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On the description of cyclic deformation processes using a more general elasto-plastic constitutive law

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To DESCRIBE large elasto-plastic deformations, a constitutive law is assumed that includes: a) a hypoelastic behaviour in the elastic range and b) a combination of "incremental theory" and "deformation theory" in the plastic range. The proposed yield condition permits an expansion as well as a translation of the yield surface. Taking into account a more general non-linear work-hardening rule some typical phenomena observed during cycling experiments can be described.

Dla opisu dużych odkształceń sprężysto-plastycznych wprowadzono równanie konstytutywne uwzględniające a) własności hiposprężyste w obszarze sprężystym, b) kombinację "teorii przyrostowej" i "teorii odkształceniowej" w obszarze plastycznym. Zaproponowany warunek plastyczności zezwala zarówno na rozszerzanie się powierzchni plastyczności jak i na jej przemieszczenie. Uwzględniając ogólniejsze, nieliniowe prawo wzmocnienia, opisać można szereg typowych zjawisk występujących w doświadczeniach z obciążeniem cyklicznym.

Предлагается определяющее уравнение, описывающее конечные упруго-пластические деформации с учетом а) гипоупругих свойств в упругой области, б) комбинации "деформационной теории" и "теории течения" в пластической области. Предложенное условие текучести описывает как расширение поверхности пластичности, так и её смещение. С учетом более общего, нелинейного закона упрочнения, можно описать ряд характерных явлений, возникающих при опытах на циклическое нагружение.

1. Introduction

ONE of the aims of present deliberations in the theory of plasticity is to select from the large number of suggested constitutive laws for large plastic deformations those which approximate most to a real material behaviour.

Therefore, two things are evidently necessary. First, the different constitutive laws have to be applied to simple deformation processes — for example tension, simple shear and pure shear. Then the special effects of these formulations can be observed.

In the second place, we must compare these theoretical results with interpretations of experiments, to be able to judge the quality of the formulation.

Let us begin with the second:

There are already some interpretations of stretching and shearing experiments which enable us to obtain at least qualitative statements of real material behaviour [1 to 5]. But useful results of experiments for a quantitative judgement of deformation processes with finite elasto-plastic deformations are not so far known.

On the theoretical side, we have better results:

Here, the research on the different formulations of constitutive laws in connection with loading processes only seems to be completed (see e.g. [6, 7](1)). Besides the effects

(1) Especially in [6], further bibliography can be found.

of higher order discussed there, now another group of effects can be noticed which are mainly caused by subjecting the tested model to — slowly proceeding — cyclic loading.

Particularly in the field of cyclic loading processes, extensive research has recently been carried out. Thus it has been shown experimentally that for cyclically varying external loads or displacements there exists a limiting steady state to which the cycles tend after an initial period of expansion (see especially Ref. [5]).

It should be emphasized here that the known hypotheses of isotropic as well as kinematic work-hardening models are not in general capable of representing this asymptotical behaviour of the cycles.

Thus, for instance, for stress cycling between prescribed limits in uniaxial tension and compression, the isotropic work-hardening model predicts transition to a purely elastic oscillation between the prescribed limits, whereas the kinematic work-hardening model predicts steady plastic cycling after the first cycle of loading. A similar situation occurs for cycling with prescribed strain amplitude involving plastic deformations.

BACKHAUS [8], BESSELING [9], IWAN [10], KAFKA [11] and MRÓZ [12, 13] try to eliminate this discrepancy between experiment and theoretical results — in part in a very different manner. All these statements, however, succeed only with great difficulty in formulating a constitutive law — representing real material behaviour; moreover, these considerations are restricted to small deformations only.

The purpose of the present paper is to compile a constitutive law from certain formulations for the description of finite elasto-plastic deformations capable of really describing cyclic loading processes.

We limit our investigation to phenomenological treatment of finite elasto-plastic deformations.

2. The constitutive law

2.1. General remarks

In what follows, elastic deformation processes will be described by the deformation-relation([2])

$$d^i_k = \underset{(el)}{d^i_k}. \tag{2.1}$$

Since it is well known that unloading of a body also runs elastically, we shall add this unloading here — without any great distinction between loading and unloading.

([2]) Here and in what follows we denote by

d^i_k tensor of strain rate,

σ^i_k stress tensor,

τ^i_k stress deviator,

$()/_0$ covariant derivation with respect to time,

$(\dot{\ })$ partial derivation with respect to time,

e.g. $\sigma^i_k|_0 = \dot{\sigma}^i_k + d^i_r\sigma^r_k - d^r_k\sigma^i_r$

etc.

