathfrak{a}\mathfrak{b}}} d_{\mathfrak{b}k}. \end{aligned} \tag{2.7}$$

Denoting $f_a = \delta^k_a f_k + \sum \delta^k_{a-3\mathfrak{a}} f^{\mathfrak{a}}_k$, we obtain from (1.1) the following form of the equations of motion:

$$\bar{\Delta}_A T_k{}^A + f_k = \frac{d}{d\tau}\frac{\partial T}{\partial \dot{\psi}^k}, \quad t^{\mathfrak{a}}_k + f^{\mathfrak{a}}_k = \frac{d}{d\tau}\frac{\partial T}{\partial \dot{d}^k_{\mathfrak{a}}}. \tag{2.8}$$

The strong conservation laws are satisfied identically and the weak conservation laws (1.10) are given by

$$\begin{gathered}\frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{\psi}^k}\right) = \Delta_A \bar{T}_k{}^A + f_k,\\ \frac{d}{d\tau}\left(\frac{\partial T}{\partial \dot{\psi}^{[k}}\psi_{l]} + \frac{\partial T}{\partial \dot{d}^{[k}_{\mathfrak{a}}} d_{\mathfrak{a}l]}\right) = \Delta_A(\bar{T}_{[k}{}^A \psi_{l]}) + f_{[k}\psi_{l]} + f^{\mathfrak{a}}_{[k} d_{\mathfrak{a}l]},\\ \frac{d}{d\tau}(T+\varepsilon) = \Delta_A(\bar{T}_k{}^A \dot{\psi}^k) + f_k \dot{\psi}^k + f^{\mathfrak{a}}_k \dot{d}^k_{\mathfrak{a}}.\end{gathered} \tag{2.9}$$

Now, let us introduce the following stress components [1]:

$$p^{A\Phi} = \frac{\partial \varepsilon}{\partial \gamma_{A\Phi}}, \quad p^{\mathfrak{a}A} = \frac{\partial \varepsilon}{\partial \gamma_{\mathfrak{a}A}}, \quad p^{\mathfrak{a}\mathfrak{b}} = \frac{\partial \varepsilon}{\partial \gamma_{\mathfrak{a}\mathfrak{b}}}. \tag{2.10}$$

Using (2.10) and (2.8), we arrive at

$$\bar{\Delta}_A(p^{A\Phi}\Delta_\Phi \psi_k + p^{\mathfrak{a}A} d_{\mathfrak{a}k}) + f_k = \frac{d}{d\tau}\frac{\partial T}{\partial \dot{\psi}^k}, \quad -p^{\mathfrak{a}A}\Delta_A \psi_k - p^{\mathfrak{a}\mathfrak{b}} d_{\mathfrak{b}k} + f^{\mathfrak{a}}_k = \frac{d}{d\tau}\frac{\partial T}{\partial \dot{d}^k_{\mathfrak{a}}}. \tag{2.11}$$

The "geometric" equations (2.6), the constitutive equations (2.10) and the equations of motion (2.11) form the alternative system of basic equations of discrete oriented elastic media. All the equations given above are also valid when $p = 0$ — i.e., when each particle of the discretized body is a free material point. Discrete oriented elastic media are analysed also in the paper [6].

3. Equations of variated states

Let there be given the motion $q^a(d, \tau)$, $d \in D$, of the given discretized body. Such motion will be called the fundamental motion. Now, we are to study the second motion

$$\overset{*}{q}{}^a(d, \tau) = q^a(d, \tau) + \epsilon\, w^a(d, \tau), \quad d \in D, \tag{3.1}$$

in which ϵ is the small parameter, i.e., the squares and the high powers of ϵ may be disregarded compared with ϵ. The set of functions $w^a(d, \tau)$ will be called the superposed motion. We assume that the superposed motion is independent of the fundamental motion. Denoting by $^*H$ an arbitrary quantity relating to the motion (3.1), we can write

$$^*H = H + \epsilon\, {}'H, \tag{3.2}$$

where H relates to the fundamental motion. From (3.2), we conclude that the variation on the elastic potential and the kinetic energy can be expressed as follows:

$${}'\varepsilon = \frac{\partial \varepsilon}{\partial q^a} w^a + \frac{\partial \varepsilon}{\partial \Delta_A q^a}\Delta_A w^a, \quad {}'T = \frac{\partial T}{\partial q^a} w^a + \frac{\partial T}{\partial \dot{q}^a}\dot{w}^a. \tag{3.3}$$

