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A minimum principle in dynamics of elastic-plastic continua at finite deformation

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THE CONCEPT of employing finite acceleration-variations in formulating a variational principle as depicted by the Gibbs-Appell principle in classical mechanics is employed to establish an absolute minimum principle in dynamics of elastic-plastic continua in finite deformation. The minimum principle is expressed in terms of Lagrangian strains, Piola-Kirchhoff stresses and the constitutive relationships developed by GREEN and NAGHDI and which include thermodynamic effects. The minimum principle is valid for continuous as well as sectionally discontinuous acceleration fields. The minimum principle may be employed to establish governing equations or to solve a problem by using a direct method of variational calculus. The application of the principle is illustrated by two examples: one on the non-linear vibrations of elastic beams and the other on the impulsive loading of rigid-plastic beams with axial constraints. The results of this analysis agree with the analytical and experimental results of the two problems available in the literature.

W pracy przedstawiono koncepcję zastosowania skończonych wariancji przyśpieszenia do sformułowania zasady wariancyjnej, odpowiadającej zasadzie Gibbsa-Appella w mechanice klasycznej; jest to zasada absolutnego minimum w dynamice ośrodków sprężysto-plastycznych przy odkształceniach skończonych. Zasadę tę wyrażono w opisie materialnym odkształceń Lagrange'a i naprężeń Pioli-Kirchhoffa za pomocą związków konstytutywnych Greena i Naghdiego, uwzględniających efekty termodynamiczne. Zasada minimum jest słuszna zarówno dla ciągłych jak i odcinkami nieciągłych pól przyśpieszenia. Zasadę tę można stosować do formułowania podstawowych równań jak i do efektywnego rozwiązywania zagadnień za pomocą metod bezpośrednich rachunku wariancyjnego. Zastosowanie tej metody ilustrują dwa przykłady dotyczące drgań nieliniowych belek sprężystych oraz udarowego obciążenia belek sztywno-plastycznych z więzami w kierunku osi belki. Wyniki analizy są zgodne z wynikami analitycznymi i eksperymentalnymi, znanymi z literatury.

В работе представлена концепция применения конечных вариаций ускорения для формулировки вариационного принципа, отвечающего принципу Гиббса-Аппеля в классической механике; это принцип абсолютного минимума в динамике упруго-пластических сред при конечных деформациях. Этот принцип выражается в материальном описании деформаций Лагранжа и напряжений Пиоли-Кирхгофа при помощи определяющих соотношений Грина и Нагди, учитывающих термодинамические эффекты. Принцип минимума справедлив так для непрерывных, как и для отрезками разрывных полей ускорения. Этот принцип можно применять так для формулировки основных уравнений, как и для эффективного решения проблем при помощи непосредственных методов вариационного исчисления. Применение этого метода иллюстрируют два примера, касающиеся нелинейных колебаний упругих балок и ударной нагрузки жестко-пластических балок со связями в направлении оси балки. Результаты анализа совпадают с аналитическими и экспериментальными результатами известными из литературы.

1. Introduction

THE VARIATIONAL methods of formulations and direct solutions have been extensively investigated in the field of solid mechanics [1]. The concept of virtual work or virtual displacement has been the basis of variational formulations in infinitesimal and finite elasticity. For static problems, the principle of virtual work leads to the principles of stationary and minimum potential energy. It has been further generalized by the introduction of Lagrange multipliers to yield a family of variational principles which includes the principle of minimum complementary energy. For dynamic problems, the Lagrange's equation of motion and the Hamilton's principle may be derived from the principle of virtual displacement. It is to be noted that, in general, the Hamilton's principle is a stationary principle and not an extremum principle for non-conservative systems.

In the theory of elasticity, whether linear or non-linear, the steps of establishing a variational principle are reasonably straightforward. Additional considerations must be given to that in the theory of plasticity. As a consequence of the irreversible property, the constitutive equations of a plastic continuum are expressed in terms of the velocities or the rates of stress and strain and not of the stresses and strains themselves. Therefore, variational and extremum principles of plasticity have been expressed in terms of velocities [2, 3, 4]. In applying these principles, the velocity or rate variations are only allowed along some permissible paths [2], such that regions of loading or unloading associated with admissible velocity fields must be agreeable with the true regions of loading or unloading. Special care is usually required in carrying out the velocity variations for a real problem.