All these quantities are relative to a body-fixed coordinate system ξ^i.

For elasto-plastic deformations a junction of elastic and plastic parts of strain rate

$$d^i_k = \underset{(el)}{d^i_k} + \underset{(pl)}{d^i_k} \tag{2.2}$$

will be used (see also [14]).

2.2. Elastic deformations

The elastic part of strain rate is described by the equation (3)

$$d^i_k = \frac{1}{2G}\tau^i_k|_0 + \frac{1}{9K}\dot{\sigma}^r_r\delta^i_k. \tag{2.3}$$

This approach coincides with a hypo-elastic behaviour of the material. It will be shown that this is at least a good approximation for all reasonable assumptions concerning the elastic part of deformations involved in the complex behaviour of material.

2.3. Elasto-plastic deformations

The plastic part of elasto-plastic deformations is regarded as incompressible. Furthermore, the deformations are assumed to be isothermal and depending only on stress or strain history.

Under these assumptions, the constitutive law consists of:

(a) a condition of plasticity, limiting the range of elastic deformations in relation to that of elasto-plastic deformations,

(b) a deformation law which connects stresses and strains (or their increments) and

(c) a hardening rule governing the changes of the yield condition during elasto-plastic deformation.

Condition of plasticity

The general form of the condition of plasticity can be described by the tensor function:

$$\begin{aligned} F(\sigma^i_k; \dots) &= \underset{(0)}{A} + \underset{(1)}{A^i_k}\tau^k_i + \underset{(2)}{A^{ir}_{ks}}\tau^k_i\tau^s_r + \dots = 0, \\ &= f(\tau^i_k; \dots) - k^2(w) = 0 \end{aligned} \tag{2.4}$$

with

$$\underset{(0)}{A} = -k^2(w)$$

and

$$f(\tau^i_k; \dots) = \underset{(1)}{A^i_k}\tau^k_i + \underset{(2)}{A^{ir}_{ks}}\tau^k_i\tau^s_r + \dots \tag{2.5}$$

(see e.g. [15]).

(3) The constants of material: Young's modulus E, shear modulus G, modulus of compression K, and Poisson ratio μ depend on each other by the well known relations:

$$K = 2G\frac{1+\mu}{3(1-2\mu)}, \quad E = 2G(1+\mu).$$

After simple conversion, this can be written in the special form:

$$F(\sigma^i_k; \ldots) = A^{ir}_{ks}(\tau^k_i - \alpha^k_i)(\tau^s_r - \alpha^s_r) - k^2(w) = 0. \tag{2.6}$$

(A special version of this condition of plasticity was first studied by BALTOV and SAWCZUK [16]). A more special form is the yield condition according to MELAN-PRAGER-SHIELD (ZIEGLER-KADASHEVITCH)NOVOZHILOV [17 to 19, 21]

$$F(\sigma^i_k; \ldots) = (\tau^i_k - \alpha^i_k)(\tau^k_i - \alpha^k_i) - k^2(w) = 0. \tag{2.7}$$

Here, the tensor α^i_k is defined by

$$\alpha^i_k|_0 = c \underset{(pl)}{d^i_k}. \tag{2.8}$$

Deformation law

As the deformation law, we choose according to LEHMANN [14]

$$\text{(a)} \quad \underset{(pl)}{d^i_k} = \dot{\lambda} \frac{\partial F}{\partial \sigma^k_i} \qquad \text{(type I)}, \tag{2.9}$$

$$\text{(b)} \quad \underset{(pl)}{d^i_k} = \tau^i_k|_0 \qquad \text{(type II)}, \tag{2.10}$$

Connecting these two formulations by means of a parameter $\varkappa$, the deformation law of elasto-plastic deformations can be represented in the form:

$$d^i_k = \dot{\lambda} \frac{\partial F}{\partial \sigma^k_i} + \varkappa \tau^i_k|_0 + \underset{(el)}{d^i_k}. \tag{2.11}$$

Substituting now the condition of plasticity (2.7) into (2.11), we obtain:

$$d^i_k = 2\dot{\lambda}(\tau^i_k - \alpha^i_k) + \varkappa \tau^i_k|_0 + \underset{(el)}{d^i_k}, \tag{2.12}$$

