Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 4, pp. 621–641, Warszawa 1973

Thermal stresses in a semi-infinite body with a cylindrical hole

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IN THIS PAPER the rotationally symmetric problem of a semi-infinite body with a cylindrical hole, with prescribed heat input along the boundary of the hole and on the plane bounding surface, is investigated. The temperature distribution in such a body may easily be obtained and from it a particular solution for the equations of linear thermo-elasticity may be derived. This solution does not satisfy the boundary conditions and the problem is reduced to the solution of a residual problem in linear elasticity. The results are obtained by applying integral transform techniques to the displacement equations of equilibrium over the region $(0, \infty)\times\times(1, \infty)$.

W pracy rozważa się obrotowo-symetryczne zagadnienie dla półnieskończonego ciała z otworem kołowym przy danym przepływie ciepła wzdłuż otworu kołowego oraz na płaskiej powierzchni ograniczającej ośrodek. Rozkład temperatury w takim ciele, łatwy do wyznaczenia, pozwala uzyskać rozwiązania szczególne układu równań liniowej termosprężystości. Rozwiązanie to nie spełnia jednak warunków brzegowych i problem sprowadza się do rozwiązania uzupełniającego zagadnienia liniowej teorii sprężystości. Wyniki uzyskuje się drogą zastosowania transformacji całkowych do przemieszczeniowych równań równowagi w obszarze $(0, \infty)\times(1, \infty)$.

В работе обсуждается кругово-симметричная проблема для полубесконечного тела с круговым отверстием при заданном течении тепла вдоль кругового отверстия, а также на плоской поверхности ограничивающей среду. Распределение температуры в таком теле, которое легко определить, позволяет получить частные решения системы уравнений линейной термоупругости. Это решение не удовлетворяет однако граничным условиям и проблема сводится к решению дополнительной проблемы линейной теории упругости. Результаты получаются путем применения интегральных преобразований к уравнениям равновесия в перемещениях в области $(0, \infty)\times(1, \infty)$.

1. Introduction

IN A RECENT paper [1], one of the present authors investigated the problem of the three-dimensional stress concentration around a cylindrical hole in a semi-infinite elastic body, subjected to a uniform plane field of stress that is parallel to the bounding plane. He found the solution by decomposing it in the form of a plane strain solution, holding for the infinite body, and a solution of a residual problem that holds in the halfspace. The boundary conditions of the residual problem were so selected that the bounding surfaces became free from stress. Owing to the complicated geometry of the body — it is bounded by two surfaces of infinite extension and different type, upon which boundary conditions have to be prescribed — the residual problem appeared to be very difficult. However, as a consequence of the existence of suitable integral transforms — the FOURIER and the WEBER — it was possible to reduce the residual problem to the solution of one integral equation for an auxiliary function and so to put it within reach of numerical analysis.

Techniques similar to those used in [1] are developed by YOUNGDAHL and STERNBERG [2] in their treatment of the same problem. YOUNGDAHL and STERNBERG give an extensive motivation for their investigation together with a review of papers dealing with the three-dimensional aspects of the plane problem.

In [1] it is pointed out that a similar method might be used for the solution of a large class of corresponding boundary value problems in linear elasticity. This class is further extended by taking into consideration problems from the theory of thermo-elasticity for the same body. This extension is based upon the possibility of decomposing in linear thermo-elasticity a solution in the form of a particular temperature-dependent solution and a residual one. The residual problems are of the same type as that discussed in [1 and 2].

In this paper, we consider the rotationally symmetric problem of a semi-infinite body, with prescribed heat input along the boundary of the hole and on the plane bounding surface. We reduce the problem to the solution of a Fredholm integral equation of the second kind for an auxiliary function and solve this equation with numerical methods. All the quantities of interest may be expressed in the auxiliary function.

The results of this investigation may be of some use for the calculation of the stress distribution that exists in a long thick pipe, conducting a hot fluid. In particular the circumferential stress on the pipe near the plane bounding surface is of interest.

We note that in most practical problems — e. g., the heat transfer problem of a fluid in a thick pipe — the wall temperature and heat input are unknowns and have to be determined. Although it is beyond the scope of the present article to enter into the details of problems of this kind, we shall derive an integral equation for the wall temperature. The solution of it may give the boundary value of the heat input for the present problem.

The analysis to be presented is rather complicated, while in a later stage of the investigation numerical methods have to be used to obtain the final results. Therefore, one might wonder whether a direct numerical analysis would not be preferable. There are two reasons for answering this question in the negative. First, it may be expected that a direct numerical treatment of the equilibrium equations of elasticity will be much more laborious than the corresponding treatment of the Fredholm integral equation. In addition, the analytical treatment is more general and can be applied to a number of different boundary value problems.

2. Statement of the problem

In the Cartesian coordinate system (x_1, x_2, x_3) we consider the region of space characterized by

$$x_3 \geqslant 0, \quad r = \sqrt{x_1^2+x_2^2} \geqslant 1, \tag{2.1}$$

occupied by an elastic solid body. The body is deformed by the action of a stationary inhomogeneous temperature field. We shall confine ourselves to an isotropic homogeneous body, with respect to both the mechanical — i. e. the shear modulus G and Poisson's ratio ν — and thermal properties, the coefficient of internal heat conduction $\varkappa$ and linear expansion α. If there are no heat sources, the stationary temperature field is determined by the Laplace equation

$$T_{,kk} = 0, \tag{2.2}$$

where T represents the increment of temperature from the initial stress-less state in which $T = 0$. We assume that the change of temperature is small, and therefore it has no influence on the mechanical and thermal properties of the body.

According to the linear theory of thermo-elasticity, the displacements u_i satisfy the equations of equilibrium:

$$u_{i,kk}+\frac{1}{1-2\nu}u_{k,ki}-\frac{2(1-\nu)}{1-2\nu}\alpha T_{,i} = 0 \tag{2.3}$$

for the case of vanishing body forces, while the stresses σ_{ij} are given by

$$\sigma_{ij} = G\left[u_{i,j}+u_{j,i}+\left(\frac{2\nu}{1-2\nu}u_{k,k}-\frac{2(1+\nu)}{1-2\nu}\alpha T\right)\delta_{ij}\right], \tag{2.4}$$

where δ_{ij} is the Kronecker symbol.

The temperature field is completely determined if we prescribe the flow of heat $-\varkappa T_{,n}$, where $T_{,n} = \partial T/\partial n$ denotes the outward normal derivative of the temperature on a surface element, at all points of the bounding surfaces $r = 1$ and $x_3 = 0$, and take $T = 0$ at infinity.

The body is free from stress at infinity and at the boundaries. This may be expressed by

$$\sigma_{ij} = 0, \quad \text{for} \quad \sqrt{r^2+z^2} = \infty, \tag{2.5}$$

and

$$\sigma_{3i} = 0, \quad \text{for} \quad x_3 = 0, \tag{2.6}$$

$$\sigma_{ij}n_j = 0, \quad \text{for} \quad r = 1, \quad i,j = 1,2,3, \tag{2.7}$$

respectively.

Assuming T to be known, we find the solution S of the problem (2.3) to (2.7) by decomposing it in the following form:

$$S = \bar{S}+\bar{\bar{S}}, \tag{2.8}$$

where $\bar{S}$ is a particular solution, derived from a thermo-elastic displacement potential according to (cf. ref. [3])

$$\bar{u}_i = \chi_{,i}, \tag{2.9}$$

and $\bar{\bar{S}}$ is the solution of the residual problem that will be formulated later on.

