

## The transient response of elastic, visco-plastic beams

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DIRECT Finite Element Analysis has been extended to analyze the transient dynamic behavior of elastic, visco-plastic cantilever beam. The salient features of strain rate theory are discussed. Results for step impulse loadings (force and velocity) and a harmonic loading are presented. The effect of visco-plasticity on the reflected wave is demonstrated.

Rozszerzono bezpośrednią metodę elementów skończonych na przypadek nieustalonych, dynamicznych stanów belek wspornikowych, wykonanych z materiału sprężystego, lepkoplastycznego. Omówiono najważniejsze aspekty teorii prędkości odkształceń. Przedstawiono rozwiązania, odpowiadające obciążeniom impulsowym typu Heaviside'a (w odniesieniu do sił i prędkości) jak również obciążeniu harmonicznemu. Pokazano wpływ lepkoplastyczności na przebieg fali odbitej.

Расширен непосредственный метод конечных элементов на случай неустановившихся, динамических состояний консольных балок изготовленных из упругого, вязкопластического материала. Обсуждены самые важные аспекты теории скорости деформаций. Представлены решения отвечающие импульсным нагрузкам типа Хевисайда (по отношению к силам и к скоростям), как тоже гармонической нагрузке. Указано влияние вязкопластичности на ход отраженной волны.

### Nomenclature

- $A$  cross-sectional area at the boundary of an element,
- $A_c$  cross-sectional area at the centroid of an element,
- $AIM$  angular impulse due to moment,
- $AIV$  angular impulse due to shear,
- $B$  beam width,
- $C$  wave speed,
- $C_1$  dilatation wave speed,
- $C_2$  shear wave speed,
- $D$  weight density,
- $dt$  real time increment,
- $dt_D$  dilatation time increment,
- $dt_R$  shear time increment,
- $dx$  length of beam element,
- $dx_c$  distance from element boundary to centroid of element,
- $E$  modulus of elasticity,
- $F$  von Mises yield condition,
- $G$  modulus of rigidity,
- $H$  height at the boundary of an element,
- $H_c$  height at the centroid of an element,
- $I$  moment of inertia at the boundary of an element,
- $I_c$  moment of inertia at the centroid of an element,
- $j$  superscript indicating  $j$ th cell quantities,
- $j_m$  number of cells into which beam is divided,

- $J_2$  second stress invariant,  
 $K_m$  number of time intervals,  
 $K_s$  shear correction factor,  
 $L$  length of beam,  
 $LIV$  linear impulse due to shear,  
 $M$  bending moment,  
 $M_y$  yield value of bending moment ( $M_y = \iint \sigma_y dydz$ ),  
 $M_0$  bending moment at free end of beam,  
 $T$  first modal period for beam,  
 $t$  time,  
 $t_D$  time clock for dilatation wave,  
 $t_R$  time clock for shear wave,  
 $t_L$  last time recorded,  
 $t_1$  dimensionless time for dilatation wave,  
 $t_2$  dimensionless time for shear wave,  
 $t_3$  dimensionless time for harmonic input,  
 $V_0$  shear force,  
 $V_y$  yield value of shear force ( $V_y = \iint \tau_0 dydz$ ),  
 $V$  shear force at free end of beam,  
 $v$  linear velocity,  
 $x$  coordinate along length of beam,  
 $y$  deflection of beam,  
 $y_B$  bending deflection of beam,  
 $y_S$  shear deflection of beam,  
 $\beta$  shear angle,  
 $\gamma$  visco-plastic attenuation factor,  
 $\delta$  exponential,  
 $\epsilon$  strain,  
 $\epsilon_y$  translation strain,  
 $\epsilon_\psi$  angular strain,  
 $\nu$  Poisson's ratio,  
 $\rho$  mass density,  
 $\sigma$  normal stress,  
 $\sigma_0$  yield value of normal stress,  
 $\tau$  shear stress,  
 $\tau_0$  yield value of shear stress,  
 $\tau_1$  rotational damping coefficient,  
 $\tau_2$  translational damping coefficient,  
 $\phi$  visco-plastic function,  
 $\Phi$  plasticity function,  
 $\psi$  bending slope of beam,  
 $\Psi$  plasticity function,  
 $\omega$  angular velocity.

## 1. Introduction

THE DETERMINATION of the dynamic stresses in a structure that is subjected to high intensity short duration loading is a fundamental engineering problem. Impact forces may arise as a result of an earth-quake, high pressure blast or by other means. Any design criteria which are established should arise from a complete study and analysis of the shock characteristics of each member of the structure. The analysis of flexural or transverse

impulsive loading is of particular interest because it is more complicated than other impulses. This type of loading is more complicated because the flexural and transverse responses are coupled together.

