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Stationary regime of a multi-mass dynamic model with inertia and force excitation

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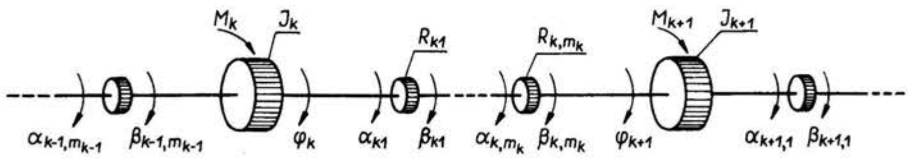
We consider the paper a stationary regime of a multi-mass machine unit with elastic links and massless reduction mechanisms, subjected to an action of kinetic and position forces.

W pracy rozpatrzono stan ustalony wielomasowego układu maszynowego z więzami sprężystymi i nieważkimi mechanizmami redukcyjnymi, poddany działaniu sił kinetycznych i pozycyjnych.

В работе рассматривается стационарный режим многомассового машинного агрегата с упругими связьями и бесмассовыми редукторными механизмами под действием кинетических и позиционных сил.

The high level of average velocities in stationary motions of most of the modern machine units converts small inertia generators (with variable inertia moments and transmission ratios) into important factors determining the dynamics of stationary regimes. Moreover, the comparative smallness of the above generators creates a possibility of establishing stable regimes of internal and external self-synchronizations for multi-mass machine units with elastic links. Consequently, the analytic methods of solving non-linear differential-algebraic systems describing the corresponding dynamic models, become especially interesting.

In this paper, we use the small parameter method to construct the successive approximations to the exact periodic solution of the system describing the motion of a multi-mass machine unit with inertia and force excitation and elastic links. We consider a rather



$$1 \leqslant k \leqslant n-1$$

FIG. 1.

general model of a machine unit, shown in Fig. 1. The model consists of n concentrated masses with reduced inertia moments $J_1, \ldots, J_n$ depending on the positions of the corresponding masses,

$$J_k(\varphi_k) = J_k(\varphi_k+\Phi) = J_{k0}[1+\varepsilon i_k(\varphi_k)], \quad k = 1, \ldots, n, \tag{1}$$

where φ_k is the generalized angular coordinate of k-th link. Each link is subjected to a reduced external moment M_k constituting a function of position and velocity

(2) $$M_k(\varphi_k, \dot{\varphi}_k) = M_k(\varphi_k+\Phi, \dot{\varphi}_k) = P_k(\varphi_k)+Q_k(\dot{\varphi}_k),$$

where

(3) $$P_k(\varphi_k) = P_k(\varphi_k+\Phi), \quad \int\limits_0^{\Phi} P_k(\varphi)d\varphi = 0, \quad k = 1, \dots, n.$$

Each pair of links [e.e. kth and $(k+1)$th] are connected by means of m_k+1 elastic shafts with reduced rigidities $C_{k1}, C_{k2}, \dots, C_{k,m_k+1}$. The motion is transferred from kth to $(k+1)$th link by means of m_k massless reduction mechanisms $R_{k1}, R_{k2}, \dots, R_{k,m_k}$. The transmission ratio ν_{ks} of the reduction mechanism R_{ks} is assumed to have the form

(4) $$\frac{\dot{\beta}_{ks}}{\dot{\alpha}_{ks}} = \nu_{ks}(\alpha_{ks}) = \nu_{ks}(\alpha_{ks}+\Phi) = N_{ks}[1+\varepsilon j_{ks}(\alpha_{ks})],$$

where α_{ks}, β_{ks} are the input and output angular parameters of the reduction mechanism R_{ks}, respectively, and Φ is the common geometric period of the functions J_k, P_k and ν_{ks}.