A variational principle is not necessarily very helpful in solving problems. The assumed velocity fields may not be close to the true one. What is required instead is an absolute maximum or minimum principle. In classical mechanics, the Gibbs-Appell variational principle [5], which can be deduced from Gauss's principle of least constraint [5], is an absolute minimum principle. The principle employs finite, variational differences in accelerations and is particularly well suited to the study of non-holonomic systems. A parallel minimum principle in dynamic plasticity has been developed by TAMUZH [6] for rigid-plastic bodies involving infinitesimal deformations.

In this paper, the concept of finite variations in accelerations is employed to establish a minimum principle in dynamics of elastic-plastic continua subject to finite deformations. The minimum principle can be used for formulations as well as for approximate solutions of problems. Two examples involving motions of beams are given as illustrations.

2. Kinematics

Consider a body of a continuum occupying in its natural state a region V and bounded by a piecewise smooth surface A. Let the initial position $\{X_K\}$ and the position $\{x_i\}$ at time t of a particle of the body be referred to a fixed system of rectangular Cartesian coordinates. Let $\{U_K\}$ be the displacement vector of the particle at time t. The history of deformation of the body under influences of external forces is then given by the functions

$$x_m = x_m(X_M, t), \quad \begin{matrix} m = 1, 2, 3, \\ M = 1, 2, 3; \end{matrix} \tag{2.1}$$

or

$$U_K = U_K(X_M, t), \quad K = 1, 2, 3. \tag{2.2}$$

In general, lower case Latin indices are associated with the coordinates $\{x_i\}$ and upper case Latin indices are associated with $\{X_M\}$. The functions x_m and U_K are assumed to be piecewise continuous and differentiable within the domain of the body.

Consider an element of initial length dS in the neighborhood of a typical point. The length at time t of the element in the configuration $\{x_i\}$ is denoted by ds. If at time t the body is conceptually unstressed in the neighborhood of the typical point and has its temperatures reduced to the initial temperature θ_0, the thermo-elastic strains will be released and only the plastic strains will remain. Upon unstressing, the length of the element becomes ds^* which may be measured in a configuration represented by the partical position $\{x_i^*\}$. It is assumed that the functions

$$x_i^* = x_i^*(X_M, t), \qquad \begin{array}{c} i = 1, 2, 3, \\ M = 1, 2, 3 \end{array} \tag{2.3}$$

and their derivatives are continuous only in the neighborhood of the typical point. The configuration $\{x_i^*\}$ is a conceptual configuration embedded in the memory of the material, only under certain conditions, which may coincide with a real configuration. The configuration $\{x_i^*\}$ is used here strictly to characterize the constitutive relationships.

The finite strains and other variables of the body may be expressed in terms of either Lagrangian $\{X_K\}$ or Eulerian $\{x_i\}$ coordinates. The Lagrangian variable rates, which contain no convective terms, are usually preferred for materials with memory. The Lagrangian strain tensor E_{KL} is given by

$$ds^2 - dS^2 = 2E_{KL}\, dX_K\, dX_L, \tag{2.4}$$

where

$$2E_{KL} = x_{k,K} x_{k,L} - \delta_{KL}, \tag{2.5}$$

or

$$2E_{KL} = U_{K,L} + U_{L,K} + U_{M,K} U_{M,L}. \tag{2.6}$$

Here, a partial differentiation of a variable with respect to X_K is designated as $(\ )_{,K}$. δ_{KL} is the Kronecker symbol and the repetition of an index in a term indicates summation. The Lagrangian strain may be divided into the elastic and plastic parts by the relationship

$$ds^2 - dS^2 = (ds^2 - ds^{*2}) + (ds^{*2} - dS^2) = 2(E'_{KL} + E''_{KL})\, dX_K\, dX_L, \tag{2.7}$$

where the elastic strain, E'_{KL}, is given by

$$2E'_{KL} = \left(\frac{\partial x_k}{\partial x_i^*} \frac{\partial x_k}{\partial x_j^*} - \delta_{ij}\right) \frac{\partial x_i^*}{\partial X_K} \cdot \frac{\partial x_j^*}{\partial X_L} \tag{2.8}$$

and the plastic strain E''_{KL} is given by

$$2E''_{KL} = x^*_{k,K} x^*_{k,L} - \delta_{KL}. \tag{2.9}$$

Equations (2.4) and (2.7) show that the finite Lagrangian strain can be expressed as