The still free parameter $\dot{\lambda}$ of this relation will be determined as follows. The secondary condition of the condition of plasticity $\dot{F} = 0$ leads us to

$$\dot{F} = 2(\tau^i_k - \alpha^i_k)(\tau^k_i|_0 - \alpha^k_i|_0) - \frac{dk^2}{dw}\dot{w} = 0. \tag{2.13}$$

On the other hand, valid for the plastic work w is

$$\dot{w} = \tau^i_k \underset{(pl)}{d^k_i}, \tag{2.14}$$

or

$$\dot{w} = 2\dot{\lambda}(\tau^i_k - \alpha^i_k)\tau^k_i + \varkappa \tau^i_k|_0 \tau^k_i. \tag{2.15}$$

Using (2.13), we can from this determine the parameter $\dot{\lambda}$ whence we finally obtain:

$$d^i_k = \frac{\left[(1-c\varkappa)(\tau^r_s - \alpha^r_s) - \dfrac{dk^2}{2dw}\tau^r_s \varkappa\right]\tau^s_r|_0}{ck^2 + \dfrac{dk^2}{2dw}(\tau^r_s - \alpha^r_s)\tau^s_r}(\tau^i_k - \alpha^i_k) + \varkappa \tau^i_k|_0 + \underset{(el)}{d^i_k}. \tag{2.16}$$

This law is now of such a form that it combines the effect of four different deformation laws.

If we put for instance: $c = 0$, then we obtain automatically $\alpha_i^k \equiv 0$ and a deformation law with isotropic hardening:

$$d_k^i = \varkappa \left\{ \left[\frac{1}{\varkappa \frac{dk^2}{dw}} - 1 \right] \frac{\tau_s^r \tau_r^s|_0}{\tau_n^m \tau_m^n} \tau_k^i + \tau_k^i|_0 \right\} + \underset{(el)}{d_k^i}, \tag{2.17}$$

which can be reduced to a formulation according to type I or type II according to the choice of the parameter.

$\varkappa = 0$

$$d_k^i = \frac{2}{\frac{dk^2}{dw}} \frac{\tau_s^r \tau_r^s|_0}{\tau_n^m \tau_m^n} \tau_k^i + \underset{(el)}{d_k^i} \quad \text{(type I)}, \tag{2.18}$$

$$\frac{\varkappa dk^2}{2dw} = 1$$

$$d_k^i = \frac{2}{\frac{dk^2}{dw}} \tau_k^i|_0 + \underset{(el)}{d_k^i} \quad \text{(type II)}. \tag{2.19}$$

The same also can be said for $c \neq 0$. According to the choice of $\varkappa$, the law will be reduced to a formulation of type I or type II.

Hardening rule

The upper bounds of the parameters $\varkappa$ and c include the hardening rule k^2, so far still undetermined, as a function of the plastic work w. Usually, this function is determined by adapting the deformation law (2.16) to the uniaxial tensile test — i.e., we define the function $k^2(w)$ in such a way that the deformation law becomes independent of the two parameters w and c.

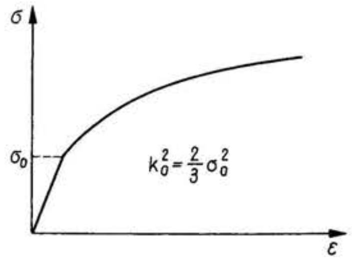

FIG. 1. Non-linear hardening-rule.

As formulation for the tensile test, we use a general statement

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\dot{\sigma}}{3B/2m} \left(\frac{\sigma}{\sigma_0} \right)^{m-2}, \tag{2.20}$$

which represents the well known linear formulation for $m = 2$, which is sublinear for $m > 2$ and hyperlinear for $m < 2$. This general formulation is very well adapted to approximate diagrams of stretching experiments — much better than would be possible by a linear formulation (see [20]).

After extensive calculations, this process of accommodation provides the relation:

$$(2.21)\qquad k^2(w) = \frac{2}{3}\sigma_0^2\left\{\left(1+\frac{3Bw}{2\sigma_0^2}\right)^{\frac{1}{m}}\left[1-\frac{cm}{B(m-1)}\left(1+\frac{3Bw}{2\sigma_0^2}\right)^{\frac{m-2}{m}}\right]+\frac{cm}{B(m-1)}\right\}^2$$

which by substitution of

$$(2.22)\qquad k_0^2 = \frac{2}{3}\sigma_0^2 \quad \text{and} \quad \zeta = \frac{cm}{B(m-1)}$$

becomes:

$$(2.23)\qquad k^2(w) = k_0^2\left\{\left(1+\frac{Bw}{k_0^2}\right)^{\frac{1}{m}}\left[1-\zeta\left(1+\frac{Bw}{k_0^2}\right)^{\frac{m-2}{m}}\right]+\zeta\right\}^2.$$

