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Yield surfaces for thin shells accounting for transverse shear

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UPPER and lower bounds to the Ilyushin-Shapiro yield surface for thin shells are derived. The extension of several approximations of this yield surface to account for transverse shear is proposed.

Określono kres górny i dolny dla powierzchni plastyczności Iliuszyna-Szapiry w teorii powłok cienkich. Zaproponowano takie uogólnienie szeregu przybliżeń tej powierzchni, które pozwoliłoby na uwzględnienie wpływu ścinania poprzecznego.

Определены верхний и нижний пределы для поверхности текучести Ильющина-Шапиро в теории тонких оболочек. Предложены такие обобщения ряда апроксиммаций этой поверхности, которые позволилибы учесть влияние поперечного сдвига.

Introduction

ILYUSHIN [1] and SAWCZUK and RYCHLEWSKI [2] have proposed parametric yield surfaces for plates and shells based on the von Mises yield criterion. SHAPIRO [3] has extended the Ilyushin yield surface to account for transverse shear, but also points out that this surface is too complicated for engineering estimates of collapse loads. A number of approximations to Ilyushin's yield surface have been proposed in the published literature. Most recently ROBINSON [4] has studied these approximations and has established bounds on the accuracy of these surfaces.

It is the purpose of the present paper to propose extensions to the Ilyushin-Shapiro yield surface approximations when transverse shear is included. We assume that the plates and shells are made of a rigid-plastic material which obeys the von Mises flow condition. It is also assumed that the plates and shells yield on the "average" and that velocities are linear through the thickness. Only the axisymmetric problem is considered, but the extension to the more general case is trivial. Our analysis follows in part the ideas of ROZENBLYUM [5].

Symbols

$\sigma_1, \sigma_2, \tau_{13}$	stresses,
T_1, T_2	membrane forces,
Q_{13}	transverse shear force,
M_1, M_2	bending moments,
h	thickness of shell wall,
z	coordinate direction from middle surfaces,
τ_0	yield stress in shear,
σ_0	yield stress in simple tension,

t_i, m_i, q	non-dimensional membrane forces, bending moments and transverse shear,
T_0	$= \sigma_0 h$,
M_0	$= \dfrac{\sigma_0 h^2}{4}$,
Q_0	$= \tau_0 h$,
$P_q^2, P_t^2, P_m^2, P_{tm}$	quadratic terms of the yield surface,
e_1, e_2, e_3, e_{13} k_1, k_2	velocities,
$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{13}$ $\xi_1, \xi_2, \xi_3, \eta_{13}$	velocity fields,
P_i	collapse loads,
P_0	collapse load based on exact Ilyushin yield surface.

1. Lower bound

Consider the statically admissible stress field for a symmetrically loaded shell element

$$\sigma_1 = \frac{T_1}{h} + \frac{4M_1}{h^2}\operatorname{sign} z, \quad \sigma_2 = \frac{T_2}{h} + \frac{4M_2}{h^2}\operatorname{sign} z, \quad \tau_{13} = \frac{Q_{13}}{h}. \tag{1.1}$$

Equation (1.1) contain the assumption of "average" yielding through the thickness. The von Mises flow condition is

$$\sigma_2^2 + \sigma_1^2 - \sigma_1\sigma_2 + 3\tau_{13}^2 = 3\tau_0^2, \quad \sqrt{3}\,\tau_0 = \sigma_0. \tag{1.2}$$

Let us use the notation

$$\begin{aligned} t_i &= \frac{T_i}{T_0}, \quad i = 1, 2, \qquad T_0 = \sigma_0 h, \\ m_i &= \frac{M_i}{M_0}, \qquad\qquad\quad M_0 = \frac{1}{4}\sigma_0 h^2, \\ q &= \frac{Q_{13}}{Q_0} = \frac{\sqrt{3}\,Q_{13}}{T_0}, \quad Q_0 = \tau_0 h. \end{aligned} \tag{1.3}$$