$$E_{KL} = E'_{KL} + E''_{KL}. \tag{2.10}$$

Consider that, at time $t = t_0$, the true displacement field $U_K^+(X_1, X_2, X_3, t_0)$ and velocity field $\dot{U}^+(X_1, X_2, X_3, t_0)$ are either predetermined or given in the body. The superscripts +

denote here and in the sequel the true fields. The dot above the letter denotes differentiation with respect to time. The true acceleration $\ddot{U}_K^+$ is to be determined by the minimum principle from a set of admissible acceleration fields, $\ddot{U}_K$, which satisfies the kinematic boundary conditions and the continuity condition of the body. The admissible strain accelerations $\ddot{E}_{KL}$ may be expressed as

$$\ddot{E}_{KL} = \frac{1}{2}(\ddot{U}_{K,L} + \ddot{U}_{L,K} + \ddot{U}_{M,K}U^+_{M,L} + \ddot{U}_{M,L}U^+_{M,K} + 2\dot{U}^+_{M,K}\dot{U}^+_{M,L}). \tag{2.11}$$

Accompanying the strain acceleration variations, there may be stress variations. It is known that prescribed external forces depending on the accelerations of a body are not admissible in Newtonian dynamics [5]. It is also recognized that the dependence of stresses on time rates of strains of higher orders is permissible by existing constitutive theories. However, the effects of strain accelerations on the general constitutive relationships of specific continua have not been theoretically or experimentally determined.

3. Constitutive relationships

The constitutive relationships of elastic-plastic continua at finite deformation have been investigated in recent years and notably by GREEN and NAGHDI [7, 8], and by LEE and LIU [9, 10]. A Lagrangian representation as well as its alternative forms are employed by the former, and a representation involving both Lagrangian and Eulerian variables is used by the latter. The Lagrangian representation is employed in the present analysis.

GREEN and NAGHDI assume that the Lagrangian strain can be divided into two parts, elastic and plastic such as it is shown in Eq. (2.10) except that E'_{KL} and E''_{KL} may be non-symmetric while E_{KL} is symmetric. E'_{KL} and E''_{KL} as given by Eqs. (2.8) and (2.9) may be physically measurable and can be the subsets of the conceptually general elastic and plastic strains postulated by GREEN and NAGHDI. The essence of their results [7, 8] are given, for convenience, as follows.

The constitutive equations developed by GREEN and NAGHDI are based on the first and second laws of thermodynamics, invariance conditions and the classical concept of a yield surface. It is postulated that the constitutive equation, in terms of Piola-Kirchhoff stress tensor S_{KL} has the form

$$S_{KL} = S_{KL}(E'_{MN}, E''_{MN}, \theta). \tag{3.1}$$

The yield surface may be described by a function of Piola-Kirchhoff stress tensor, plastic strain E''_{KL} and temperature θ such as

$$f(S_{KL}, E''_{KL}, \theta) = \varkappa, \tag{3.2}$$

where f is a regular (continuously differentiable) function of its variables and $\varkappa$ is a scalar which depends in some way on the history of motion. It is assumed that

$$\dot{\varkappa} = h_{KL}(S_{MN}, E''_{KN}, \theta)\dot{E}''_{KL}, \tag{3.3}$$

where h_{KL} are tensor functions of S_{MN}, E''_{MN} and θ. The constitutive relationships may be expressed as

$$\dot{E}''_{KL} = \lambda\beta_{KL}\left(\frac{\partial f}{\partial S_{MN}}\dot{S}_{MN} + \frac{\partial f}{\partial \theta}\dot{\theta}\right), \tag{3.4$_1$}$$

when $f = \varkappa(\dot{\varkappa} \neq 0)$ and $\frac{\partial f}{\partial S_{MN}}\dot{S} \quad + \frac{\partial f}{\partial \theta}\dot{\theta} > 0$

and

$$\dot{E}''_{KL} = 0. \tag{3.4$_2$}$$

when

$$f = \varkappa(\dot{\varkappa} = 0) \text{ and } \frac{\partial f}{\partial S_{MN}}\dot{S}_{MN} + \frac{\partial f}{\partial \theta}\dot{\theta} \leqslant 0$$

or when $f < \varkappa$ with $\dot{\varkappa} = 0$.