Substituting (2.9) in (2.3), we obtain:

$$\chi_{,ikk}+\frac{1}{1-2\nu}\chi_{,kki}-\frac{2(1+\nu)}{1-2\nu}\alpha T_{,i} = 0, \tag{2.10}$$

and these equations can be integrated with respect to x_i. We find:

$$\chi_{,kk} = mT, \tag{2.11}$$

where

$$m = \frac{1+\nu}{1-\nu}\alpha. \tag{2.12}$$

From (2.4) and (2.11) we calculate the stress field, belonging to the solution $\bar{S}$, as

$$\bar{\sigma}_{ij} = 2G(\chi_{,ij} - \delta_{ij}\chi_{,kk}). \tag{2.13}$$

For the representation of $\bar{S}$ and $\bar{\bar{S}}$, cylindrical coordinates (r, φ, z) defined by the mapping

$$\begin{gathered} x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \quad x_3 = z, \\ 0 \leqslant r \leqslant \infty, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant z \leqslant \infty), \end{gathered} \tag{2.14}$$

are more convenient.

Since the problems under discussion are rotationally symmetric, (2.2) takes the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{2.15}$$

and (2.11) may be written as

$$\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial z^2} = mT. \tag{2.16}$$

The stresses derived from the thermo-elastic potential are given by

$$\begin{gathered} \bar{\sigma}_r = -2G\left(\chi_{,zz} + \frac{1}{r}\chi_{,r}\right), \quad \bar{\sigma}_\varphi = -2G(\chi_{,zz} + \chi_{,rr}), \\ \bar{\sigma}_z = -2G\left(\frac{1}{r}\chi_{,r} + \chi_{,rr}\right), \quad \bar{\tau}_{r\varphi} = \bar{\tau}_{z\varphi} = 0, \quad \bar{\tau}_{rz} = 2G\chi_{,rz}. \end{gathered} \tag{2.17}$$

The solution $\bar{S}$ does not satisfy the boundary conditions (2.5), (2.6) and (2.7). Therefore, we superpose a solution $\bar{\bar{S}}$ of (2.3), with $T = 0$, so that for S defined by (2.8) these boundary conditions are met.

3. The temperature field

A solution for (2.15) in the region (2.1) which satisfies the boundary conditions

$$-2\pi r\frac{\partial T}{\partial r} = \frac{q(z)}{\varkappa}, \quad \text{at} \quad r = 1, \tag{3.1}$$

$$\frac{\partial T}{\partial z} = 0, \quad \text{at} \quad z = 0, \quad \text{and} \quad T = 0, \quad \text{for} \quad \sqrt{r^2+z^2} = \infty, \tag{3.2}$$

is assumed in the form:

$$T(r, z) = \int_0^\infty A(\lambda)K_0(\lambda r)\cos\lambda z\, d\lambda, \tag{3.3}$$

where the heat input $q(z)$ is a given function that is continuous and integrable in $[0, \infty)$. In (3.3) $K_0(\lambda r)$ is the modified Bessel function of the second kind of order zero.

From (3.1) and (3.3) we find an expression for $q(z)$:

$$\frac{q(z)}{2\pi\varkappa} = \int_0^\infty \lambda K_1(\lambda)A(\lambda)\cos\lambda z\, d\lambda. \tag{3.4}$$

Inverting this result, we have:

$$\lambda K_1(\lambda)A(\lambda) = \frac{1}{\pi^2\varkappa}\int_0^\infty q(z)\cos\lambda z\,dz. \tag{3.5}$$

It follows from the representations (3.4) and (3.5) that $\lambda K_1(\lambda)A(\lambda)$ is ultimately decreasing, bounded and integrable in $[0, \infty)$. We derive from these properties that the integral (3.3) exists and represents indeed a bounded function $T(r, z)$, as

$$\int_0^\infty A(\lambda)K_0(\lambda r)d\lambda = \int_0^\infty \frac{K_0(\lambda r)}{\lambda K_1(\lambda)}[\lambda K_1(\lambda)A(\lambda)]d\lambda < \infty. \tag{3.6}$$

Note that we have suitably restricted the function $q(z)$ in order to obtain simple conditions for the existence of the integral representations. Of course, some of the restrictions may be weakened. Further we can formulate the problem for the case of prescribed temperature at $r = 1$ under appropriate conditions.

For future reference we give the inversion formula of (3.3):

$$A(\lambda) = \frac{2}{\pi}\frac{1}{K_0(\lambda)}\int_0^\infty T(1, z)\cos\lambda z\,dz. \tag{3.7}$$

4. The particular solution

The temperature field being known, we first derive a particular solution of (2.10). The function $\chi(r, z)$, represented by

$$\chi(r, z) = -\frac{m}{2}\int_0^\infty \lambda^{-2}A(\lambda)[\lambda rK_1(\lambda r)\cos\lambda z - \lambda K_1(\lambda)]d\lambda, \tag{4.1}$$

satisfies (2.16), and may be used for the derivation of the stresses of $\bar{S}$ by (2.17). We obtain the following expressions:

$$\frac{1}{2G}\bar{\sigma}_r = -\frac{m}{2}\int_0^\infty A(\lambda)\cos\lambda z\{K_0(\lambda r)+\lambda rK_1(\lambda r)\}d\lambda, \tag{4.2}$$

$$\frac{1}{2G}\bar{\sigma}_\varphi = -\frac{m}{2}\int_0^\infty A(\lambda)K_0(\lambda r)\cos\lambda z\,d\lambda, \tag{4.3}$$

$$\frac{1}{2G}\bar{\sigma}_z = -\frac{m}{2}\int_0^\infty A(\lambda)\cos\lambda z\{2K_0(\lambda r)-\lambda rK_1(\lambda r)\}d\lambda, \tag{4.4}$$

$$\frac{1}{2G}\bar{\tau}_{rz} = -\frac{m}{2}r\int_0^\infty \lambda A(\lambda)K_0(\lambda r)\sin\lambda z\,d\lambda. \tag{4.5}$$

We note that the integrals (4.1) to (4.5) converge for $z \geqslant 0$ and $r \geqslant 1$.

It is obvious that the decomposition (2.8) is not unique. But now we have chosen the function $\chi(r, z)$ by (4.1), the residual problem $\bar{\bar{S}}$ can be formulated and it has a unique solution. We proceed with the discussion of this problem.

5. Reduction of the residual problem

In cylindrical coordinates the displacements $\bar{\bar{u}}$ and $\bar{\bar{w}}$ are governed by the equations:

(5.1)
$$\Delta\bar{\bar{u}}-\frac{u}{r^2}+\frac{1}{1-2\nu}\frac{\partial}{\partial r}\left(\frac{\partial\bar{\bar{u}}}{\partial r}+\frac{\bar{\bar{u}}}{r}+\frac{\partial\bar{\bar{w}}}{\partial z}\right)=0,\quad \Delta\bar{\bar{w}}+\frac{1}{1-2\nu}\frac{\partial}{\partial z}\left(\frac{\partial\bar{\bar{u}}}{\partial r}+\frac{\bar{\bar{u}}}{r}+\frac{\partial\bar{\bar{w}}}{\partial z}\right)=0.$$

By means of Love's function L (cf. [3]), that satisfies

$$\Delta\Delta L = 0, \tag{5.2}$$

we can represent the solutions of (5.1) in the following form:

$$\bar{\bar{u}} = -\frac{\partial^2 L}{\partial r\,\partial z},\quad \bar{\bar{w}} = 2(1-\nu)\Delta L-\frac{\partial^2 L}{\partial z^2}. \tag{5.3}$$