Next, we shall obtain:

$$'T_a{}^{\Lambda} = \frac{\partial' \varepsilon}{\partial \Delta_{\Lambda} q^a}, \quad 't_a = -\frac{\partial' \varepsilon}{\partial q^a}. \tag{3.4}$$

Hence, if we denote

$$K_{ab}^{\Lambda\Phi} = \frac{\partial^2 \varepsilon}{\partial \Delta_{\Lambda} q^a \partial \Delta_{\Phi} q^b}, \quad L_{ab}^{\Lambda} = \frac{\partial^2 \varepsilon}{\partial \Delta_{\Lambda} q^a \partial q^b}, \quad M_{ab} = \frac{\partial^2 \varepsilon}{\partial q^a \partial q^b}, \tag{3.5}$$

where the quantities (3.5) are given for each fundamental motion, we can transform (3.4) to the form:

$$\begin{aligned} 'T_a{}^{\Lambda} &= K_{ab}^{\Lambda\Phi} \Delta_{\Phi} w^b + L_{ab}^{\Lambda} w^b, \\ 't_a &= -L_{ba}^{\Phi} \Delta_{\Phi} w^b - M_{ab} w^b. \end{aligned} \tag{3.6}$$

Using (3.2) and (3.4), we can write

$$\bar{\Delta}_{\Lambda}{}' T_a{}^{\Lambda} + 't_a + 'f_a = 'r_a, \tag{3.7}$$

where

$$'r_a = \frac{d}{d\tau} \frac{\partial 'T}{\partial \dot{q}^a} - \frac{\partial 'T}{\partial q^a} = \frac{d}{d\tau}\left(\frac{\partial a_{ac}}{\partial q^b} q^c w^b + a_{ab} \dot{w}^b \right) - \left(\frac{\partial^2 a_{cd}}{\partial q^a \partial q^b} \dot{q}^c \dot{q}^d w^b + \frac{\partial a_{bc}}{\partial q^a} \dot{q}^c \dot{w}^b \right). \tag{3.8}$$

The Eqs. (3.6) are said to be the constitutive equations for the superposed motion and the Eqs. (3.7) are called the equations of the superposed motion. These equations with the initial conditions, which can be assumed in the homogeneous form

$$w^a(d, \tau_0) = 0, \quad \dot{w}^a(d, \tau_0) = 0, \quad d \in D, \tag{3.9}$$

govern the problem of the superposed motion. We are able to solve this problem if the fundamental motion is known.

Using the equations of variational states (3.6), (3.7) and (3.9), we can formulate the problem of stability in the discrete elasticity. Let us suppose that the fundamental motion reduces to an equilibrium state — i.e. $\dot{q}^a(d, \tau) = 0$ for each $d \in D$ and τ. Moreover, let $'f_a = 0$, and let us assume that each particle of the discretized body is the holonomic scleronomic dynamic system. It follows that $'r_a = a_{ab}\ddot{w}^b$, and from (3.6), (3.7) we obtain:

$$\bar{\Delta}_{\Lambda}(K_{ab}^{\Lambda\Phi} \Delta_{\Phi} w^b + L_{ab}^{\Lambda} w^b) - L_{ab}^{\Lambda} \Delta_{\Lambda} w^b - M_{ab} w^b = a_{ab} \ddot{w}^b. \tag{3.10}$$

An equilibrium state described by the functions $q^a(d, \tau) = q^a(d, \tau_0)$ is said to be stable if the amplitude of the superposed motion is always vanishingly small when the disturbance itself is sufficiently small. Substituting $w^a(d, \tau) = u^a(d) e^{i\breve{\omega}\tau}$ into (3.10), we obtain

$$\bar{\Delta}_{\Lambda}(K_{ab}^{\Lambda\Phi} \Delta_{\Phi} u^b + L_{ab}^{\Lambda} u^b) - L_{ba}^{\Lambda} \Delta_{\Lambda} u^b - M_{ab} u^b = -\breve{\omega}^2 a_{ab} u^b. \tag{3.11}$$

The equilibrium is stable if the small oscillations $w(d, \tau) = u^a(d) e^{i\omega\tau}$ are limited for each τ. It follows that the general criterion of stability has the known form $\mathrm{Im}\,\breve{\omega} \geqslant 0$. Using the statical criterion of stability, we put $\breve{\omega} = 0$ into (3.11); if there exists only a trivial solution of the problem, then the state of equilibrium is said to be stable. Other problems of the theory of variational states in discrete elasticity are studied in [7].

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