In Eq. $(3.4)_1$, λ is a scalar function of S_{MN}, E''_{MN} and θ and β_{KL} is a symmetric tensor function which may be determined by the condition that during loading $\dot{f} = \dot{\varkappa}$ and consequently

$$\lambda\beta_{KL}\left(h_{KL} - \frac{\partial f}{\partial E''_{KL}}\right) = 1. \tag{3.5}$$

In order to satisfy the Clausius-Duhem entropy production inequality, the elastic strains follow the relationship

$$S_{KL} = \varrho_0 \frac{\partial A}{\partial E'_{KL}}, \tag{3.6}$$

where ϱ_0 is the initial mass density and A is the Helmholtz free energy function which is a function of E'_{KL}, E''_{KL} and θ. Equation (3.6) can be used to express E'_{KL} in terms of S_{KL}.

Although the constitutive relationship by GREEN and NAGHDI as just described is of a "rate type", the effects of strain rates are not included. It is known experimentally [11, 12] that, in one-dimensional cases, the strain rate does influence the yield stress. Empirical relationships between yield stress and strain rate for one-dimensional cases have been suggested [13, 14]. General constitutive relationships for rate-dependent materials, excluding thermal effects and at infinitesimal deformations, have also been suggested [14, 15]. However, the effects of strain rates and temperature on the constitutive relationships at finite strains require further investigations. In the subsequent analysis, it is assumed that the constitutive relationship is not influenced by the strain acceleration but may be influenced by the strain velocity such as

$$S_{KL} = S_{KL}(E'_{MN}, E''_{MN}, \dot{E}''_{MN}, \theta). \tag{3.7}$$

4. Minimum principle

The true accelerations $\ddot{U}^+_K$ and $\ddot{E}^+_{KL}$ are distinguished from all possible ones by satisfying the equations of motion in the Lagrangian coordinates:

$$[S^+_{KL}(\delta_{ML} + U^+_{M,L})]_{,K} + \varrho_0(F_M - \ddot{U}^+_M) = 0, \tag{4.1}$$

where F_M is the body force per unit mass. The true Piola-Kirchhoff stresses, S^+_{KL}, satisfy the boundary conditions

$$S^+_{KL}(\delta_{ML} + U^+_{M,L})N_K = T_M \tag{4.2}$$

on that part of the initial surface area A_T, where the surface force per unit area T_M is prescribed. N_K is the outward unit normal to A.

Consider a class of arbitrary small acceleration increments, $\delta\ddot{U}_M$, which are continuous triply differentiable over the domain V, and which vanish over the boundary surface A_U, where the displacements are prescribed. Multiplying Eq. (4.1) by $\delta\ddot{U}_M$, integrating the products over V and employing Eq. (4.2) and the Gauss theorem, it is found that

$$(4.3)\qquad \int\limits_{A_T} T_M\,\delta\ddot{U}_M\,dA - \int\limits_V S^+_{KL}\,\delta\ddot{E}_{KL}\,dV + \int\limits_V \varrho_0 F_M\,\delta\ddot{U}\,dV - \int\limits_V \varrho_0\,\ddot{U}^+_M\,\delta\ddot{U}_M\,dV = 0.$$

Equations (4.3) may be restated as

$$(4.4)\qquad \delta_{acc}J = 0,$$

where

$$(4.5)\qquad J = \int\limits_V \varrho_0\frac{\ddot{U}^2_M}{2}\,dV + \int\limits_V S_{KL}\ddot{E}_{KL}\,dV - \int\limits_{A_T} T_M\ddot{U}_M\,dA - \int\limits_V \varrho_0 F_M\,\ddot{U}_M\,dV$$

and $\delta_{acc}J$ is to be interpreted as the first variation of the quantity J as the acceleration is varied in accordance with the compatibility and boundary conditions. Equation (4.4) states a variational principle, that is, of all accelerations satisfying the given boundary conditions, those which satisfy the equations of motion are distinguished by a stationary value of the functional J. Furthermore, it may be shown that the true accelerations are those which minimize the functional J.