By differentiation with respect to w, we find finally:

$$(2.24)\qquad \frac{dk^2}{dw} = \frac{2B}{m}\left\{\left(1+\frac{Bw}{k_0^2}\right)^{\frac{1}{m}}\left[1-\zeta\left(1+\frac{Bw}{k_0^2}\right)^{\frac{m-2}{m}}\right]+\zeta\right\}\times$$
$$\times\left\{1-\zeta(m-1)\left(1+\frac{Bw}{k_0^2}\right)^{\frac{m-2}{m}}\right\}\left(1+\frac{Bw}{k_0^2}\right)^{\frac{1-m}{m}}.$$

3. Examples

3.1. General remarks

The constitutive law thus investigated will now be applied to three simple deformation processes:

(a) to an uniaxial tensile test,
(b) to simple shear and
(c) to pure shear.

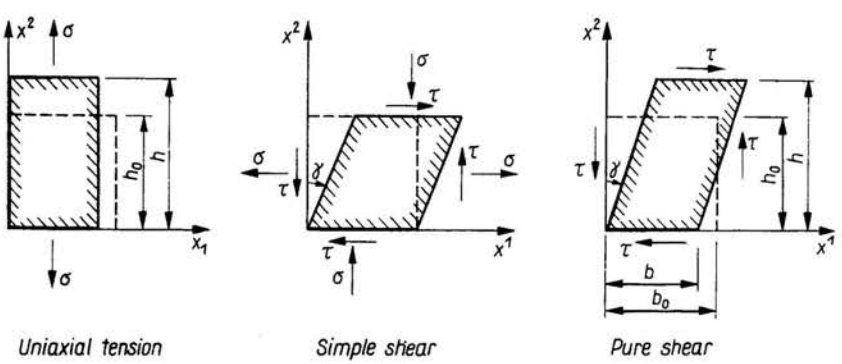

FIG. 2. Deformation processes.

Thereby we denote:

ϑ the tangent of shear angle,

ε the normal strain in tension and in pure shear,

σ the normal stress in tension and in simple shear,

τ the shear stress.

First, we shall study the tensile test.

3.2. Tensile test

In formulating the constitutive law, we had to adjust the hardening rule to the tensile test in order to determine the free parameters.

Vice versa now, seeking a general solution of tensile test it must be possible to formulate this in closed form by the general hardening rule (2.20).

For loading only this result will obviously be trivial — and therefore is not further dealt with here. More difficulties will arise from cyclic loading processes.

In this case, the solution (2.20) only can be used for the first half-cycle of the deformation process. All subsequent cycles are relatively undetermined as to their shape. Careful consideration of the particular deformation history enables us to describe them easily in the form of a more general recurrence formula.

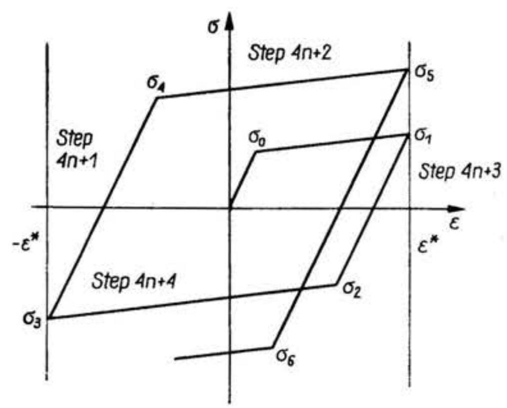

FIG. 3. Cyclic tension-compression curve.

For the nth cycle, we have

Step $4n+1$

$$\varepsilon = \frac{\sigma}{E} + \frac{2m}{3(m-1)}\frac{\sigma_0}{B}\left\{-\sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-1}}{\sigma_0}\right|\right]^{m-1} + \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-2}}{\sigma_0}\right|\right]^{m-1} + \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-3}}{\sigma_0}\right|\right]^{m-1} - \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-4}}{\sigma_0}\right|\right]^{m-1}\right\};$$

