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The plane micropolar strain of orthotropic elastic solids(*)

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THE PRESENT paper is concerned with the static theory of plane micropolar strain for a homogeneous and orthotropic elastic solid. The uniqueness theorems, existence theorems and the reduction of the boundary value problems to integral equations for which the Fredholm's basic theorems are valid, are derived.

W pracy zajęto się statyką płaskiego stanu odkształcenia jednorodnego, mikropolarnego, ortotropowego ciała sprężystego. Wyprowadzono twierdzenia o jednoznaczności i istnieniu rozwiązań oraz o sprowadzeniu zagadnień brzegowych do równań całkowych, dla których obowiązują podstawowe twierdzenia Fredholma.

В работе занимаются статикой плоского деформационного состояния однородного, микрополярного, ортотропного упругого тела. Выведены теоремы однозначности и существования решений, а также о сведении краевых задач к интегральным уравнениям, для которых обязывают основные теоремы Фредгольма.

1. Introduction

THE PLANE problem in the linear theory of micropolar elasticity for isotropic solids has been considered in various papers (see, e.g. [1–17]). Some existence theorems in the static theory of plane micropolar strain were derived in [10]. In [18] were given the constitutive equations for an orthotropic micropolar elastic solid. In the present paper, we consider the static problem of plane micropolar strain for a homogeneous and orthotropic elastic solid. The uniqueness theorems and existence theorems are derived. We give a Galerkin representation and introduce the elastic potentials. By means of the method of potentials [19], we reduce the boundary value problems to singular integral equations for which Fredholm's basic theorems are valid.

2. Basic equations

Throughout this paper a rectangular coordinate system (x_1, x_2) is employed. The indices denoted by small Greek letters take the values 1, 2.

We consider a finite regular plane region Σ occupied by a micropolar elastic material, whose boundary is L.

The basic equations in the static theory of the plane strain of a homogeneous and orthotropic elastic solid, are:

equilibrium equations

$$t_{\beta\alpha,\beta}+f_\alpha = 0, \quad m_{\alpha 3,\alpha}+\epsilon_{\alpha\beta 3}t_{\alpha\beta}+l = 0, \tag{2.1}$$

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constitutive equations

$$
(2.2)\qquad
\begin{aligned}
t_{11} &= A_{11}\varepsilon_{11}+A_{12}\varepsilon_{22}, \quad t_{22} = A_{12}\varepsilon_{11}+A_{22}\varepsilon_{22},\\
t_{12} &= A_{77}\varepsilon_{12}+A_{78}\varepsilon_{21}, \quad t_{21} = A_{78}\varepsilon_{12}+A_{88}\varepsilon_{21},\\
m_{13} &= B_{66}\,\varphi_{,1}, \quad m_{23} = B_{44}\,\varphi_{,2},
\end{aligned}
$$

geometrical equations

$$
(2.3)\qquad \varepsilon_{\alpha\beta} = u_{\beta,\alpha}+\epsilon_{\beta\alpha 3}\varphi .
$$

In these relations, we have used the following notations: $t_{\alpha\beta}$ — components of the stress tensor, $m_{\alpha 3}$ — components of the couple stress tensor, f_α — components of the body force, l — body couple, $\varepsilon_{\alpha\beta}$ — components of the micropolar strain tensor, u_α — components of the displacement vector, φ — component of microrotation vector, ϵ_{ijk} — alternating symbol, $A_{\alpha\beta}, A_{77}, A_{78}, A_{88}, B_{44}, B_{66}$ — characteristic constants of the material, the comma denotes partial derivation with respect to the variables x_α.

The surface tractions and surface moment acting at a point $x(x_\alpha)$ on the curve L are given by

$$
(2.4)\qquad t_\alpha = t_{\beta\alpha}n_\beta, \quad m = m_{\alpha 3}n_\alpha,
$$

where $n_\alpha = \cos(n_x, x_\alpha)$, n_x being the unit vector of the outward normal to L at x.

From (2.1)–(2.3), we obtain the field equations of the plane strain for orthotropic solids in the form:

$$
(2.5)\qquad
\begin{aligned}
&\left(A_{11}\frac{\partial^2}{\partial x_1^2}+A_{88}\frac{\partial^2}{\partial x_2^2}\right)u_1+(A_{12}+A_{78})\frac{\partial^2 u_2}{\partial x_1\partial x_2}-k_1\frac{\partial\varphi}{\partial x_2}+f_1 = 0,\\
&(A_{12}+A_{78})\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\left(A_{77}\frac{\partial^2}{\partial x_1^2}+A_{22}\frac{\partial^2}{\partial x_2^2}\right)u_2-k_2\frac{\partial\varphi}{\partial x_1}+f_2 = 0,\\
&\left(B_{66}\frac{\partial^2}{\partial x_1^2}+B_{44}\frac{\partial^2}{\partial x_2^2}-\varkappa\right)\varphi+k_1\frac{\partial u_1}{\partial x_2}+k_2\frac{\partial u_2}{\partial x_1}+l = 0,
\end{aligned}
$$

where

$$
(2.6)\qquad k_1 = A_{78}-A_{88}, \quad k_2 = A_{77}-A_{78}, \quad \varkappa = k_2-k_1 .
$$

The system (2.5) can be written in a matrix form. The vector $v = (v_1, \ldots, v_m)$ will be considered as a column-matrix. Thus, the product of the matrix $A = \|a_{ij}\|_{mxm}$ and the vector v is an m-dimensional vector. The vector v multiplied by the matrix A will denote the matrix product between the row matrix $v = \|v_1, \ldots, v_m\|$ and the matrix A.