If we take L in the form

$$L = P+z\frac{\partial Q}{\partial z}, \tag{5.4}$$

with

$$\Delta P = \Delta Q = 0, \tag{5.5}$$

the displacement equations (5.3) become:

$$\bar{\bar{u}} = -\frac{\partial}{\partial r}\left(\frac{\partial P}{\partial z}+\frac{\partial Q}{\partial z}+z\frac{\partial^2 Q}{\partial z^2}\right),\quad \bar{\bar{w}} = -\frac{\partial}{\partial z}\left[\frac{\partial P}{\partial z}-(3-4\nu)\frac{\partial Q}{\partial z}+z\frac{\partial^2 Q}{\partial z^2}\right]. \tag{5.6}$$

From (5.6) we derive by means of (2.4) the stresses of $\bar{\bar{S}}$:

$$\begin{aligned}
\frac{1}{2G}\bar{\bar{\sigma}}_r &= \left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\frac{\partial P}{\partial z}+\left[\frac{1}{r}\frac{\partial}{\partial r}+(1+2\nu)\frac{\partial^2}{\partial z^2}+\frac{z}{r}\frac{\partial^2}{\partial r\,\partial z}+z\frac{\partial^3}{\partial z^3}\right]\frac{\partial Q}{\partial z},\\
\frac{1}{2G}\bar{\bar{\sigma}}_\varphi &= -\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial P}{\partial z}-\left(\frac{1}{r}\frac{\partial}{\partial r}-2\nu\frac{\partial^2}{\partial z^2}+\frac{z}{r}\frac{\partial^2}{\partial r\,\partial z}\right)\frac{\partial Q}{\partial z},\\
\frac{1}{2G}\bar{\bar{\sigma}}_z &= -\frac{\partial^2}{\partial z^2}\frac{\partial P}{\partial z}+\left[(1-2\nu)\frac{\partial^2}{\partial z^2}-z\frac{\partial^3}{\partial z^3}\right]\frac{\partial Q}{\partial z},\\
\frac{1}{2G}\bar{\bar{\tau}}_{rz} &= -\frac{\partial^2}{\partial r\,\partial z}\frac{\partial P}{\partial z}-\left(2\nu\frac{\partial^2}{\partial r\,\partial z}+z\frac{\partial^3}{\partial r\,\partial z^2}\right)\frac{\partial Q}{\partial z},\\
\frac{1}{2G}\bar{\bar{\tau}}_{r\varphi} &= \frac{1}{2G}\bar{\bar{\tau}}_{z\varphi} = 0.
\end{aligned} \tag{5.7}$$

It appears that the displacements and the stresses are only dependent on the derivatives $\partial P/\partial z$ and $\partial Q/\partial z$. Therefore, we may confine ourselves to giving only the integral representations for these functions:

$$\frac{\partial P}{\partial z} = c\log r+(1-2\nu)\int\limits_0^\infty \lambda^{-2}B(\lambda)[K_0(\lambda r)\cos\lambda z-K_0(\lambda)+\lambda K_1(\lambda)\log r]d\lambda$$
$$+\int\limits_0^\infty \lambda^{-2}B(\lambda)[\lambda r K_1(\lambda^{r})\cos\lambda z-\lambda K_1(\lambda)]d\lambda+z\int\limits_0^\infty \lambda^{-1}B(\lambda)K_0(\lambda r)\sin\lambda z\,d\lambda$$
$$-\int\limits_0^\infty \lambda^{-2}C(\lambda)K_0(\lambda r)\cos\lambda z\,d\lambda+\nu\pi\int\limits_0^\infty \lambda^{-2}G(\lambda)\left[\Phi_0(r,\lambda)e^{-\lambda z}+\frac{2}{\pi}(1-\lambda z)\right]d\lambda, \tag{5.8}$$

and

$$\frac{\partial Q}{\partial z} = \int\limits_0^\infty \lambda^{-2}B(\lambda)[K_0(\lambda r)\cos\lambda z-K_0(\lambda)+\lambda K_1(\lambda)\log r]d\lambda$$
$$-\frac{\pi}{2}\int\limits_0^\infty \lambda^{-2}G(\lambda)\left[\Phi_0(r,\lambda)e^{-\lambda z}+\frac{2}{\pi}(1-\lambda z)\right]d\lambda. \tag{5.9}$$

In (5.8) and (5.9) have been introduced the unknown coefficient functions $B(\lambda)$, $C(\lambda)$ and $G(\lambda)$. Further, a new function $\Phi_0(r, \lambda)$ is used, defined by

$$\Phi_0(r,\lambda) = \lambda Y_1(\lambda)J_0(\lambda r)-\lambda J_1(\lambda)Y_0(\lambda r), \tag{5.10}$$

where J_n and Y_n are Bessel functions of the first and second kind, respectively.

For the function $\Phi_0(r, \lambda)$ the relation

$$\Phi_0(1,\lambda) = -\frac{2}{\pi}. \tag{5.11}$$

holds.

We also shall make use of the function $\Phi_1(r, \lambda)$, related to the derivative of $\Phi_0(r, \lambda)$ with respect to r

$$\Phi_1(r,\lambda) = -\frac{1}{\lambda}\frac{\partial}{\partial r}\Phi_0(r,\lambda) = \lambda Y_1(\lambda)J_1(\lambda r)-\lambda J_1(\lambda)Y_1(\lambda r), \tag{5.12}$$

that satisfies

$$\Phi_1(1,\lambda) = 0. \tag{5.13}$$

The coefficient functions $B(\lambda)$, $C(\lambda)$ and $G(\lambda)$ are assumed to behave in such a way that the integrals in (5.8) and (5.9) may be differentiated a sufficient number of times under the integral sign. Formulation for the conditions is postponed.

The representations of $\partial P/\partial z$ and $\partial Q/\partial z$ are broken up into separate integral forms in order to obtain simple expressions for the stresses at the boundaries. An elementary solution is added to (5.8) to meet the boundary condition (2.7).

We derive from (5.7) to (5.9) the following formulae for the stresses:

$$(5.14)\qquad \frac{1}{2G}\bar{\bar{\sigma}}_r = \frac{c}{r^2} - \frac{2(1-\nu)}{r^2}\int_0^\infty \lambda^{-2}B(\lambda)[\lambda r K_1(\lambda r)\cos\lambda z - \lambda K_1(\lambda)]d\lambda$$
$$-\int_0^\infty B(\lambda)[K_0(\lambda r)+\lambda r K_1(\lambda r)]\cos\lambda z\,d\lambda + \frac{1}{r^2}\int_0^\infty \lambda^{-2}C(\lambda)\lambda r K_1(\lambda r)\cos\lambda z\,d\lambda$$
$$+\int_0^\infty C(\lambda)K_0(\lambda r)\cos\lambda z\,d\lambda + \frac{\pi}{2r}\int_0^\infty \lambda^{-1}G(\lambda)[(1-2\nu)-\lambda z]\Phi_1(r,\lambda)e^{-\lambda z}d\lambda$$
$$-\frac{\pi}{2}\int_0^\infty G(\lambda)(1-\lambda z)\Phi_0(r,\lambda)e^{-\lambda z}d\lambda,$$

$$(5.15)\qquad \frac{1}{2G}\bar{\bar{\sigma}}_\varphi = -\frac{c}{r^2} + \frac{2(1-\nu)}{r^2}\int_0^\infty \lambda^{-2}B(\lambda)[\lambda r K_1(\lambda r)\cos\lambda z - \lambda K_1(\lambda)]d\lambda$$
$$+(1-2\nu)\int_0^\infty B(\lambda)K_0(\lambda r)\cos\lambda z\,d\lambda - \frac{1}{r}\int_0^\infty \lambda^{-1}C(\lambda)K_1(\lambda r)\cos\lambda z\,d\lambda$$
$$-\frac{\pi}{2r}\int_0^\infty \lambda^{-1}G(\lambda)[(1-2\nu)-\lambda z]\Phi_1(r,\lambda)e^{-\lambda z}d\lambda - \nu\pi\int_0^\infty G(\lambda)\Phi_0(r,\lambda)e^{-\lambda z}d\lambda,$$

