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Remarks on internal variable and history descriptions of material

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CONSIDERED are the deformation history and the internal parameter descriptions of material in dynamic processes in the interval of time $(-\infty, t)$. It has been shown the constitutive equation for simple material can be supplied with the information corresponding to the initial value of internal parameters. Conditions under which both the approaches give the same stress for the same history of deformation are formulated.

1. Introduction

IN NOLL's simple material [3], actual state of stress depends on total history of deformation. There is another approach (see for example [4, 5]) in which the state of stress is known when the actual value of deformation and of additional parameters as well are prescribed. These parameters, called internal variables, are introduced by solution of an initial value problem. The solution is obtained in terms of the deformation history and an initial value of the parameters. The main aim of this paper is to consider similarity between these two approaches coming from the common dependence on the deformation history.

2. Internal variable and history approaches

Let us consider a deformable body $\mathscr{B}$ referred to any arbitrary reference configuration $\varkappa$ with prescribed mass density $\varrho_\varkappa$. We assume $\chi: \varkappa(\mathscr{B}) \times R \to E_3$ to be the function of motion for $\mathscr{B}$.

A local dynamic process $(t_p, t_k) \subset R$ at a particle $X \in \mathscr{B}$ for the motion χ is the collection $\mathscr{P}_X$ of functions

$$\mathscr{P}_X = \{F(X, t), T(X, t)\}, \tag{2.1}$$

given for every time $t \in (t_p, t_k)$, which satisfies the first Cauchy law of motion

$$\operatorname{div} T + \varrho b = \varrho \ddot{\chi} \tag{2.2}$$

at all $t \in (t_p, t_k)$, where $F(X, t)$ is the deformation gradient, $T(X, t)$ is the Cauchy symmetric stress tensor, ϱ denotes the actual mass density and b — prescribed in advance body forces.

Formally the notion of the dynamic process as well as the other notions introduced in this paper come, as a particular case, from the thermodynamic description of materials formulated in the works [1, 4]. However, it has to be noticed that in the present formulation we do not make use of the internal dissipation inequality as the limitation imposed on the possible dynamic processes in the body $\mathscr{B}$. We rather postulate here it is whole the class of solutions of dynamic processes that has to be limited to those which satisfy the condition of non negative internal dissipation.

A mechanical state of the particle X at the time t is said to be the value of the function $\mathscr{P}_X$ at this time.

The state description consists of the value $F(X, t)$ and of the preparation method of this value.

What concerns the material structure of the body $\mathscr{B}$ we admit the following assumptions.

The method of preparation is a unique solution of the initial value problem

$$(2.3) \qquad \dot{\alpha}(X, t) = A_{\varkappa}\big(F(X, t), \alpha(X, t); X\big), \quad \alpha(X, t_0) = \alpha_0(X),$$

where the values $\alpha(X, t)$ lie in m — dimensional normed vector space V_m.

The principle of determinism is expressed by the constitutive equation

$$(2.4) \qquad T(X, t) = \mathscr{T}_{\varkappa}\big(F(X, t), \alpha(X, t); X\big).$$

The Eqs. (2.3) and (2.4) are valid for generally nonhomogeneous body $\mathscr{B}$.

The quantity $\alpha(X, t)$ represents the set of internal variables the choice of which depends on the physical phenomena accompanying the deformation of the body under consideration. The above assumptions limit the whole class of dynamic processes in the body $\mathscr{B}$ to admissible ones.

In order to be sure that there exists a unique solution of the problem (2.3), we complement our theory by the subsequent mathematical assumptions.

For each particle X, $A_{\varkappa}(\cdot, \cdot; X)$ is a continuous function of its variables (1).

There exists $K > 0$ such that for each two pairs (F, α_1) and (F, α_2) and for every particle $X \in \mathscr{B}$ the inequality

$$(2.5) \qquad ||A_{\varkappa}(F, \alpha_1; X) - A_{\varkappa}(F, \alpha_2; X)|| \leqslant K||\alpha_1 - \alpha_2||$$

is satisfied.