The functional J is a function of acceleration $\ddot{U}_M$, velocity $\dot{U}_M$, displacement U_M and its history. Consider that the body has a prescribed or predetermined configuration and velocity field at time t_0. Let $\ddot{U}_M$ be any kinematically admissible acceleration field distinct from the true acceleration field $\ddot{U}^+_M$. The difference between $J^+(U^+_M, \dot{U}^+_M, \ddot{U}^+_M)$ and $J(U^+_M, \dot{U}^+_M, \ddot{U}_M)$ may be expressed as

$$(4.6)\qquad J^+ - J = \int\limits_V \frac{\varrho_0}{2}(\ddot{U}^{+2}_M - \ddot{U}^2_M)\,dV + \int\limits_V S^+_{KL}(\ddot{E}^+_{KL} - \ddot{E}_{KL})\,dV$$
$$- \int\limits_V \varrho_0 F_M(\ddot{U}^+_M - \ddot{U}_M)\,dV - \int\limits_{V_T} T_M(\ddot{U}^+_M - \ddot{U}_M)\,dA.$$

The first term at the right-hand side of Eq. (4.6) may be transformed to

$$(4.7)\qquad \int \frac{\varrho_0}{2}(\ddot{U}^{+2}_M - \ddot{U}^2_M)\,dV = \int \varrho_0\,\ddot{U}^+_M(\ddot{U}^+_M - \ddot{U}_M)\,dV - \int\limits_V \frac{\varrho_0}{2}(\ddot{U}^+_M - \ddot{U}_M)^2\,dV.$$

Using Eqs. (2.11), (4.1), (4.2) and the symmetry property of S_{KL}, the following integrals may be written as:

$$(4.8)\qquad \int\limits_V \varrho_0\,\ddot{U}^+_M(\ddot{U}^+_M - \ddot{U}_M)\,dV - \int\limits_V \varrho_0 F_M(\ddot{U}^+_M - \ddot{U}_M)\,dV - \int\limits_{A_T} T_M(\ddot{U}^+_M - \ddot{U}_M)\,dA$$
$$= \int\limits_V \{\varrho_0(\ddot{U}^+_M - F_M) - [S^+_{KL}(\delta_{ML} + U^+_{M,L})]_{,K}\}(\ddot{U}^+_M + \ddot{U}_M)\,dV$$
$$- \int\limits_V S^+_{KL}(\delta_{ML} + U^+_{M,L})(\ddot{U}^+_M - \ddot{U}_M)_{,K}\,dV = -\int\limits_V S^+_{KL}(\ddot{E}^+_{KL} - \ddot{E}_{KL})\,dV.$$

Substitution of Eqs. (4.7) and (4.8) into Eq. (4.6) yields

$$J^+ - J = -\int_V \frac{\varrho_0}{2}(\ddot{U}_M^+ - \ddot{U}_M)^2\,dV. \tag{4.9}$$

In the foregoing evaluation, it is assumed that the Piola-Kirchhoff stresses are functions of elastic strain, plastic strain, strain velocity, temperature and their history but not strain acceleration. As the integral at the right-hand side of Eq. (4.9) is positive definite, an absolute minimum principle is established, i.e.

$$J^+ - J \leqslant 0. \tag{4.10}$$

If Piola-Kirchhoff stresses depend on strain accelerations, the minimum principle remains valid for a limited class of acceleration fields subject to the kinematic boundary constraints and the requirements that

$$\ddot{E}_{KL}(S_{KL}^+ - S_{KL}) \leqslant 0 \text{ in } V$$

and

$$S_{KL} = S_{KL}^+ \quad \text{on} \quad A_T. \tag{4.11}$$

The minimum principle given by Eq. (4.10) shows that the acceleration field satisfying the equations of motion and kinematic conditions at each instant is unique.

5. Discontinuous fields

It is possible to generalize the minimum principle depicted above to the acceleration fields which have discontinuities on the surface dividing the body into a finite number of regions inside which the accelerations are continuous. Such a generalization is necessary since in practical problems the spatial derivatives of accelerations may be discontinuous. The value of the functional J for the entire region of a body is equal to the sum of that of its sub-regions such as

$$J = \sum_i J_i, \tag{5.1}$$

where J_i is the value of the functional for the ith region which may be bounded by a true discontinuity surface σ^+, and/or an assumed discontinuity surface σ. Here, a discontinuity surface is a material interface on which the displacement, velocity and stress traction must be continuous and they are either prescribed or pre-determined. Thus, by Eq. (4.10), the value of the functional J_i^+ of the true acceleration field for the ith region is less than that of any other corresponding admissible field or

$$J_i^+ - J_i \leqslant 0. \tag{5.2}$$

Eq. (5.2) shows that

$$J^+ - J = \sum_i (J_i^+ - J_i) \leqslant 0. \tag{5.3}$$

Therefore, the minimum principle remains valid for a body having discontinuous acceleration fields. Using the Green-Gauss Theorem, Eq. (5.1) may be expressed as