$$(5.16)\qquad \frac{1}{2G}\bar{\bar{\sigma}}_z = \int_0^\infty B(\lambda)[\lambda r K_1(\lambda r) - 2K_0(\lambda r)]\cos\lambda z\,d\lambda$$
$$-\int_0^\infty C(\lambda)K_0(\lambda r)\cos\lambda z\,d\lambda - \frac{\pi}{2}\int_0^\infty G(\lambda)(1+\lambda z)\Phi_0(r,\lambda)e^{-\lambda z}d\lambda,$$

and

$$(5.17)\qquad \frac{1}{2G}\bar{\bar{\tau}}_{rz} = -\int_0^\infty B(\lambda)\lambda r K_0(\lambda r)\sin\lambda z\,d\lambda + \int_0^\infty C(\lambda)K_1(\lambda r)\sin\lambda z\,d\lambda$$
$$-\frac{\pi z}{2}\int_0^\infty \lambda G(\lambda)\Phi_1(r,\lambda)e^{-\lambda z}d\lambda.$$

6. Boundary conditions at $r = 1$

The boundary conditions (2.7) require the vanishing of the stresses σ_r and τ_{rz} at $r = 1$. We can derive from (4.2), (4.5), (5.14) and (5.17) formulas for these stresses. Before doing so, we introduce the function $g(\lambda)$:

$$(6.1)\qquad g(\lambda) = \frac{4}{\pi}\int_0^\infty \frac{\lambda^3}{(\lambda^2+\eta^2)^2}\eta G(\eta)d\eta.$$

With (6.1) we are able to transform the Laplace integral in (5.14) into a Fourier integral, using the relation:

$$(6.2)\qquad \int_0^\infty (1-\eta z)e^{-\eta z}G(\eta)d\eta = \int_0^\infty \lambda^{-1}g(\lambda)\cos\lambda z\, d\lambda.$$

The proof of (6.2) is provided by substituting (6.1) in (6.2) and interchanging the order of integration.

By means of (6.1), we arrive at the following expressions for the stresses at $r = 1$:

$$(6.3)\qquad \frac{1}{2G}\sigma_r(1, z) = \int_0^\infty \lambda^{-2}\cos\lambda z\Big\{-\frac{m}{2}[\lambda+K(\lambda)]\lambda^2 K_1(\lambda)A(\lambda)$$
$$-[2(1-\nu)+\lambda^2+\lambda K(\lambda)]\lambda K_1(\lambda)B(\lambda)+[1+\lambda K(\lambda)]\lambda K_1(\lambda)C(\lambda)$$
$$+\lambda g(\lambda)\Big\}d\lambda+2(1-\nu)\int_0^\infty \lambda^{-1}K_1(\lambda)B(\lambda)d\lambda+c,$$

and

(6.4)

$$\frac{1}{2G}\tau_{rz}(1, z) = \int_0^\infty \sin\lambda z\Big\{-\frac{m}{2}\lambda K(\lambda)K_1(\lambda)A(\lambda)-\lambda K(\lambda)K_1(\lambda)B(\lambda)+K_1(\lambda)C(\lambda)\Big\}d\lambda,$$

where the function $K(\lambda)$ is defined by

$$(6.5)\qquad K(\lambda) = \frac{K_0(\lambda)}{K_1(\lambda)}.$$

We can satisfy (2.7) for each value of z by putting equal to zero the coefficient of $\cos\lambda z$ in (6.3) and of $\sin\lambda z$ in (6.4) and by taking $-c$ equal to the last integral of (6.3). This leads to the following expressions for $B(\lambda)$, $C(\lambda)$ and c, formulated in terms of the known function $A(\lambda)$ and the unknown $g(\lambda)$:

$$(6.6)\qquad B(\lambda) = -\frac{m}{2}\lambda[1-K^2(\lambda)]\Lambda(\lambda)A(\lambda)+\frac{\Lambda(\lambda)}{\lambda K_1(\lambda)}g(\lambda),$$

$$(6.7)\qquad C(\lambda) = (1-\nu)mK(\lambda)\Lambda(\lambda)A(\lambda)+\frac{K(\lambda)\Lambda(\lambda)}{K_1(\lambda)}g(\lambda),$$

and

$$(6.8)\qquad c = -2(1-\nu)\int_0^\infty \lambda^{-1}B(\lambda)K_1(\lambda)d\lambda.$$

In (6.6) to (6.8) a new function $\Lambda(\lambda)$ is introduced, defined by

$$(6.9)\qquad \Lambda(\lambda) = \frac{\lambda}{2(1-\nu)+\lambda^2-\lambda^2K^2(\lambda)}.$$

This function is monotonic and lies between zero and one. In fact we have

$$\Lambda(\lambda) = \frac{\lambda}{2(1-\nu)} + 0(\lambda^2), \quad \lambda \to 0, \tag{6.10}$$

while

$$\Lambda(\lambda) \to 1, \quad \lambda \to \infty. \tag{6.11}$$

7. Boundary conditions at $z = 0$

From (4.4), (4.5), (5.16) and (5.17), we find for the total stresses σ_z and τ_{rz} at the boundary $z = 0$:

$$\frac{1}{2G}\sigma_z(r,0) = \frac{m}{2}\int_0^\infty [2K_0(\lambda r) - \lambda r K_1(\lambda r)]A(\lambda)\,d\lambda - \int_0^\infty [2K_0(\lambda r) - \lambda r K_1(\lambda r)]B(\lambda)\,d\lambda - \int_0^\infty C(\lambda)K_0(\lambda r)\,d\lambda - \frac{\pi}{2}\int_0^\infty G(\lambda)\phi_0(r,\lambda)\,d\lambda, \tag{7.1}$$

and

$$\frac{1}{2G}\tau_{rz}(r,0) = 0. \tag{7.2}$$

Eliminating $B(\lambda)$ and $C(\lambda)$ from (7.1) by means of (6.6) and (6.7), and substituting in the boundary condition (2.6), we obtain the integral equation:

$$-(1-\nu)m\int_0^\infty \frac{\Lambda(\lambda)}{\lambda}\{[2+\lambda K(\lambda)]K_0(\lambda r) - \lambda r K_1(\lambda r)\}A(\lambda)\,d\lambda - \int_0^\infty \frac{\Lambda(\lambda)}{\lambda K_1(\lambda)} \times \{[2+\lambda K(\lambda)]K_0(\lambda r) - \lambda r K_1(\lambda r)\}g(\lambda)\,d\lambda - \frac{\pi}{2}\int_0^\infty G(\lambda)\Phi_0(r,\lambda)\,d\lambda = 0. \tag{7.3}$$

As will be shown later on, the function $G(\lambda)$ behaves at infinity as

$$G(\lambda) \to c_1\lambda^{-1} + 0(\lambda^{-2}), \quad \lambda \to \infty, \tag{7.4}$$

while, as follows from (6.1)

$$g(\lambda) \to c_1 + 0(\lambda^{-1}), \quad \lambda \to \infty, \tag{7.5}$$

where c_1 is a constant.

