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A shearing crack in a semi-space under plane strain conditions

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A TWO-DIMENSIONAL static problem for an arbitrarily situated crack in a half-plane is solved by introducing a singular integral equation for displacement jump derivative on the crack line. The integral equation is obtained using the solution for a single dislocation in a half-plane and is then reduced to a set of simultaneous linear equations. Two cases are distinguished: one, when the crack intersects the free boundary, and the other when it does not. As an example, stress-intensity factors and the displacement jumps versus a distance along the crack for some initial stress conditions are obtained.

W pracy został rozwiązany problem statyczny szczeliny ścinania w półprzestrzeni sprężystej w warunkach płaskiego stanu odkształcenia. Wprowadzono osobliwe równanie całkowe dla pochodnej skoku przemieszczenia na linii szczeliny, które uzyskano wykorzystując rozwiązanie dla pojedynczej dyslokacji w półpłaszczyźnie, a następnie przekształcono je w układ równań liniowych. Układ ten rozwiązano numerycznie. Uwzględniono oba możliwe położenia szczeliny, tzn. szczelinę, wychodzącą na powierzchnię pod dowolnym kątem, oraz dowolnie położoną szczelinę wewnętrzną. Na zakończenie pracy wykonano kilka przykładów liczbowych, obliczając współczynniki intensywności naprężeń oraz rozkład przemieszczenia u wzdłuż szczeliny, w polu stałych naprężeń ścinających. Obliczenia dotyczyły zarówno szczelin powierzchniowych, jak i wewnętrznych, a ich celem było przede wszystkim zbadanie zbieżności uzyskanych rozwiązań, co zostało w pełni potwierdzone.

В работе дано решение плоской задачи для трещины среза в упругом полупространстве. Пользуясь решением для единственной дислокации в полупространстве введено сингулярное интегральное уравнение для производной скачка смещения на линии трещины, которое преобразовано в систему линейных уравнений. Эту систему решено нумерически. Задача решена для обоих возможных случаев положения трещины, т. е. для трещины, выходящей на свободную поверхность — и для внутренней трещины. В заключении работы сделано несколько нумерических примеров, вычисляя коэффициенты интенсивности напряжений и расположение смещения u на трещине при постоянных напряжениях, с целью исследования сходимости полученного в работе решения.

Introduction

THE FACT that a great number of crack problems have been solved reflects their importance in the field of fracture mechanics (see, e.g., Fracture, 1968; PANASJUK, 1968). Most of the solutions are limited to the case of tensile rupture which is of great importance for engineering applications. Some problems of longitudinal shear cracks have also been solved for reasons of their mathematical simplicity. In geophysical applications (i.e., earthquake mechanics), however, the shear cracks are of great importance because of the well-known fact that compressive stresses in the earth's interior must be very high. We shall not enter into discussion of possible mechanisms of sliding deep in the interior of the earth, but we shall mention here that the shear crack with friction may be a very good model for faulting in earthquakes sources (AKI, 1971, BURRIDGE and HALLIDAY, 1971). When considering corresponding problems we must take into account surface of the earth which is free of tractions. A somewhat usual case is, when the fault is strongly elongated and parallel to the earth's surface so that the problem may be reduced to the two-dimensional one.

Recently, such a problem was studied for the case of strike-slip fault — i.e., for a longitudinal shear model [5, 14, 15, 16]. For dip-slip fault, when displacements are in the vertical plane, the corresponding problem refers to the plane strain. Although methods for solving plane crack problems in a half-space have been developed (BOWIE, SAVIN, SIH, PARIS, ERDOGAN), they are useful only for tensile cracks because they are based on the conformal mapping, and for shear crack the normal components of displacement and stress vectors must be continuous across the crack surface, which after the mapping cannot be reformulated as a local condition. Owing to the lack of a proper mathematical technique, some simplified models were considered — that of a single dislocation line within a half-space or some prescribed distribution of dislocations on plane of the fault (CHINNERY and PETRAK, 1968).

This suggests the following approach to the crack problem:

(a) introduce an initially unknown distribution of edge dislocations, slip planes of which coincide with the crack plane;

(b) solve the problem of a given distribution of dislocations for the shear stress on the crack plane;

(c) requiring the stress to be the prescribed stress on the crack, obtain an integral equation for the dislocation density;

(d) then, solving the equation numerically, obtain the solution of the crack problem.

Such an approach is realized below.

1. Mathematical formulation

Consider a homogeneous elastic half-space having Lame's constants λ and μ and containing a strip-line crack which reaches the surface (surface crack, Fig. 1a) or is embedded into the half-space (internal crack, Fig. 1b).

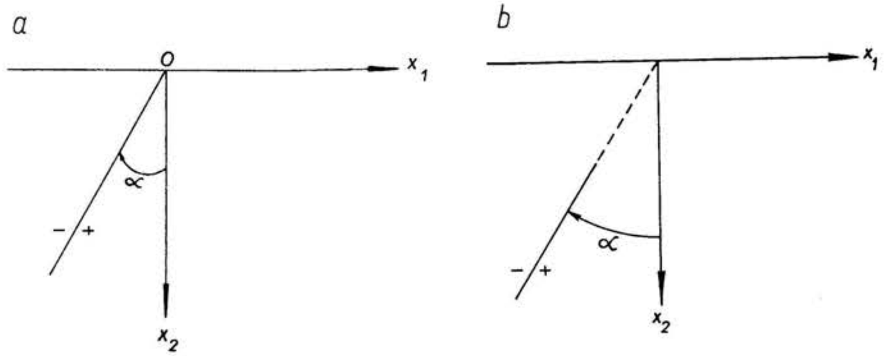

FIG. 1.

Denote by α the angle between the surface of the half-space, and normal to the crack. Let us assume that external loads would create in a half-space without a crack a stress field σ_{ij}^0, which would not depend on the coordinate along the crack edge (x_3). Then we

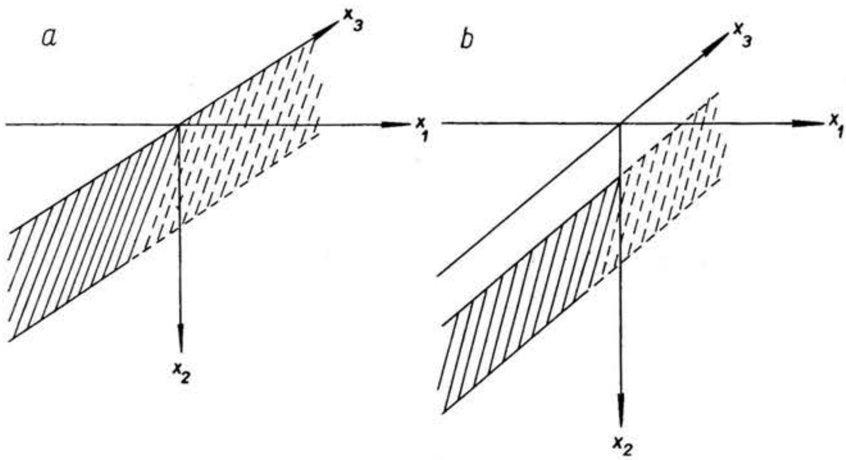

FIG. 2.

obtain a plane strain problem, when no quantities depend of x_3 (Fig. 2a and 2b), and the crack surface is given by

$$x_1 = -s\sin\alpha, \quad x_2 = s\cos\alpha, \quad 0 \leqslant s \leqslant b \tag{1.1}$$

for surface crack, or

$$x_1 = -s\sin\alpha, \quad x_2 = s\cos\alpha, \quad a \leqslant s \leqslant b \tag{1.1'}$$

for internal crack.

