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BRIEF NOTES

Reduced forms for constitutive equations of transversely-isotropic materials with memory

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REDUCED forms for the constitutive equation of transversely-isotropic simple solids and of the heat flux whose isotropy group is the group of rotations arround an axis are obtained without any assumption on the smoothness of constitutive functionals.

1. Introduction

THE THEORY of simple materials is based upon the following constitutive equation [1]

$$\mathbf{T}(t) = \mathop{\mathscr{H}}_{s=0}^{\infty}\big(\mathbf{F}_t(t-s); \mathbf{F}(t)\big), \tag{1.1}$$

where $\mathbf{T}$ is the stress tensor, $\mathbf{F}_t(t-s)$ is the history of the relative deformation gradient, $\mathbf{F}(t)$ is the deformation gradient at the present time, taken relatively to a fixed arbitrary local reference configuration, and $\mathscr{H}$ is a functional.

Let u be the unimodular group. Then the group, defined by

$$u^* = \Big\{\mathbf{H}|\mathbf{H} \in u, \mathop{\mathscr{H}}_{s=0}^{\infty}\big(\mathbf{F}_t(t-s);\mathbf{F}\mathbf{H}\big) = \mathop{\mathscr{H}}_{s=0}^{\infty}\big(\mathbf{F}_t(t-s);\mathbf{F}\big)\Big\}, \tag{1.2}$$

is called the isotropy group of the material represented by the functional $\mathscr{H}$. The isotropy group depends on the local reference configuration. By definition, for anisotropic simple solids there is at least one local reference configuration such that the corresponding isotropy group is a subgroup of the orthogonal group. Such a configuration is called an undistorted state of the simple solid [1].

The first purpose of this work is to obtain reduced forms of constitutive equations for transversely-isotropic simple solids when the local reference configuration is an undistorted state. Then, by using the same method (used also by COLEMAN in [2]), one infers a representation for the heat flux of a simple thermodynamic material whose isotropy group is the group of rotations around an axis. These representations are obtained without any assumption concerning the smoothness of constitutive functionals.

2. Constitutive equations for transversely-isotropic simple solids

The isotropy group of transversely-isotropic simple solids, when the local reference configuration is an undistorted state, can be written as ([1, 3]; see also [5])

$$\mathbf{H} = \exp\left\{\begin{bmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\} \otimes \{\mathbf{1}, -\mathbf{1}\} = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \{\mathbf{1}, -\mathbf{1}\}, \tag{2.1}$$

where $\mathbf{1}$ is the unity tensor and $\otimes$ is the direct product of the two subgroups of the orthogonal group.

To obtain the constitutive equation of transversely-isotropic simple solids, we shall use the following Coleman's theorem [2].

THEOREM 1. *Let* $\mathbf{H} \in u$ *and let* $\hat{\mathbf{F}}, \mathbf{F} \in l$ *and* l *being the general linear group. If*

$$\mathbf{He} = \pm \alpha \mathbf{e}, \tag{2.2}$$

where $\mathbf{e}$ *is a vector and* α *is a scalar, then*

$$\hat{\mathbf{F}} = \mathbf{FH}, \tag{2.3}$$

if and only if,

$$\frac{\hat{\mathbf{F}}\mathbf{e}}{|\hat{\mathbf{F}}\mathbf{e}|} = \pm \frac{\mathbf{Fe}}{|\mathbf{Fe}|} \tag{2.4}$$

and

$$\det \hat{\mathbf{F}} = \pm \det \mathbf{F}. \tag{2.5}$$

Remark 1. If $\mathbf{e}$ is a complex vector, then (2.4) is equivalent to

$$\begin{aligned} \mathrm{Re}(\hat{\mathbf{F}}\mathbf{e})/(|\hat{\mathbf{F}}\mathbf{e}|) &= \pm \mathrm{Re}(\mathbf{Fe})/(|\mathbf{Fe}|), \\ \mathrm{Im}(\hat{\mathbf{F}}\mathbf{e})/(|\hat{\mathbf{F}}\mathbf{e}|) &= \pm \mathrm{Im}(\mathbf{Fe})/(|\mathbf{Fe}|). \end{aligned} \tag{2.6}$$

If $\alpha = 1$, then (2.4) is equivalent to

$$\hat{\mathbf{F}}\mathbf{e} = \pm \mathbf{Fe}. \tag{2.7}$$

From (2.1), we can obtain the eigenvectors of tensors which belong to this isotropy group. Hence we can write

$$\mathbf{He}_1 = \pm \mathbf{e}_1, \quad \mathbf{He}_2 = \pm \exp(ix)\mathbf{e}_2, \quad \mathbf{He}_3 = \pm \exp(-ix)\mathbf{e}_3, \tag{2.8}$$

where

$$[\mathbf{e}_1] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{e}_2] = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \quad [\mathbf{e}_3] = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}. \tag{2.9}$$

It is clear that, conversely, the set of relations (2.8) determines the isotropy group (2.1).