As a consequence of (7.4) and (7.5) the integrals

$$\int_0^\infty G(\lambda)\Phi_0(r,\lambda)\,d\lambda$$

and

$$\int_0^\infty \frac{\Lambda(\lambda)}{\lambda K_1(\lambda)}\{[2+\lambda K(\lambda)]K_0(\lambda r) - \lambda r K_1(\lambda r)\}g(\lambda)\,d\lambda,$$

taken separately are divergent at $r = 1$. However, it appears that the integrals taken together converge, as can easily be seen.

For the solution of the Eq. (7.3), we use the following inversion theorem for the Weber integral (cf. [4], pp. 86–88): If

$$\int_0^\infty S(\lambda)\Phi_0(r, \lambda)d\lambda = \Sigma(r), \tag{7.6}$$

then,

$$\lambda[J_1^2(\lambda)+Y_1^2(\lambda)]S(\lambda) = \int_1^\infty r\Sigma(r)\Phi_0(r, \lambda)dr, \tag{7.7}$$

under the condition that

$$\int_1^\infty |\Sigma(r)|\sqrt{r}dr < \infty. \tag{7.8}$$

To be able to apply this theorem we first write (7.3) in another form. The integrals (cf. [5])

$$\int_1^\infty r\Phi_0(r, s)K_0(\lambda r)dr = -\frac{2}{\pi}\frac{\lambda K_1(\lambda)}{(s^2+\lambda^2)}, \tag{7.9}$$

and

$$\int_1^\infty r\Phi_0(r, s)\,\lambda r K_1(\lambda r)dr = -\frac{2}{\pi}\frac{\lambda^2 K_0(\lambda)}{(s^2+\lambda^2)} - \frac{4}{\pi}\frac{\lambda^3 K_1(\lambda)}{(s^2+\lambda^2)^2}, \tag{7.10}$$

are considered as Weber transforms and after inverting them we can derive the relation:

$$[2+\lambda K(\lambda)]K_0(\lambda r) - \lambda r K_1(\lambda r) = -\frac{4}{\pi}\int_0^\infty \frac{\eta}{[J_1^2(\eta)+Y_1^2(\eta)]}\frac{\lambda K_1(\lambda)}{(\lambda^2+\eta^2)^2}\Phi_0(r, \eta)d\eta. \tag{7.11}$$

We introduce (7.11) into (7.3) and obtain:

$$\begin{aligned}
-(1-\nu)m\int_0^\infty \frac{\Lambda(\lambda)}{\lambda}\{[2+\lambda K(\lambda)]K_0(\lambda r) - \lambda r K_1(\lambda r)\}A(\lambda)d\lambda \\
+\frac{4}{\pi}\int_0^\infty \Lambda(\lambda)g(\lambda)\left\{\int_0^\infty \frac{\eta}{[J_1^2(\eta)+Y_1^2(\eta)]}\frac{1}{(\lambda^2+\eta^2)^2}\Phi_0(r, \eta)d\eta\right\}d\lambda \\
-\frac{\pi}{2}\int_0^\infty G(\lambda)\Phi_0(r, \lambda)d\lambda = 0.
\end{aligned} \tag{7.12}$$

The order of integration in the second term of (7.12) may be interchanged for $r > 1$, whereby (7.12) takes the form:

$$(7.13)\qquad -(1-\nu)m\int\limits_0^\infty \frac{\Lambda(\lambda)}{\lambda}\{[2+\lambda K(\lambda)]K_0(\lambda r)-\lambda r K_1(\lambda r)\}A(\lambda)d\lambda$$

$$+\int\limits_0^\infty \Phi_0(r,\lambda)\left\{-\frac{\pi}{2}G(\lambda)+\frac{4}{\pi}\frac{\lambda}{J_1^2(\lambda)+Y_1^2(\lambda)}\int\limits_0^\infty \frac{\Lambda(\eta)g(\eta)}{(\lambda^2+\eta^2)^2}d\eta\right\}d\lambda = 0.$$

We note that in the limit for $r\to 1$ the integrals in (7.13) converge, so that we can consider (7.13) as an integral equation in the whole interval. To proceed, we multiply this equation by $r\Phi_0(r, s)$ and integrate over r from 1 to infinity. We again apply Weber's inversion theorem and find:

$$(7.14)\qquad \frac{\pi^2 s^2}{8}[J_1^2(s)+Y_1^2(s)]G(s) = \int\limits_0^\infty \frac{s^3}{(s^2+\lambda^2)^2}\Lambda(\lambda)g(\lambda)d\lambda$$

$$+(1-\nu)m\int\limits_0^\infty \frac{s^3}{(s^2+\lambda^2)^2}\Lambda(\lambda)K_1(\lambda)A(\lambda)d\lambda.$$

Substituting (6.1) in (7.14) and interchanging the order of integration, we arrive at the following definite form of the integral equation for the unknown $G(s)$:

$$(7.15)\qquad \frac{\pi^3 s^2}{32}[J_1^2(s)+Y_1^2(s)]G(s) = \int\limits_0^\infty \eta G(\eta)\left\{\int\limits_0^\infty \frac{s^3\lambda^3\Lambda(\lambda)}{(s^2+\lambda^2)^2(\eta^2+\lambda^2)^2}d\lambda\right\}d\eta$$

$$+\frac{\pi}{4}(1-\nu)m\int\limits_0^\infty \frac{s^3}{(s^2+\lambda^2)^2}\Lambda(\lambda)K_1(\lambda)A(\lambda)d\lambda.$$

8. Discussion of the integral equation (7.15)

We may consider our boundary value problem (2.3) to (2.7) to be solved if we succeed to obtain a solution for the Fredholm Eq. (7.15) that leads to convergent integral representations for the stresses. Because of the complicated character of this equation, an analytic solution is out of the question. However, a numerical solution can only be obtained if we can find some general data as to the behaviour of the function $G(s)$. To this end, we write (7.15) in the form:

$$(8.1)\qquad G(s) = \int\limits_0^\infty R(\eta, s)G(\eta)d\eta+\alpha(s)D(s),$$

where

$$(8.2)\qquad \alpha(s) = \frac{32}{\pi^3}\frac{1}{s^2[J_1^2(s)+Y_1^2(s)]},$$

(8.2) [cont.]

$$R(\eta, s) = \alpha(s)\eta \int_0^\infty \frac{s^3\lambda^3}{(s^2+\lambda^2)^2(\eta^2+\lambda^2)^2} \Lambda(\lambda)d\lambda,$$

$$D(s) = \frac{\pi}{4} m(1-\nu) \int_0^\infty \frac{s^3}{(s^2+\lambda^2)^2} \Lambda(\lambda)K_1(\lambda)A(\lambda)d\lambda.$$

In the neighbourhood of $s = 0$, these functions behave as follows:

$$\alpha(s) = \frac{8}{\pi} + 0(s^2), \qquad D(s) = \frac{\pi^2}{32} mA(0) + 0(s), \tag{8.3}$$

while it can easily be seen that the integral in (8.1) tends to zero. We conclude that

$$G(s) = \frac{\pi}{4} mA(0) + 0(s). \tag{8.4}$$

If s tends to infinity we have:

$$\alpha(s) \to \frac{16}{\pi^2} \frac{1}{s} + 0(s^{-3}), \tag{8.5}$$

from which we derive:

$$\alpha(s) D(s) \to 0(s^{-2}). \tag{8.6}$$

From this result we conclude that the function $G(s)$ goes to zero as

$$G(s) \to \frac{c_1}{s} + 0(s^{-2}), \tag{8.7}$$

because only in this case we have

$$\int_0^\infty R(\eta, s)G(\eta)d\eta \to \frac{c_1}{s} + 0(s^{-2}). \tag{8.8}$$

From the definition (6.1) we find for $g(\lambda)$

$$g(\lambda) = \frac{2}{\pi} \lambda + 0(\lambda^2), \qquad \lambda \to 0, \tag{8.9}$$

$$g(\lambda) \to c_1 + 0(\lambda^{-1}), \qquad \lambda \to \infty. \tag{8.10}$$

With the data (8.4) and (8.7) we were able to solve the integral equation (8.1) on the computer. We note that c_1 was an unknown and had to be determined in the course of the process.