We introduce the matrical differential operator:

$$
(2.7)\qquad A\left(\frac{\partial}{\partial x}\right) = -\left\| D_{ij}\left(\frac{\partial}{\partial x}\right)\right\|_{3\times 3},
$$

where

$$
(2.8)\qquad
\begin{aligned}
D_{11} &= A_{11}\frac{\partial^2}{\partial x_1^2}+A_{88}\frac{\partial^2}{\partial x_2^2}, \quad D_{12} = D_{21} = (A_{12}+A_{78})\frac{\partial^2}{\partial x_1\partial x_2},\\
D_{13} &= -D_{31} = -k_1\frac{\partial}{\partial x_2}, \quad D_{22} = A_{77}\frac{\partial^2}{\partial x_1^2}+A_{22}\frac{\partial^2}{\partial x_2^2},\\
D_{23} &= -D_{32} = -k_2\frac{\partial}{\partial x_1}, \quad D_{33} = B_{66}\frac{\partial^2}{\partial x_1^2}+B_{44}\frac{\partial^2}{\partial x_2^2}-\varkappa .
\end{aligned}
$$

We denote

(2.9) $$u = (u_1, u_2, \varphi), \quad f = (f_1, f_2, l), \quad t = (t_1, t_2, m)$$

The system (2.5) can be written in the form:

(2.10) $$Au = f.$$

In what follows we consider two kinds of boundary conditions:
the first boundary value problem

(2.11) $$u_\alpha = \tilde{u}_\alpha, \quad \varphi = \tilde{\varphi} \quad \text{on} \quad L,$$

the second boundary value problem

(2.12) $$t_\alpha = \tilde{t}_\alpha, \quad m = \tilde{m} \quad \text{on} \quad L,$$

where $\tilde{u}_\alpha, \tilde{\varphi}, \tilde{t}_\alpha, \tilde{m}$ are prescribed functions. Other boundary value problems might be considered (see, e.g. [20]), but we shall restrict ourselves to the cases considered above.

3. Uniqueness theorems

We introduce the notations:

(3.1) $$2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha}, \quad 2r = u_{2,1} - u_{1,2}.$$

Obviously,

(3.2) $$\varepsilon_{11} = e_{11}, \ \varepsilon_{22} = e_{22}, \ \varepsilon_{12} = e_{12} + r - \varphi, \ \varepsilon_{21} = e_{12} - (r - \varphi).$$

If $\varphi = r$, we obtain the theory of couple stress with constrained rotation. In what follows we assume $\varphi \neq r$.

Let us establish the uniqueness theorems for the boundary value problems (2.10), (2.11) and (2.10), (2.12). We assume that the internal energy density

(3.3) $$2U = t_{\alpha\beta}\varepsilon_{\alpha\beta} + m_{\alpha 3}\varphi_{,\alpha} = A_{11}\varepsilon_{11}^2 + 2A_{12}\varepsilon_{11}\varepsilon_{22} + A_{22}\varepsilon_{22}^2 + A_{77}\varepsilon_{12}^2 + 2A_{78}\varepsilon_{12}\varepsilon_{21} + A_{88}\varepsilon_{21}^2 + B_{66}(\varphi_{,1})^2 + B_{44}(\varphi_{,2})^2,$$

is a positive definite quadratic form. It is easy to show that

$$2U = \frac{1}{A_{11}}(A_{11}e_{11} + A_{12}e_{22})^2 + \frac{1}{A_{11}}(A_{11}A_{22} - A_{12}^2)e_{22}^2 + \frac{1}{\varkappa}[\varkappa(r-\varphi) + (A_{77} - A_{78})e_{12}]^2 + \left[A_{77} + A_{88} + 2A_{78} - \frac{1}{\varkappa}(A_{77} - A_{88})^2\right]e_{12}^2 + B_{66}(\varphi_{,1})^2 + B_{44}(\varphi_{,2})^2.$$

The necessary and sufficient conditions for the internal energy to be positive definite are

(3.4) $$A_{11} > 0, \quad A_{11}A_{22} - A_{12}^2 > 0, \quad \varkappa = A_{77} + A_{88} - 2A_{78} > 0,$$
$$A_{88}A_{77} - A_{78}^2 > 0, \quad B_{66} > 0, \quad B_{44} > 0.$$

Taking into account the relations (2.1), (2.3), (2.4) and using the Green-Gauss theorem, we obtain

(3.5) $$\int_L (t_\alpha u_\alpha + m\varphi)\,ds + \int_\Sigma (f_\alpha u_\alpha + l\varphi)\,d\sigma = 2\int_\Sigma U\,d\sigma.$$

Let $u_\alpha^{(\varrho)}$, $\varphi^{(\varrho)}$ be two solutions of the boundary value problems considered. We denote

$$u_\alpha^* = u_\alpha^{(1)} - u_\alpha^{(2)}, \quad \varphi^* = \varphi^{(1)} - \varphi^{(2)}. \tag{3.6}$$

According to the linearity of the problem, the differences considered satisfy the basic equations and boundary conditions in their homogeneous form, and from (3.5), we obtain:

$$\int_\Sigma U^* d\sigma = 0,$$

where U^* is the internal energy density corresponding to the system (3.6). Because U^* is a positive definite quadratic form, it follows that $\varepsilon_{\alpha\beta}^* = \varphi_{,\alpha}^* = 0$ and using (3.2), we obtain

$$e_{\alpha\beta}^* = 0, \quad r^* = \varphi^*, \quad \varphi_{,\alpha}^* = 0. \tag{3.7}$$

From (3.7) it follows that

$$u_\alpha^* = a\,\epsilon_{\alpha\beta 3} x_\beta + b_\alpha, \quad \varphi^* = -a, \tag{3.8}$$

where a and b_α are arbitrary constants.

In the case of the boundary conditions (2.11), we obtain:

$$u_\alpha^* = 0, \quad \varphi^* = 0. \tag{3.9}$$

Thus we have:

THEOREM 3.1. *The boundary value problem* (2.10), (2.11) *admits at most one solution.*

THEOREM 3.2. *The solution of the boundary value problem* (2.10), (2.12) *is determined to within an additive rigid-displacement of the form* (3.8).

In the case of isotropic solids, the uniqueness theorems were derived in [5].