Using the notation (1.3), we substitute (1.1) into (1.2) and obtain

$$P_q^2 + P_t^2 + P_m^2 + 2|P_{tm}| = 1, \tag{1.4}$$

where

$$\begin{aligned} P_t^2 &= t_1^2 - t_1 t_2 + t_2^2, \\ P_m^2 &= m_1^2 - m_1 m_2 + m_2^2, \quad P_q^2 = q^2, \\ 2P_{tm} &= 2t_1 m_1 + 2t_2 m_2 - t_1 m_2 - t_2 m_1. \end{aligned}$$

Since $P_t P_m \geqq |P_{tm}|$, it follows that a lower bound is given by

$$P_q^2 + P_t^2 + P_m^2 + 2P_t P_m = 1. \tag{1.5}$$

Equation (1.5), without consideration of transverse shear, has been first derived by ROZENBLYUM [5]. The yield surface described by (1.5) is shown in Fig. 1.

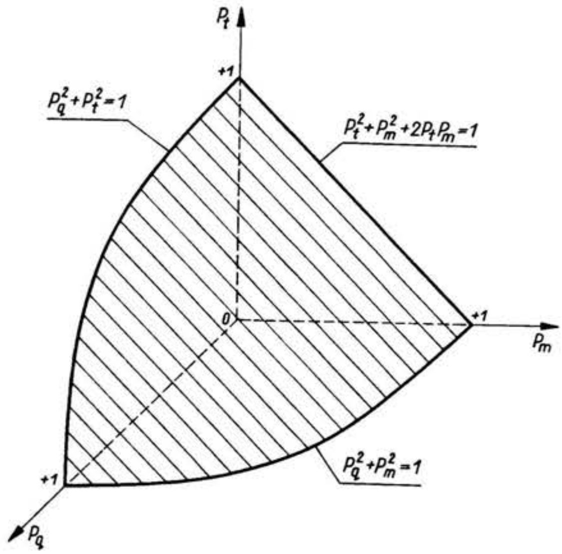

FIG. 1. Lower bound for yield surface.

2. Upper bound

Assume a kinematically admissible velocity field of the form

$$\begin{aligned} \varepsilon_1 &= e_1+zk_1, & \varepsilon_3 &= -(\varepsilon_1+\varepsilon_2), \\ \varepsilon_2 &= e_2+zk_2, & \varepsilon_{13} &= e_{13}. \end{aligned} \tag{2.1}$$

Equations (2.1) contain the assumption of linear velocity distribution through the shell thickness.

Using the minimum properties of the functional [6],

$$J = \tau_0 \int_v H dv - \int_s (X_n u + Y_n v + Z_n w) ds, \tag{2.2}$$

where

$$H = \sqrt{\frac{2}{3}\left[(\xi_1-\xi_2)^2+(\xi_2-\xi_3)^2+(\xi_3-\xi_1)^2+\frac{3}{2}\eta_{13}^2\right]}^{1/2},$$

and considering the velocity field (2.1), i.e.

$$\xi_1 = \varepsilon_1, \quad \xi_2 = \varepsilon_2, \quad \xi_3 = \varepsilon_3, \quad \eta_{13} = \varepsilon_{13}$$

the conditions

$$\frac{\partial J}{\partial e_i} = 0, \quad \frac{\partial J}{\partial k_i} = 0, \quad \frac{\partial J}{\partial e_{13}} = 0$$

lead to Shapiro's [3] equations

$$\begin{aligned} T_1 &= \frac{\sigma_0}{\sqrt{3}}[(2e_1+e_2)I_1+(2k_1+k_2)I_2], \\ T_2 &= \frac{\sigma_0}{\sqrt{3}}[(e_1+2e_2)I_1+(k_1+2k_2)I_2], \end{aligned} \tag{2.3}$$

$$Q_{13} = \frac{\sigma_0 e_{13}}{2\sqrt{3}} I_1,$$

(2.3) [cont.]