Our objective is to find the stress perturbation which is connected with introduction of the crack into the half-space. The corresponding displacement vector (u_1, u_2) will be connected with the stress field perturbed by the crack formation as follows:

$$\sigma_{ij} = \sigma_{ij}^0 + \lambda u_{k,k} + 2\mu u_{i,j}, \quad i, j, k = 1, 2, \tag{1.2}$$

where the comma denotes partial differentiation with respect to the Cartesian coordinates and the summation convention is assumed. Equation (1.2) may be rewritten in the form:

$$\tau_{ij} = \lambda u_{k,k} + 2\mu u_{i,j}, \quad i, j, k = 1, 2, \tag{1.2'}$$

where $\tau_{ij} = \sigma_{ij} - \sigma_{ij}^0$ is the stress disturbance created by the crack.

We suggest that the external forces do not change during the crack formation.

Then τ_{ij} will satisfy the homogeneous equilibrium equations

$$\tau_{ij,j} = 0 \tag{1.3}$$

and, independently from external tractions applied to the surface $x_2 = 0$, disturbance τ_{ij} must satisfy the following boundary condition:

$$\tau_{i2} = 0 \quad \text{for} \quad x_2 = 0, \quad i = 1, 2. \tag{1.4}$$

Let us assume, that the confining pressure corresponding to the initial stress σ_0^{ij} is sufficiently high to prevent any opening of the crack — i.e., the crack is a purely shear one. Consequently, we obtain the following condition on the crack line:

$$u_n^+ - u_n^- = 0 \quad \text{on the crack surface,} \tag{1.5}$$

where $u_n = u_i n_i = u_1 \cos\alpha + u_2 \sin\alpha$, and "plus" and "minus" superscripts denote here and hereinafter quantities on different sides of the crack, as in Fig. 2.

Also, the normal component of the stress must be continuous on the crack, which gives

$$\tau_n^+ - \tau_n^- = 0, \tag{1.6}$$

because all components of the initial stress σ_{ij}^0 are continuous. Here

$$\tau_n = \tau_{ij} n_i n_j = \tau_{11} \cos^2\alpha + \tau_{22} \sin^2\alpha + 2\tau_{12} \cos\alpha \sin\alpha.$$

For the sake of simplicity, we assume that there is no friction between the crack sides. Then shear stress on the crack line must be zero, or

$$\tau_t = -\sigma_t^0 \equiv f(s), \tag{1.7}$$

where

$$\tau_t = \frac{1}{2}(\tau_{22} - \tau_{11}) \sin 2\alpha + \tau_{12} \cos 2\alpha, \quad \text{and}$$

$$f(s) \equiv -\sigma_t^0 = \frac{1}{2}(\sigma_{11}^0 - \sigma_{22}^0) \sin 2\alpha - \sigma_{12}^0 \cos 2\alpha \tag{1.8}$$

is a given function on a crack line [Eq. (1.1) or (1.1′)].

Since the displacement u_i must be singlevalued, one has the following conditions at infinity:

$$u_i \to 0 \quad \text{as} \quad r \to \infty, \tag{1.9}$$

where $r = \sqrt{x_1^2 + x_2^2}$. Then, for stress components, it follows that

$$\tau_{ij} = 0(r^{-2}) \quad \text{as} \quad r \to \infty. \tag{1.9′}$$

For a surface crack [Eq. (1.1)] an additional condition must be stated at the point $r = 0$ which provides the uniqueness of the solution:

$$\tau_{ij} = 0(r^{\lambda}) \quad \text{as} \quad r \to 0, \quad \lambda > -1. \tag{1.10}$$

Equations (1.2′), (1.3), together with the boundary conditions (1.4) to (1.7) and the additional conditions (1.9) to (1.10), constitute complete mathematical formulation of the problem under consideration.

Unfortunately, there does not exist any straightforward analytical approach to the problem. There exists, however, a closely related problem, solution of which may be obtained very easily — namely, the problem of Somigliana's dislocation. This problem differs from that just formulated in that the condition (1.7) is relaxed to

$$\tau_t^+ - \tau_t^- = 0 \quad \text{on the crack line}, \tag{1.11}$$

and an additional condition which specifies the tangential displacement jump is formulated:

$$u_t^+ - u_t^- = u(s) \quad \text{on the crack line}. \tag{1.12}$$

Here $u_t = -u_1 \sin\alpha + u_2 \cos\alpha$, and the function $u(s)$ is assumed to be known. Suppose that the analytical solution for the last problem is constructed for an arbitrary function

$u(s)$. Then, if we were able to find such a function $u(s)$ that the corresponding shear stress distribution would be $f(s)$, our main problem would be solved. Next, the problem of Somigliana dislocation may be reduced to the Volterra one, which is characterized by the specific function $u(s)$:

$$u(s) = u_V(s, s_1) = \begin{cases} 1 & \text{for} \quad s < s_1, \\ 0 & \text{for} \quad s > s_1. \end{cases} \tag{1.13}$$

In fact, if $\tau_{ij}^{(V)}(x_1, x_2, s_1)$ is the stress solution for the function (1.13), then, from the linearity of the problem, the stress solution for the arbitrary function $u(s)$ will be given by

$$\tau_{ij}(x_1, x_2) = -\int_0^\infty u'(s_1)\,\tau_{ij}^{(V)}(x_1, x_2, s_1)\,ds_1, \tag{1.14}$$

where the prime denotes the derivative with respect to s_1.

Now observe, that in our crack problem the displacement jump and its derivative outside the crack equal zero. Then, using the condition (1.7), we obtain the following equation for $u'(s_1)$:

$$\int_b^a \tau_t^{(V)}(s, s_1)u'(s_1)\,ds_1 = -f(s), \tag{1.15}$$

where $\tau_t^{(V)}(s, s_1)$ is the shear stress on the crack line corresponding to the Volterra dislocation situated at $s = s_1$ (or, more precisely, at the point $x_1 = -s_1 \sin\alpha$, $x_2 = s_1 \cos\alpha$).

Since the Eq. (1.15) contains only the displacement jump derivative $u'(s_1)$, in the case of an internal crack the following condition ensuring displacement continuity outside the crack line must be added to the equation:

$$u(a) - u(b) = 0 \tag{1.16a}$$

or

$$\int_a^b u'(s)\,ds = 0. \tag{1.16b}$$

In the next section, we obtain a convenient expression for $\tau_t^{(V)}(s, s_1)$.