Step $4n+2$

$$\varepsilon = \frac{\sigma}{E} + \frac{2m}{3(m-1)}\frac{\sigma_0}{B}\left\{\left[\left|\frac{\sigma}{\sigma_0}\right|\right]^{m-1} - \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i}}{\sigma_0}\right|\right]^{m-1} - \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-1}}{\sigma_0}\right|\right]^{m-1} + \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-2}}{\sigma_0}\right|\right]^{m-1} + \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-3}}{\sigma_0}\right|\right]^{m-1}\right\};$$

Step $4n+3$

$$\varepsilon = \frac{\sigma}{E} + \frac{2m}{3(m-1)}\frac{\sigma_0}{B}\left\{\sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i+1}}{\sigma_0}\right|\right]^{m-1} - \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i}}{\sigma_0}\right|\right]^{m-1} - \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-1}}{\sigma_0}\right|\right]^{m-1} + \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-2}}{\sigma_0}\right|\right]^{m-1}\right\};$$

Step $4n+4$

$$\varepsilon = \frac{\sigma}{E} + \frac{2m}{3(m-1)}\frac{\sigma_0}{B}\left\{-\left[\left|\frac{\sigma}{\sigma_0}\right|\right]^{m-1} + \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i+2}}{\sigma_0}\right|\right]^{m-1} + \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i+1}}{\sigma_0}\right|\right]^{m-1} - \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i}}{\sigma_0}\right|\right]^{m-1} - \sum_{i=0}^{n}\left[\left|\frac{\sigma_{4i-1}}{\sigma_0}\right|\right]^{m-1}\right\}.$$

Thereby, we have the initial conditions:

$$\frac{\sigma_i}{\sigma_0} = \begin{cases} 0, & i < 0, \\ 1, & i = 0, \end{cases}$$
$$\alpha = 0, \qquad i \leqslant 0.$$

Moreover, the condition of constant yield surface during elastic deformation processes leads to some transition conditions. Thus,

$$\sigma_{4i} = 3\alpha_{4i-1} - \sigma_{4i-1},$$
$$\sigma_{4i+2} = 3\alpha_{4i+1} - \sigma_{4i+1}.$$

Now, the results of these considerations are presented in Figs. 4 and 5[4], linear hardening hypothesis ($m = 2$) in Fig. 4 and non-linear hardening hypothesis ($m = 2.83$) in Fig. 5 within the cyclic tension-compression behaviour between fixed limits of deformation.

Thereby it is shown very clearly (Fig. 4) that in the case of linear hardening neither an isotropic work-hardening model ($cm/B = 0$) nor a kinematic hardening model ($cm/B = 1$) is able to represent real material behaviour. An isotropic hardening model describes pure elastic behaviour after a transitory period of permanent increase between the fixed limits

(4) The results presented in this paper are based on brass alloy 70/30 with data:
$G = 300$ Mp/cm², $\mu = 0.3$.
$B = 52.9$ Mp/cm²,
$\sigma_0 = 0.9$ Mp/cm²,

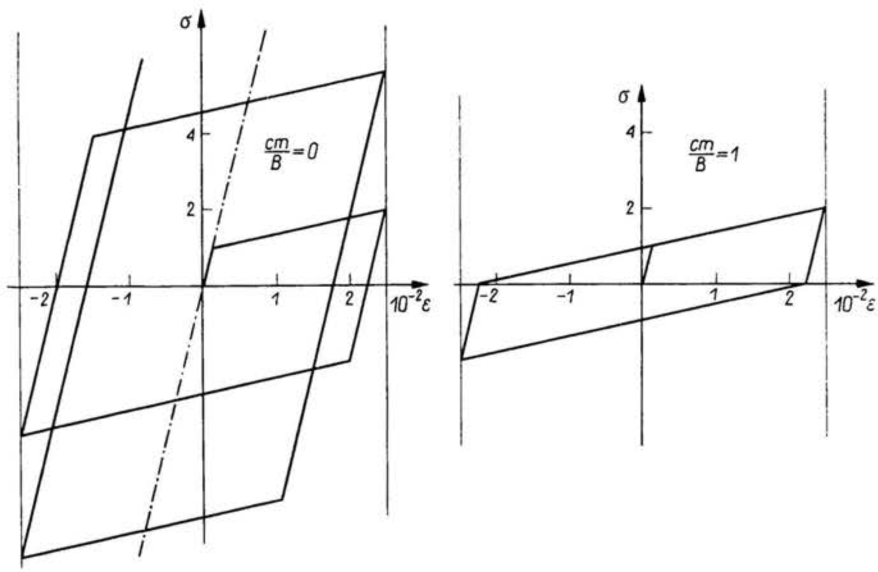

FIG. 4. Tensile test under cyclic loading $m = 2$.