Previous studies by DAVIDS [1] and KOENIG [2] have shown how the Direct Analysis Method may be utilized in the analysis of uniform finite beams and plates which possess internal damping. The undamped transient and static behaviors of these bodies were studied. Direct Analysis is extended herein to include the effects of visco-plasticity.

There are numerous theories concerned with analyzing anelastic impact. Limited experimental work combined with several different theoretical models have prevented the acceptance of any unified theory for anelastic dynamic analysis. The constitutive equations used in this paper represent the modern trend of using strain rate theory. Strain rate theory considers an elastic, visco-plastic media where the important factor is the rate of loading rather than the magnitude of loading.

MALVERN [3, 4] investigated the elastic, visco-plastic properties of a steel bar by using the method of characteristics to solve the longitudinal wave propagation problem. His analysis prompted further investigations designed to expand on his significant development.

PLASS [5] investigated the problem of flexural impulses on an elastic, visco-plastic beam. He used the method of characteristics to solve the dynamic response in this medium. Reflected waves were not considered. The following constitutive equations were used to describe the bending and shear visco-plastic effects respectively,

$$EI \frac{\partial \omega}{\partial x} = \frac{\partial M}{\partial t} + \gamma(M - M_y)$$

and

$$GK_s A \left( \frac{\partial V}{\partial x} + \omega \right) = \frac{\partial V}{\partial t} + \gamma(V - V_y),$$

where  $M_y$  and  $V_y$  are determined from the von Mises yield condition. It should be noted that these strain rate terms, when included in an elastic stress-strain relationship, cannot be classified as constitutive equations. True constitutive equations are independent of the initial loading conditions [6]. When  $M_y$  and  $V_y$  are dependent on the initial loading the relationship is really a structure equation. PLASS solved the flexural wave problem by neglecting the effect of shear visco-plasticity.

BEJDA [6] solved the same problem as PLASS, using different constitutive relations. The bending and shear constitutive equations BEJDA used are,

$$-I \frac{\partial \omega}{\partial x} = \frac{1}{E} \frac{\partial M}{\partial t} + \frac{2}{3} \frac{\gamma}{\sqrt{J_2}} \phi(F) M$$

and

$$K_s A \left( \frac{\partial V}{\partial x} - \omega \right) = \frac{1}{G} \frac{\partial V}{\partial t} + \frac{\gamma}{\sqrt{J_2}} \phi(F) V,$$

where  $F = \sqrt{(M/M_y)^2 + (V/V_y)^2} - 1$  is the von Mises static yield condition, when  $M_y$  and  $V_y$  are related by a simple tension test.  $J_2$  is the second stress invariant and  $\phi(F) = F^3$ . The unloading condition is  $F \leq 0$ . The problem was solved using the method of charac-

teristics. Although a technique was postulated to solve the reflected wave problem, no numerical results were presented.

PERZYNA [7] examined strain rate theory for a generalized state of stress based on the von Mises yield condition. He derived the form of the visco-plastic terms employed by BEJDA and discussed the motivation for using this particular form.

PERZYNA and BEJDA [8] analyzed one dimensional wave propagation. They used known test data to determine the numerical values for the parameters in the constitutive relationships determined by PERZYNA. The same numerical values were used by BEJDA for his research and are employed herein.

## 2. Strain rate theory

The following discussion on strain rate theory is initially restricted to describing stress wave propagation in a one dimensional bar. This restriction is imposed to simplify the discussion of strain rate theory. A more detailed description of strain rate theory may be found in a discussion by CRISTESCU [9].

It is well known that the strain in a bar is composed of two components, an elastic portion  $\varepsilon_e$  and a plastic portion  $\varepsilon_p$ . The constitutive equation for elastic behavior is Hooke's Law

$$\frac{\partial \varepsilon_e}{\partial t} = \frac{1}{E} \frac{\partial \sigma}{\partial t}$$

while the constitutive equation for plastic behavior is described by two arbitrary functions [9]

$$\frac{\partial \varepsilon_p}{\partial t} = \Phi(\sigma, \varepsilon) \frac{\partial \sigma}{\partial t} + \Psi(\sigma, \varepsilon), \quad \varepsilon = \varepsilon_e + \varepsilon_p.$$

