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# A lower bound theorem for dynamically loaded rigid-viscoplastic structures

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A LOWER bound on permanent dynamics deformations of impulsively loaded rigid-plastic structures was presented in the author's earlier papers. In the present paper, a proof is given that similar lower bound exists also in the case of a rigid-viscoplastic material. This result is of a particular value not only because most of the structural materials exhibit some degree of strain rate sensitivity but also because corresponding upper bound could not be derived without some additional information about the response of the structure. The method is explained on the example of a beam for which the exact solution is known.

Dolne oszacowanie na trwałe dynamiczne deformacje impulsywnie obciążonych sztywno-plastycznych konstrukcji było przedstawione we wcześniejszej pracy autora. W obecnej pracy udowodniono, że podobne dalsze oszacowanie istnieje również w przypadku sztywno lepkoplastycznego materiału. Rezultat ten ma znaczenie nie tylko dlatego, że większość materiałów konstrukcyjnych wykazuje wrażliwość na prędkość odkształcenia ale również dlatego, że odpowiednie górne oszacowanie nie mogłoby być wyprowadzone bez dodatkowych informacji o własnościach konstrukcji. Metoda została wyjaśniona na przykładzie belki, dla której znane jest rozwiązanie w postaci zamkniętej.

В предыдущей работе автора была дана нижняя оценка остаточных деформаций в динамически мгновенно нагруженном жестко-пластическом сооружении. Данная работа содержит доказательство того, что аналогичные оценки можно построить также в случае жестко-вязкопластического материала. Этот результат имеет существенное значение не только потому, что большинство конструктивных материалов обладает чувствительностью к скорости деформирования, но также и потому, что соответствующие верхние оценки не могли бы быть найдены без дополнительной информации о свойствах сооружения. Метод иллюстрируется на примере балки, для которой известны замкнутые решения.

## 1. Introduction

IN EARLIER papers [1, 2], a technique was presented to bound from below the permanent dynamic deformations of impulsively loaded rate insensitive structures. The technique complemented the displacement upper bound theorem developed by MARTIN [3, 4] providing, through relatively simple calculations, a way of solving a number of problems of current interest whose solution, if at all possible, may require involved numerical computation [5].

However, the results of experimental and theoretical investigations of dynamically loaded cantilever beams [6] suggest that, while elastic vibrations do not have much effect on the permanent deformations when the ratio of input kinetic energy to maximum possible elastic energy is of the order of 10, the influence of strain rate on the material yield stress was primarily responsible for the deviations between the elementary rigid-plastic theory and experiment. The study further showed that the effect of strain hardening decreased

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with increasing strain rate. The technique presented below provides a mean of computing lower bounds on the deformations of certain class of dynamically loaded time-dependent inelastic structures.

## 2. Lower bound theorem

Consider a time-dependent inelastic body of volume $V$ and surface $S$ which at time $t < 0$ is assumed to be at rest. Let a velocity $\dot{u}_i^0$ be prescribed at all points in the continuum at time $t = 0$, and for times $t > 0$ it is assumed that the displacement rates $\dot{u}_i$ are zero on the portion of the surface $S_u$ and tractions $T_i$ are zero on the portion of the surface $S_F$. Furthermore, it is assumed that the effect of body forces $F_i$ is negligible in the process of deformation.

In order to generalize the uniaxial experimental observations to combined state of stress, DRUCKER [7] formulated a criterion of stability for a large class of engineering materials in terms of the work done by the stress increment on the plastic deformation increment. Denoting the generalized stresses acting at a section of the body by $Q_k$ $(k = 1, \ldots, n)$ and the associated generalized strain rates by $\dot{q}_k$ $(k = 1, \ldots, n)$, if $dQ_k$ represents a stress increment and $d\dot{q}_k$ the corresponding strain rate increment from some initial state $Q_k^0$, $\dot{q}_k^0$, Drucker's definition of a stable plastic material requires that,

$$dQ_k\, d\dot{q}_k \geqslant 0. \tag{2.1}$$

Hence, if the state of stress of a material element is changed from $Q_k^0$ to $Q_k^*$ with an associated change in strain rate from $\dot{q}_k^0$ to $\dot{q}_k^*$, Eq. (2.1) may be written as,

$$\int\limits_{\dot{q}_k^0}^{\dot{q}_k^*} (Q_k - Q_k^0)\, d\dot{q}_k \geqslant 0. \tag{2.2}$$