$$M_1 = \frac{\sigma_0}{\sqrt{3}} [(2e_1+e_2)I_2+(2k_1+k_2)I_3],$$

$$M_2 = \frac{\sigma_0}{\sqrt{3}} [(e_1+2e_2)I_2+(k_1+2k_2)I_3].$$

Here we have

$$I_1 = \int_{-a_1}^{a_1} \frac{dz}{\sqrt{P}}, \quad I_2 = \int_{-a_1}^{a_1} \frac{z\,dz}{\sqrt{P}}, \quad I_3 = \int_{-a_1}^{a_1} \frac{z^2 dz}{\sqrt{P}}, \quad \left(a_1 = \frac{h}{2}\right),$$

$$P = P_e+2zP_{ek}+z^2P_k,$$

$$P_e = e_1^2+e_2^2+e_1e_2+\frac{1}{4}e_{13}^2,$$

$$P_{ek} = e_1k_1+e_2k_2+\frac{1}{2}e_1k_2+\frac{1}{2}e_2k_1,$$

$$P_k = k_1^2+k_2^2+k_1k_2.$$

Equations (2.3) represent the yield surface in parametric form, i.e. $F(N_i, M_i, Q_{13}) = 0$. Without consideration of transverse shear they were first presented by ILYUSHIN [1]. We shall refer to (2.3) as the Ilyushin-Shapiro yield surface.

2.1. Membrane action

For this case of constant deformation the velocity field is

$$\xi_1 = e_1, \quad \xi_2 = e_2, \quad \xi_3 = -(e_1+e_2).$$

Equations (2.3) reduce to

$$T_1 = \frac{\sigma_0}{\sqrt{3}}(2e_1+e_2)I_1, \quad T_2 = \frac{\sigma_0}{\sqrt{3}}(e_1+2e_2)I_2, \tag{2.4}$$

where

$$I_1 = \int_{-a_1}^{a_1} (e_1^2+e_2^2+e_1e_2)^{-\frac{1}{2}} dz.$$

Eliminating e_1 and e_2 from (2.4) leads to the relation

$$t_1^2-t_1t_2+t_2^2 = 1$$

or

$$P_t^2 = 1. \tag{2.5}$$

Equation (2.5) is shown in Fig. 2.

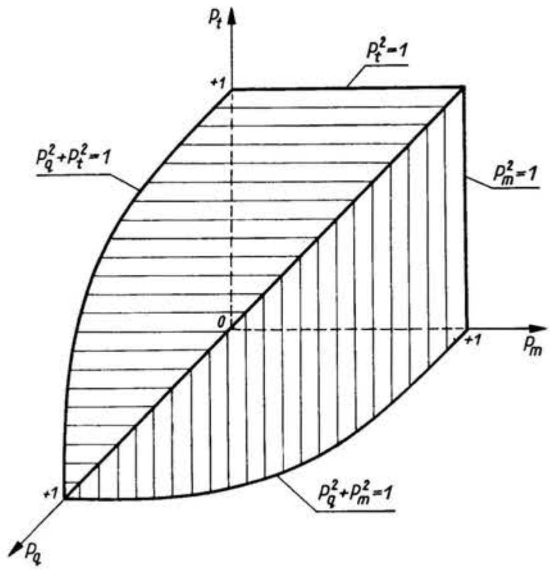

FIG. 2. Upper bound for yield surface.

2.2. Moment action

The velocity field is given as

$$\xi_1 = zk_1, \quad \xi_2 = zk_2, \quad \xi_3 = -(\xi_1+\xi_2).$$

Using the same procedure as before, we obtain the relation

$$m_1^2-m_1m_2+m_2^2 = 1$$

or

$$P_m^2 = 1. \tag{2.6}$$

Equation (2.6) is shown in Fig. 2. Note that Eqs. (2.5) and (2.6) have been first proposed by ROZENBLYUM [5] and are given here for the sake of completeness only.

2.3. Membrane-shear interaction

The velocity field is

$$\xi_1 = e_1, \quad \xi_2 = e_2, \quad \xi_3 = -(\xi_1+\xi_2), \quad \eta_{13} = e_{13}.$$

From Eq. (2.3)

$$T_1 = \frac{\sigma_0}{\sqrt{3}}(2e_1+e_2)I_1, \quad T_2 = \frac{\sigma_0}{\sqrt{3}}(e_1+2e_2)I_1, \quad Q_{13} = \frac{\sigma_0 e_{13}}{2\sqrt{3}}I_1, \tag{2.7}$$

where

$$I_1 = \int\limits_{-a_1}^{a_1}\left(e_1^2+e_2^2+e_2e_1+\frac{1}{4}e_{13}^2\right)^{-\frac{1}{2}}dz.$$

The ratios $a = e_1/e_{13}$ and $b = e_2/e_{13}$ can be eliminated from Eq. (2.7).