4. Existence theorems

Let us consider a body subjected to two different systems of elastic loadings and the two corresponding elastic configurations $u_\alpha^{(\varrho)}$, $\varphi^{(\varrho)}$. Using (2.1)–(2.4) and the Green-Gauss theorem, we obtain:

$$\int_L (t_\alpha^{(1)} u_\alpha^{(2)} + m^{(1)} \varphi^{(2)}) ds + \int_\Sigma (f_\alpha^{(1)} u_\alpha^{(2)} + l^{(1)} \varphi^{(2)}) d\sigma = 2 \int_\Sigma U_{12} d\sigma, \tag{4.1}$$

where

$$\begin{aligned} 2U_{12} = t_{\alpha\beta}^{(1)} \varepsilon_{\alpha\beta}^{(2)} + m_{\alpha 3}^{(1)} \varphi_{,\alpha}^{(2)} = {} & A_{11} \varepsilon_{11}^{(1)} \varepsilon_{11}^{(2)} + A_{12} (\varepsilon_{11}^{(1)} \varepsilon_{22}^{(2)} + \varepsilon_{11}^{(2)} \varepsilon_{22}^{(1)}) + A_{22} \varepsilon_{22}^{(1)} \varepsilon_{22}^{(2)} \\ & + A_{77} \varepsilon_{12}^{(1)} \varepsilon_{12}^{(2)} + A_{78} (\varepsilon_{12}^{(1)} \varepsilon_{21}^{(2)} + \varepsilon_{12}^{(2)} \varepsilon_{21}^{(1)}) + A_{88} \varepsilon_{21}^{(1)} \varepsilon_{21}^{(2)} + B_{66} \varphi_{,1}^{(1)} \varphi_{,1}^{(2)} + B_{44} \varphi_{,2}^{(1)} \varphi_{,2}^{(2)}. \end{aligned} \tag{4.2}$$

If we introduce the notations

$$\begin{aligned} u &= (u_1^{(1)}, u_2^{(1)}, \varphi^{(1)}), \quad v = (u_1^{(2)}, u_2^{(2)}, \varphi^{(2)}), \quad U_{12} = U(u, v), \\ t(u) &= (t_1^{(1)}, t_2^{(1)}, m^{(1)}), \quad t(v) = (t_1^{(2)}, t_2^{(2)}, m^{(2)}), \end{aligned} \tag{4.3}$$

the relation (4.1) can be written in the form:

$$\int_L v t(u) ds + \int_\Sigma v A u d\sigma = 2 \int_\Sigma U(u, v) d\sigma. \tag{4.4}$$

From (4.2) it follows that $U(u, v) = U(v, u)$, $U(u, u) = U$, so that from (4.4), we obtain:

$$\tag{4.5} \begin{aligned} \int_{\Sigma} (vAu - uAv) d\sigma &= \int_{L} [ut(v) - vt(u)] ds, \\ \int_{\Sigma} uAu d\sigma &= -\int_{L} ut(u) ds + 2\int_{\Sigma} U(u, u) d\sigma. \end{aligned}$$

In what follows, we establish certain existence theorems using the results from [21]. We consider homogeneous boundary conditions and assume that Σ is C^{∞}-smooth [21, p. 61]. We have the equation

$$\tag{4.6} Au = f,$$

with the boundary conditions

$$\tag{4.7} u = 0 \quad \text{on} \quad L,$$

or

$$\tag{4.8} t(u) = 0 \text{ on } L.$$

Taking into account the conditions (4.7), (4.8), from (4.5), we obtain:

$$\tag{4.9} \int_{\Sigma} uAu d\sigma = 2\int_{\Sigma} U(u, u) d\sigma.$$

In order to prove the existence of the solution of the boundary value problem (4.6) (4.7) we need to prove that [21, p. 62]

$$\tag{4.10} 2\int_{\Sigma} U(u, u) d\sigma \geqslant c_0 ||u||_1^2,$$

for any $u = (u_1, u_2, \varphi) \in \mathring{H}_1(\Sigma)$, c_0 being a positive constant. By $\mathring{H}_1(\Sigma)$ is denoted [21, p. 17] the Hilbert function space obtained by functional completion of $\mathring{C}^1(\Sigma)$ with respect to the scalar product

$$(u, v)_1 = \int_{\Sigma} D^s u D^s v \, d\sigma, \quad 0 \leqslant |s| \leqslant 1.$$

The form (3.3) is a positive definite quadratic form — i.e., there exists a positive constant c such that

$$\tag{4.11} 2U(u, u) \geqslant c \sum_{\alpha, \beta=1}^{2} [\varepsilon_{\alpha\beta}^2 + (\varphi_{,\alpha})^2].$$

Taking into account (3.2), we can write

$$\sum_{\alpha, \beta=1}^{2} \varepsilon_{\alpha\beta}^2 = \sum_{\alpha, \beta=1}^{2} [e_{\alpha\beta}^2 + 2(r-\varphi)^2],$$

so that

$$\tag{4.12} 2U(u, u) \geqslant c \sum_{\alpha, \beta=1}^{2} [e_{\alpha\beta}^2 + (\varphi_{,\alpha})^2].$$

If we use the first Korn's inequality

$$\int\limits_{\Sigma} \sum_{\alpha,\beta=1}^{2} e_{\alpha\beta}^2 d\sigma \geqslant c_1 ||u^{(1)}||_1^2, \quad c_1 > 0,$$

where $u^{(1)} = (u_1, u_2, 0)$, and the Poincaré inequality [21, p. 19]

$$||u^{(2)}||_1^2 \leqslant c_2 \sum_{\alpha=1}^{2} \int\limits_{\Sigma} (\varphi_{,\alpha})^2 d\sigma, \quad c_2 > 0,$$

where $u^{(2)} = (0, 0, \varphi)$, from (4.12) we arrive at:

$$2\int\limits_{\Sigma} U(u, u) d\sigma \geqslant c_0(||u^{(1)}||_1^2 + ||u^{(2)}||_1^2) = c_0||u||_1^2. \tag{4.13}$$

Thus we have:

THEOREM 4.1. *Given $f \in C^\infty(\overline{\Sigma})$, there exists one and only one solution of the boundary value problem* (4.6), (4.7) *which belongs to $C^\infty(\overline{\Sigma})$.*

To prove the existence theorem for the boundary value problem (4.6), (4.8), as in [21, p. 91], we consider the system

$$Au + p_0 u = f, \tag{4.14}$$

where p_0 is any positive constant. First, we consider the boundary value problem (4.14), (4.8). The inequality to be proved in this case is

$$\int\limits_{\Sigma} \sum_{\alpha,\beta=1}^{2} [e_{\alpha\beta}^2 + (\varphi_{,\alpha})^2] d\sigma + \int\limits_{\Sigma} u^2 d\sigma \geqslant c_3 ||u||_1^2, \quad c_3 > 0, \tag{4.15}$$

for any $u \in H_1(\Sigma)$.