Now, let us notice that when $F(X, t)$ is a continuous function of t on the interval $[t_0, t_k)$, then the problem formulated by (2.3) leads to the solution of an ordinary differential equation. Thus making use of the well known theory of those equations we have the theorem.

THEOREM 1. *For each continuous function* $F(X, t)$ *on the interval* $[t_0, t_k)$ *and for each vector* $\alpha_0(X)$ *there exists the function* $\alpha(X, t)$ *on* $[t_0, t_k)$ *which is the solution of the problem* (2.3), i.e.:

1) *the function* $\alpha(X, t)$ *is continuously differentiable in* $[t_0, t_k)$,
2) *the pair* $\big(F(X, t), \alpha(X, t)\big)$ *satisfies Eq.* (2.3) *for each* $t \in (t_0, t_k)$,
3) *the function* $\alpha(X, t)$ *fulfils the initial condition* $\alpha(X, t_0) = \alpha_0(X)$.

Furthermore, for the fixed function $F(X, t)$ *on* $[t_0, t_k)$ *and for the fixed vector* $\alpha_0(X)$ *the function* $\alpha(X, t)$ *is unique on* $[t_0, t_k)$ (2).

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(1) The domain of the definition of the function $A_{\varkappa}(\cdot, \cdot; X)$ is assumed to be a cone in finite dimensional normed vector space.

(2) It may be proved that there exists a unique solution of (2.3) when $F(X, t)$ is even a regulated function of t.

On the basis of above theorem, we infer that there must exist the functional $\mathscr{F}_{\varkappa}$ such that

$$\alpha(X, t) = \mathop{\mathscr{F}_{\varkappa}}_{t_0}^{t}\big(F(X, \tau), \alpha_0(X); X\big), \qquad t \in (t_0, t_k), \tag{2.6}$$

with the property

$$\mathop{\mathscr{F}_{\varkappa}}_{t_0}^{t_0}\big(F(X, \tau), \alpha_0(X); X\big) = \alpha_0(X), \tag{2.7}$$

It has been shown by J. LUBLINER [2] that the value of $\alpha(X, t)$ does not depend on the value of the deformation gradient at the time t, but it does depend on its past history $F_r(X, \tau)$, $\tau \in (t_0, t)$. Thus, instead of the Eq. (2.6), we have

$$\alpha(X, t) = \mathop{\mathscr{F}_{\varkappa}}_{t_0}^{t}\big(F_r(X, \tau), \alpha_0(X); X\big). \tag{2.6'}$$

By substitution of the Eq. (2.7) into the Eq. (2.4), we arrive at the constitutive equation for stress in the form

$$T(X, t) = \mathscr{T}_{\varkappa}\big(F(X, t), \alpha(X, t); X\big) = \mathscr{T}_{\varkappa}\big(F(X, t), \mathop{\mathscr{F}_{\varkappa}}_{t_0}^{t}\big(F_r(X, \tau), \alpha(X, t_0); X\big). \tag{2.8}$$

Let us assume that $(t_p, t_k) = (-\infty, t_k)$ and denote

$$\lim_{t \to -\infty} \alpha(X, t) = \alpha_\infty(X). \tag{2.9}$$

Now, consider the dynamic process $(-\infty, t_k)$ with the deformation gradient history $F(X, \tau)$, $\tau \in (-\infty, t_k)$, for which it is reasonable to specify the initial value of the parameters α arbitrarily for in the past. Then the solution of the problem (2.3) but with the initial condition (2.9) has the form

$$\alpha(X, t) = \mathop{\mathscr{F}_{\varkappa}}_{-\infty}^{t}\big(F_r(X, \tau), \alpha_\infty(X); X\big) \tag{2.10}$$

with the property [cf. (2.7)]

$$\lim_{t \to -\infty} \mathop{\mathscr{F}_{\varkappa}}_{-\infty}^{t}\big(F_r(X, \tau), \alpha_\infty(X); X\big) = \alpha_\infty(X). \tag{2.11}$$