2. Solution for a single dislocation

We start from the general solution of the Eqs. (1.2′) and (1.3) in the Kolosov-Muskhelishvili form (MUSKHELISHVILI [10]):

$$\begin{aligned} \tau_{11} + \tau_{22} &= 2\left(\Phi(z) + \overline{\Phi(z)}\right), \\ \tau_{22} - \tau_{11} + 2i\tau_{12} &= 2\left(\bar{z}\Phi'(z) + \Psi(z)\right); \end{aligned} \tag{2.1}$$

$$2\mu(u_1 + iu_2) = \varkappa\varphi(z) - z\overline{\Phi(z)} - \overline{\psi(z)}, \tag{2.2}$$

where $z = x_1 + ix_2$, the prime means derivative with respect to z, the bar denotes complex conjugation, and

$$\varkappa = \frac{\lambda + 3\mu}{\lambda + \mu}. \tag{2.3}$$

The function Φ, Ψ and φ, ψ are related by

$$\Phi(z) = \varphi'(z); \quad \Psi(z) = \psi'(z), \tag{2.4}$$

and all the functions must be holomorphic whereever the medium is assumed to be continuous.

Therefore, in our problem of single dislocation Φ and Ψ must be holomorphic everywhere in the half-plane $\operatorname{Im} z > 0$, excluding the point $z = z_1$,

$$z_1 = -s_1 \sin\alpha + is_1 \cos\alpha = is_1 e^{i\alpha}, \tag{2.5}$$

where the dislocation must be situated. In the vicinity of this point, the stress components may have singularity of the first order. From the Eq. (2.1) it follows then that $\Phi(z)$ may have at $z = z_1$ a simple pole, whereas Ψ may have a pole of second order. The corresponding representation for $\Phi(z)$ and $\Psi(z)$ would have the form

$$\begin{aligned} \Phi(z) &= \frac{a}{z-z_1} + \Phi_1(z), \\ \Psi(z) &= \frac{b}{z-z_1} + \frac{a\bar{z}_1}{(z-z_1)^2} + \Psi_1(z), \end{aligned} \tag{2.6}$$

where $\Phi_1(z)$ and $\Psi_1(z)$ are holomorphic in the half-plane, and a and b are complex numbers.

The functions φ and ψ, and consequently the displacement components, will not be singlevalued within the half-space, but if we make a cut from $z = z_1$ to $z = 0$, then the displacements will be single-valued and will have a jump across the cut.

From the conditions (1.5) and (1.13) it follows that the complex displacement jump must be equal to $ie^{i\alpha}$. Then from (2.2), (2.4) and (2.6), we have:

$$2\pi i(\bar{b} + \varkappa a) = -2\mu i e^{i\alpha}. \tag{2.7}$$

From the condition that no external force is applied on the dislocation point we obtain (MUSKHELISHVILI, 1966):

$$2\pi i(a - \bar{b}) = 0. \tag{2.8}$$

It follows then that

$$a = -\frac{\mu e^{i\alpha}}{\pi(\varkappa+1)}, \quad b = -\frac{\mu e^{-i\alpha}}{\pi(\varkappa+1)}. \tag{2.9}$$

It now remains only to calculate the functions Φ_1 and Ψ_1, holomorphic everywhere within the half-space. To this end, we use the boundary conditions on the half-space surface (1.4) or

$$\tau_{22} + i\tau_{12} = 0 \quad \text{for} \quad x_2 = 0. \tag{2.10}$$

Using the Eqs. (2.1) and (2.6), we obtain:

$$\tau_{22} + i\tau_{12} = \Phi_1(x_1) + \overline{\Phi_1(x_1)} + x_1\Phi_1'(x_1) + \Psi_1(x_1) - \frac{\mu}{\pi(\varkappa+1)} \left\{ \frac{e^{i\alpha} + e^{-i\alpha}}{x_1 - z_1} + \frac{e^{-i\alpha}}{x_1 - \bar{z}_1} - \frac{(x_1 - \bar{z}_1)e^{i\alpha}}{(x_1 - z_1)^2} \right\} = 0.$$

Note that $\overline{\Phi_1(x_1)}$ is a boundary value of a function which is holomorphic within the half-plane $\operatorname{Im} z < 0$, and rewrite this equation in the following form:

$$(2.11)\quad \overline{\Phi_1(x_1)} - \frac{\mu}{\pi(\varkappa+1)}\left\{\frac{2\cos\alpha}{x_1-z_1} - \frac{x_1-\bar{z}_1}{(x_1-z_1)^2}e^{i\alpha}\right\} = -\Phi_1(x_1) - x_1\Phi_1'(x_1) - \Psi_1(x_1) + \frac{\mu}{\pi(\varkappa+1)}\cdot\frac{e^{-i\alpha}}{x_1-\bar{z}_1}.$$

Both sides of the Eq. (2.11) are boundary values of functions which are holomorphic within the upper and lower half-planes, respectively, and consequently are holomorphic everywhere. It follows from the condition (1.9′) that the functions must tend to zero at infinity. We conclude then that both sides of the Eq. (2.11) are equal to zero. Now, we easily find that

$$(2.12)\quad \Phi_1(z) = \frac{\mu}{\pi(\varkappa+1)}\left\{\frac{2\cos\alpha}{z-\bar{z}_1} - \frac{z-z_1}{(z-\bar{z}_1)^2}\cdot e^{-i\alpha}\right\},$$

$$(2.13)\quad \Psi_1(z) = \frac{\mu}{\pi(\varkappa+1)}\frac{e^{-i\alpha}}{z-\bar{z}_1} - \Phi_1(z) - z\Phi_1'(z).$$

The Eqs. (2.6), (2.9), (2.12) and (2.13) constitute the solution sought for. From (2.6) it follows that the normal and shear components of stress on the crack line are given by

$$(2.14)\quad \tau_n^{(V)} - i\tau_t^{(V)} = \Phi(z) - \overline{\Phi(z)} - e^{2i\alpha}(\bar{z}\varphi'(z) + \Psi(z)).$$

Now, using the Eqs. (2.6), (2.9) and (2.12) to (2.14) and evaluating the real and imaginary parts of the Eq. 2.14, we obtain:

$$(2.15)\quad \frac{\pi(\varkappa+1)}{2\mu}\tau_t^{(V)} = \frac{1}{s-s_1} - \frac{s+s_1\cos2\alpha}{s^2+s_1{}^2+2ss_1\cos2\alpha} - \frac{4s_1\cos\alpha(s-s_1)(s^3\cos3\alpha+3s^2s_1\cos\alpha+3s_1^2s\cos\alpha+s_1^3\cos3\alpha)}{(s^2+s_1^2+2ss_1\cos2\alpha)^3}$$

and

$$(2.16)\quad \frac{\pi(\varkappa+1)}{2\mu}\tau_n^{(V)} = -\frac{4s_1^2\cos^2\alpha(s^3\sin4\alpha+3s^2s_1\sin2\alpha-s_1^3\sin2\alpha)}{(s^2+s_1^2+2ss_1\cos2a)^3}.$$

Now, we are in a position to return to our basic crack problem.