Using the second Korn's inequality

$$\int\limits_{\Sigma} \sum_{\alpha,\beta=1}^{2} e_{\alpha\beta}^2 d\sigma + \int\limits_{\Sigma} (u^{(1)})^2 d\sigma \geqslant c_4 ||u^{(1)}||_1^2, \quad c_4 > 0,$$

and the relation

$$\sum_{\alpha=1}^{2} \int\limits_{\Sigma} (\varphi_{,\alpha})^2 d\sigma + \int\limits_{\Sigma} \varphi^2 d\sigma = ||u^{(2)}||_1^2,$$

it is easy to derive (4.15). It follows that (4.14), (4.8) has only one solution which is C^∞ in $\overline{\Sigma}$. The system considered is formally self adjoint, so that a C^∞ solution of the following system

$$Au + p_0 u - \lambda u = f, \tag{4.16}$$

with the boundary condition (4.8) exists when and only when

$$\int\limits_{\Sigma} f \mathring{u} d\sigma = 0, \tag{4.17}$$

where $\mathring{u} = (\mathring{u}_1, \mathring{u}_2, \mathring{\varphi})$ is any C^∞ solution of the problem (4.16), (4.8) with $f = 0$.

In the case $\lambda = p_0$, the only C^∞ solution of the homogeneous system is

$$\mathring{u}_\alpha = a\, \epsilon_{\alpha\beta 3} x_\beta + b_\alpha, \quad \mathring{\varphi} = -a, \tag{4.18}$$

where a, b_α are arbitrary constants. Thus we have:

THEOREM 4.2. *The boundary value problem* (4.6), (4.8) *has solutions belonging to* $C^\infty(\bar{\Sigma})$ *if and only if the* C^∞ *vector* $f = (f_1, f_2, l)$ *satisfies the condtitions*

$$\int_\Sigma f_\alpha d\sigma = 0, \quad \int_\Sigma (x_1 f_2 - x_2 f_1 + l) d\sigma = 0. \tag{4.19}$$

The above results are valid for inhomogeneous bodies [21] and can be extended under more general hypotheses on f and Σ [21, 22].

5. Galerkin representation

Using the associated matrices method [23], as in [24], we obtain the following representation of Galerkin type:

$$\begin{aligned}
u_1 &= \left(D_{22}D_{33} + k_2^2 \frac{\partial^2}{\partial x_1^2}\right)\Gamma_1 - \frac{\partial^2}{\partial x_1 \partial x_2}[(A_{12}+A_{78})D_{33} + k_1 k_2]\Gamma_2 + \\
&\quad + \frac{\partial}{\partial x_2}\left\{[k_1 A_{77} - k_2(A_{12}+A_{78})]\frac{\partial^2}{\partial x_1^2} + k_1 A_{22}\frac{\partial^2}{\partial x_2^2}\right\}\Gamma_3, \\
u_2 &= -\frac{\partial^2}{\partial x_1 \partial x_2}[(A_{12}+A_{78})D_{33} + k_1 k_2]\Gamma_1 + \left(D_{11}D_{33} + k_1^2 \frac{\partial^2}{\partial x_2^2}\right)\Gamma_2 \\
&\quad + \frac{\partial}{\partial x_1}\left\{k_2 A_{11}\frac{\partial^2}{\partial x_1^2} + [k_2 A_{88} - k_1(A_{12}+A_{78})]\frac{\partial^2}{\partial x_2^2}\right\}\Gamma_3, \\
\varphi &= -\frac{\partial}{\partial x_2}\left\{[k_1 A_{77} - k_2(A_{12}+A_{78})]\frac{\partial^2}{\partial x_1^2} + k_1 A_{22}\frac{\partial^2}{\partial x_2^2}\right\}\Gamma_1 \\
&\quad - \frac{\partial}{\partial x_1}\left\{k_2 A_{11}\frac{\partial^2}{\partial x_1^2} + [k_2 A_{78} - k_1(A_{12}+A_{78})]\frac{\partial^2}{\partial x_2^2}\right\}\Gamma_2 + \left\{A_{11}A_{77}\frac{\partial^4}{\partial x_1^4}\right. \\
&\quad \left. + [A_{11}A_{22} + A_{77}A_{88} - (A_{12}+A_{78})^2]\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + A_{22}A_{88}\frac{\partial^4}{\partial x_2^4}\right\}\Gamma_3,
\end{aligned} \tag{5.1}$$

where D_{ij} are defined in (2.8).

The functions $\Gamma_i(x_1, x_2)$, $(i = 1, 2, 3)$, satisfy the equations

$$\mathfrak{M}\Gamma_\alpha = -f_\alpha, \quad \mathfrak{M}\Gamma_3 = -l, \tag{5.2}$$

where

$$\begin{aligned}
\mathfrak{M} &= \left\{A_{11}A_{77}\frac{\partial^4}{\partial x_1^4} + [A_{11}A_{22} + A_{77}A_{88} - (A_{12}+A_{78})^2]\frac{\partial^4}{\partial x_1^2 \partial x_2^2}\right. \\
&\quad \left. + A_{22}A_{88}\frac{\partial^4}{\partial x_2^4}\right\}\left(B_{66}\frac{\partial^2}{\partial x_1^2} + B_{44}\frac{\partial^2}{\partial x_2^2}\right) + A_{11}(A_{78}^2 - A_{77}A_{88})\frac{\partial^4}{\partial x_1^4} \\
&\quad + \left[k_1^2 A_{77} + k_2^2 A_{88} - \varkappa(A_{11}A_{22} + A_{77}A_{88}) + \varkappa(A_{12}+A_{78})^2\right. \\
&\quad \left. - 2k_1 k_2(A_{12}+A_{78})\right]\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + A_{22}(A_{78}^2 - A_{77}A_{88})\frac{\partial^4}{\partial x_2^4}.
\end{aligned} \tag{5.3}$$