In the nine-dimensional stress space, the components of the stress vector $Q_k$ can be visualized as the component of a stress or "force" vector in an $n$-dimensional Euclidian stress space. Therefore, in the stress space, $Q_k$ is represented by a point while $(Q_k^* - Q_k^0)$ is represented by a path. Consider now the path from a third stress point $Q_k^s$ to $Q_k^*$ passing through $Q_k^0$ as shown in Fig. 1. Therefore, from Eq. (2.2),

$$\int\limits_{\dot{q}_k^s}^{\dot{q}_k^*} (Q_k - Q_k^s)\, d\dot{q}_k = \int\limits_{\dot{q}_k^s}^{\dot{q}_k^0} (Q_k - Q_k^s)\, d\dot{q}_k + \int\limits_{\dot{q}_k^0}^{\dot{q}_k^*} (Q_k - Q_k^s)\, dq_k \geqslant 0. \tag{2.3}$$

But if $Q_k^0$ and $Q_k^s$ are two stress points which lie within the yield surface, the path between the two points is reversible. Therefore,

$$\int\limits_{\dot{q}_k^s}^{\dot{q}_k^0} (Q_k - Q_k^s)\, d\dot{q}_k = \int\limits_{\dot{q}_k^0}^{\dot{q}_k^s} (\dot{q}_k - \dot{q}_k^0)\, dQ_k \tag{2.4}$$

and

$$\int\limits_{\dot{q}_k^0}^{\dot{q}_k^*} (Q_k - Q_k^s)\, d\dot{q}_k = \int\limits_{\dot{q}_k^0}^{\dot{q}_k^*} (Q_k - Q_k^0)\, d\dot{q}_k - (Q_k^s - Q_k^0)(\dot{q}_k^* - \dot{q}_k^0), \tag{2.5}$$

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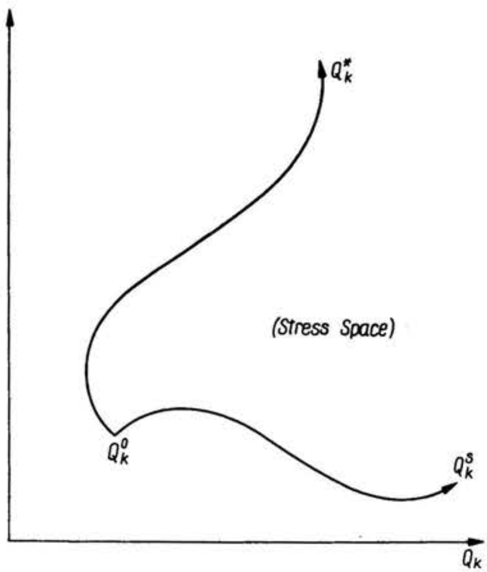


FIG. 1.

where $Q_k^0$ remains constant along integration path to stress point $Q_k^*$ lying outside yield surface.

Replacing Eqs. (2.4) and (2.5) into (2.3)

$$(2.6)\qquad \int\limits_{Q_k^0}^{Q_k^s} (\dot{q}_k-\dot{q}_k^0)dQ_k+\int\limits_{\dot{q}_k^0}^{\dot{q}_k^*}(Q_k-Q_k^0)d\dot{q}_k \geqslant (Q_k^s-Q_k^0)(\dot{q}_k^*-\dot{q}_k^0).$$

The integrals in Eq. (2.6) denote integration along paths from stress point $Q_k^0$ to $Q_k^s$ and $Q_k^*$, respectively. It is to be noted that the integration can be carried out independently of each other and hence in stress space the above equation represents two independent stress paths from an initial state $Q_k^0$. If the material is assumed to be stressed from the virgin state, $Q_k^0$ and $\dot{q}_k^0$ are set equal to zero and Eq. (2.6) becomes

$$(2.7)\qquad \int\limits_{0}^{Q_k^s} \dot{q}_k dQ_k+\int\limits_{0}^{\dot{q}^*} Q_k\, d\dot{q}_k \geqslant Q_k^s\dot{q}_k^*.$$

The result shown in Eq. (2.7) enabled MARTIN [4] to establish a minimum principle for viscous continua which was then used to develop the displacement upper bound theorem by noting that the $s$ and * system are completely independent of each other. The same condition will be employed to derive the displacement lower bound theorem for dynamically loaded rigid-viscoplastic continua.