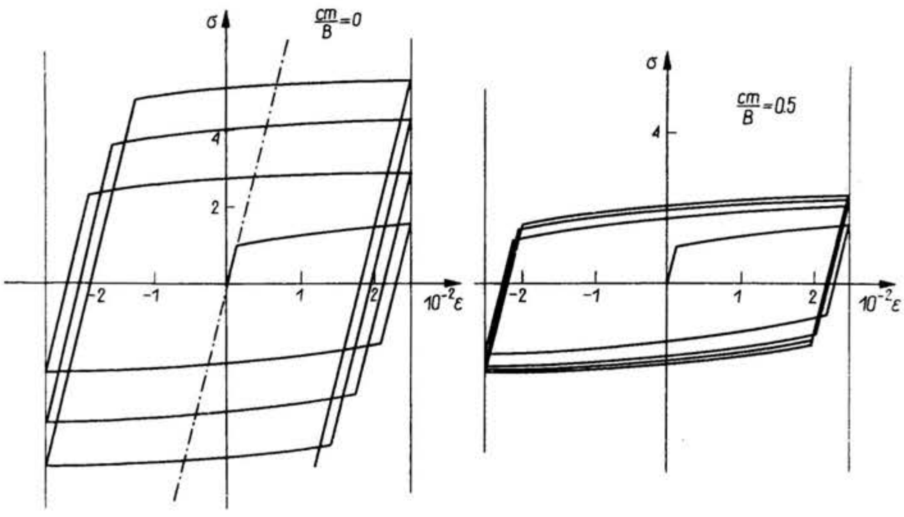

FIG. 5. Tensil test under cyclic loading $m = 2.83$.

(dash-dotted line), whereas the rule of kinematic work-hardening implies a steady state after the first cycle of loading. For every intermediate value of cm/B, there will also appear a behaviour as for $cm/B = 0$ (isotropic behaviour).

It is otherwise, however in sublinear hardening (Fig. 5). Indeed, for $cm/B = 0$, the effect above described of approaching the pure elastic deformation process can be seen. But every change of the parameter cm/B (here $cm/B = 0.5$) makes the loading cycle go to a limit state after an initial phase of increase.

3.3. Pure shear, simple shear

For the simple geometric conditions of a uniaxial tensile test, a closed solution of this problem may be found. In the cases of pure shear and simple shear, however, a numerical integration of a system of 5 non-linear differential equations of 1st order is necessary. The results of these extensive calculations are shown in Figs. 6 to 9. Fig. 6 and Fig. 7 represent pure shear, Fig. 8 and Fig. 9 — simple shear.

Thus in Fig. 6 and Fig. 8, the final values of stresses or strains in each of the half-cycles are plotted versus the number of these half-cycles. That happens alike for a deformation law of type I ($\varkappa B/m = 0$) and for a combination of type I+II ($\varkappa B/m = 0.1$). In Fig. 7 and Fig. 9, the progresses of the second-order effects are finally plotted against the variable shear angle (here only for a combination of type I+II).

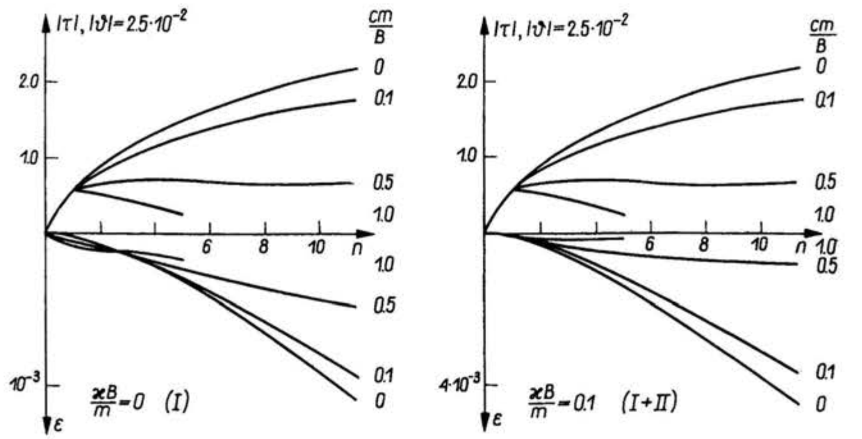

FIG. 6. Amplitudes of loading (pure shear) $m = 2.83$.