3. Integral equation for the crack problem

Introducing the expression for $\tau_t^{(V)}$ from the Eq. (2.15) into the Eq. (1.15), we obtain the following singular integral equation for $u'(s)$, the derivative of the displacement jump:

$$(3.1)\quad \frac{1}{\pi}\int_a^b\frac{u'(s_1)ds_1}{s_1-s} + \frac{1}{\pi}\int_a^b K(s,s_1)u'(s_1)ds_1 = \frac{\varkappa+1}{2\mu}f(s)\quad \text{for}\quad a<s<b,$$

where

$$(3.2)\quad K(s,s_1) = \frac{s+s_1\cos2\alpha}{(s^2+s_1^2+2ss_1\cos2\alpha)} + \frac{4s_1\cos\alpha(s-s_1)(s^3\cos3\alpha+3s^2s_1\cos\alpha+3s_1^2s\cos\alpha+s_1^3\cos3\alpha)}{(s^2+s_1^2+2ss_1\cos2\alpha)^3}.$$

For the surface crack, we obtain the same integral equation, where only a must be replaced by zero.

Though we have restricted ourselves to the case in which there is no friction between crack surfaces, we note here that friction may easily be accounted for by a slight modification of the above equations. Consider, for example, the case of Coulomb friction. Then in place of the condition (1.7), we should have:

$$\sigma_t + k\sigma_n = 0, \tag{3.3}$$

where k is the coefficient of friction. Introducing into this equation the stress perturbations τ_t and τ_n, we obtain:

$$\tau_t - k\tau_n = -\sigma_t^0 - k\sigma_n^0. \tag{3.4}$$

Now, after examining the previous considerations, we conclude that to account for friction we must put

$$f(s) = -\sigma_t^0 - k\sigma_n^0 \tag{3.5}$$

and

$$K(s, s_1) = \frac{s + s_1 \cos 2\alpha}{(s^2 + s_1^2 + 2ss_1 \cos 2\alpha)} + \frac{4s_1 \cos\alpha (s - s_1)(s^3 \cos 3\alpha + 3s^2 s_1 \cos\alpha + 3s_1^2 s \cos\alpha + s_1^3 \cos 3\alpha)}{(s^2 + s_1^2 + 2ss_1 \cos 2\alpha)^3} + k\frac{4s_1^2 \cos^2\alpha (s^3 \sin 4\alpha + 3s^2 s_1 \sin 2\alpha - s_1^3 \sin 2\alpha)}{(s^2 + s_1^2 + 2ss_1 \cos 2\alpha)^3}. \tag{3.6}$$

We shall not go into greater detail of the friction case, but one must understand that in this case essential properties of the integral equations will be the same as for the case without friction, which will be explained below.

The properties of the integral equation (3.1) are quite different, depending on whether the crack is internal or the surface one.

If $a \neq 0$ (internal crack), the Eq. (3.1) is a common singular integral equation of the type thoroughly studied elsewhere (see, e.g., MUSKHELISHVILI). Without going into detail, let us note here that the Eq. (3.1), together with the condition (1.16b), has a unique solution, which has the inverse square root singularities at $s = a$ and $s = b$:

$$u'(s) = 0\left((s-a)^{-\frac{1}{2}}\right) \quad \text{as } s \to a+, \quad \text{and} \tag{3.7}$$
$$u'(s) = 0\left((b-s)^{-\frac{1}{2}}\right) \quad \text{as } s \to b-.$$

At the other points between a and b, $u'(s)$ has the same smoothness as the right-hand side of the Eq. (3.1). This suggests the following representation for $u'(s)$:

$$u'(s) = \frac{v(s)}{\sqrt{(b-s)(s-a)}} \frac{\varkappa + 1}{2\mu}, \tag{3.8}$$

where $v(s)$ is a smooth function of s.

Then the Eq. (3.1) takes the following form:

$$\frac{1}{\pi}\int_a^b \frac{v(s_1)ds_1}{(s_1-s)\sqrt{(b-s_1)(s_1-a)}}+\frac{1}{\pi}\int_a^b K(s,s_1)v(s_1)\frac{ds_1}{\sqrt{(b-s_1)(s_1-a)}}=f(s). \tag{3.9}$$

For fracture mechanics, of the greatest importance are stress intensity factors — i.e., the coefficients in the asymptotic representation of the stress (in the case under consideration the shear stress τ_t), having the form:

$$\begin{cases} \tau_t \approx \dfrac{K(b)}{\sqrt{2(b-s)}} \quad \text{as} \quad s \to b+, \quad \text{and} \\ \tau_t \approx \dfrac{K(a)}{\sqrt{2(s-a)}} \quad \text{as} \quad s \to a-. \end{cases} \tag{3.10}$$

We easily obtain from the Eq. (3.9) the expressions for $K(a)$ and $K(b)$:

$$K(a)=\sqrt{\frac{2}{b-a}}\cdot v(a), \quad K(b)=-\sqrt{\frac{2}{b-a}}\cdot v(b). \tag{3.11}$$

Now, the case of surface crack ($a = 0$) is more complicated. In this case, the kernel $K(s, s_1)$ has a singularity at the point $s = 0$, $s_1 = 0$, so that the usual theory of singular integral equations cannot be applied. The scope of this paper does not permit us to study the equation thoroughly. Using the fact that the crack problem with the additional condition (1.10) has a unique solution, we can prove that the solution of the equation is also unique. A proof of the existence of the solution would be rather difficult. So we restrict ourselves here to investigation of the behaviour of the solution at the end points $s = 0$ and $s = b$, assuming its existence. Usual methods give the following asymptotic representation for the vicinity of $s = b$:

$$u'(s) \approx \frac{\text{const}}{\sqrt{b-s}} \quad \text{as} \quad s \to b-. \tag{3.12}$$

Obtainment of an asymptotic representation of the solution near the point $s = 0$ needs rather sophisticated treatment. However, owing to the great importance of this point, we describe it in greater detail in the Appendix.

As is shown there, the asymptotic representation of $u'(s)$ near the point $s = 0$ has the form:

$$u'(s) = \operatorname{Re}\sum_n u_n s^{\beta_n}, \quad (\operatorname{Re}\beta_n > -1), \tag{3.13}$$

where u_n are unknown constants and β_n are roots of the following equation:

$$2\cos\pi\beta-\beta(\beta+1)\cos 2\alpha\beta-2\beta(\beta+2)\cos 2\alpha(\beta+1)-(\beta+1)(\beta+2)\cos 2\alpha(\beta+2) = 0. \tag{3.14}$$

It may be shown that this equation has no roots between -1 and 0. So the main term of the expansion (3.13) is

$$u'(s) = \operatorname{Re} u_1 s^{\beta_1}, \tag{3.15}$$

where $\operatorname{Re}\beta_1 > 0$ and $\operatorname{Im}\beta_1 \neq 0$.