On the basis of the reasonable assumption

(5) $$\max_{\varphi\in I}|i_k(\varphi)| \ll 1, \quad \max_{\alpha\in I}|j_{ks}(\alpha)| \ll 1, \quad I = [0, \Phi],$$
$$k = 1, \dots, n, \quad s = 1, \dots, m_k,$$

we introduce into (1) and (4) a fictitious small parameter ε which will eventually be taken to be equal to unity.

Without affecting the generality, we may assume that $N_{ks} = 1$, $k = 1, \dots, n-1$, $s = 1, \dots, m_k$ since this can always be achieved by an appropriate reduction. Then, on the basis of (4), we obtain

(6) $$\beta_{ks} = \alpha_{ks}+\delta_{ks}+\varepsilon\gamma_{ks}(\alpha_{ks}), \quad \begin{matrix} k = 1, \dots, n-1, \\ s = 1, \dots, m_k, \end{matrix}$$
$$\delta_{ks} = \beta_{ks}(t_0)-\eta_{ks},$$

where

(7) $$\gamma_{ks}(\alpha_{ks}) = \int\limits_{\eta_{ks}}^{\alpha_{ks}} j_{ks}(\alpha)d\alpha, \quad \eta_{ks} = \alpha_{ks}(t_0).$$

It can be shown (1) that when the degree of non-uniformity of rotation of seperate masses is comparatively small, then to within first order of smallness, in the case of stationary regimes, we have the relations

(8) $$P_k(\varphi_k) = \varepsilon P_k(\varphi_k), \quad k = 1, \dots, n.$$

The differential system describing the motion of the model has the form

(9)

$$J_k\ddot{\varphi}_k = M_k(\varphi_k, \dot{\varphi}_k)+C_{k1}(\alpha_{k1}-\varphi_k)+C_{k-1,m_k+1}(\beta_{k-1,m_k}-\varphi_k)-\frac{1}{2}J_k'\dot{\varphi}_k^2, \quad k = 1, \dots, n,$$

where

(10) $$C_{0,m_0+1} = C_{n1} = 0, \quad J'_k = \frac{dJ_k}{d\varphi_k}, \quad k = 1, \ldots, n.$$

Here α_{k1} and β_{k-1,m_k} depend on $\varphi_1, \ldots, \varphi_n$ through the remaining variables α_{ks}, β_{ks}; thus, to transform the right-hand side of (9) to an explicit form, it is necessary to determine all parameters α_{ks}, β_{ks} the number of which is $2p$ and $p = \sum_{k=1}^{n-1} m_k$. In accordance with (6), we have p explicit relations between α_{ks} and β_{ks}. The remaining p equations for the determination of α_{ks} and β_{ks} are obtained on the basis of the law of conservation of the power transmitted between the neighbouring shafts, namely

(11) $$C_{ks}(\alpha_{ks}-\beta_{k,s-1})\dot{\alpha}_{ks} = C_{k,s+1}(\alpha_{k,s+1}-\beta_{ks})\dot{\beta}_{ks},$$
$$k = 1, \ldots, n-1, \quad s = 1, \ldots, m_k,$$

where $\beta_{k0} = \varphi_k$, $\alpha_{k,m_k+1} = \varphi_{k+1}$.

Thus, we arrive at the expressions for the elastic torsion moments

(12) $$F_{ks} = F_{k,s+1}\nu_{ks}, \quad k = 1, \ldots, n-1, \quad s = 1, \ldots, m_k+1$$

acting on sth shaft situated between the kth and $(k+1)$th links,

(13) $$F_{ks} = \begin{cases} C_{k1}(\alpha_{k1}-\varphi_k), & s = 1, \\ C_{ks}(\alpha_{ks}-\beta_{k,s-1}), & 2 \leqslant s \leqslant m_k, \\ C_{k,m_k+1}(\varphi_{k+1}-\beta_{k,m_k+1}), & s = m_k+1. \end{cases}$$