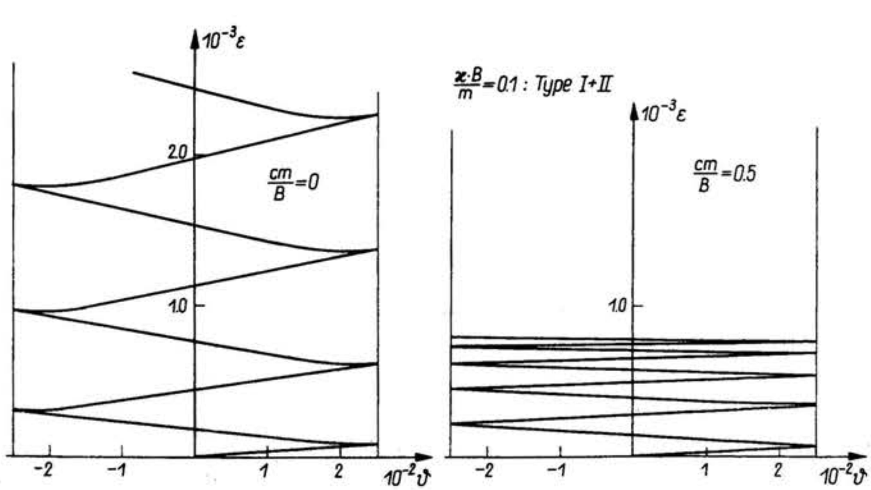

FIG. 7. Normal strain under cyclic loading (pure shear) $m = 2.83$.

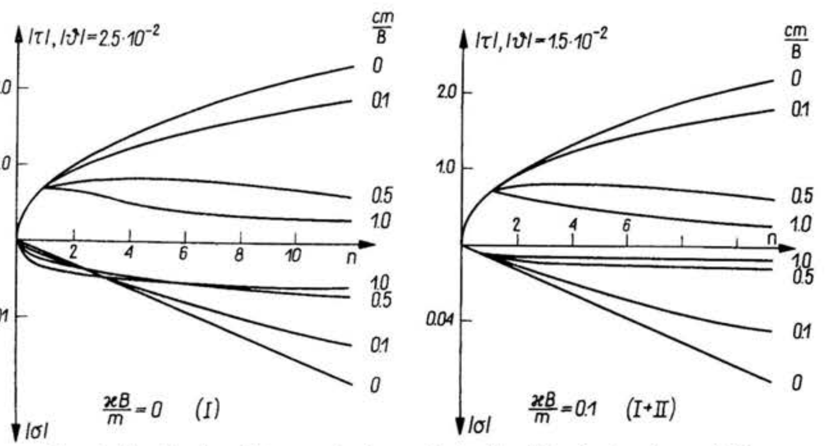

FIG. 8. Amplitudes of stresses during cyclic loading (simple shear) $m = 2.83$.

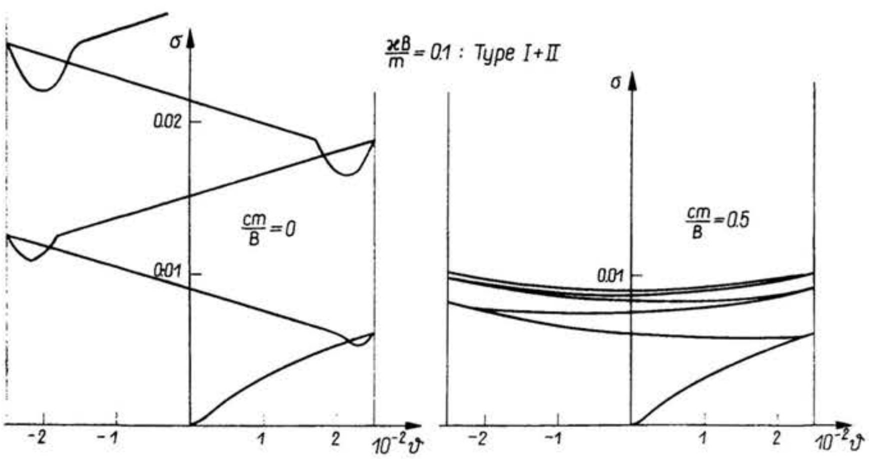

FIG. 9. Normal stresses under cyclic loading (simple shear) $m = 2.83$.

We must emphasize in particular that the curves of Fig. 7 are obviously in qualitative accord with the experimental results of ROSE and STÜWE [4].