From (3.15) it is seen that $u'(s)$ tends to zero as $s \to 0$. This fact will be of substantial use for the algorithm which is developed in the following section.

4. Reduction to a set of simultaneous linear equations

4.1. Internal crack

For numerical solution of the Eq. (3.9), we shall replace it approximately by a set of simultaneous linear equations. For this purpose, we shall use the Hermite quadrature formula to represent the integral

$$\frac{1}{\pi}\int_a^b K(s, s_1)\frac{v(s_1)s\,ds_1}{\sqrt{(s-a)(s-b)}},$$

which gives

$$\frac{1}{\pi}\int_a^b K(s, s_1)\frac{v(s_1)ds_1}{\sqrt{(s-a)(s-b)}} = \frac{1}{n}\sum_{m=1}^{n} K(s, s_m)v(s_m), \tag{4.1}$$

where

$$s_m = \frac{a+b}{2}+\frac{b-a}{2}x_m, \quad m = 1, \dots, n. \tag{4.2}$$

Here, x_m are the zeros of $T_n(x)$, the Tschebyscheff polynomial of order n, first kind—i.e.,

$$x_m = \cos\frac{2m-1}{2n}\pi. \tag{4.3}$$

A similar representation for a singular integral

$$\frac{1}{\pi}\int_a^b \frac{v(s_1)ds_1}{(s_1-s)\sqrt{(s-a)(s-b)}}$$

may be obtained as a specific case of the general formula derived by KORNEICHUK [8], which in our case gives:

$$\frac{1}{\pi}\int_a^b \frac{v(s_1)ds_1}{(s_1-s)\sqrt{(s-a)(s-b)}} = \frac{1}{n}\sum_{m=1}^{n}\frac{v(s_m)}{(s_m-s)}\left[1-\frac{U_{n-1}\left(\frac{2s}{b-a}-\frac{b+a}{b-a}\right)}{U_{n-1}(x_m)}\right], \tag{4.}$$

where $U_{n-1}(x)$ is the Tschebyscheff polynomial of second kind, and s_m, x_m are defined in the Eq. (4.2), (4.3).

So the Eq. (3.9) can be written in the approximate form:

$$\frac{1}{n}\sum_{m=1}^{n}\frac{v(s_m)}{s_m-s}\left[1-\frac{U_{n-1}\left(\frac{2s}{b-a}-\frac{b+a}{b-a}\right)}{U_{n-1}(x_m)}\right] + \frac{1}{n}\sum_{m=1}^{n} v(s_m)K(s, s_m) = f(s). \tag{4.5}$$

To obtain from this equation a set of simultaneous algebraic equations, it is convenient to set

$$s = t_k = \frac{a+b}{2}+\frac{b-a}{2}y_k, \quad k = 1, \dots, n-1, \tag{4.6}$$

where y_k are the roots of $U_{n-1}(y)$ — i.e.,

$$y_k = \cos\frac{k\pi}{n}, \quad k = 1, \ldots, n-1.$$

Then the second term in brackets in the Eq. (4.5) vanishes and we have:

$$\frac{1}{n}\sum_{m=1}^{n} v(s_m)\left[\frac{1}{s_m - t_k} + K(t_k, s_m)\right] = f(t_k), \quad k = 1, \ldots, n-1. \tag{4.7}$$

To complete the set of equations, we must represent the additional condition (1.16b) using the expression (3.8) and Hermite's formula, which gives

$$\frac{1}{n}\sum_{m=1}^{n} v(s_m) = 0. \tag{4.8}$$

This formula gives just the last, n-th equation of the set.

Now, once the set is solved, the values of $v(s)$ at any point may be obtained by interpolation. The corresponding expression is

$$v(s) = \frac{2}{n}\sum_{k=0}^{n-1}\left(\sum_{m=1}^{n} v(s_m) T_k(x_m)\right) T_k\left(\frac{2s-b-a}{b-a}\right); \tag{4.9}$$

for $s = a$ and $s = b$, this expression simplifies to

$$v(b) = \frac{1}{n}\sum_{m=1}^{n} v(s_m)\big(1 + (1-x_m)^{-1} T_{n-1}(x_m)\big) \tag{4.10}$$

and

$$v(a) = \frac{1}{n}\sum_{m=1}^{n} v(s_{n-m+1})\big(1 + (1-x_m)^{-1} T_{n-1}(x_m)\big). \tag{4.11}$$

We may easily obtain the appropriate formulae for coefficients of intensity $K(a)$ and $K(b)$, substituting these equations into the expressions (3.11).

The formula for the displacement jump $u(s)$ is obtained by integration of the Eq. (3.8), using the expression (4.9), which gives

$$\begin{aligned} u(s) = &-\frac{\varkappa+1}{2\mu}\left(\sum_{m=1}^{\infty} v(s_m)\right)\arccos\frac{2s-a-b}{b-a} \\ &-\frac{\varkappa+1}{2\mu}\sum_{k=1}^{n-1}\frac{2}{k}\left(\sum_{m=0}^{n} v(s_m) T_k(x_m)\right) U_{k-1}\left(\frac{2s-a-b}{b-a}\right)\frac{\sqrt{(b-s)(s-a)}}{b-a}. \end{aligned} \tag{4.12}$$

Owing to the condition (4.8), the first term vanishes, and for the case of internal crack with which we are dealing, we have finally:

$$u(s) = -\frac{\varkappa+1}{2\mu}\sum_{k=1}^{n-1}\frac{2}{k}\left(\sum_{m=0}^{n} v(s_m) T_k(x_m)\right) U_{k-1}\left(\frac{2s-a-b}{b-a}\right)\frac{\sqrt{(b-s)(s-a)}}{b-a}. \tag{4.13}$$

4.2. Surface crack

It is rather difficult to achieve a good numerical approximation for the behaviour of the solution as described by the Eq. (3.15). We choose here a somewhat simplified way. As may be seen from the asymptotic representation of the solution near the point $s = 0$, $u'(s)$ is finite and even $u'(s) = 0$ at the point $s = 0$; thus it could be expanded into Fourier series by means of Tschebyscheff polynomials.

Now, after replacing $v(s)$ by the part of this expansion and then using interpolation formulae, we obtain the same set of equations (4.7) as for the inner crack. Instead of the Eq. (4.8) there must remain the following equation, obtained from the Eq. (4.11) and the condition that $u'(s)$ tends to zero as $s \to 0$:

$$\frac{1}{n}\sum_{m=0}^{n} v(s_m)\left(1+(-1)^{n+m}(1+x_m)^{-1}T_{n-1}(x_m)\right) = 0. \tag{4.14}$$

With this equation we obtain, of course, $K(0) = 0$. Physically speaking this condition means that there exist no bonds between the crack sides at the point $s = 0$ — i.e., on the free surface.

Note that to calculate the displacement jump the Eq. (4.12) must be used — not the Eq. (4.13).