In accordance with the small parameter method, we seek the solution of the differential system (9) and the algebraic system (6), (11) [or the differential system (9) and the algebraic system (6), (12)] in the form

$$\varphi_k = \sum_{\nu=0}^{\infty} \varepsilon^\nu \varphi_k^{(\nu)}, \quad k = 1, \ldots, n,$$

$$F_{ks} = \sum_{\nu=0}^{\infty} \varepsilon^\nu F_{ks}^{(\nu)}, \quad \begin{matrix} k = 1, \ldots, n-1, \\ s = 1, \ldots, m_k+1, \end{matrix}$$

(14) $$\alpha_{ks} = \sum_{\nu=0}^{\infty} \varepsilon^\nu \alpha_{ks}^{(\nu)}, \quad \begin{matrix} k = 1, \ldots, n-1, \\ s = 1, \ldots, m_k, \end{matrix}$$

$$\beta_{ks} = \sum_{\nu=0}^{\infty} \varepsilon^\nu \beta_{ks}^{(\nu)}, \quad \begin{matrix} k = 1, \ldots, n-1, \\ s = 1, \ldots, m_k. \end{matrix}$$

Substituting (14) into (9), (6) and (11) (or into (9), (6) and (12)), we obtain a family of linear systems of differential equations for the determination of $\varphi_k^{(0)}, \varphi_k^{(1)}, \ldots, k = 1, \ldots, n$. The system for the zeroth approximation has the form

(15) $$J_{k0}\ddot{\varphi}_k^{(0)} = Q_k(\dot{\varphi}_k^{(0)}) + C_k(\varphi_{k+1}^{(0)} - \varphi_k^{(0)}) + C_{k-1}(\varphi_{k-1}^{(0)} - \varphi_k^{(0)}), \quad k = 1, \ldots, n,$$

where $C_0 = C_n = 0$ and

$$C_k = \left(\sum_{s=1}^{m_k+1} \frac{1}{C_{ks}}\right)^{-1}, \quad k = 1, \ldots, n-1 \tag{16}$$

are the reduced equivalent rigidity coefficients between the kth and $(k+1)$th links.

The only rotational solution of the system (15), to within an additive constant in determining any of the variables $\varphi_k^{(0)}$ (e.g. $\varphi_1^{(0)}$), has the form

$$\varphi_k^{(0)} = \omega_e t - \frac{1}{C_{k-1}} \sum_{s=1}^{k-1} Q_s(\omega_e), \quad k = 1, \ldots, n. \tag{17}$$

Here, the mean energy velocity ω_e is the solution of the equation

$$\sum_{s=1}^{n} Q_s(\omega_e) = 0. \tag{18}$$

The zeroth approximation $\alpha_{ks}^{(0)}$, $\beta_{ks}^{(0)}$ and $F_{ks}^{(0)}$ of the solution of the algebraic system (6), (11), (12) is to be determined from the equations

$$\delta_{ks} + \alpha_{ks}^{(0)} = \beta_{ks}^{(0)} = \left[1 - C_k \sum_{s=1}^{k+1} \frac{1}{C_{ks}}\right] \varphi_k^{(0)} + \varphi_{k+1}^{(0)} C_k \sum_{s=1}^{k+1} \frac{1}{C_{ks}}, \tag{19}$$

$$k = 1, \ldots, n-1, \quad s = 1, \ldots, m_k;$$

$$F_{ks}^{(0)} = C_k(\varphi_{k+1}^{(0)} - \varphi_k^{(0)}), \quad \begin{matrix} k = 1, \ldots, n-1, \\ s = 1, \ldots, m_k+1. \end{matrix}$$