We can now prove the following:

THEOREM 2. *Suppose that the local reference configuration of a transversely-isotropic simple solid is an undistorted state. Then its constitutive equation may be written in the form*

$$\mathbf{T}(t) = \mathop{\mathscr{H}}_{s=0}^{\infty}\left(\mathbf{F}_t(t-s); \frac{\mathbf{F}_1(t)}{K(t)}, \frac{\mathbf{F}_2(t)}{K(t)}, \mathbf{F}_3(t), \varrho(t)\right), \tag{2.10}$$

where $\mathbf{F}_k(t)$, $k = 1, 2, 3$, *is a vector whose components are* (f_{1k}, f_{2k}, f_{3k}), $f_{kl} = f_{kl}(t)$ $k, l = 1, 2, 3$, *being the components of the deformation gradient* $\mathbf{F}(t)$. $\varrho(t)$ *is the mass, density in the present configuration and*

$$K(t) \equiv \sqrt{(f_{11}+f_{12})^2+(f_{21}+f_{22})^2+(f_{31}+f_{32})^2}. \tag{2.11}$$

P r o o f. If the simple material is a transversely-isotropic simple solid referred to an undistorted state, then

$$\mathop{\mathscr{H}}_{s=0}^{\infty}\left(\mathbf{F}(t-s); \hat{\mathbf{F}}\right) = \mathop{\mathscr{H}}_{s=0}^{\infty}\left(\mathbf{F}_t(t-s); \mathbf{F}\right) \tag{2.12}$$

when $\hat{\mathbf{F}} = \mathbf{FH}$, $\mathbf{H}$ being of the form (2.1). From (2.8), (2.9), (2.7) and (2.6), $\hat{\mathbf{F}} = \mathbf{FH}$ if and only if

$$\begin{gathered}\hat{\mathbf{F}}\mathbf{e}_1 = \pm\mathbf{F}\mathbf{e}_1,\ \mathrm{Re}(\hat{\mathbf{F}}\mathbf{e}_2)/(|\hat{\mathbf{F}}\mathbf{e}_2|) = \pm\mathrm{Re}(\mathbf{F}\mathbf{e}_2)/(|\mathbf{F}\mathbf{e}_2|),\\ \mathrm{Im}(\hat{\mathbf{F}}\mathbf{e}_2)/(|\hat{\mathbf{F}}\mathbf{e}_2|) = \pm\mathrm{Im}(\mathbf{F}\mathbf{e}_2)/(|\mathbf{F}\mathbf{e}_2|),\end{gathered} \tag{2.13}$$

and

$$|\det\hat{\mathbf{F}}| = |\det\mathbf{F}|. \tag{2.14}$$

It follows that the response functional $\mathscr{H}$ must depend on $\mathbf{F}\mathbf{e}_1$, $\mathrm{Re}(\mathbf{F}\mathbf{e}_2)/(|\mathbf{F}\mathbf{e}_2|)$, $\mathrm{Im}(\mathbf{F}\mathbf{e}_2)/(|\mathbf{F}\mathbf{e}_2|)$ and $|\det\mathbf{F}|$. If we denote by ϱ_R the mass density in the local reference configuration, then [1]

$$\varrho(t) = \frac{\varrho_R}{|\det\mathbf{F}|} \tag{2.15}$$

and the assertion of the theorem is proved.

R e m a r k 2. Let us denote

$$\mathbf{v}_1 \equiv \frac{\mathbf{F}_1}{K}, \quad \mathbf{v}_2 \equiv \frac{\mathbf{F}_2}{K}, \quad \mathbf{v}_3 \equiv \mathbf{F}_3\,. \tag{2.16}$$

The principle of material objectivity requires that the response functional $\mathscr{H}$ obey the identity

$$\begin{aligned}&\mathop{\mathscr{H}}_{s=0}^{\infty}\left(\mathbf{Q}(s)\mathbf{F}_t(t-s)\mathbf{Q}(0)^T; \mathbf{Q}(0)\mathbf{v}_1, \mathbf{Q}(0)\mathbf{v}_2, \mathbf{Q}(0)\mathbf{v}_3, \varrho(t)\right)\\ &\qquad = \mathbf{Q}(0)\mathop{\mathscr{H}}_{s=0}^{\infty}\left(\mathbf{F}_t(t-s); \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \varrho(t)\right)\mathbf{Q}(0)^T,\end{aligned} \tag{2.17}$$

for each function $\mathbf{Q}(\cdot)$ whose values are orthogonal tensors. Then there exists [1] a symmetric tensor-valued functional $\mathscr{L}$ such that

$$\mathbf{T}(t) = \mathop{\mathscr{L}}_{s=0}^{\infty}\left(\mathbf{C}_t(t-s); \mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t), \varrho(t)\right), \tag{2.18}$$

where

$$C_t(t-s) = F_t(t-s)F_t(t-s)^T \tag{2.19}$$

is the history of the relative right Cauchy-Green tensor. The functional $\mathscr{L}$ satisfies