These ratios are

$$a = \frac{2t_1 - t_2}{2\sqrt{3}\,q}, \quad b = \frac{2t_2 - t_1}{2\sqrt{3}\,q},$$

and the resulting relation satisfying Eq. (2.7) is

$$t_1^2 - t_1 t_2 + t_2^2 + q^2 = 1$$

or

$$P_t^2 + P_q^2 = 1. \tag{2.8}$$

Equation (2.8) coincides with the lower bound, thus we have obtained the exact solution for this intersection.

2.4. Moment-shear interaction

An admissible velocity field is

$$\xi_1 = zk_1, \quad \xi_2 = zk_2, \quad \xi_3 = -(\xi_1 + \xi_2), \quad \eta_{13} = e_{13}.$$

Equations (2.3) become

$$M_1 = \frac{\sigma_0}{\sqrt{3}}(2k_1 + k_2) I_3, \quad M_2 = \frac{\sigma_0}{\sqrt{3}}(k_1 + 2k_2) I_3, \quad Q_{13} = \frac{\sigma_0 e_{13}}{2\sqrt{3}} I_1, \tag{2.9}$$

where

$$I_1 = \int_{-a_1}^{a_1} \left[\frac{1}{4} e_{13}^2 + z^2(k_1^2 + k_2^2 + k_1 k_2)\right]^{-\frac{1}{2}} dz,$$

$$I_3 = \int_{-a_1}^{a_1} \left[\frac{1}{4} e_{13}^2 + z^2(k_1^2 + k_2^2 + k_1 k_2)\right]^{-\frac{1}{2}} z^2\, dz.$$

Equations (2.9) lead to

$$a = b\,\frac{m_2 - 2m_1}{m_1 - 2m_2}, \tag{2.10}$$

where

$$a = \frac{k_1 h}{e_{13}} \text{ and } b = \frac{k_2 h}{e_{13}}. \tag{2.11}$$

With the aid of (2.10) and (2.11) we can reduce (2.9) to the quadratic equation

$$Ab^2 - Bb + C = 0. \tag{2.12}$$

In the process of obtaining (2.12) we used the square of the resulting equation, thus only the value of b of which its single value is applicable. The "non-uniqueness" of b obtained in (2.12) is therefore a consequence of the structure of the equations.

In (2.12), we have used the notation

$$A = 3(m_1^2 - m_1 m_2 + m_2^2)(m_1^2 - m_1 m_2 + m_2^2 - 1)(m_1 - 2m_2)^{-2},$$
$$B = 2\sqrt{3}q(m_1^2 - m_1 m_2 + m_2^2)(m_1 - 2m_2)^{-1},$$
$$C = q^2 - 1.$$

The solution of (2.12) can be written as

$$b = \frac{1}{2A}[-B \pm (B^2-4CA)^{1/2}]. \tag{2.13}$$

Obviously, b is single-valued if

$$B^2 = 4CA. \tag{2.14}$$

If we now substitute A, B, and C into (2.14), we obtain the relation

$$m_1^2 - m_1 m_2 + m_2^2 + q^2 = 1$$

or

$$P_q^2 + P_m^2 = 1. \tag{2.15}$$

Again (2.15) coincides with the lower bound and is therefore the exact solution. SHAPIRO [3] has stated that the lower bound and upper bound approaches for the exact solution coincide, which has been shown for the $P_q - P_m$ and $P_q - P_t$ intersections in the present paper. Equation (2.15) is shown in Fig. 2.

3. Comparison of yield surfaces

ROBINSON [4] has presented an excellent comparison of yield surfaces for thin shells. We shall examine these surfaces and propose their extension when transverse shear is included.

3.1. Rozenblyum-Shapiro-Schroeder yield surface

ROZENBLYUM [5] has suggested that a reasonable approximation to the yield surface (2.3) would be

$$P_t^2 + P_m^2 = 1. \tag{3.1}$$

SHAPIRO [3] has stated that inclusion of transverse shear would lead to

$$P_q^2 + P_t^2 + P_m^2 = 1. \tag{3.2}$$

More recently SCHROEDER [7] has derived (3.2) in a somewhat more consistant manner.