The numerical solution of the surface crack problem so obtained should be convergent with the exact one with $n \to \infty$. In the next section, we shall consider numerical solutions for some particular examples which would suggest that sufficient accuracy may be reached with not very large n.

5. Computational results

As an example of the mathematical method presented above, we computed numerically the displacement jump dependence and stress-intensity factors for a number of surface

LINE	1	2	3	4
n	10	20	30	40
K(b)	1,44	1,47	1,49	1.50

FIG. 3.

and internal cracks with different lengths, in a field of given constant shear stress acting along a crack line. Our main object was to study the convergence of the numerical solution to the exact one with $n \to \infty$. We have restricted ourselves here to the mathematical problem mentioned above, leaving for future research such physical questions as the dependence of the displacement jump and stress-intensity factors on the length and position of the crack, stress field and material constants.

Comparison of the numerical results obtained for $n = 10$, 20, 30 and 40 with all the other parameters — such as stress field, length and position of the crack and material constants being fixed (Fig. 3) — clearly shows their convergence. It is obvious that the higher is n, the better the coincidence between the results obtained and the exact solution with $n \to \infty$. But let us recall here that in increasing the number of equations n we increase above all the number of points near the crack ends (see our definition of x_m, 4.3). It can be seen from our results that $n = 40$ gives sufficient accuracy of the solution and there is no use in further increasing this value.

6. Conclusions

The solution presented above for an arbitrarily situated shearing crack in a semi-space under plane strain conditions opens up a prospect, for future applications of the results in different branches of science. It would be particularly valuable in the physics of the earth's interior, and, especially, in earthquakes mechanics. In this last application, however, one should probably consider a crack with some kind of friction between its surfaces, so that it is necessary to take in mind that in such a case the kernel $K(s, s_1)$ described by the Eq. (3.2) must be replaced by the Eq. (3.6). In this case, the essential properties of the integral equations will be the same as for the case without friction, which we have solved above.

Appendix

Investigation of the asymptotic behaviour of the displacement jump derivative for a surface crack

Here, we shall study the asymptotic behaviour (for small s) of the solution for the case of surface crack when the integral equation has the form:

$$\text{(A.1)} \qquad \frac{1}{\pi}\int_0^b u'(s_1)\left[\frac{1}{s_1-s}+K(s,s_1)\right]ds_1 = f(s)\frac{\varkappa+1}{2\mu}.$$

Consider a number $0 < \varepsilon \ll b$ and rewrite the equation for $0 < s < \varepsilon$ as follows:

$$\text{(A.2)} \qquad \frac{1}{\pi}\int_0^\varepsilon u'(s_1)\left[\frac{1}{s_1-s}+K(s,s_1)\right]ds_1 = f(s)\frac{\varkappa+1}{2\mu} - \frac{1}{\pi}\int_\varepsilon^b u'(s_1)\left[\frac{1}{s_1-s}+K(s,s_1)\right]ds_1.$$

Assume that for $s < \varepsilon$ the function $f(s)$ may be expanded into a power series — i.e.,

$$f(s)\frac{\varkappa+1}{2\mu} = \sum_{n=0}^{\infty} f_n s^n, \tag{A.3}$$

and suppose that for the same values of s there exists an asymptotic expansion for the solution

$$u'(s_1) = \sum_{n=0}^{\infty} s_1^{n+\lambda_n} u_n(s_1), \tag{A.4}$$

where λ_n are real numbers such as

$$-1 < \lambda_n \leqslant 0 \tag{A.5}$$

and

$$u_n(s_1) = 0(1) \quad \text{as} \quad s_1 \to 0. \tag{A.6}$$

This expansion is consistent with the additional condition (4.8).

Now, for $s < \varepsilon < s_1$, we have the expansions

$$\frac{1}{s_1 - s} = \sum_{n=0}^{\infty} \frac{s^n}{s_1^{n+1}} \tag{A.7}$$

and

$$K(s, s_1) = \sum_{n=0}^{\infty} c_n \frac{s^n}{s_1^{n+1}}. \tag{A.8}$$

The last equation is easily obtained from the expression (3.11) for $K(s, s_1)$.

Introducing the expansions (A.3) to (A.8) into the Eq. (A.2), we have

$$\sum_{n=0}^{\infty} \frac{1}{\pi} \int_0^{\varepsilon} s_1^{n+\lambda_n} u_n(s_1) \left[\frac{1}{s_1 - s} + K(s, s_1)\right] ds_1 = \sum_{n=0}^{\infty} \left\{-\frac{1}{\pi}(1+c_n) \int_{\varepsilon}^{b} u'(s_1) \frac{ds_1}{s_1^{n+1}} + f_n\right\} s^n. \tag{A.9}$$

Now observe that $K(s, s_1)$ has the form:

$$K(s, s_1) = \frac{Q_0(s, s_1)}{P(s, s_1)}, \tag{A.10}$$

where $Q_0(s, s_1)$ and $P(s, s_1)$ are the homogeneous polynomials of orders five and six, respectively:

$$\begin{aligned} Q_0(s, s_1) &= \sum_{k=0}^{5} a_k^{(0)} s_1^{5-k} s^k, \\ P(s, s_1) &= \sum_{k=0}^{6} b_k s_1^{6-k} s^k. \end{aligned} \tag{A.11}$$

It is easy to prove the following identity:

$$s_1^n K(s, s_1) = s^n \frac{Q_n(s, s_1)}{P(s, s_1)} + \sum_{m=0}^{n-1} c_m s_1^{n-m-1} s^m, \tag{A.12}$$

where

$$Q_n(s, s_1) = \sum_{k=0}^{5} a_k^{(n)} s_1^{5-k} s^k \tag{A.13}$$

and

$$c_m = \frac{a_0^{(m)}}{b_0}, \tag{A.14}$$

$$a_k^{(m+1)} = a_{k+1} - c_m b_{k+1}, \quad k = 0, 1, \ldots, 4, \tag{A.15}$$

$$a_5^{(m+1)} = -c_m b_6. \tag{A.16}$$

Observing that for $s < s_1$

$$\lim_{n\to\infty} \frac{s^n}{s_1^n} \frac{Q_n(s, s_1)}{P(s, s_1)} = 0,$$

and comparing (A.12) with (A.8), we conclude that c_m as defined by (A.14) is the same as in the expansion (A.8).