Let the $s$-system represent a statically admissible stress field and the *-system be a kinematically admissible strain field. Furthermore, let the response of the system for times $t > 0$ be characterized by displacement, velocity, acceleration, generalized stresses and strain rates given by $u_i$, $\dot{u}_i$, $\ddot{u}_i$, $Q_k$, $\dot{q}_k$, respectively. Since the complete solution is both

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statically and kinematically admissible, the true state can be associated with either the $s$ or the * system. Let the true state be associated with the $s$-system and let $\dot{u}_i^*$, $\dot{q}_i^*$ be any time dependent kinematically admissible field. Integrating Eq. (2.7) over the volume $V$ of the continuum, we hawe

$$\int_V dV \int_0^{Q_k} \dot{q}_k \, dQ_k + \int_V dV \int_0^{\dot{q}_k^*} Q_k \, d\dot{q}_k \geqslant \int_V Q_k \dot{q}_k^* dV. \tag{2.8}$$

From the principle of virtual velocities,

$$\int_V Q_k \dot{q}_k^* dV = \int_S T_i \dot{u}_i^* dS + \int_V F_i \dot{u}_i^* dV - \int_V \varrho \ddot{u}_i \dot{u}_i^* dV \tag{2.9}$$

and since $F_i = T_i = 0$,

$$\int_V Q_k \dot{q}_k \, dV = - \int_V \varrho \ddot{u}_i \dot{u}_i^* dV. \tag{2.10}$$

From (2.10), Eq. (2.8) become

$$\int_V dV \int_0^{Q_k} \dot{q}_k \, dQ_k + \int_V dV \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k \geqslant - \int_V \varrho \ddot{u}_i \dot{u}_i^* dV. \tag{2.11}$$

Integrating Eq. (2.11) from $t = 0$ to $t = t_f$, the response time of the structure,

$$\int_0^{t_f} dt \int_V dV \int_0^{Q_k} \dot{q}_k dQ_k + \int_0^{t_f} dt \int_V dV \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k \geqslant - \int_V dV \int_0^{t_f} \varrho \ddot{u}_i \dot{u}_i^* dt. \tag{2.12}$$

Adding a positive definite quantity to the left side of Eq. (2.12) will not affect the inequality, and since

$$\int_0^{t_f} dt \int_V dV \int_0^{Q_k} \dot{q}_k dQ_k + \int_0^{t_f} dt \int_V dV \int_0^{\dot{q}_k} Q_k d\dot{q}_k = \int_0^{t_f} dt \int_V Q_k \dot{q}_k dV \tag{2.13}$$

then,

$$\int_0^{t_f} dt \int_V Q_k \dot{q}_k dV \geqslant \int_0^{t_f} dt \int_V dV \int_0^{Q_k} \dot{q}_k dQ_k \tag{2.14}$$

for all terms in Eq. (2.13) are positive definite quantities.

Hence, from Eqs. (2.14) and (2.12)

$$\int_0^{t_f} dt \int_V D(\dot{q}_k) dV + \int_0^{t_f} dt \int_V dV \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k \geqslant - \int_V dV \int_0^{t_f} \varrho \ddot{u}_i \dot{u}_i^* dt, \tag{2.15}$$

where $D(\dot{q}_k)$ is the rate of dissipation of internal energy and is defined by

$$D(\dot{q}_k) = Q_k \dot{q}_k. \tag{2.16}$$

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Since a rigid-viscoplastic continuum is a totally dissipative medium, the internal and external dissipation rates can be equated

$$\frac{d}{dt}\int_V \frac{1}{2}\varrho\dot{u}_i\dot{u}_i dV = -\int_V D(\dot{q}_k)dV. \tag{2.17}$$

Integrating (2.17) between $t = 0$ and $t = t_f$,

$$\frac{1}{2}\int_V \varrho\dot{u}_i^0\dot{u}_i^0 dV = \int_0^{t_f} dt \int_V D(\dot{q}_k)dV, \tag{2.18}$$

since by definition $\dot{u}_i = 0$ at $t = t_f$ and $\dot{u}_i = \dot{u}_i^0$ at $t = 0$.