We have shown in the preceding section that the yield surface (3.2) reduces to the exact surface if either $P_t^2 = 0$ or $P_m^2 = 0$. When $P_q^2 = 0$, the Eq. (3.2) is an approximate solution to (2.3), the accuracy of which has been examined by ROBINSON [4].

3.2. Mroz-Bing-Ye yield surface

The yield surface recommended by MRÓZ and BING-YE [8] is

$$P_t^2 + P_m = 1. \tag{3.3}$$

With the inclusion of transverse shear (3.3) would become

$$P_q^2 + P_t^2 + P_m = 1. \tag{3.4}$$

The intersection in the $P_q - P_m$ plane is exact, according to (2.8). For the intersection in the $P_q - P_m$ plane to be exact, we require that b in (2.12) be single valued. If $P_q^2 + P_m = 1$ were a unique solution of (2.9), then for b to be single-valued, we must satisfy the condition $P_q^2 = 1$ or $P_q^2 + P_m^2 = 1$, which is contradictory to (3.4). Therefore, we conclude that for bending-shear interaction this yield surface is not exact, except for the pure shear case $P_q^2 = 1$ and shear-membrane interaction. The use of (3.4) results in a lower bound on yield surface (3.2) when shear-bending interaction is considered.

3.3. Ilyushin and Rozenblyum yield surface

The approximation to the exact yield surface (2.3) of ILYUSHIN [1] is written with the transverse shear term included as

$$P_q^2 + P_t^2 + P_m^2 + \frac{1}{\sqrt{3}} |P_{tm}| = 1. \tag{3.5}$$

Similarly, the extension of ROZENBLYUM'S [5] lower bound, which has been derived in the first section of the present paper, is

$$P_q^2 + P_t^2 + P_m^2 + 2|P_{tm}| = 1. \tag{3.6}$$

Both these surfaces are exact, if either $P_m^2 = 0$ or $P_t^2 = 0$, as has been shown in the second section of the present paper. We conclude therefore that any surface of the form

$$P_q^2 + P_t^2 + P_m^2 + K|P_{tm}| = 1$$

is exact, if shear-bending or shear-membrane interaction is considered.

3.4. Ivanov yield surface

IVANOV [9] has given some higher order approximations for the yield surface (2.3). If transverse shear is concluded, they are

$$P_q^2 + P_t^2 + \frac{1}{2} P_m^2 + \left(\frac{1}{4} P_m^4 + P_{tm}^2\right)^{1/2} = 1 \tag{3.7}$$

and

$$P_q^2 + P_t^2 + \frac{1}{2} P_m^2 - \frac{1}{4} (P_t^2 P_m^2 - P_{tm}^2)(P_t^2 + 0.48 P_m^2)^{-1} + \left(\frac{1}{4} P_m^4 + P_{tm}^2\right)^{1/2} = 1. \tag{3.8}$$

According to the second section of the present paper, these yield surfaces are exact, if either $P_m^2 = 0$ or $P_t^2 = 0$. ROBINSON [4] has also given error estimates on the collapse load for the yield surfaces discussed above, when shear is not included. In Table 1, we have given a summary of the yield surfaces examined by ROBINSON and we show the modified surfaces when transverse shear is included. In Fig. 3, we show schematically all yield surfaces of Table 1.

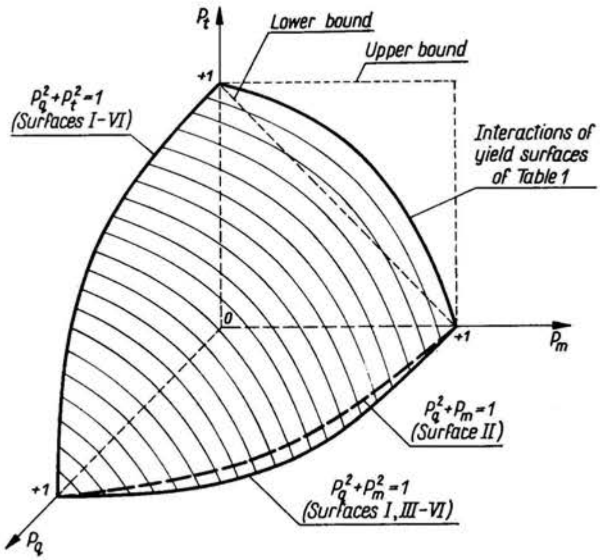

FIG. 3. Yield surfaces of Table 1.