For the numerical calculations we have rewritten (8.1) in the form:

$$G(s) = \Omega(s) + \alpha(s) D(s) \tag{8.11}$$

with

$$\Omega(s) = \int_0^\infty R(\eta, s)G(\eta)d\eta = \frac{\pi}{4} \alpha(s) \int_0^\infty \frac{s^3}{(\lambda^2+s^2)^2} \Lambda(\lambda)g(\lambda)d\lambda, \tag{8.12}$$

from (8.2) and (6.1). We now write (6.1) as

$$g(\lambda) = \frac{4}{\pi}\int_0^N \frac{\lambda^3}{(\lambda^2+\eta^2)^2}\,\eta G(\eta)\,d\eta + \frac{4c_1}{\pi}\int_N^\infty \frac{\lambda^3}{(\lambda^2+\eta^2)^2}\,d\eta$$
$$= \frac{4}{\pi}\int_0^N \frac{\lambda^3}{(\lambda^2+\eta^2)^2}\,\eta G(\eta)\,d\eta + \frac{4c_1}{\pi} Z(\lambda, N), \tag{8.13}$$

with

$$Z(\lambda, N) = \int_N^\infty \frac{\lambda^3}{(\lambda^2+\eta^2)^2}\,d\eta = \frac{\pi}{4} - \frac{1}{2}\arctan\left(\frac{N}{\lambda}\right) - \frac{N\lambda}{2(N^2+\lambda^2)}. \tag{8.14}$$

In (8.13) and (8.14) we have introduced the number N, such that

$$N \gg 1. \tag{8.15}$$

With

$$M \gg 1, \tag{8.16}$$

we write for $\Omega(s)$:

$$\Omega(s) = \frac{\pi}{4}\alpha(s)\int_0^M \frac{s^3}{(s^2+\lambda^2)^2}\Lambda(\lambda) g(\lambda)\,d\lambda + \frac{\pi}{4}\alpha(s) c_1 \int_M^\infty \frac{s^3}{(s^2+\lambda^2)^2}\,d\lambda$$
$$= \frac{\pi}{4}\alpha(s)\int_0^M \frac{s^3}{(s^2+\lambda^2)^2}\Lambda(\lambda)\left\{\frac{4}{\pi}\int_0^N \frac{\lambda^3}{(\lambda^2+\eta^2)^2}\,\eta G(\eta)\,d\eta + \frac{4c_1}{\pi} Z(\lambda, N)\right\} d\lambda$$
$$+ \frac{\pi}{4}\alpha(s) c_1 Z(s, M) = \alpha(s)\int_0^N G(\eta)\left\{\eta \int_0^M \frac{s^3\lambda^3}{(s^2+\lambda^2)^2(\eta^2+\lambda^2)^2}\Lambda(\lambda)\,d\lambda\right\} d\eta$$
$$+ c_1\alpha(s)\int_0^M \frac{s^3}{(s^2+\lambda^2)^2}\Lambda(\lambda) Z(\lambda, N)\,d\lambda + \frac{\pi}{4} c_1 \alpha(s) Z(s, M)$$
$$= \int_0^N R_M(\eta, s) G(\eta)\,d\eta + c_1\alpha(s)\left[V(s) + \frac{\pi}{4} Z(s, M)\right], \tag{8.17}$$

with

$$R_M(\eta, s) = \alpha(s)\eta\int_0^M \frac{s^3\lambda^3}{(s^2+\lambda^2)^2(\eta^2+\lambda^2)^2}\Lambda(\lambda)\,d\lambda, \tag{8.18}$$

and

$$V(s) = \int_0^M \frac{s^3}{(s^2+\lambda^2)^2}\Lambda(\lambda) Z(\lambda, N)\,d\lambda. \tag{8.19}$$

We have now reduced the integral equation (8.1) to an integral equation with finite bounds:

$$G(s) = \int_0^N R_M(\eta, s)G(\eta)d\eta + E(s), \tag{8.20}$$

with

$$E(s) = \alpha(s)\left[D(s)+c_1\left\{V(s)+\frac{\pi}{4}Z(s, M)\right\}\right]. \tag{8.21}$$

We note that the contribution of $E(s)$ to $G(s)$ can not be neglected. In our derivation we have used the asymptotic formulas (8.7) and (8.10) and as a consequence our result depends on the choice of N and M. Considering our $G(s)$ found from (8.20) as a first approximation, we may improve our result by calculating further terms in the asymptotic expansions. However, we cannot get rid of truncation errors due to the infinite bound in (8.1).

The constant c_1 in (8.7) is found from (8.20) by continuity considerations. In fact, we have in this process:

$$G(N) = \int_0^N R_M(\eta, N)G(\eta)d\eta + E(N), \tag{8.22}$$

determining c_1

We can give an explicit expression for c_1 by writing (6.2) in the form:

$$\int_0^\infty \left[G(\lambda)e^{-\lambda z} - \frac{g(\lambda)}{\lambda}\cos\lambda z\right]d\lambda = z\int_0^\infty \lambda G(\lambda)e^{-\lambda z}d\lambda. \tag{8.23}$$

If we take the limit $z \to 0$ in (8.23) we arrive at:

$$c_1 = \int_0^\infty \left\{G(\lambda) - \frac{g(\lambda)}{\lambda}\right\}d\lambda, \tag{8.24}$$

for which derivation we have used the asymptotic expression (8.7). For our numerical results the Eq. (8.24) is of great value.

From the behaviour of the functions $G(\lambda)$ and $g(\lambda)$ at $\lambda \to 0$ and $\lambda \to \infty$, according to (8.4), (8.7), (8.9) and (8.10), follow the limit values of $B(\lambda)$ and $C(\lambda)$ by (6.6) and (6.7).

We have

$$\begin{aligned} B(\lambda) &= 0(\lambda^2), \quad \lambda \to 0, \\ \lambda K_1(\lambda)B(\lambda) &= 0(1), \quad \lambda \to \infty, \end{aligned} \tag{8.25}$$

$$\begin{aligned} C(\lambda) &= 0(\lambda^2 \log\lambda), \quad \lambda \to 0, \\ K_1(\lambda)C(\lambda) &= 0(1), \quad \lambda \to \infty. \end{aligned} \tag{8.26}$$

From the results of this paragraph it can easily be seen that the relevant integrals containing the coefficient functions do converge and may be differentiated under the integral sign.