Substituting (2.18) into (2.16),

$$\frac{1}{2}\int_V \varrho\dot{u}_i^0\dot{u}_i^0 dV + \int_0^{t_f} dt \int_V dV \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k \geqslant -\int_V dV \int_0^{t_f} \varrho\ddot{u}_i\dot{u}_i^* dt. \tag{2.19}$$

The term on the right side of Eq. (2.19) is next integrated twice by parts with respect to the time variable as follows,

$$-\int_0^{t_f} \varrho\ddot{u}_i\dot{u}_i^* dt = -\varrho\dot{u}_i\dot{u}_i^*\Big|_0^{t_f} + \varrho\ddot{u}_i^* u_i\Big|_0^{t_f} - \int_0^{t_f} \varrho\dddot{u}_i^* u_i dt. \tag{2.20}$$

But at $t = 0, u_i = 0, \dot{u}_i = \dot{u}_i^0$. Hence,

$$-\int_0^{t_f} \varrho\ddot{u}_i\dot{u}_i^* dt = \varrho\dot{u}_i^0\dot{u}_i^*\Big|_{t=0} + \varrho\ddot{u}_i^* u_i\Big|_{t=t_f} - \int_0^{t_f} \varrho\dddot{u}_i^* u_i dt. \tag{2.21}$$

Replacing Eq. (2.21) in (2.19),

$$\frac{1}{2}\int_V \varrho\dot{u}_i^0\dot{u}_i^0 dV + \int_0^{t_f} dt \int_V dV \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k \geqslant \int_V \varrho\dot{u}_i^0\dot{u}_i^*\Big|_{t=0} dV + \int_V \varrho\ddot{u}_i^* u_i\Big|_{t=t_f} dV - \int_0^{t_f} \varrho\dddot{u}_i^* u_i dt. \tag{2.22}$$

The assumed kinematically admissible velocity field $\dot{u}_i^*$ is then chosen in such a way as to cause the vanishing of the last term on the right side of (2.22). If $\dot{u}_i^*$ is assumed to be representable by a product of a time-independent function $U_i^*(x_i)$ and a time-dependent function $\dot{T}^*(t)$, then

$$\dot{u}_i^* = U_i^*(x_i)\dot{T}^*(t). \tag{2.23}$$

If $\dot{T}^*$ is chosen to be of the form,

$$\dot{T}^* = (t_f^* - t)/t_f^*, \quad 0 \leqslant t \leqslant t_f^*, \tag{2.24}$$

$$\dot{T}^* = 0, \quad t \geqslant t_f^*,$$

where $t_f^*$ is a constant yet to be determined, $\dddot{u}_i^*$ will vanish.

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Substituting Eq. (2.24) into Eq. (2.23),

$$(2.25)\quad \int_V \varrho U_i^* u_i \Big|_{t=t_f} dV \geqslant t_f^* \left\{ \int_V \varrho \dot{u}_i^0 U_i^* dV - \frac{1}{2} \int_V \varrho \dot{u}_i^0 \dot{u}_i^0 dV - \int_0^{t_f} dt \int_V dV \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k \right\}.$$

Since the objective of this theorem is to obtain a lower bound on the maximum permanent deformation that a structure undergoes at time $t = t_f$ when subjected to a dynamic loading, i.e.,

$$(2.26)\qquad (U_i)_{\max}\Big|_{t=t_f} \geqslant \textit{lower bound.}$$

Equation (2.26) can be put in this form by recalling the extended mean value theorem of integral calculus [8], that if $f(x)$ and $g^*(x)$ are two continuous functions in the interval $a \leqslant x \leqslant b$ and if the maximum value of $f(x)$ in this tange is denoted by $M$, then, provided $g^*(x)$ does not change sign in $[a, b]$,

$$(2.27)\qquad \int_a^b f(x) g^*(x) dx \leqslant M \int_a^b g^*(x) dx,$$

where $M \geqslant f(x)$ for all $x$ in $[a, b]$.