6. Fundamental solutions

To obtain the fundamental solutions of the system (2.5), we use the representation (5.1) and the fundamental solution $\Phi(x, y)$ of the equation

$$\mathfrak{M}\omega = 0. \tag{6.1}$$

If we know the fundamental solution $\Phi(x, y)$, then from (5.1), for $\Gamma_1 = \Phi$, $\Gamma_2 = \Gamma_3 = 0$, we obtain:

$$\begin{aligned} u_1^{(1)}(x, y) &= \left(D_{22} D_{33} + k_2^2 \frac{\partial^2}{\partial x_1^2}\right)\Phi, \\ u_2^{(2)}(x, y) &= -\frac{\partial^2}{\partial x_1 \partial x_2} [(A_{12}+A_{78}) D_{33} + k_1 k_2]\Phi, \\ \varphi^{(1)}(x, y) &= -\frac{\partial}{\partial x_2}\left\{[k_1 A_{77} - k_2(A_{12}+A_{78})]\frac{\partial^2}{\partial x_1^2} + k_1 A_{22}\frac{\partial^2}{\partial x_2^2}\right\}\Phi. \end{aligned} \tag{6.2}$$

For $\Gamma_1 = \Gamma_3 = 0$, $\quad \Gamma_2 = \Phi$, we obtain:

$$\begin{aligned} u_1^{(2)}(x, y) &= -\frac{\partial^2}{\partial x_1 \partial x_2} [(A_{12}+A_{78}) D_{33} + k_1 k_2]\Phi, \\ u_2^{(2)}(x, y) &= \left(D_{11} D_{33} + k_1^2 \frac{\partial^2}{\partial x_2^2}\right)\Phi, \\ \varphi^{(2)}(x, y) &= -\frac{\partial}{\partial x_1}\left\{k_2 A_{11}\frac{\partial^2}{\partial x_1^2} + [k_2 A_{88} - k_1(A_{12}+A_{78})]\frac{\partial^2}{\partial x_2^2}\right\}\Phi, \end{aligned} \tag{6.3}$$

and for $\Gamma_1 = \Gamma_2 = 0$, $\quad \Gamma_3 = \Phi$:

$$\begin{aligned} u_1^{(3)}(x, y) &= \frac{\partial}{\partial x_2}\left\{[k_1 A_{77} - k_2(A_{12}+A_{78})]\frac{\partial^2}{\partial x_1^2} + k_1 A_{22}\frac{\partial^2}{\partial x_2^2}\right\}\Phi, \\ u_2^{(3)}(x, y) &= \frac{\partial}{\partial x_1}\left\{k_2 A_{11}\frac{\partial^2}{\partial x_1^2} + [k_2 A_{88} - k_1(A_{12}+A_{78})]\frac{\partial^2}{\partial x_2^2}\right\}\Phi, \\ \varphi^{(3)}(x, y) &= \left\{A_{11} A_{77}\frac{\partial^4}{\partial x_1^4} + [A_{11} A_{22} + A_{77} A_{88} - (A_{12}+A_{78})^2]\frac{\partial^4}{\partial x_1^2 \partial x_2^2} \right. \\ &\qquad \left. + A_{22} A_{88}\frac{\partial^4}{\partial x_2^4}\right\}\Phi. \end{aligned} \tag{6.4}$$

The matrix of the fundamental solutions is

$$\Gamma(x, y) = \begin{bmatrix} u_1^{(1)} & u_1^{(2)} & u_1^{(3)} \\ u_2^{(1)} & u_2^{(2)} & u_2^{(3)} \\ \varphi^{(1)} & \varphi^{(2)} & \varphi^{(3)} \end{bmatrix}. \tag{6.5}$$

Let us consider the characteristic equation corresponding to the elliptic equation (6.1):

$$\{A_{22} A_{88}\alpha^4 + [A_{11} A_{22} + A_{77} A_{88} - (A_{12}+A_{78})^2]\alpha^2 + A_{11} A_{77}\} (B_{44}\alpha^2 + B_{66}) = 0. \tag{6.6}$$

The roots of the first factor of the Eq. (6.6) have one of the following forms:

(a) $\alpha_k = ib_k, \quad \bar{\alpha}_k = -ib_k, \quad b_k > 0,$

(b) $\alpha_k = (-1)^{k-1}a+ib, \quad \bar{\alpha}_k = (-1)^{k-1}a-ib, \quad b > 0,$

(c) $\alpha_k = ib, \quad \alpha_k = -ib, \quad b > 0, \quad k = 1, 2.$

In what follows we consider the case (a). The other cases can be treated in a similar way. Therefore, the roots of the Eq. (6.6) have the form:

$$\alpha_k = ib_k, \quad \bar{\alpha}_k = -ib_k, \quad b_k > 0, \quad k = 1, 2, 3; \tag{6.7}$$
$$b_3 = \sqrt{\frac{B_{66}}{B_{44}}}, \quad b_3 \neq b_1, b_2.$$

Let us consider the function [25]

$$\Psi(x, y) = a\,\mathrm{Im}\sum_{k=1}^{3} d_k\sigma_k^4 \ln\sigma_k, \tag{6.8}$$

where

$$\sigma_k = (x_1-y_1)+\alpha_k(x_2-y_2), \quad a = -\frac{1}{12B_{44}A_{22}A_{88}}, \tag{6.9}$$

and d_k is cofactor of α_k^5 from the determinant

$$d = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 & \alpha_1^4 & \alpha_1^5 \\ 1 & \bar{\alpha}_1 & \bar{\alpha}_1^2 & \bar{\alpha}_1^3 & \bar{\alpha}_1^4 & \bar{\alpha}_1^5 \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 & \alpha_2^4 & \alpha_2^5 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \bar{\alpha}_3 & \bar{\alpha}_3^2 & \bar{\alpha}_3^3 & \bar{\alpha}_3^4 & \bar{\alpha}_3^5 \end{vmatrix},$$

divided by d.

The fundamental solution of the Eq. (6.1) has the form [25]:

$$\Phi(x, y) = \Psi(x, y)+\Omega(x, y), \tag{6.10}$$

where the function $\Omega(x, y)$ and its derivatives, for $x = y$, have a singularity of a lower order than the function $\Psi(x, y)$ and the corresponding derivatives. The explicit form of the function $\Phi(x, y)$ can be obtained using the method from [26].