Now, defining the kernel $K_n(s, s_1)$ by

$$K_n(s, s_1) = \frac{Q_n(s, s_1)}{P(s, s_1)}, \tag{A.17}$$

it is easy to see, that $K_n(s, s_1)$ has the same properties as $K(s, s_1)$, and that there exists for $s < s_1$ the following expansion:

$$K_n(s, s_1) = \sum_{m=0}^{\infty} c_{m+n} \frac{s^m}{s_1^{m+1}}. \tag{A.18}$$

Using (A.12), (A.17) and the obwious identity

$$\frac{s_1^n}{s_1 - s} = \sum_{m=0}^{n-1} s_1^{n-m-1} s^m + \frac{s^n}{s_1 - s}, \tag{A.19}$$

and changing the summation order, we obtain from (A.9):

$$\sum_{n=0}^{\infty} \frac{1}{\pi} \left\{ s^n \int_0^{\varepsilon} s_1^{\lambda_n} u_n(s_1) \left[\frac{1}{s_1 - s} + K_n(s, s_1) \right] ds_1 + s^n \sum_{m=n+1}^{\infty} (1 + c_n) \int_0^{\varepsilon} s_1^{m+\lambda_n - n - 1} u_m(s_1) ds_1 \right\} = \sum_{n=0}^{\infty} s^n \left\{ f_n - \frac{1}{\pi}(1 + c_n) \int^{b} u'(s_1) s_1^{-n-1} ds_1 \right\}. \tag{A.20}$$

Now, using the expansion (A.4) we can obtain the obvious identity

$$\text{(A.21)} \qquad \sum_{m=n+1}^{\infty} \int_0^{\varepsilon} s_1^{m+\lambda_n-n-1} u_m(s_1)\, ds_1 + \sum_{m=0}^{n} \text{f. p.} \int_0^{\varepsilon} s_1^{m+\lambda_n-n-1} u_m(s_1)\, ds_1 = \text{f. p.} \int_0^{\varepsilon} u'(s_1) s_1^{-n-1} ds_1,$$

where f.p. denotes the finite part of a diverging integral in the sense of HADAMARD [COURANT and HILBERT].

Substracting (A.21) multiplied by $(1+c_n)$ from each term of (A.20), we obtain:

$$\text{(A.22)} \qquad \sum_{n=0}^{\infty} \frac{1}{\pi} s^n \Bigg\{ \int_0^{\varepsilon} s_1^{\lambda_n} u_n(s_1) \left[\frac{1}{s_1 - s} + K_n(s, s_1) \right] ds_1$$

$$- \sum_{m=0}^{n} (1+c_n) \text{f. p.} \int_0^{\varepsilon} s_1^{m+\lambda_m-n-1} u_m(s_1)\, ds_1 \Bigg\} = \sum_{n=0}^{\infty} s^n \left\{ f_n - \frac{1}{\pi}(1+c_n) \text{f. p.} \int_0^{b} u'(s_1) s_1^{-n-1} ds_1 \right\}.$$

Let us change the summation order in the left side of the Eq. (A.22). Then,

$$\text{(A.23)} \qquad \sum_{n=0}^{\infty} \frac{1}{\pi} \Bigg\{ s^n \int_0^{\varepsilon} s_1^{\lambda_n} u_n(s_1) \left[\frac{1}{s_1 - s} + K_n(s, s_1) \right] ds_1$$

$$- \sum_{m=n}^{\infty} (1+c_m) s^m \text{f. p.} \int_0^{\varepsilon} s_1^{n+\lambda_n-m-1} u_n(s_1)\, ds_1 \Bigg\} = \sum_{n=0}^{\infty} s^n \Bigg\{ f_n$$

$$- \frac{1}{\pi}(1+c_n) \text{f. p.} \int_0^{b} u'(s_1) s_1^{-n-1} ds_1 \Bigg\}.$$

Introducing new variables by

$$s = \varepsilon\xi, \qquad s_1 = \varepsilon\eta,$$

we have

$$\text{(A.24)} \qquad \sum_{n=0}^{\infty} \frac{1}{\pi} \varepsilon^{n+\lambda_n} \Bigg\{ \xi^n \int_0^{1} \eta^{\lambda_n} u_n(\varepsilon\eta) \left[\frac{1}{\eta - \xi} + K_n(\xi, \eta) \right] d\eta$$

$$- \sum_{m=n}^{\infty} (1+c_m) \xi^m \text{f. p.} \int_0^{1} \eta^{n+\lambda_m-m-1} u_n(\varepsilon\eta)\, d\eta \Bigg\} = \sum_{m=0}^{\infty} \varepsilon^n \xi^n \Bigg\{ f_n$$

$$- \frac{1}{\pi}(1+c_n) \text{f. p.} \int_0^{b} u'(s_1) s_1^{-n-1} ds_1 \Bigg\}.$$

Since the powers of ε are linearly independent, we conclude that if $\lambda_n \neq 0$, then

$$\text{(A.24a)} \quad \frac{1}{\pi}\int_0^1 \eta^{\lambda_n} u_n(\varepsilon\eta)\left[\frac{1}{\eta-\xi}+K_n(\xi,\eta)\right]d\eta - \frac{1}{\pi}\sum_{m=0}^{\infty}(1+c_{m+n})\xi^m \text{f. p.}\int_0^1 \eta^{\lambda_n} u_n(\varepsilon\eta)\frac{d\eta}{\eta^{m+1}} = 0,$$

$$\text{(A.24b)} \quad \frac{1}{\pi}\text{f. p.}\int_0^b u'(s_1)s_1^{-n-1}ds_1 = f_n/(1+c_n).$$

In the case, if $\lambda_n = 0$, we obtain only one equation:

$$\frac{1}{\pi}\int_0^1 u_n(\varepsilon\eta)\left[\frac{1}{\eta-\xi}+K_n(\xi,\eta)\right]d\eta - \frac{1}{\pi}\sum_{m=0}^{\infty}(1+c_{m+n})\xi^m \text{f. p.}\int_0^1 u_n(\varepsilon\eta)\frac{d\eta}{\eta^{m+1}} = f_n - \frac{1}{\pi}(1+c_n)\text{f. p.}\int_0^b u'(s_1)s_1^{-n-1}ds_1.$$

However, because ε is an arbitrary constant, it may be proved that a solution of the last equation may be obtained only if the right-hand term is equal to zero. So for $\lambda_n = 0$, we return to the pair of equations (A.24a), (A.24b).

Next, we shall solve the Eq. (A.24a). At this point let us observe first that the equation is homogeneous, and consequently it permits the trivial solution $u_n \equiv 0$. It is understood that a nontrivial solution may exist only for certain discrete values of λ_n. Later, we shall construct an equation the roots of which will be equal to λ_n.

Let us demonstrate here that in the Eq. (A.24a) an arbitrary number $R > 1$.