The instantaneous plastic properties of a material are associated with the  $\Phi$  function. The  $\Phi$  function is a measure of the part of the strain directly proportional to an increase in stress. In this way an increase in stress results in an immediate increase in strain. The  $\Phi$  function affects the stress wave speed as demonstrated by examining the *instantaneous* stress strain relation.

$$\frac{\partial \varepsilon}{\partial t} = \left( \frac{1}{E} + \Phi \right) \frac{\partial \sigma}{\partial t}.$$

The  $\Psi$  function is indicative of the non-instantaneous properties of the material under consideration. The  $\Psi$  function is a measure of the visco-plastic strain and depends on the actual value of the stress rather than on the increase in stress. The relationship between the  $\Psi$  function and the plastic strain is time dependent and, therefore, the wave speed is not affected by the existence of the  $\Psi$  function [9]. The wave speed is a function of the instantaneous stress-strain relationship as it cannot depend on the non-instantaneous stress-strain function because by its very nature the  $\Psi$  function is influential only after the stress has been in existence for a period of time.

The vanishing of the  $\Phi$  function is indicative of high speed *elastic* loading of ductile metals [9]. High speed elastic loading when combined with visco-plastic effects is commonly

known as strain rate theory. Thus, for high rates of loading on ductile metals the  $\Phi$  function can be assumed to vanish [9]. This assumption results in the following stress-strain equation,

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{E} \frac{\partial \sigma}{\partial t} + \Psi(\sigma, \varepsilon).$$

The above equation is the standard strain rate equation for a one-dimensional bar. This equation when combined with the longitudinal equation of motion and the kinematic relation yields the speed of propagation of the stress wave. The propagation speed is equal to the elastic wave speed (independent of the  $\Psi$  function).

An important characteristic of strain rate theory is the absence of permanent set. This characteristic can be readily verified by examining the time integrated strain rate equation for the condition existing when the variables do not vary with time. For this condition, the strain rate equation degenerates to Hooke's Law validating the above contention.

The theory of strain rate cannot be used successfully for static loading as it is a dynamic characteristic of high speed loading. Strain rate theory is compatible with impact analysis and was, therefore, selected to solve the elastoplastic segment of this paper.

The constitutive equations used in this work are the same constitutive equations that were used by BEJDA [6].

### 3. Physical laws

Four equations completely describe the problem of stress wave propagations in beams. They are the angular and translational impulse-momentum laws and the bending moment and shear force constitute equations.

*Angular impulse-momentum law*

$$\frac{d\omega}{dt} = \frac{\frac{1}{2} (V^j dx_c^j + V^{j+1} (dx^j - dx_c^j)) + M^j - M^{j+1}}{\rho I dx^j},$$

where the centroidal variables ( $c$  subscript) are computed by the standard procedures resulting in the following equation:

$$dx_c^j = \frac{1}{2} dx^j.$$

*Translational impulse-momentum law*

$$\frac{dv}{dt} = \frac{V^{j+1} - V^j}{\rho A dx^j}.$$

*Constitutive equation for bending moment (Bejda's equation)*

$$-I \frac{\partial \omega}{\partial x} = \frac{1}{E} \frac{\partial M}{\partial t} + \frac{2}{3} \frac{\gamma}{\sqrt{J_2}} \phi(F) M.$$

If the second term on the right side of the above equation is ignored the elastic beam bending equation results. Similarly, if the first term on the right side of the equation is

ignored the visco-plastic beam bending equation results. The elastic beam bending equation, in direct analysis notation, is

$$M^{j+1} = -IE\varepsilon_{\psi}^j,$$

where the bending strain is defined across two adjoining elements:

$$\varepsilon_{\psi}^j = \frac{\psi^{j+1} - \psi^j}{(dx^j - dx_c^j) + dx_c^{j+1}}.$$

Substituting the elastic beam bending equation into the visco-plastic beam bending equation and writing the resulting equation in direct analysis notation results in the following equation

$$\frac{d\omega}{dt} = -\frac{\gamma}{3} E \left( \frac{\phi^j}{\sqrt{J_2^j}} + \frac{\phi^{j+1}}{\sqrt{J_2^{j+1}}} \right) \omega.$$

*Constitutive equation for shear force (Bejda's equation)*

$$K_s A \left( \frac{\partial v}{\partial x} - \omega \right) = \frac{1}{G} \frac{\partial V}{\partial t} + \frac{\gamma}{\sqrt{J_2}} \phi(F) V.$$