We have

$$d = 8ib_1b_2b_3(b_1^2-b_2^2)(b_2^2-b_3^2)(b_3^2-b_1^2),$$
$$d_1 = -\frac{i}{2b_1(b_2^2-b_1^2)(b_3^2-b_1^2)}, \quad d_2 = -\frac{i}{2b_2(b_1^2-b_2^2)(b_3^2-b_2^2)}, \tag{6.11}$$
$$d_3 = -\frac{i}{2b_3(b_1^2-b_3^2)(b_2^2-b_3^2)}.$$

In what follows, we shall use the relations:

$$\sum_{k=1}^{3} d_k = -\frac{i(b_1+b_2+b_3)}{2b_1b_2b_3(b_1+b_2)(b_2+b_3)(b_3+b_1)}, \quad \sum_{k=1}^{3}\alpha_k d_k = 0, \tag{6.12}$$

(6.12) [cont.]
$$\sum_{k=1}^{3} \alpha_k^2 d_k^2 = -\frac{i}{2(b_1+b_2)(b_2+b_3)(b_3+b_1)}, \quad \sum_{k=1}^{3} \alpha_k^3 d_k = 0,$$
$$\sum_{k=1}^{3} \alpha_k^4 d_k = -\frac{i(b_1 b_2+b_2 b_3+b_3 b_1)}{2(b_1+b_2)(b_2+b_3)(b_3+b_1)}, \quad \sum_{k=1}^{3} \alpha_k^5 d_k = \frac{1}{2}.$$

Using (6.2)–(6.4), (6.8), (6.10), the matrix $\Gamma(x, y)$ can be written in the form:

$$\Gamma(x, y) = \operatorname{Im} \sum_{k=1}^{3} \begin{bmatrix} A_k & B_k & 0 \\ B_k & C_k & 0 \\ 0 & 0 & D_k \end{bmatrix} \ln \sigma_k + \Lambda(x, y), \tag{6.13}$$

in which we have pointed out the terms with singularities and used the following notations

$$\begin{aligned} A_k &= 24a(A_{77}+A_{22}\alpha_k^2)(B_{66}+B_{44}\alpha_k^2)d_k, \\ B_k &= -24a(A_{12}+A_{78})(B_{66}+B_{44}\alpha_k^2)\alpha_k d_k, \\ C_k &= 24a(A_{11}+A_{88}\alpha_k^2)(B_{66}+B_{44}\alpha_k^2)d_k, \\ D_k &= 24a\{A_{11}A_{77}+[A_{11}A_{22}+A_{77}A_{88}-(A_{12}+A_{78})^2]\alpha_k^2+A_{22}A_{88}\alpha_k^4\}d_k. \end{aligned} \tag{6.14}$$

Obviously, $A_3 = B_3 = C_3 = D_1 = D_2 = 0$.

We have

$$\Gamma(x, y) = \Gamma^*(x, y), \tag{6.15}$$

where Γ^* is the transposed matrix of Γ. We denote by $\Gamma^{(k)}$ $(k = 1, 2, 3)$ the columns of the matrix $\Gamma(x, y)$.

Let us introduce the matricial differential operator

$$H\left(\frac{\partial}{\partial x}, n\right) = \left\| H_{ij}\left(\frac{\partial}{\partial x}, n_x\right)\right\|_{3\times 3}, \tag{6.16}$$

where

$$\begin{aligned} H_{11} &= A_{11}n_1\frac{\partial}{\partial x_1}+A_{88}n_2\frac{\partial}{\partial x_2}, \quad H_{12} = A_{12}n_1\frac{\partial}{\partial x_2}+A_{78}n_2\frac{\partial}{\partial x_1}, \\ & \qquad H_{13} = (A_{88}-A_{78})n_2, \\ H_{21} &= A_{12}n_2\frac{\partial}{\partial x_1}+A_{78}n_1\frac{\partial}{\partial x_2}, \quad H_{22} = A_{77}n_1\frac{\partial}{\partial x_1}+A_{22}n_2\frac{\partial}{\partial x_2}, \\ & \qquad H_{23} = (A_{78}-A_{77})n_1, \\ H_{13} &= H_{23} = 0, \quad H_{33} = B_{66}n_1\frac{\partial}{\partial x_1}+B_{44}n_2\frac{\partial}{\partial x_2}. \end{aligned} \tag{6.17}$$

Using the notations (2.9), the relations (2.4) can be written in the form:

$$t = H\left(\frac{\partial}{\partial x}, n_x\right)u. \tag{6.18}$$

Let $H_i\left(\frac{\partial}{\partial x}, n_x\right)$ be the row-matrix with the elements $H_{ij}\left(\frac{\partial}{\partial x}, n_x\right)$.

We introduce the operators

$$T_{\alpha}^{(x)}u = H_{\alpha}\left(\frac{\partial}{\partial x}, n_x\right)u, \quad M^{(x)}u = H_3\left(\frac{\partial}{\partial x}, n_x\right)u, \tag{6.19}$$

and the matrix

$$\mathcal{T}_y \Gamma(x, y) = H\left(\frac{\partial}{\partial y}, n_y\right)\Gamma^*(x, y). \tag{6.20}$$

From (6.2)–(6.4), (6.8)–(6.10), (6.19), we obtain:

$$\begin{aligned}
T_1^{(y)}\Gamma^{(1)} &= -\operatorname{Im}\sum_{k=1}^{3}[(A_{11}A_k+A_{12}B_k\alpha_k)n_1+(A_{78}B_k+A_{88}A_k\alpha_k)n_2]\frac{1}{\sigma_k}+\pi_{11},\\
T_2^{(y)}\Gamma^{(1)} &= -\operatorname{Im}\sum_{k=1}^{3}[(A_{77}B_k+A_{78}A_k\alpha_k)n_1+(A_{12}A_k+A_{22}B_k\alpha_k)n_2]\frac{1}{\sigma_k}+\pi_{12},\\
T_1^{(y)}\Gamma^{(2)} &= -\operatorname{Im}\sum_{k=1}^{3}[(A_{11}B_k+A_{12}\alpha_k C_k)n_1+(A_{78}C_k+A_{88}B_k\alpha_k)n_2]\frac{1}{\sigma_k}+\pi_{21},\\
T_2^{(y)}\Gamma^{(2)} &= -\operatorname{Im}\sum_{k=1}^{3}[(A_{77}C_k+A_{78}B_k\alpha_k)n_1+(A_{12}B_k+A_{22}C_k\alpha_k)n_2]\frac{1}{\sigma_k}+\pi_{22},\\
M^{(y)}\Gamma^{(3)} &= -\operatorname{Im}\sum_{k=1}^{3}[B_{66}D_k n_1+B_{44}D_k\alpha_k n_2]\frac{1}{\sigma_k}+\pi_{33},\\
T_\alpha^{(y)}\Gamma^{(3)} &= \pi_{3\alpha}, \quad M^{(y)}\Gamma^{(\alpha)} = \pi_{\alpha 3},
\end{aligned} \tag{6.21}$$