Substituting this solution into the Eq. (2.8), we obtain the constitutive equation

$$T(X, t) = \mathscr{T}_{\varkappa}\big(F(X, t), \mathop{\mathscr{F}_{\varkappa}}_{-\infty}^{t}\big(F_r(X, \tau), \alpha_\infty(X); X\big); X\big) \equiv \mathop{\mathscr{H}_{\varkappa}}_{-\infty}^{t}\big(F(X, \tau), \alpha_\infty(X); X\big). \tag{2.12}$$

Now, let us make a comparison between the Eq. (2.12) and the constitutive equation

$$T(X, t) = \mathop{\mathscr{G}'_{\varkappa}}_{-\infty}^{t}\big(F(X, \tau); X\big) \equiv \mathop{\mathscr{G}_{\varkappa}}_{-\infty}^{t}\big(F(X, t), F_r(X, \tau); X\big) \tag{2.13}$$

for simple material in the sense of W. NOLL [3]. In both the equations, the state of stress in the course of dynamic process $(-\infty, t)$ depends on the total history of the deformation gradient. In the material with internal variables approach [Eq. (2.12)], the state of stress depends additionally in explicite form on the values of the internal parameters at the time $t = -\infty$. This is obviously an additional information about the physical properties of the material which must be given in this procedure. It turns out that by means of particular assumptions about form of the constitutive functional in the relation (2.13), the constitutive equation for simple material can also be supplied with the information corresponding to α_∞.

To show this, let us restrict the constitutive equations (2.13) to a class of equations for which there exist a function $\mathcal{S}_\varkappa$ and a functional $\mathcal{K}_\varkappa$ such that

$$\overset{t}{\underset{-\infty}{\mathcal{G}_\varkappa}}\big(F(t), F_r(\tau); X\big) = \mathcal{S}_\varkappa\big(F(t), \overset{t}{\underset{-\infty}{\mathcal{K}_\varkappa}}(F_r(\tau); X); X\big) \tag{2.14}$$

and at the same time there exists the following limit

$$\lim_{t\to-\infty} \overset{t}{\underset{-\infty}{\mathcal{K}_\varkappa}}\big(F_r(\tau); X\big) = k_\infty(X), \tag{2.15}$$

for each history $F(\cdot)$ and $t \in (-\infty, t_k)$.

It results from the Eqs. (2.11)–(2.12) and (2.14)–(2.15) that the limit quantity k_∞ may play the same role in simple material approach as α_∞ does in the internal variable procedure.

Let us assume that the values of the functional $\mathcal{K}_\varkappa$ are in the same set as the internal parameters α. Now, if for each history $F(\cdot)$

$$\overset{t}{\underset{-\infty}{\mathcal{K}_\varkappa}}\big(F_r(\tau); X\big) = \overset{t}{\underset{-\infty}{\mathcal{F}_\varkappa}}\big(F_r(\tau); k_\infty(X); X\big), \qquad t \in (-\infty, t_k), \tag{2.16}$$

and the functions $\mathcal{T}_\varkappa$ and $\mathcal{S}_\varkappa$ are equal, i.e.

$$\mathcal{T}_\varkappa \equiv \mathcal{S}_\varkappa, \tag{2.17}$$

then we have

$$\mathcal{T}_\varkappa\Big(F(t), \overset{t}{\underset{-\infty}{\mathcal{F}_\varkappa}}(F_r(\tau),\ k_\infty(X); X);\ X\Big) = \mathcal{S}_\varkappa\big(F(t), \overset{t}{\underset{-\infty}{\mathcal{K}_\varkappa}}(F_r(\tau); X); X\big) \tag{2.18}$$

for each history $F(\cdot)$ and $t \in (-\infty, t_k)$.

The assumptions (2.14)–(2.16) lead to the conclusion that k_∞ and α_∞ are the same physical quantities. Hence by (2.18) we arrive at the theorem.

THEOREM 2. *In the course of any dynamic process* $(-\infty, t_k)$ *the stress in the simple material* (2.13) *under the restriction* (2.14)–(2.17) *is the same as in the material with the internal parameters* (2.4) *in which the parameters* α *are determined by the differential equation* (2.3) *but with the condition*

$$\lim_{t\to-\infty} \alpha(X, t) = k_\infty(X). \tag{2.19}$$

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