4. Summary and discussion

We have extended the approximate yield surfaces examined by ROBINSON [4] to account for transverse shear. In Table 1 is shown a summary of Robinson's results and our proposed extensions when transverse shear is considered. It may be of interest to point out that some of these yield surfaces have been used to find collapse loads when transverse shear is included in the analysis. A summary of such solutions is given in Ref. [10].

ROBINSON [4] has made a study of the accuracy of the yield surfaces of Table 1 (without transverse shear) and has suggested the error bounds given in this table. Although these error bounds apply only to the P_t-P_m intersection (Fig. 3), we should like to suggest that the total range of these bounds, when transverse shear is included, will not be exceeded for each particular surface. This conclusion may not apply to surface II. For this surface, the inclusion of transverse shear, as proposed herein, may result in more conservative collapse loads as shown in Table 1. Therefore we conclude that the use of the modified yield surfaces of Table 1 will give collapse loads which are within the bounds proposed by ROBINSON [4].

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Table 1

	Yield surfaces with transverse shear included	Yield surface without transverse shear, Robinson [4]	Error bounds on collapse load, Robinson [4]	References
I	$P_q^2+P_t^2+P_m^2 = 1$	$P_t^2+P_m^2 = 1$	$0.955P_0 \leqslant P_1 \leqslant 1.155P_0$	ROZENBLYUM [5], SHAPIRO [3] SCHROEDER [7]
II	$P_q^2+P_t^2+P_m = 1$	$P_t^2+P_m = 1$	$0.833P_0 \leqslant P_2 \leqslant P_0$	MROZ-BING-YE [8]
III	$P_q^2+P_t^2+P_m^2+\frac{1}{\sqrt{3}}\lvert P_{tm}\rvert = 1$	$P_t^2+P_m^2+\frac{1}{\sqrt{3}}\lvert P_{tm}\rvert = 1$	$0.939P_0 \leqslant P_3 \leqslant 1.034P_0$	ILYUSHIN [1]
IV	$P_q^2+P_t^2+P_m^2+2\lvert P_{tm}\rvert = 1$	$P_t^2+P_m^2+2\lvert P_{tm}\rvert = 1$	$0.8P_0 \leqslant P_4 \leqslant P_0$	ROZENBLYUM [5]
V	$P_q^2+P_t^2+\frac{1}{2}P_m^2+\left[\frac{1}{4}P_m^4+P_{tm}^2\right]^{1/2} = 1$	$P_t^2+\frac{1}{2}P_m^2+\left[\frac{1}{4}P_m^4+P_{tm}^2\right]^{1/2} = 1$	$0.955P_0 \leqslant P_5 \leqslant P_0$	IVANOV [9]
VI	$P_q^2+P_t^2+\frac{1}{2}P_m^2-\frac{\frac{1}{4}[P_t^2P_m^2-P_{tm}^2]}{P_t^2+0.48P_m^2}+\left[\frac{1}{4}P_m^4+P_{tm}^2\right]^{1/2} = 1$	$P_t^2+\frac{1}{2}P_m^2-\frac{\frac{1}{4}(P_t^2P_m^2-P_{tm}^2)}{P_t^2+0.48P_m^2}+\left[\frac{1}{4}P_m^4+P_{tm}^2\right]^{1/2} = 1$	$0.999P_0 \leqslant P_6 \leqslant 1.005P_0$	IVANOV [9]

Note added in proof

Since the writing of this paper it has been pointed put to us (P. G. HODGE, Jr., personal communication) that Eq. (2.15) is a lower bound on Eq. (2.3) since the substitution into Eq. (2.9) does not lead to an identity. This fact may result in slight changes of the range for collapse loads as given in Table 1, when shear is included in the yield surface.

We feel however, that these changes are insignificant for engineering applications.

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