where the terms π_{ij} have "weak" singularities (by comparison with the main one). Using the relations

$$\begin{aligned}
A_{11}A_k+A_{12}B_k\alpha_k &= -\alpha_k(A_{78}B_k+A_{88}A_k\alpha_k),\\
A_{77}B_k+A_{78}A_k\alpha_k &= -\alpha_k(A_{12}A_k+A_{22}B_k\alpha_k),\\
A_{11}B_k+A_{12}C_k\alpha_k &= -\alpha_k(A_{78}C_k+A_{88}B_k\alpha_k),\\
A_{77}C_k+A_{78}B_k\alpha_k &= -\alpha_k(A_{12}B_k+A_{22}C_k\alpha_k), \quad B_{66}D_k = -B_{44}D_k\alpha_k^2,\\
\frac{\partial}{\partial s_y}\ln\sigma_k &= \frac{1}{\sigma_k}[\cos(n_y, x_2)-\alpha_k\cos(n_y, x_1)],
\end{aligned} \tag{6.22}$$

we obtain:

$$\begin{aligned}
T_1^{(y)}\Gamma^{(1)} &= \operatorname{Im}\sum_{k=1}^{3}L_k\frac{\partial\ln\sigma_k}{\partial s_y}+\pi_{11}, \quad T_2^{(y)}\Gamma^{(1)} = \operatorname{Im}\sum_{k=1}^{3}M_k\frac{\partial\ln\sigma_k}{\partial s_y}+\pi_{12},\\
T_1^{(y)}\Gamma^{(2)} &= \operatorname{Im}\sum_{k=1}^{3}N_k\frac{\partial\ln\sigma_k}{\partial s_y}+\pi_{21}, \quad T_2^{(y)}\Gamma^{(2)} = \operatorname{Im}\sum_{k=1}^{3}P_k\frac{\partial\ln\sigma_k}{\partial s_y}+\pi_{22},\\
M^{(y)}\Gamma^{(3)} &= \operatorname{Im}\sum_{k=1}^{3}R_k\frac{\partial\ln\sigma_k}{\partial s_y}+\pi_{33},
\end{aligned} \tag{6.23}$$

where

$$
\begin{aligned}
&L_k = -(A_{78}B_k+A_{88}A_k\alpha_k), \quad M_k = -(A_{12}A_k+A_{22}B_k\alpha_k),\\
(6.24)\qquad &N_k = -(A_{78}C_k+A_{88}B_k\alpha_k), \quad P_k = -(A_{12}B_k+A_{22}C_k\alpha_k),\\
&R_k = -B_{44}D_k\alpha_k.
\end{aligned}
$$

We denote by $\Lambda(x, y)$ the matrix obtained from (6.20) by interchanging the rows and columns. Using (6.15) we can write:

$$
(6.25)\qquad \Lambda(x, y) = \left[H\left(\frac{\partial}{\partial y}, n_y\right)\Gamma^*(x, y)\right]^* = \left[H\left(\frac{\partial}{\partial y}, n_y\right)\Gamma(y, x)\right]^*.
$$

Taking into account (6.23), we have:

$$
(6.26)\qquad \Lambda(x, y) = \mathrm{Im}\sum_{k=1}^{3}\begin{bmatrix} L_k & M_k & 0\\ N_k & P_k & 0\\ 0 & 0 & R_k\end{bmatrix}\frac{\partial \ln\sigma_k}{\partial s_y} + \pi(x, y),
$$

where $\pi(x, y)$ is the matrix with the elements π_{ij}.

From (6.9), (6.12), (6.14), we obtain:

$$
\sum_{k=1}^{3} L_k = \sum_{k=1}^{3} P_k = \sum_{k=1}^{3} R_k = 1, \quad \sum_{k=1}^{3} M_k = iM, \quad \sum_{k=1}^{3} N_k = -iN,
$$

$$
M = \frac{p}{(b_1+b_2)(b_2+b_3)(b_3+b_1)}, \quad N = \frac{q}{(b_1+b_2)(b_2+b_3)(b_3+b_1)},
$$

$$
p = (b_1b_2+b_1b_3+b_2b_3)\frac{A_{78}}{A_{88}} - \left(\frac{A_{12}A_{77}}{A_{22}A_{88}} - \frac{A_{78}B_{66}}{A_{88}B_{44}}\right) - \frac{A_{12}A_{77}B_{66}(b_1+b_2+b_3)}{A_{22}A_{88}B_{44}b_1b_2b_3},
$$

$$
q = \frac{A_{11}A_{78}B_{66}(b_1+b_2+b_3)}{A_{22}A_{88}B_{44}b_1b_2b_3} + \frac{A_{11}A_{78}}{A_{22}A_{88}} - \frac{A_{12}B_{66}}{A_{22}B_{44}} - \frac{A_{12}}{A_{22}}(b_1b_2+b_2b_3+b_3b_1).
$$

It is easy to verify that the columns of the matrix $\Lambda(x, y)$ satisfy the homogeneous system (2.10) at the point x.