$$(5.4)\qquad J = \sum_i J_i = \int_{V-\sigma} \frac{\varrho_0}{2} \ddot{U}_M^2 dV + \int_{V-\sigma} S_{KL} \ddot{E}_{KL} dV - \int_{V-\sigma} \varrho_0 F_K \ddot{U}_K dV - \int_{A_T} T_K \ddot{U}_K dA + \int_{\sigma} [S_{KL} N_K (\delta_{ML} + U_{M,L}) \ddot{U}_M] d\sigma,$$

whereby the last integral is extended over all discontinuity surface σ. Here and in the sequel, quantities enclosed by a boldface bracket indicate the jump, that is, the difference between their values from the positive and negative sides of the discontinuity surface.

The minimum principle obtained above may be employed to derive the exact or approximate field equations of any non-holonomic problem. However, the exact solution of a non-linear or non-conservative problem is usually difficult to obtain. Nevertheless, the minimum principle in conjunction with a direct variational method can be applied to obtain an approximate solution of a problem. Two examples are given as follows.

6. Non-linear vibration of an elastic beam

Consider the large-amplitude, free vibrations of a flexible rectangular beam of uniform width b, and thickness h and of length l. The beam is pinned at the ends to a rigid base. The beam is initially straight and subjected to an initial axial tensile force of N_0. The plane motion of the beam may be described by the displacements (U_1, U_3) in the directions of the axial and transverse coordinates (X, Z), respectively. Employing the Bernoulli-Euler assumption, the displacements may be expressed as

$$(6.1)\qquad \begin{aligned} U_1(X, Z, t) &= U(X, t) - Z W_{,X}, \\ U_3(X, Z, t) &= W(X, t), \end{aligned}$$

where (U, W) are the displacements of a point at the centroidal axis of the beam and $W_{,X} = \partial W/\partial X$. Assuming that a linear relationship between Piola-Kirchhoff stress and Lagrangian strain prevails and that

$$(6.2)\qquad U(X, t) = 0,$$

the functional J, by Eq. (4.5), for the beam in the absence of surface tractions and body forces, may be expressed as

$$(6.3)\qquad J = \int_0^l \Big\{ \frac{1}{2} EA W_{,X}^2 (W_{,X} + \ddot{W}_{,X} \dot{W}_{,X}^2) + EI \Big[W_{,XX} \ddot{W}_{,XX} + \frac{1}{2} W_{,X}^2 (W_{,XX} \ddot{W}_{,XX} + \dot{W}_{,XX}^2) + \frac{1}{2} W_{,XX}^2 (W_{,X} \ddot{W}_{,X} + \dot{W}_{,X}^2) \Big] + \frac{3}{40} EIh^2 W_{,XX}^2 (W_{,XX} \ddot{W}_{,XX} + \dot{W}_{,XX}^2) + N_0 \Big[W_{,X} \ddot{W}_{,X} + \dot{W}_{,X}^2 + \frac{1}{12} h^2 (W_{,XX} \ddot{W}_{,XX} + \dot{W}_{,XX}^2) \Big] + \frac{A\varrho_0}{2} \ddot{W}^2 + \frac{I\varrho_0}{2} \ddot{W}_{,X}^2 \Big\} dX,$$

where E is Young's modulus, A and I are the area and moment of inertia of the cross-section, respectively.

A variation of the functional J with respect fo $\ddot{W}$ may yield an equation of motion of the beam. However, it is tedious to obtain a solution of the non-linear equation. Instead, an approximate solution may be obtained by assuming that

$$W = a\sin\frac{\varkappa\pi X}{l}\sin\omega t, \quad \varkappa = 1, 2, 3 \tag{6.4}$$

and

$$\delta\ddot{W} = -\omega^2\sin\frac{\varkappa\pi X}{l}\sin\omega t\delta a\,.$$

Using Eq. (6.4) in the condition $\delta_{acc}J = 0$, omitting the second and higher order terms, and equating the coefficient of $\sin\omega t\,\delta a$ to zero, it is found that

$$\omega^2 = \omega_0^2\left[1+\frac{9}{4}\left(\frac{a}{h}\right)^2+N_0\frac{1+\frac{h^2}{12}\left(\frac{\varkappa\pi}{l}\right)^2}{EI\left(\frac{\varkappa\pi}{l}\right)^2}\right], \tag{6.5}$$

where

$$\omega_0^2 = \frac{EI}{\varrho_0 A\left[1+\left(\frac{\varkappa\pi}{12}\right)^2\left(\frac{h}{l}\right)^2\right]}\left(\frac{\varkappa\pi}{l}\right)^4. \tag{6.6}$$

The approximate relationship between the frequency ω and the amplitude ratio a/h, as given by Eq. (6.5), agrees very well with the analytical and experimental results by Ray and Bert [16].