Dividing the equation into parts, as was done for the bending equation, results in the elastic shear force equation:

$$V^{j+1} = K_s A G (\varepsilon_y^j - \psi^j),$$

where the translational strain is defined across two adjoining elements:

$$\varepsilon_y^j = \frac{y^{j+1} - y^j}{(dx^j - dx_c^j) + dx_c^{j+1}}$$

and the visco-plastic shear force equation is:

$$\frac{dv}{dt} = -\frac{\gamma}{2} G \left( \frac{\phi^j}{\sqrt{J_2^j}} + \frac{\phi^{j+1}}{\sqrt{J_2^{j+1}}} \right) v.$$

The two wave speeds are defined in the following form

$$C_1 = \left( \frac{E}{\rho} \right)^{1/2} \quad \text{and} \quad C_2 = \left( \frac{K_s G}{\rho} \right)^{1/2}.$$

The boundary conditions used herein correspond to those of a cantilever beam; namely, zero deflection and zero bending rotation (no restriction on shear rotation).

#### 4. Method of solution

It is the purpose of this section to illustrate, in detail, the procedure used to solve the problems presented in this paper. Only one computer program is needed to solve all the different problems investigated in this work.

As in most problems the first step is to specify the parameters needed to solve the problem.

- (1) Specify time parameters  $dt$ ,  $K_m$ .
- (2) Specify beam parameters  $K_s$ ,  $B$ ,  $L$ ,  $E$ ,  $G$ ,  $\rho$ ,  $H$ .

After specifying the above parameters the following parameters must be calculated:

(3) Calculate  $C_1, C_2, dx^j, dx_c^j, A, I, j_m$ .

The value of  $dx^j$  is selected to fulfil the requirements of the characteristic assumption. For a transverse impulse the shear wave speed is used to satisfy the characteristic assumption so that the excess of momenta does not have to be corrected along this discontinuity. An analogous situation exist for a flexural impulse. For a combined flexural and transverse impulse the characteristic assumption is satisfied by arbitrarily using the dilatation wave speed. If neither wave speed satisfied the characteristic assumption the flexural variables would have to be corrected more often than the transverse variables. All other considerations being equal, it seems reasonable to eliminate the momentum correction from the wave that is corrected most frequently.

Separate clocks must be established in each element to measure the propagation of the dilatation and shear waves.

(4) Determine clocks  $dt_D^j = dx^j/C_1, dt_R^j = dx^j/C_2$ .

The problem is solved for a predetermined time interval ( $t = K_m dt$ ). The following procedure (steps 5 through 9) must be repeated  $K_m$  times until the time interval has been achieved.

The problem can be solved for various loading conditions including either flexural or transverse impulses (or a combination of the two types). Four different impulse functions were programmed to be analyzed. They are step, ramp, parabolic and harmonic functions.

(5) Specify loading conditions.

The clock method is employed to test whether to propagate the dilatation or the shear wave for each element.

(6) Test to determine which wave is to be propagated (each element has to be individually examined). The cyclic procedure is divided into two segments; namely, the dilatation cycle and shear cycle.

#### 1) Dilatation cycle

(7) Propagate wave across elements selected under condition 6.

- |  |  |
|--|--|
| a) $AIM^j = AIM^j + (M^j - M^{j+1}) dt_D^j$  | angular impulse due to moment                  |
| b) $AIV^j = AIV^j + (V^j dx_c^j + V^{j+1}(dx^{j+1} - dx_c^j)) \times (t_D^j + dt_D^j - t_L^j)$   | angular impulse due to shear                   |
| c) $LIV^j = LIV^j + (V^{j+1} - V^j)(t_D^j + dt_D^j - t_L^j)$   | linear impulse due to shear                    |
| d) $\omega^j = \omega^j \left( 1 - \frac{dt_D^j}{2\tau_1} \right)$   | damping of angular velocity (first portion)    |
| e) $d\omega = (AIM^j + AIV^j)/\rho I^j dx^j$   | angular impulse-momentum law                   |
| f) $\omega^j = \omega^j + d\omega$   | cumulate angular velocity                      |
| g) $\omega^j = \omega^j \left( 1 - \frac{\gamma}{3} dt_D^j \left( \frac{1}{\sqrt{J_2^j}} \phi^j E + \frac{1}{\sqrt{J_2^{j+1}}} \phi^{j+1} E \right) \right)$ | contribution due to rotational viscoplasticity |