Denoting the three components of $u_i$ and $U_i^*$ as $u, g, w$ and $U^*, G^*, W^*$, respectively, the left side of (2.28) can be written as

$$(2.28)\qquad \int_V \varrho U_i^* u_i \Big|_{t=t_f} dV = \int_V \varrho U^* u \Big|_{t=tf} dV + \int_V \varrho G^* g \Big|_{t=t_f} dV + \int_V \varrho W^* w \Big|_{t=t_f} dV.$$

In order to obtain lower bounds on each of the three components of $u_i$, three separate choices of the components of the assumed kinematically admissible field $U_i^*$ must be made. For example, if a bound for $w$ is desired, $U_i^*$ can be assumed to have components $U^* = G^* = 0$ and $W^* \neq 0$. Under these conditions, Eq. (2.28) becomes

$$(2.29)\qquad \int_V \varrho U_i^* u_i \Big|_{t=t_f} dV = \int_V \varrho W^* w \Big|_{t=t_f} dV.$$

Applying the result of Eq. (2.27) to Eq. (2.29),

$$(2.30)\qquad \int_V \varrho W^* w \Big|_{t=t_f} dV \leqslant w_{\max}\Big|_{t=t_f} \int_V \varrho W^* dV.$$

Hence, from Eqs. (2.25) and (2.30), a lower bound for $w_{\max}$ is obtained from

$$(2.31)\quad w_{\max}\Big|_{t=t_f} \geqslant \left(1/\int_V \varrho W^* dV\right) t_f^* \left\{ \int_V \varrho \dot{u}_k^0 U_k^* dV - \frac{1}{2} \int_V \varrho \dot{u}_k^0 \dot{u}_k^0 dV - \int_0^{t_f} dt \int_V dV \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k \right\}.$$

Two other similar expressions are needed to bound the $u_{\max}$ and $g_{\max}$ components of de-

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formation of a body resulting from an impulsive loading. Therefore, the result of Eq. (2.31) can be generalized to

(2.32)

$$(u_i)_{\max}\Big|_{t=t_f} \geqslant \Big(1/\int\limits_V \varrho U_i^* dV\Big) t_f^* \Big\{\int\limits_V \varrho \dot{u}_k^0 U_k^* dV - \frac{1}{2}\int\limits_V \varrho \dot{u}_k^0 \dot{u}_k^0 dV - \int\limits_0^{t_f} dt \int\limits_V dV \int\limits_0^{\dot{q}_k^*} Q_k d\dot{q}_k\Big\},$$

if the stated limitations on the assumed velocity components are recognized.

In order to use the previous expression, information regarding the last term is needed, for at this stage both $t_f$ and $t_f^*$ are unknown. However, MARTIN [4] obtained a lower bound on the response time in the form

$$t_f \geqslant \Big(1/\int\limits_V dV \int\limits_0^{\dot{q}^s} Q_k d\dot{q}_k\Big)\Big\{\int\limits_V \varrho \dot{u}_i^0 \dot{u}_i^s dV - \frac{1}{2}\int\limits_V \varrho \dot{u}_i^0 \dot{u}_i^0 dV\Big\}, \tag{2.33}$$

where $\dot{u}_i^s$ denotes a time-independent kinematically admissible velocity field and $\dot{q}_k^s$ the corresponding generalized time-independent strain rate field. If $t_f^*$ is chosen equal to the right side of Eq. (2.31), i.e.,

$$t_f^* = \Big(1/\int\limits_V dV \int\limits_0^{\dot{q}_k^s} Q_k d\dot{q}_k\Big)\Big\{\int\limits_V \varrho \dot{u}_i^0 \dot{u}_i^s dV - \frac{1}{2}\int\limits_V \varrho \dot{u}_i^0 \dot{u}_i^0 dV\Big\}, \tag{2.34}$$