9. The stresses

From (4.2) to (4.5) and (5.14) to (5.17) and by means of (6.6) to (6.9), the following relations for the stresses can be deduced:

(9.1)
$$\frac{1}{2G}\sigma_r(r,z) = -m(1-\nu)\int_0^\infty A(\lambda)\Lambda(\lambda)\Big\{[1-\lambda K(\lambda)]\frac{K_0(\lambda r)}{\lambda} + [\lambda r^2-\lambda-K(\lambda)+\lambda K^2(\lambda)]\frac{K_1(\lambda r)}{\lambda r}\Big\}\cos\lambda z d\lambda - \int_0^\infty \frac{\Lambda(\lambda)}{\lambda K_1(\lambda)}g(\lambda)\Big\{[1-\lambda K(\lambda)]K_0(\lambda r) + [2(1-\nu)+\lambda^2 r^2-\lambda K(\lambda)]\frac{K_1(\lambda r)}{\lambda r}\Big\}\cos\lambda z d\lambda + \frac{\pi}{2r}\int_0^\infty \frac{G(\lambda)}{\lambda}[(1-2\nu) - \lambda z]\Phi_1(r,\lambda)e^{-\lambda z}d\lambda - \frac{\pi}{2}\int_0^\infty G(\lambda)(1-\lambda z)\Phi_0(r,\lambda)e^{-\lambda z}d\lambda,$$

(9.2)
$$\frac{1}{2G}\sigma_\varphi(r,z) = -m(1-\nu)\int_0^\infty A(\lambda)\Lambda(\lambda)\Big\{[1+\lambda^2-\lambda^2K^2(\lambda)]\frac{K_0(\lambda r)}{\lambda} + [K(\lambda)+\lambda-\lambda K^2(\lambda)]\frac{K_1(\lambda r)}{\lambda r}\Big\}\cos\lambda z d\lambda + \int_0^\infty \frac{\Lambda(\lambda)}{\lambda K_1(\lambda)}g(\lambda)\Big\{(1-2\nu)]K_0(\lambda r) + [2(1-\nu)-\lambda K(\lambda)]\frac{K_1(\lambda r)}{\lambda r}\Big\}\cos\lambda z d\lambda - \frac{\pi}{2r}\int_0^\infty \frac{G(\lambda)}{\lambda}[(1-2\nu) - \lambda z]\Phi_1(r,\lambda)e^{-\lambda z}d\lambda - \nu\pi\int_0^\infty G(\lambda)\Phi_0(r,\lambda)e^{-\lambda z}d\lambda,$$

(9.3)
$$\frac{1}{2G}\sigma_z(r,z) = -m(1-\nu)\int_0^\infty \frac{A(\lambda)\Lambda(\lambda)}{\lambda}\{[2+\lambda K(\lambda)]K_0(\lambda r) - \lambda r K_1(\lambda r)\}\cos\lambda z d\lambda - \int_0^\infty \frac{\Lambda(\lambda)}{\lambda K_1(\lambda)}g(\lambda)\{[2+\lambda K(\lambda)]K_0(\lambda r) - \lambda r K_1(\lambda r)\}\cos\lambda z d\lambda - \frac{\pi}{2}\int_0^\infty G(\lambda)(1+\lambda z)\Phi_0(r,\lambda)e^{-\lambda z}d\lambda,$$

and

(9.4)
$$\frac{1}{2G}\tau_{rz}(r,z) = -m(1-\nu)\int_0^\infty \frac{A(\lambda)\Lambda(\lambda)}{\lambda}\{\lambda r K_0(\lambda r)-\lambda K(\lambda)K_1(\lambda r)\}\sin\lambda z d\lambda - \int_0^\infty \frac{\Lambda(\lambda)}{\lambda K_1(\lambda)}g(\lambda)\{\lambda r K_0(\lambda r)-\lambda K(\lambda)K_1(\lambda r)\}\sin\lambda z d\lambda - \frac{\pi}{2}\int_0^\infty \lambda z G(\lambda)\Phi_1(r,\lambda)e^{-\lambda z}d\lambda.$$

It can be shown that the stresses satisfy the boundary conditions for $z = 0$, $r > 1$ and $r = 1$, $z > 0$. In the cornerpoint $z = 0$, $r = 1$, the stress component τ_{rz} also meets these conditions.

If we approach the cornerpoint along $z = 0$ from $r > 1$, the stress component σ_z also satisfies the prescribed boundary condition. A similar behaviour is shown by the stress component σ_r, provided we approach the cornerpoint along $r = 1$ from $z > 0$. There is a jump in these stresses if the cornerpoint is reached along any other path. The values of these jumps are proportional to c_1.

10. Numerical results

We have solved the integral equation (7.15) for the special case

$$q(z) = \frac{Q}{(1+z^2)^{3/2}}, \tag{10.1}$$

where Q is the total heat input:

$$Q = \int_0^\infty q(z)\,dz. \tag{10.2}$$

For this choice (3.5) gives

$$A(\lambda) = \frac{Q}{\varkappa\pi^2}, \tag{10.3}$$

from which follows by (8.4)

$$G(0) = \frac{mQ}{4\pi\varkappa}. \tag{10.4}$$

In Fig. 1 is shown the graph of $G(s)$. For Poisson's ratio ν the value 0.3 is taken.

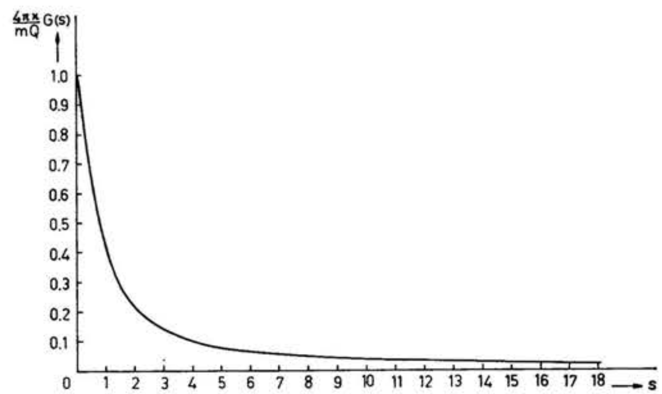

FIG. 1. Solution of integral Eq. (7.15) for $q(z)$ according to (10.1) and $\nu = 0.3$.

The most interesting stress component is the circumferential stress σ_φ, especially along the pipe. We have from (9.2):

$$\frac{1}{2G}\sigma_\varphi(1,z) = -\frac{mQ(1-\nu)}{\varkappa\pi^2}\int\limits_0^\infty \Big[1+\frac{1}{\lambda}(2+\lambda^2)K(\lambda)-K^2(\lambda) - \lambda K^3(\lambda)\Big]\Lambda(\lambda)K_1(\lambda)\cos\lambda z\,d\lambda + \int\limits_0^\infty \frac{\Lambda(\lambda)g(\lambda)}{\lambda}[\lambda K^2(\lambda)-2\nu K(\lambda) - \lambda]\cos\lambda z\,d\lambda + \int\limits_0^\infty G(\lambda)[(1+2\nu)-\lambda z]e^{-\lambda z}d\lambda. \tag{10.5}$$

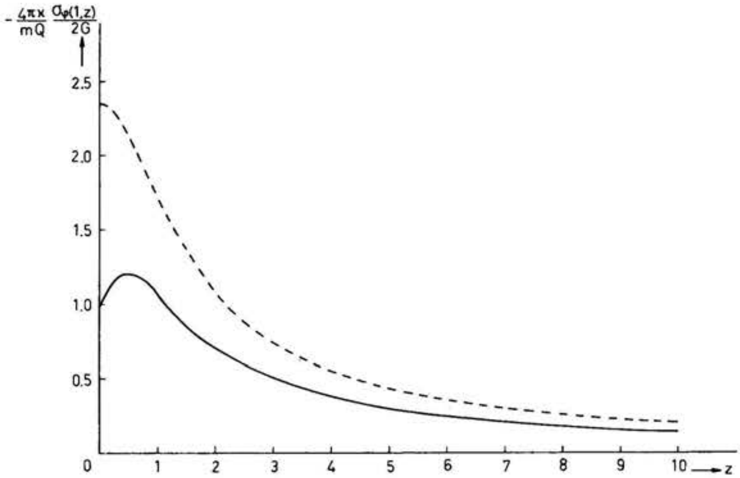

FIG. 2. Circumferential stresses along the pipe as a function of z.