Hence

$$\delta_{ks} + \alpha_{ks}^{(0)} = \beta_{ks}^{(0)} = \omega_e t - \mu \sum_{s=1}^{k} \frac{1}{C_{ks}}, \tag{20}$$

$$k = 1, \ldots, n-1, \quad s = 1, \ldots, m_k, \quad \mu = Q_1(\omega_e).$$

The system for the determination of the first corrections $\varphi_k^{(1)}$, $k = 1, \ldots, n$, to the solution (17) of the generating system (15) can be written in the form

$$J_{k0} \ddot{\varphi}_k^{(1)} = \lambda_k \dot{\varphi}_k^{(1)} + F_{k1}^{(1)} - F_{k-1, m_{k-1}+1}^{(1)} - \frac{1}{2} \omega_e^2 J_{k0} i_k'(\varphi_k^{(0)}) + P_k(\varphi_k^{(0)}), \quad k = 1, \ldots, n, \tag{21}$$

where

$$\lambda_k = \left. \frac{dQ_k(\omega)}{d\omega} \right|_{\omega=\omega_e}, \quad k = 1, \ldots, n,$$

$$F_{k1}^{(1)} = C_k(\varphi_{k+1}^{(1)} - \varphi_k^{(1)}) - C_k \sum_{\nu=1}^{m_k} \gamma_{k\nu}(\alpha_{k\nu}^{(0)}) - \mu \sum_{\nu=1}^{m_k} j_{k\nu}(\alpha_{k\nu}^{(0)}) \sum_{s=\nu+1}^{m_k+1} \frac{C_k}{C_{ks}}, \quad k = 1, \ldots, n-1, \tag{22}$$

$$F^{(1)}_{k,m_k+1} = C_k(\varphi^{(1)}_{k+1}-\varphi^{(1)}_k)-C_k\sum_{\nu=1}^{m_k}\gamma_{k\nu}(\alpha^{(0)}_{k\nu})-\mu\sum_{\nu=1}^{m_k}j_{k\nu}(\alpha^{(0)}_{k\nu})\times$$
$$\times\left(-1+\sum_{s=\nu+1}^{m_k+1}\frac{C_k}{C_{ks}}\right), \quad k = 1, \ldots, n-1.$$

The first corrections $F^{(1)}_{ks}$, $s = 2, \ldots, m_k$ to the reduced torsion moments F_{ks}, $s = 2, \ldots, m_k$, are to be found from the equations

$$F^{(1)}_{ks} = F^{(1)}_{k1}+\mu\sum_{\nu=1}^{s-1}j_{k\nu}(\alpha^{(0)}_{k\nu}) = F^{(1)}_{k,m_k+1}-\mu\sum_{\nu=s}^{m_k}j_{k\nu}(\alpha^{(0)}_{k\nu}). \tag{23}$$

The solution of the system (21) in a closed form encounters no principal difficulties, even if some of the physical-geometric constants of the unit are given in a parametric form.

It is noteworthy that in investigating dynamics of a stationary regime it is very important to determine the torsion moments as functions of time $F_{ks} = F_{ks}(t)$; $k = 1, \ldots, n-1$; $s = 1, \ldots, m_k+1$, since this enables us to calculate the overloading coefficients.

There is another important factor in the investigations of the stationary regime, namely the determination of the functions $\omega_k = \omega_k(t)$; $\omega_k = \dot{\varphi}_k$ and the coefficients of non-uniformity of rotation δ_k,

$$\delta_k = \frac{1}{\omega_e}(\omega_{k,\max}-\omega_{k,\min}).$$

By means of the above-presented formulae the functions $F_{ks}(t)$ and $\omega_k(t)$ can be determined with a sufficient engineering accuracy.

The presented computational schemes with slight modifications can also be used in the cases of weakly non-linear elastic links of the form

$$C_{ks}(\tau_{ks}) = C^*_{ks}\tau_{ks}+\varepsilon C^{**}_{ks}(\tau_{ks}), \quad c^*_{ks} = \text{const},$$

where $\tau_{ks} = \alpha_{ks}-\beta_{k,s-1}$.