All four figures show very clearly that both parameters ($\varkappa$ and ζ) can be so determined that stresses as well as strains in pure and simple shear aim at a fixed limit with an increasing number of half-cycles n.

4. Concluding remarks

Taking a special form of deformation law which comprises the deformation laws studied so far ([5]) by suitable choice of the parameters $\left(\varkappa \text{ and } \zeta = \frac{m}{m-1}\frac{c}{B}\right)$ (the pa-

([5]) Except those deformation laws based on already initially anisotropic yield conditions. This will be dealt with a further paper.

rameter $\varkappa$ represents the influence of type II in the deformation law, the parameter ζ denotes existing kinematic hardening), and taking a more general (non-linear) hardening model, we can describe in a simple manner real material behaviour in such deformation processes as uniaxial tension, pure shear and simple shear.

Further quantitative experiments are necessary for a more exact determination of the parameters $\varkappa$ and ζ.

References

1. D. S. DUGDALE, *Stess-strain cycles of large amplitude*, J. Mech. Phys. Solids, **7**, 135–142, 1959.
2. P. P. BENHAM, *Some observations of cyclic strain-induced creep in mild steel at room temperature*, Int. J. Mech. Sci., **7**, 81–86, 1965.
3. A. M. FREUDENTHAL and M. RONAY, *Second-order effects in dissipative media*, Proc. Roy. Soc., ser. A **292**, 14–50, 1966.
4. W. ROSE and H. P. STÜWE, *Der Einfluss der Textur auf die Längenänderung im Torsionsversuch*, Z. Metallkunde., **59**, 396–399, 1968.
5. L. F. COFFIN, *Low-cycle fatigue: a review*, Appl. Mat. Res., **1**, 129–141, 1962.
6. TH. LEHMANN, *Einige Bemerkungen zu einer allgemeinen Klasse von Stoffgesetzen für grosse elasto-plastische Formänderungen*, Ing.-Archiv 41, 297–310, 1972.
7. TH. LEHMANN, *Vergleich einiger Stoffgesetze für plastische Formänderungen im Hinblick auf Effekte zweiter Ordnung* [unpublished].
8. G. BACKHAUS, *Zur Fliessgrenze bei allgemeiner Verfestigung*, Z. Angew. Math. Mech. (ZAMM), **48**, 99–108, 1968.
9. I. F. BESSELING, *An theory of plastic flow for anisotropic hardening in plastic deformation of an initially isotropic material*, Natn. Aero. Res. Inst. Rep. S-410, 1953.
10. W. D. IWAN, *On a class of models for the yielding behaviour of continuous and composite systems*, J. Appl. Mech., **34**, 612–617, 1967.
11. V. KAFKA, *Der Einfluss der mikroskopischen Heterogenität auf die elastischen Verformungsgesetze*, Z. Angew. Math. Mech. (ZAMM), **46**, pp. 501–509, 1966.
12. Z. MRÓZ, *On the description of anisotropic workhardening*, J. Mech. Phys. Solids, **15**, 163–175, 1967.
13. Z. MRÓZ, *An attempt to describe the behaviour of metals under cyclic loads using a more general work-hardening model*, Acta Mechanica, **7**, 199–212, 1969.
14. TH. LEHMANN, *Zur Beschreibung grosser plastischer Formänderungen unter Berücksichtigung der Werkstoffverfestigung*, Rheol. Acta, **2**, 247–254, 1962.
15. TH. LEHMANN, *Anisotrope plastische Formänderungen*, Rheol. Acta, **3**, 281–285, 1964.
16. A. BALTOV and A. SAWCZUK, *A rule of anisotropic hardening*, Acta Mechanica, **1**, 81–92, 1965.
17. E. MELAN, *Zur Plastizität des räumlichen Kontinuums*, Ing.-Archiv, **9**, 116–126, 1938.
18. W. PRAGER, *The theory of plasticity: A survey of recent achievments* (James Clayton Lecture), Proc. Instn. Mech. Engrs., **169**, 41–57, 1955.
19. R. T. SHIELD and H. ZIEGLER, *On Prager's hardening rule*, Z. Angew. Math. Phys. (ZAMP), **9**, 260–276, 1958.
20. O. BRUHNS, *Die Berücksichtigung einer isotropen Werkstoffverfestigung bei der elastisch-plastischen Blechbiegung mit endlichen Formänderungen*, Ing-Archiv, **39**, 63–72, 1970.
21. YU. KADASHEVICH and V. V. NOVOZHILOV, Prikl. Math., **22**, 78, 1958.

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