In Fig. 2, $\left(-\frac{\sigma_\varphi(1,z)}{2G}\right)$ is represented as a function of z. We see from this figure that the stress concentration at the point $r = 1$, $z = 0$

$$\frac{\sigma_\varphi(1.0)}{2G} = -0.993\frac{mQ}{4\pi\varkappa}. \tag{10.6}$$

To show the influence of the free surface $z = 0$ on the circumferential stress σ_φ, we have also calculated this stress for the infinite body with a cylindrical hole under the same, symmetrical heat loading. The solution of this problem is obtained in a trivial way from our more general equations. In Fig. 2 we have plotted this stress by a dotted line.

11. Extension of the problem

We have discussed a special problem associated with the temperature field (3.3), since a general discussion is impossible. However, on the same lines, many temperature fields may be treated for the region $(0, \infty)\times(1, \infty)$. The following possibilities may be noted:

1. The loading need not to be rotationally symmetric. We may expand the boundary values in Fourier series with respect to φ. Each of the terms is treated similarly. Weber's formulae have to be adapted correspondingly.

2. We may prescribe stresses at the bounding surfaces. Only the residual problem will change accordingly.

3. We may prescribe zero temperature at $z = 0$. In this case we use a sine integral instead of (3.3). The residual problem can be treated by taking sine integrals in (5.8) and (5.9).

4. Other temperature fields, corresponding to mixed boundary value problems, may be considered by expanding in the associated Fourier transform and formulating the surface bounding values by the appropriate Weber transforms.

5. We may even admit some classes of body forces.

12. The boundary values

In the problem under consideration, the heat flow at $r = 1$ was prescribed and we were especially concerned with the stress field that occurs in the solid body. However, in many problems of practical interest the heat flow at $r = 1$ is not given a priori, but has to be determined, whether by calculation or by measurement. In such a problem, we consider the heat-transfer problem of a hot fluid, flowing in the pipe $z \geqslant 0$, $r < 1$. We denote the temperature increment by θ and assume that it is governed by the equation

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} - \frac{2v}{\beta}(1-r^2)\frac{\partial\theta}{\partial z} = 0, \quad r < 1, \tag{12.1}$$

where v is the mean velocity and β is the coefficient of temperature conductivity of the fluid. The coefficient β satisfies

$$\beta = \frac{\varkappa_1}{\varrho c}, \tag{12.2}$$

where $\varkappa_1$ is the coefficient of internal heat conductivity, c is the specific heat and ϱ the density. The boundary conditions for (12.1) are:

$$\theta = \theta_0 \quad \text{for} \quad z = 0,\ r < 1, \tag{12.3}$$

$$\theta = f(z) \quad \text{for} \quad r = 1,\ z > 0. \tag{12.4}$$

In this problem, $f(z)$ of (12.4) has to be identified with $\int\limits_0^\infty A(\lambda)K_0(\lambda)\cos\lambda z\,d\lambda$ and is an unknown function. We further have the continuity of the heat flow:

$$\varkappa_1\frac{\partial\theta}{\partial r} = \varkappa\frac{\partial T}{\partial r}, \quad \text{at } r = 1,\ z > 0. \tag{12.5}$$

To simplify the analysis we suppose θ_0 to be constant. We transform (12.1) into an equation for the Laplace transform θ, defined by

$$\bar{\theta} = \int\limits_0^\infty e^{-\lambda z}\theta\,dz, \tag{12.6}$$

by multiplying it by $e^{-\lambda z}$ and integrating over z from zero to infinity. We obtain:

$$\frac{\partial^2\bar{\theta}}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{\theta}}{\partial r}-\frac{2v\lambda}{\beta}(1-r^2)\bar{\theta} = -\frac{2v}{\beta}(1-r^2)\theta_0. \tag{12.7}$$

The solution of (12.7) can be written:

$$\bar{\theta} = \frac{\theta_0}{\lambda}+D(\lambda)\psi(\lambda, r), \tag{12.8}$$

where $\psi(\lambda, r)$ is defined as the solution of

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{2v\lambda}{\beta}(1-r^2)\psi = 0, \tag{12.9}$$

which is bounded at $r = 0$ and is normalized according to

$$\psi(\lambda, 1) = 1, \tag{12.10}$$

while $D(\lambda)$ is a factor which has to be determined.

If we represent the function $f(z)$ in the form $\int\limits_0^\infty A(\lambda)K_0(\lambda)\cos\lambda z\,d\lambda$, where $A(\lambda)$ is now an unknown function, and apply Laplace's transform, the boundary condition (12.4) becomes:

$$\frac{\theta_0}{\lambda}+D(\lambda) = \lambda\int\limits_0^\infty A(\mu)K_0(\mu)\frac{d\mu}{\lambda^2+\mu^2}. \tag{12.11}$$

Repeating this procedure we find from (12.5):

$$\varkappa_1 D(\lambda)\frac{\partial\psi(\lambda, 1)}{\partial r} = -\varkappa\lambda\int\limits_0^\infty A(\mu)K_1(\mu)\frac{\mu\,d\mu}{\mu^2+\lambda^2}. \tag{12.12}$$

Eliminating $D(\lambda)$ from (12.11) and (12.12) yields the integral equation for $A(\mu)$:

$$\varkappa_1\frac{\partial\psi(\lambda, 1)}{\partial r}\left\{\lambda\int\limits_0^\infty A(\mu)K_0(\mu)\frac{d\mu}{\lambda^2+\mu^2}\right\} +\varkappa\lambda\int\limits_0^\infty A(\mu)K_1(\mu)\frac{\mu\,d\mu}{\lambda^2+\mu^2} = \varkappa_1\frac{\partial\psi(\lambda, 1)}{\partial r}\frac{\theta_0}{\lambda}. \tag{12.13}$$

With

$$\varkappa/\varkappa_1 = p, \tag{12.14}$$

this equation takes the form:

$$\lambda^2\int\limits_0^\infty\frac{A(\mu)K_0(\mu)}{\lambda^2+\mu^2}\left\{\frac{\partial\psi(\lambda, 1)}{\partial r}-p\bar{K}(\mu)\right\}d\mu = \theta_0\frac{\partial\psi(\lambda, 1)}{\partial r}, \tag{12.15}$$

which can be written as an integral equation for $f(z)$ from (3.7)

$$\lambda^2 \int_0^\infty f(z) S(z, \lambda) dz = \theta_0 \frac{\partial \psi(\lambda, 1)}{\partial r}, \tag{12.16}$$

where $S(z, \lambda)$ has been defined by

$$S(z, \lambda) = \frac{2}{\pi} \int_0^\infty \frac{\cos \mu z}{\lambda^2 + \mu^2} \left\{ \frac{\partial \psi(\lambda, 1)}{\partial r} - p \bar{K}(\mu) \right\} d\mu \tag{12.17}$$

$$= \frac{1}{\lambda} e^{-\lambda z} \frac{\partial \psi(\lambda, 1)}{\partial r} - \frac{2p}{\pi} \int_0^\infty \frac{\bar{K}(\mu) \cos \mu z}{\lambda^2 + \mu^2} d\mu.$$

For numerical purposes, (12.15) seems to be preferable to (12.16).

In (12.15) and (12.17) an abbreviation $\bar{K}(\mu)$ has been introduced, defined by

$$\bar{K}(\mu) = -\mu \frac{K_1(\mu)}{K_0(\mu)}. \tag{12.18}$$

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Received June 17, 1972