7. Impulsive loading of a rigid-plastic beam with axial constraints

As another example, the subject problem solved by Symonds and Mentel [17] is considered. A rectangular beam ($b\times h$) of length $2l$ and mass m per unit length is subjected to a very short pulse of uniform pressure that imparts a uniform velocity v_0 to the beam, with zero initial displacement. The ends of the beam are pinned to immovable supports. It is assumed that the Piola-Kirchhoff stress versus Lagrangian strain relationship of the material is rigid-plastic and that the strain-rate effect is negligible. At any section, there is an axial force N and a bending moment M. Plastic deformations occur at regions or surfaces of discontinuities satisfying the plasticity condition [17]

$$f(M/M_0, N/N_0) = M/M_0+N^2/N^2{}_0-1 = 0, \tag{7.1}$$

where M_0 is the limit moment in pure bending and N_0 is the axial force at yield in simple tension or compression. The corresponding flow rule [17] takes the form

$$\frac{N_0\dot{\varepsilon}}{M_0\dot{\psi}} = 2\frac{N}{N_0}, \tag{7.2}$$

where ε is the strain at the beam center-line and ψ is the curvature; dots indicate time rates.

An approximate solution of the problem may be obtained by assuming that Eqs. (6.1) and (6.2) hold for the present problem and an admissible velocity distribution, which is symmetric and given by

(7.3)
$$\dot{W} = \frac{X}{\xi l} v \quad \text{for} \quad 0 \leqslant X \leqslant \xi l,$$
$$W = \delta \text{ and } \dot{W} = v \quad \text{for} \quad \xi l \leqslant X \leqslant l,$$

where $X = \xi l$ is a surface of discontinuity in $\ddot{W}$. Here v, ξ and δ are functions of time with the initial conditions $t = 0$, $v = v_0$, $\xi = 0$ and $\delta = 0$. For the rigid-plastic beam, $\ddot{E}_{KL} = 0$ everywhere except at the surfaces of discontinuity, it is assumed that the square of the slope of the deflection curve is small as compared with unity and that the axial force N may be taken as constant along the beam. With these assumptions and the velocity field by Eq. (7.3), the functional J, by Eq. (5.4), for this problem may be expressed as

(7.4)
$$J = \frac{m}{3} \frac{(\dot{v}\xi - \dot{\xi}v)^2 l}{\xi} + m\dot{v}^2 l(1-\xi) + 2(M - N\delta) \frac{\dot{v}\xi - \dot{\xi}v}{\xi^2 l}.$$

Equating the derivatives of J with respect to $\dot{v}$ and $\dot{\xi}$ to zero, the following two equations are obtained:

(7.5)
$$\dot{v} = 0$$

and

(7.6)
$$\dot{\xi} = \frac{M - N\delta}{\xi l} \frac{3}{mvl}.$$

Equation (7.5) shows that

$$v = v_0 = \text{constant}$$

and

(7.7)
$$\delta = v_0 t \quad \text{for} \quad 0 < t < t^*,$$

where t^* is the time when ξ first reaches the value of unity. If has been shown by SYMONDS and MENTEL [17] that Eq. (7.2) may be expressed as

(7.8)
$$\frac{N}{N_0} = \frac{N_0 \delta}{2M_0}.$$

Employing this equation and the plasticity condition Eq. (7.1), the solution of ξ from Eq. (7.6) is found to be

(7.9)
$$\xi^2 = \frac{6M_0}{mv_0 l^2} \left(1 + \frac{N_0^2 v_0^2 t^2}{12 M_0}\right) t.$$

The time t^* may be determined by Eq. (7.9) by setting $\xi = 1$.