then

$$t_f \geqslant t_f^*. \tag{2.35}$$

Then, since $\dot{q}_k^*$ is zero for $t \geqslant t_f^*$, the last term of Eq. (2.32) can be evaluated as,

$$\int\limits_0^{t_f} dt \int\limits_V dV \int\limits_0^{\dot{q}_k^*} Q_k d\dot{q}_k = \int\limits_0^{t_f^*} dt \int\limits_V dV \int\limits_0^{\dot{q}_k^*} Q_k d\dot{q}_k + \int\limits_{t_f^*}^{t_f} dt \int\limits_V dV \int\limits_0^{\dot{q}_k^*} Q_k d\dot{q}_k = \int\limits_0^{t_f^*} dt \int\limits_V dV \int\limits_0^{\dot{q}_k^*} Q_k d\dot{q}_k. \tag{2.36}$$

Therefore, the lower bound theorem becomes,

(2.37)

$$(u_i)_{\max}\Big|_{t=t_f} \geqslant \Big(1/\int\limits_V \varrho U_i^* dV\Big) t_f^* \Big\{\int\limits_V \varrho \dot{u}_k^0 U_k^* dV - \frac{1}{2}\int\limits_V \varrho \dot{u}_k^0 \dot{u}_k^0 dV - \int\limits_0^{t_f^*} dt \int\limits_V dV \int\limits_0^{\dot{q}_k^*} Q_k d\dot{q}_k\Big\}.$$

## 3. Example problem

Consider a cantilever beam of length $L$ and mass $m$ per unit length with a point mass $M$ attached at the tip. At time $t = 0$, the mass $M$ is given an initial velocity $V_0$ in the vertical

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direction, while the remainder of the structure remains stationary. The beam will be assumed to be rigid-viscoplastic with constitutive equation

$$\begin{aligned} \frac{\dot{k}}{\dot{k}_0} &= \left(\frac{M}{M_0}-1\right)^5 \quad &\text{for} \quad M \geqslant M_0, \\ \dot{k}_0 &= 0 \quad &\text{for} \quad M \leqslant M_0, \end{aligned} \tag{3.1}$$

where the curvature rate $\dot{k}$ and bending moment $M$ are the only non-zero generalized strain-rate and stress, respectively, and $\dot{k}_0$, $M_0$ are constants having the dimensions of curvature rate and bending moment, respectively.

To determine a lower bound on the permanent deformation, a kinematically admissible velocity field $U^*$ must be assumed. Thus, if the assumed mode shape $U^*$ is given by

$$U^* = A\left(1-\cos\frac{\pi x}{2L}\right), \tag{3.2}$$

where $A$ is a constant amplitude,

$$\dot{k}^* = \dot{k}^s \dot{T}^* = \frac{d^2\dot{u}^*}{dx^2} = \dot{T}^*\frac{d^2U^*}{dx^2} = A\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L}\left(1-\frac{t}{t_f^*}\right). \tag{3.3}$$

Then

$$\int_V dV \int_0^{t_f^*} dt \int_0^{\dot{q}_k^*} Q_k d\dot{q}_k = \int_0^L dx \int_0^{t_f^*} dt \int_0^{\dot{k}^*} M d\dot{k} = \int_0^L dx \int_0^{t_f^*} M_0 \dot{k}_0 \left\{\frac{5}{6}\left(\frac{\dot{k}^*}{\dot{k}_0}\right)^{6/5} + \left(\frac{\dot{k}^*}{\dot{k}_0}\right)\right\} dt = M_0\dot{k}_0 t^* \int_0^L \left\{\frac{25}{66}\left(\frac{\dot{k}^s}{\dot{k}_0}\right)^{6/5} + \frac{1}{2}\left(\frac{\dot{k}^s}{\dot{k}_0}\right)\right\} dx, \tag{3.4}$$

$$\int_V dV \int_0^{t_f^*} dt \int_0^{\dot{q}_k} Q_k d\dot{q}_k = M_0 \dot{k}_0 L t_f^* \left\{\frac{5}{22}\left(\frac{\pi}{4}\right)^{12/5}\left(\frac{A}{V_0}\right)^{6/5}\left(\frac{V_0}{\dot{k}_0 L^2}\right)^{6/5} + \frac{\pi}{4}\left(\frac{A}{V_0}\right)\left(\frac{V_0}{\dot{k}_0 L^2}\right)\right\}. \tag{3.5}$$