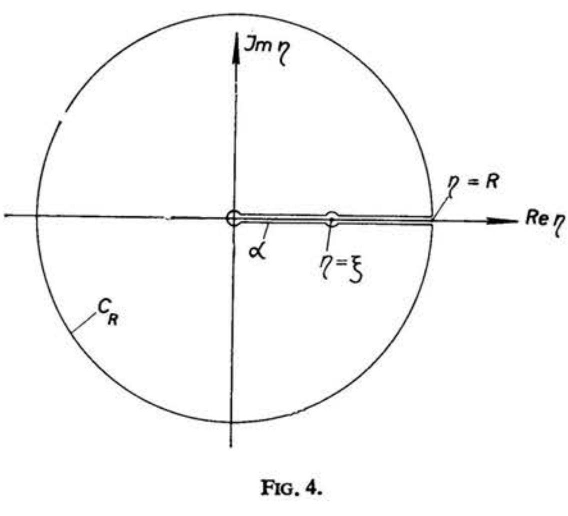

FIG. 4.

may be chosen for the upper limit of integration. In fact, we consider the integral

$$\frac{1}{\pi}\int_1^R \eta^{\lambda_n} u_n(\varepsilon\eta)\left[\frac{1}{\eta-\xi}+K(\xi,\eta)\right]d\eta$$

for $\xi < 1$. Here, the expression in brackets may be expanded into the power series in terms of ξ/η, which gives

$$\frac{1}{\pi}\int_1^R \eta^{\lambda_n} u_n(\varepsilon\eta)\left[\frac{1}{\eta-\xi}+K(\xi,\eta)\right]d\eta = \frac{1}{\pi}\sum_{m=0}^{\infty}(1+c_{m+n})\xi^m\int_1^R \eta^{\lambda_m} u_m(\varepsilon\eta)\frac{d\eta}{\eta^{m+1}}.$$

We can now easily obtain the conclusion desired.

With this fact we begin from defining a function $U_n(\eta)$ of the complex variable η, which is regular everywhere outside the cut along the positive real axis, and such that

$$U_n(\eta+iO)-U_n(\eta-iO) = \eta^{\lambda_n}U_n(\eta) \tag{A.25}$$

for all positive η. Then, the integrals in the appropriately prepared Eq. (A.24a) may be reduced to contour integrals along a loop (Fig. 4) around the part of the real axis from 0 to η, and the equation can then be rewritten in the form:

$$\frac{1}{\pi}\text{ V. P.}\int_{\mathscr{L}} U_n(\varepsilon\eta)\left[\frac{1}{\eta-\xi}+K_n(\xi,\eta)\right]-\frac{1}{\pi}\sum_{m=0}^{\infty}(1+c_{m+n})\xi^m \times\int_{\mathscr{L}} U_n(\varepsilon\eta)\frac{d\eta}{\eta^{m+1}} = 0. \tag{A.26}$$

Now, in each term of the series, the contour of the integration may be deformed into the circular path C_R of radius R. On C_R the modulus of η is equal to unity and for $\xi < R$ we can exchange the order of integration and summation, which in view of the expansions (A.18) and (A.7) gives in place of (A.26) the following expression:

$$\frac{1}{\pi}\text{ V. P.}\int_{\mathscr{L}_R} U_n(\varepsilon\eta)\left[\frac{1}{\eta-\xi}+K_n(\xi,\eta)\right]d\eta = 0, \tag{A.27}$$

where $\mathscr{L}_R$ is a closed contour consisting of the loop $\mathscr{L}$ together with the path C_R.

From (A.17), (A.12) and (A.19) we have the following identity:

$$\frac{1}{s_1-s}+K_n(s,s_1) = \frac{s_1^n}{s^n}\left(\frac{1}{s_1-s}+K(s,s_1)\right)-\sum_{m=1}^{n-1}(1+c_m)s_1^{m-1}s^{-m}. \tag{A.28}$$

Introducing this into the Eq. (A.27), and observing that the last sum is regular within $\mathscr{L}_R$, we obtain:

$$\frac{1}{\pi}\text{ V. P.}\int_{\mathscr{L}_R}(\varepsilon\eta)^n U_n(\varepsilon\eta)\left[\frac{1}{\eta-\xi}+K(\xi,\eta)\right]d\eta = 0. \tag{A.29}$$

Using the fact that the solution must be a homogeneous function, it is easy to prove that the solution of this equation may have only the power form:

$$(\varepsilon\eta)^n U_n(\varepsilon\eta) = C(\varepsilon\eta)^{\beta}, \tag{A.30}$$

where C is an arbitrary constant and β is some complex number to be determined later. From (A.27), (A.30) we observe that the real part of β is equal to $n+\lambda_n$.

Now, the kernel $K(\xi, \eta)$ as determined in the Eq. (3.2) may be represented as follows:

$$K(\xi, \eta) = \frac{1}{2}\left\{\frac{e^{2i\alpha}}{\eta+\xi e^{2i\alpha}} + \frac{e^{-2i\alpha}}{\eta+\xi e^{-2i\alpha}} + e^{3i\alpha}\frac{4\eta\cos\alpha(\xi-\eta)}{(\eta+\xi e^{2i\alpha})^3} + e^{-3i\alpha}\frac{4\eta\cos\alpha(\xi-\eta)}{(\eta+\xi e^{-2i\alpha})^3}\right\}. \tag{A.31}$$

It is seen that $K(\xi, \eta)$ is a regular function of η everywhere except the poles at $\eta = -\xi e^{2\alpha i}$ and $\eta = -\xi e^{-2\alpha i}$. Thus, introducing (A.24) and (A.31) into (A.29), the left side of this equation is reduced to the sum of residues at $\eta = \xi$, $\eta = -\xi e^{2\alpha i}$ and $\eta = -\xi e^{-2\alpha i}$. After some rearangement, we obtain:

$$iCe^{i\pi\beta}[2\cos\pi\beta-\beta(\beta+1)\cos 2\alpha\beta-2\beta(\beta+2)\cos 2\alpha(\beta+1) -(\beta+1)(\beta+2)\cos 2\alpha(\beta+2)](\varepsilon\xi)^{\beta} = 0. \tag{A.32}$$

Since the equation must be satisfied for any ξ, the expression in brackets ought to be zero — i.e.,

$$\frac{1}{\sin\pi\beta}[2\cos\pi\beta-\beta(\beta+1)\cos 2\alpha\beta-2\beta(\beta+2)\cos 2\alpha(\beta+1) -(\beta+1)(\beta+2)\cos 2\alpha(\beta+2)] = 0, \tag{A.33}$$

where the factor $(\sin\pi\beta)^{-1}$ is introduced to accent the fact that integer roots of the equation are unsatisfactory. This is the equation determining λ_n. Now, the asymptotic expansion may be rewritten in more convenient form:

$$u'(s) = \mathrm{Re}\,\Sigma u_n s^{\beta_n}, \quad \mathrm{Re}\,\beta_n > 1, \tag{A.34}$$

where u_n are arbitrary constants and β_n are the roots of the Eq. (A.33).

The Eqs. (A.33), (A.34) and (A.35) complete the result sought for.

For the sake of completeness we note here that the Eq. (A.24) with the Eq. (A.1), rewritten for the interval $<\varepsilon, b>$ only

$$\frac{1}{\pi}\int_{\varepsilon}^{b} u'(s_1)\left[\frac{1}{s_1-s}+K(s, s_1)\right]ds_1 = f(s)\frac{\varkappa+1}{2\mu} - \frac{1}{\pi}\sum \mathrm{Re}\left(u_n\int_0^{\varepsilon} s_1^{\beta_n}\left[\frac{1}{s_1-s}+K(s, s_1)\right]ds_1\right), \quad \mathrm{Re}\,\beta_n > -1, \quad \varepsilon < s < b, \tag{A.35}$$

may be used for determination of the constants u_n.

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