For $t \geqslant t^*$, $\xi = 1 =$ constant, there is one surface of discontinuity in acceleration at the center of the beam. Assuming that

(7.10)
$$\dot{W} = \frac{Xv}{l} \qquad \text{for} \quad 0 \leqslant X \leqslant l.$$

The functional J by Eq. (5.4) may now be written as

$$J = 2\left[\frac{ml\dot{v}^2}{6} + (M+N\delta)\frac{\dot{v}}{l}\right]. \tag{7.11}$$

Minimization of J with respect to $\dot{v}$ yields an equation which leads to the solution

$$v^2 = v_0^2 - \frac{6}{M_0 ml^2}\left[M_0^2(\delta-\delta^*) + \frac{N_0^2}{12}(\delta^3-\delta^{*3})\right], \tag{7.12}$$

where $\delta^* = v_0 t^*$. The maximum deflection which occurs at $v = 0$, may be readily determined by Eq. (7.12). The foregoing results are identical to those by SYMONDS and MENTEL [17].

8. Conclusions

An absolute minimum principle, which is based on the concept of acceleration-variations, has been developed for dynamics of elastic-plastic continua at finite deformation. It has been shown that the principle is valid for continua whose constitutive relationships are independent of strain-accelerations; for such continua, there are no stress variations accompanying strain-acceleration variations. The limitations of the principle for strain-acceleration dependent continua may be further evaluated when specific constitutive relationships are known.

The minimum principle is valid for continuous as well as sectionally discontinuous acceleration fields. The minimum principle may be employed to establish governing equations or to solve a problem by using a direct method of variational calculus.

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References

1. K. WASHIZU, *Variational methods in elasticity and plasticity*, Pergamon Press, Oxford 1968.
2. D. C. DRUCKER, *Variational principles in the mathematical theory of plasticity*, Proc. Symposia in Appl. Math., **8**, Calculus of variations and its applications, McGraw Hill Book Co., 7–22, 1958.
3. R. HILL, *A general theory of uniqueness and stability in elastic-plastic solids*, J. Mech. Phys. Solids, **6**, 236–249, 1958.
4. L. H. N. LEE and L. M. MURPHY, *Inelastic axisymmetric buckling of ring stiffened cylindrical shells under external pressure*, Technical Report No. Themis-UND-69-2 (ONR Contract No. N0014-68-A-0152), University of Notre Dame, 94, May 1969.
5. L. A. PARS, *A treatise on analytical dynamics*, Heinemann, pp. 12, 40, 42, 200–201, 1965.
6. V. P. TAMUZH, *On a minimum principle in dynamics of rigid-plastic bodies*, PMM, **26**, 4, 715–715, 1962.
7. A. E. GREEN and P. M. NAGHDI, *A general theory of an elastic-plastic continuum*, Arch. Rat. Mech. Anal., **18**, 251–281, 1965.
8. A. E. GREEN and P. M. NAGHDI, *A thermodynamic development of elastic-plastic continua*, Proc. of Symposia of IUTAM on the Irreversible Aspects of Continuum Mechanics held in June 1966 in Vienna, Springer Verlag, 117–131, New York 1968.

9. E. H. LEE and D. T. LIU, *Finite-strain elastic-plastic theory with application to plane-wave analysis*, J. Appl. Phys., **38**, 1, 19–27, 1967.
10. E. H. LEE, *Elastic-plastic deformation at finite strains*, J. Appl. Mech., ASME, 1–6, 1969.
11. P. PERZYNA and T. WIERZBICKI, *Temperature dependent and strain rate sensitive materials*, Arch. Mech. Stos., **16**, 1, 135–143, 1964.
12. D. J. LLOYD, *Inverse strain-rate effects*, J. Appl. Phys., **41**, 8, 3910, 1970.
13. T. C. T. TING, *The plastic deformation of a contilever beam with strain rate sensitivity under impulsive loading*, TR 70, Brown University, Contract No 562 (10), July 1961.
14. N. CRISTESCU, *Dynamic plasticity*, John Wiley and Sons, Inc. Chapter X, New York 1967.
15. J. N. JOHNSON, *Constitutive relation for rate-dependent plastic flow in polycrystalline metals*, J. Appl. Phys., **40**, 5, 2287–2293, 1969.
16. J. D. RAY and C. W. BERT, *Nonlinear vibrations of a beam with pinned ends*, J. Engin. Industry, ASME, **91**, Ser. AB, 997–100:, 1969.
17. P. S. SYMONDS and T. J. MENTEL, *Impulsive loading of plastic beams with axial constraints*, J. Mech. Phys. Solids, **6**, 186–202, 1958.

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