Neglecting the mass of the beam compared to the point mass $M$,

$$\frac{1}{2}\int_V \varrho \dot{u}_k^0 \dot{u}_k^0 dV = \frac{1}{2} M V_0^2, \qquad \int_V \varrho U_k^* \dot{u}_k^0 dV = AMV_0. \tag{3.6}$$

Substituting Eqs. (3.5), $(3.6)_1$ and $(3.6)_2$ into (2.37),

$$u \geqslant \frac{t_f^*}{AM}\left\{AMV_0 - \frac{1}{2}MV_0^2 - M_0\dot{k}Lt_f^*\left[\frac{5}{22}\left(\frac{\pi}{4}\right)^{12/5}\left(\frac{A}{V_0}\right)^{6/5}\left(\frac{V_0}{\dot{k}_0 L^2}\right)^{6/5} + \frac{\pi}{4}\left(\frac{A}{V_0}\right)\left(\frac{V_0}{\dot{k}_0 L^2}\right)\right\}. \tag{3.7}$$

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Utilizing the response time lower bound theorem, Eq. (2.33), MARTIN [4] obtained

$$t_f \geqslant 0.552 MLV_0/M_0, \tag{3.8}$$

where

$$\left(\frac{V_0}{\dot{k}_0 L^2}\right) = 3.62\times 10^{-3}. \tag{3.9}$$

Hence, from Eq. (2.35),

$$t_f^* = 0.552 MLV_0/M_0. \tag{3.10}$$

Therefore, substituting (3.10) into (3.7),

$$u \geqslant \frac{0.552ML}{\left(\frac{A}{V_0}\right)M_0} V_0^2 \left\{\left(\frac{A}{V_0}\right) - \frac{1}{2} - 0.552\left\{\frac{5}{22}\left(\frac{\pi}{4}\right)^{12/5}\left(\frac{A}{V_0}\right)^{6/5}\left(\frac{V_0}{\dot{k}_0 L^2}\right)^{6/5} + \frac{\pi}{4}\left(\frac{A}{V_0}\right)\right\}\right\}. \tag{3.11}$$

Eq. (3.11) can be optimized numerically with respect to $(A/V)$ utilizing Eq. (3.9)

$$u \geqslant 0.28\frac{ML}{M_0}V_0^2. \tag{3.12}$$

The upper bound on u has been obtained by MARTIN as

$$u \leqslant 0.36\frac{ML}{M_0}V_0^2. \tag{3.13}$$

Therefore,

$$0.28\frac{ML}{M_0}V_0^2 \leqslant u \leqslant 0.36\frac{ML}{M_0}V_0^2. \tag{3.14}$$

The exact solution to this problem was given by COWPER and SYMONDS [9] as

$$u = 0.33\frac{ML}{M_0}V_0^2. \tag{3.15}$$

Utilizing the mode approximation technique introduced by MARTIN and SYMONDS [10], an approximate end deflection was determined by SYMONDS [11] as

$$u = 0.30\frac{ML}{M_0}V_0^2. \tag{3.16}$$

## 4. Conclusions

Through the use of energy methods and basic inequalities inherent in the theory of plasticity, a technique is presented to bound from below the permanent deformations of impulsively loaded visco-plastic structures. This method complements the displacement upper bound theorem introduced by MARTIN providing by means of relatively simple calculations a way to bracket the response of a number of engineering problems whose

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solutions, if at all possible, require long running numerical solutions. It is in this respect where this technique becomes a powerful tool for preliminary design.

The lower bound theorem is applied to a sample problem and the result compared to the exact solution, Martin's upper bound result and to the amplitude of the mode approximation technique. Good results are obtained with both the bounding and the approximate methods. However, without denying the usefulness of the approximate technique, the bounding method constitutes in many instances a more powerful method to a number of engineering problems requiring preliminary assessment of the capabilities of a structure and possible design for the upper bound yields a consistent conservative estimate and the lower bound allows for the close bracketing of the exact structural response.

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