$$\underset{s=0}{\overset{\infty}{\mathscr{L}}}\big(\mathbf{Q}\mathbf{C}_t(t-s)\mathbf{Q}^T;\ \mathbf{Q}\mathbf{v}_1, \mathbf{Q}\mathbf{v}_2, \mathbf{Q}\mathbf{v}_3, \varrho(t)\big) = \mathbf{Q}\underset{s=0}{\overset{\infty}{\mathscr{L}}}\big(\mathbf{C}_t(t-s);\ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \varrho(t)\big)\mathbf{Q}^T \tag{2.20}$$

for every constant orthogonal tensor $\mathbf{Q}$. By Cauchy's representation theorem the Eq. (2.18) may be written in the form

$$\mathbf{T}(t) = \underset{s=0}{\overset{\infty\, ij}{\mathscr{L}}}\big(C_t^{kl}(t-s), I_{v_1, v_2, v_3}, \varrho(t)\big)\mathbf{v}_i\otimes\mathbf{v}_j, \quad i, j, k, l = 1, 2, 3, \tag{2.21}$$

where $C_t^{kl}\,(t-s)$ are the components of $\mathbf{C}_t(t-s)$ relative to the basis $\{\mathbf{v}_i(t)\}$ and I_{v_1, v_2, v_3} represents the set of the following invariants:

$$|\mathbf{v}_i(t)|, \quad i = 1, 2, 3; \quad \mathbf{v}_i\cdot\mathbf{v}_j, \quad i \neq j, i, \quad j = 1, 2, 3. \tag{2.22}$$

Remark 3. From (2.21), we see that the constitutive equation of transversely-isotropic elastic solids, when the chosen reference configuration is an undistorted state, is of the form

$$\mathbf{T}(t) = l^{ij}\big(I_{v_1, v_2, v_3}, \varrho(t)\big)\mathbf{v}_i(t)\otimes\mathbf{v}_j(t), \quad i, j = 1, 2, 3, \tag{2.23}$$

where l^{ij}, $i, j = 1, 2, 3$, are scalar-valued functions.

Remark 4. It is worth noticing that all the tensors in the isotropy group (2.1) have the same eigenvectors. In fact, we have here a dilatation group [4] in a generalized sense; the eigenvalues of tensors in this group are not real numbers. In our previous paper [5], we have found an exponential representation of isotropy groups of simple solids when the local reference configuration is distorted. The specialization of those formulae leads to an exponential representation of the subgroups of the orthogonal group, which are also isotropy groups for simple solids, when the chosen reference configuration is an undistorted state. Taking into account the well-known result that every two tensors with distinct eigenvalues have the same eigenvectors if and only if they commute, it can easily be seen from the above-mentioned representations that not every isotropy group of simple solids has the property to possess common eigenvectors for all tensors belonging to it. An example is the full orthogonal group. This is the reason why we cannot employ the method used in the Theorem 2 to find reduced forms for constitutive equations of all types of simple solids.

3. Representation for the heat flux of a simple thermodynamic material

The heat flux of a simple thermodynamic material is defined by

$$\mathbf{h}(t) = \underset{s=0}{\overset{\infty}{\mathscr{P}}}\big(\mathbf{F}_t(t-s); \mathbf{F}(t), \theta(t-s), \mathbf{g}(t-s)\big), \tag{3.1}$$

where $\theta(t-s)$ is the history of the temperature, $\mathbf{g}(t-s)$ the history of the temperature gradient and $\mathscr{P}$ an arbitrary vector valued functional. For the heat flux one defines an isotropy group in an analogous way as for the stress tensor of simple materials.

Let us suppose that the isotropy group is of the form (2.1). Then, by using the Theorem 1 and the Remark 1, we have the following:

THEOREM 3. *The constitutive equation for the heat flux of a simple thermodynamic material whose isotropy group is* (2.1) *may be written in the form*

$$\mathbf{h}(t) = \mathop{\mathscr{P}}_{s=0}^{\infty}\left(\mathbf{F}_t(t-s); \frac{\mathbf{F}_1}{K}, \frac{\mathbf{F}_2}{K}, \mathbf{F}_3, \theta(t-s), \mathbf{g}(t-s), \varrho(t)\right). \tag{3.2}$$

Remark 5. In view of the objectivity principle, by using again the Cauchy's representation theorem, one obtains the following expression for the heat flux of a thermoelastic material whose isotropy group is (2.1).

$$\mathbf{h}(t) = \sum_{k=1}^{4} p_k\big(l_{v_1, v_2, v_3}, \theta(t), \varrho(t)\big)\mathbf{v}_k(t). \tag{3.3}$$

Here, $p_k, k = 1, 2, 3, 4$ are scalar valued functions, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are given by (2.16), $\mathbf{v}_4 \equiv \mathbf{g}(t)$ and l_{v_1, v_2, v_3, v_4} represents the set of the following invariants

$$|\mathbf{v}_i(t)|, \quad i = 1, 2, 3, 4; \quad \mathbf{v}_i \cdot \mathbf{v}_j \quad i \neq j, \quad i, j = 1, 2, 3, 4. \tag{3.4}$$

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