7. Reduction of the boundary value problems to integral equations

Let Σ_i be a finite domain bounded by a closed Liapunov curve L, and Σ_e the complementary of Σ_i+L to the entire plane. The reciprocity relation (4.5) for the region Σ_i can be written in the form:

$$
(7.1)\qquad \int_{\Sigma_i}(vAu-uAv)\,d\sigma = \int_L\left[uH\left(\frac{\partial}{\partial x}, n_x\right)v - vH\left(\frac{\partial}{\partial x}, n_x\right)u\right]ds.
$$

Let $\sigma(y, \varepsilon)$ be a circle with centre in y and with radius ε. Let $y \in \Sigma_i$ and let ε be so small that σ be entirely contained in Σ_i. Then the formula (7.1) can be applied in $\Sigma_i-\sigma$ to some regular vector $u(x)$ and to vector $v(x) = \Gamma^{(k)}(x, y)$, $(k = 1, 2, 3)$. As in [19, 27], we obtain:

$$(7.2)\qquad 2\pi u_k(y) = \int_L \left[u(x) H\left(\frac{\partial}{\partial x}, n_x\right) \Gamma^{(k)}(x, y) - \Gamma^{(k)}(x, y) H\left(\frac{\partial}{\partial x}, n_x\right) u(x) \right] ds_x - \int_{\Sigma_i} \Gamma^{(k)}(x, y) A u \, d\sigma_x,$$

where by u_k we have indicated the components of the vector u.

The relation (7.2) can be written in the form:

$$(7.3)\qquad 2\pi u(y) = \int_L \left\{ \left[H\left(\frac{\partial}{\partial x}, n_x\right) \Gamma(x, y) \right]^* u(x) - \Gamma^*(x, y) H\left(\frac{\partial}{\partial x}, n_x\right) u(x) \right\} ds_x - \int_{\Sigma_i} \Gamma^*(x, y) A u \, d\sigma_x.$$

Taking into account (6.15), (6.25), from (7.3) we obtain:

$$(7.4)\qquad 2\pi u(x) = \int_L \left[\Lambda(x, y) u(y) - \Gamma(x, y) H\left(\frac{\partial}{\partial y}, n_y\right) u(y) \right] ds_y - \int_{\Sigma_i} \Gamma(x, y) A u(y) \, d\sigma_y.$$

Let $\psi(x)$ be a vector satisfying Holder's condition. We introduce the potential of a single layer:

$$(7.5)\qquad V(x; \psi) = \frac{1}{\pi} \int_L \Gamma(x, y) \psi(y) \, ds_y,$$

and the potential of a double layer:

$$(7.6)\qquad W(x; \psi) = \frac{1}{\pi} \int_L \Lambda(x, y) \psi(y) \, ds_y.$$

As in [27–29], we can prove:

THEOREM 7.1. *The potential of a single layer is continuous throughout.*

THEOREM 7.2. *The potential of a double layer tends to finite limits when the point x tends to $z \in L$, both from within and from without, and these limits are respectively equal to*

$$W_i(z; \psi) = \psi(z) + \frac{1}{\pi} \int_L \Lambda(z, y) \psi(y) \, ds_y,$$

$$W_e(z; \psi) = -\psi(z) + \frac{1}{\pi} \int_L \Lambda(z, y) \psi(y) \, ds_y.$$

THEOREM 7.3. *The $H\left(\frac{\partial}{\partial x}, n_x\right)$ operator of the single-layer potential $V(x; \psi)$ tends to finite limits, when the point x tends to the boundary point $z \in L$ from within or from without and these limits are respectively equal to*

$$\left[H\left(\frac{\partial}{\partial z}, n_z\right) V(z; \psi) \right]_i = -\psi(z) + \frac{1}{\pi} \int_L \left[H\left(\frac{\partial}{\partial z}, n_z\right) \Gamma(z, y) \right] \psi(y) \, ds_y,$$

$$\left[H\left(\frac{\partial}{\partial z}, n_z\right) V(z; \psi) \right]_e = \psi(z) + \frac{1}{\pi} \int_L \left[H\left(\frac{\partial}{\partial z}, n_z\right) \Gamma(z, y) \right] \psi(y) \, ds_y.$$

We consider the homogeneous system (2.10) and the boundary conditions (2.11) or (2.12), written in the form

(I) $$\lim_{x\to z} u(x) = \tilde{u}(z),$$

(II) $$\lim_{x\to z} H\left(\frac{\partial}{\partial x}, n_x\right)u(x) = \tilde{t}(z),$$

where $x \in \Sigma_i$, $z \in L$ and $\tilde{u}$, $\tilde{t}$ are given vectors satisfying Holder's condition.

We seek the solution of the first boundary value problem in the form of a double-layer potential and the solution of the second boundary value problem in the form of a single-layer potential. Using Theorems 7.2, 7.3 and the relation (6.25), we obtain for the unknown density, the following singular integral equations:

(I) $$\psi(z) + \frac{1}{\pi}\int_L \Lambda(z, y)\psi(y)ds_y = \tilde{u}(z),$$

(II) $$-\psi(z) + \frac{1}{\pi}\int_L \Lambda^*(y, z)\psi(y)ds_y = \tilde{t}(z).$$

Taking into account the relations

$$r^2 = (z_1-y_1)^2+(z_2-y_2)^2, \quad \sigma = (z_1-y_1)+i(z_2-y_2), \quad \frac{\partial \ln r}{\partial s_y}ds_y = \frac{dr}{r} = \frac{dt}{t-t_0} - id\theta.$$

$$\frac{\partial \ln \sigma_k}{\partial s_y} = \frac{\partial}{\partial s_y}\ln\frac{\sigma_k}{r} + \frac{\partial \ln r}{\partial s_y} = \frac{\partial \ln r}{\partial s_y} + \frac{i-\alpha_k}{\sigma\sigma_k} r\cos(r, n_y) - \frac{i\cos(r, n_y)}{r},$$

where t and t_0 are the affixes of the points y and z, and pointing out the characteristic part of the singular operator, the system (I) can be written in the form:

$$\psi(t_0) + \frac{1}{\pi}\begin{vmatrix} 1 & M & 0 \\ -N & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \int_L \frac{\psi(t)dt}{t-t_0} + K\psi = \tilde{u}(t_0).$$

For a general micropolar elastic solid, the index [19] of the system (I) is zero, so that this system is a system of singular integral equations of the normal type for which Fredholm's basic theorems are valid. It can be proved, in a similar way, that for the system (II) the Fredholm's basic theorems are valid.

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