In view of the representation (8), the regimes of the external and internal self-synchronization are equivalent to within the first order of smallness. The conditions for a stable self-synchronization regime in the general case are subject to certain restrictions on the coefficients $\lambda_1, \ldots, \lambda_n$ and $C_1, \ldots, C_{n-1}$. Observe that in practice, the number of masses subject to the action of external moments depending on the velocity is not greater than two [1, 2].

Consider, for instance, the case when $\lambda_k \neq 0$ and $\lambda_1 = \ldots = \lambda_{k-1} = \lambda_{k+1} = \ldots = \lambda_n = 0$. By induction we can prove that the necessary and sufficient conditions of stability of an excited motion of the system (9) have the form

$$\lambda_k < 0, \quad C_k > 0, \quad k = 1, \ldots, n-1. \tag{24}$$

These conditions ensuring a stable regime of dynamic self-synchronization correspond to the stable segment of the mechanical characteristic of the engine.

As an example consider an application of our computational schemes to a determination of the stationary regime of a two-mass machine unit with a universal joint transmission and constant inertia moments (Fig. 2), subject to an action of kinetic forces only. It is known [3], that the non-linear terms in the transmission ratio of the universal joint mech-

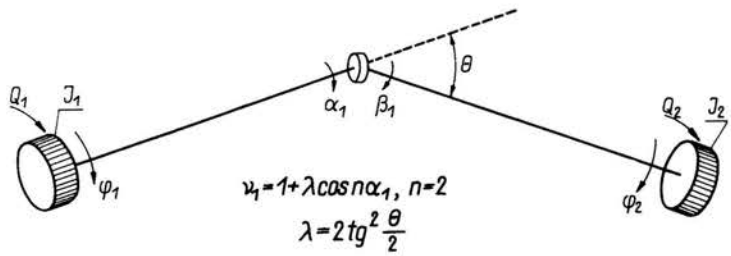


FIG. 2.

anism can be multiplied by a natural small parameter λ. The system of differential equations describing the first approximation has the form

$$J_1 n^2 \omega_e^2 \varphi_1^{(1)''} = \lambda_1 n\omega_e \varphi_1^{(1)'} + c(\varphi_2^{(1)} - \varphi_1^{(1)}) - \frac{\mu c}{c_2}\cos\tau - \frac{c}{n}\sin\tau,$$

(25) $$J_2 n^2 \omega_e^2 \varphi_2^{(1)''} = \lambda_2 n\omega_e \varphi_2^{(1)'} - c(\varphi_2^{(1)} - \varphi_1^{(1)}) + \frac{\mu c}{c_1}\cos\tau + \frac{c}{n}\sin\tau,$$

$$\left(' \equiv \frac{d}{d\tau}\right), \quad \tau = n\alpha_0 = n\beta_0 = n\omega_e t - \frac{n\mu}{c_1}.$$

Bearing in mind that

$$\varphi_k \approx \varphi_k^{(0)} + \lambda \varphi_k^{(1)}, \quad k = 1, 2, \ .$$

(26) $$\varphi_1^{(0)} = \omega_e t, \quad \varphi_2^{(0)} = \omega_e t - \frac{\mu}{c};$$

we observe that the system (25) enables us both to analyse the stationary regime of the machine unit with one universal joint transmission and to carry out a dynamic synthesis in accordance with a prescribed degree of non-uniformity of each of the masses. Observe that here the dynamic synthesis can be performed also in accordance with the coefficients of overloading. Thus, for instance, for the coefficient of non-uniformity of the first mass when $Q_2 = \text{const}$, $C_1 = \infty$, we obtain

(27) $$\delta_1^2 = (2\lambda)^2 \frac{c^2(1+\chi_2)^2 + (n\mu)^2}{c^2(\chi_2 - \chi_1^{-1})^2 + (n\lambda_1\omega_e)^2},$$

where

(28) $$\chi_k = \frac{c}{J_k(n\omega_e)^2 - c}, \quad k = 1, 2.$$

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