- h)  $\omega^j = \omega^j \left( 1 - \frac{dt_D^j}{2\tau_1} \right)$  damping of angular velocity  
(second portion)
- i)  $AIM^j = 0$  initialize angular impulse
- j)  $AIV^j = 0$  initialize angular impulse
- (8) Calculate forces due to impulses for each selected element.
- a)  $d\varepsilon_\psi = (\omega^{j+1} - \omega^j) dt_D^j / (dx^j - dx_c^j + dx_c^{j+1})$  angular strain
- b)  $dM = -EI d\varepsilon_\psi$  moment constitutive equation
- c)  $M^{j+1} = M^{j+1} + dM$  cumulate moment
- d)  $dV = -K_s AG \omega^{j+1} dt_D^j$  shear constitutive equation (dilatation portion)
- e)  $V^{j+1} = V^{j+1} + dV$  cumulate shear
- f)  $\psi^j = \psi^j + \omega^j dt_D^j$  cumulate rotational deformation
- g)  $t_D^j = t_D^j + dt_D^j$  advance dilatation wave front
- h)  $t_L^j = t_D^j$  record last time
- II) Shear cycle
- (7) Propagate wave across element selected.
- a)  $AIV^j = AIV^j + (V^j dx^j + V^{j+1} (dx^j - dx_c^j)) \times (t_R^j + dt_R^j - t_L^j)$  angular impulse due to shear
- b)  $LIV^j = LIV^j + (V^{j+1} - V^j) (t_R^j + dt_R^j - t_L^j)$  linear impulse due to shear
- c)  $v^j = v^j \left( 1 - \frac{dt_R^j}{2\tau_2} \right)$  damping of linear velocity (first portion)
- d)  $dv = LIV^j / \rho A dx^j$  linear impulse-momentum law
- e)  $v^j = v^j + dv$  cumulate linear velocity
- f)  $v^j = v^j \left( 1 - \frac{\gamma}{2} dt_R^j \left( \frac{1}{\sqrt{J_2^j}} \phi^j G + \frac{1}{\sqrt{J_2^{j+1}}} \phi^{j+1} G \right) \right)$  contribution due to translational viscoplasticity
- g)  $v^j = v^j \left( 1 - \frac{dt_R^j}{2\tau_2} \right)$  damping of linear velocity (second portion)
- h)  $LIV^j = 0$  initialize linear momentum
- (8) Calculate forces due to impulses for each selected elements.
- a)  $d\varepsilon_y = (v^{j+1} - v^j) dt_R^j / (dx^j - dx_c^j + dx_c^{j+1})$  linear strain
- b)  $dV = K_s AG d\varepsilon_y$  shear constitutive equation (shear portion)
- c)  $V^{j+1} = V^{j+1} + dV$  cumulate shear
- d)  $y^j = y^j + v^j dt_R^j$  cumulate deflection
- e)  $t_R^j = t_R^j + dt_R^j$  advance shear wave front
- f)  $t_L^j = t_R^j$  record last time

At the completion of the propagation cycle the boundary conditions must be satisfied; for a cantilever beam the boundary conditions are satisfied automatically. This automatic process is obtained by analyzing the strain over the half element at the restrained boundary. For all other types of beams a boundary condition must be imposed.



## (9) Apply boundary conditions.

This cyclic propagation procedure continues until the allotted time expires.

The damping terms and visco-plastic terms are listed in their proper order and may be included or omitted as desired. No distinction is made between right running and left running waves as in other numerical procedures such as the method of characteristics. At the termination of a time increment both waves (left and right running waves) are propagated across the cell together. In this manner, all waves are considered simultaneously.

For all cases considered in this work, the following dimensionless variables are used for presenting the results.

$$\begin{array}{ll} \bar{L} = 1 & \bar{V} = (VL^2/EI) \times 100 \\ \bar{x} = x/L & \bar{M} = (ML/EI) \times 100 \\ \bar{y} = (x/L) \times 1000 & t_1 = tC_1/L \\ \bar{\psi} = (\psi) \times 1000 & t_2 = tC_2/L \\ \bar{v} = (v/C_2) \times 100 & t_3 = t/T. \\ \bar{\omega} = (\omega L/C_1) \times 100 & \end{array}$$

The following numerical values were assigned to the beam parameters in this paper:

$$\begin{array}{ll} E = 30 \times 10^6 \text{ psi} & H = 1.0 \text{ in} \\ D = \rho g = 0.31 \text{ lb/in}^3 & \gamma = 450 \text{ sec}^{-1} \\ \nu = 0.3 & \delta = 1.0 \\ L = 1.5 \text{ in} & \tau_0 = 34700 \text{ psi.} \\ B = 1.0 \text{ in} & \end{array}$$

It takes 7.76 microseconds for a dilatation wave to travel 1.5 inches in a steel beam. It takes 13.71 microseconds for a shear wave to transverse an identical path.

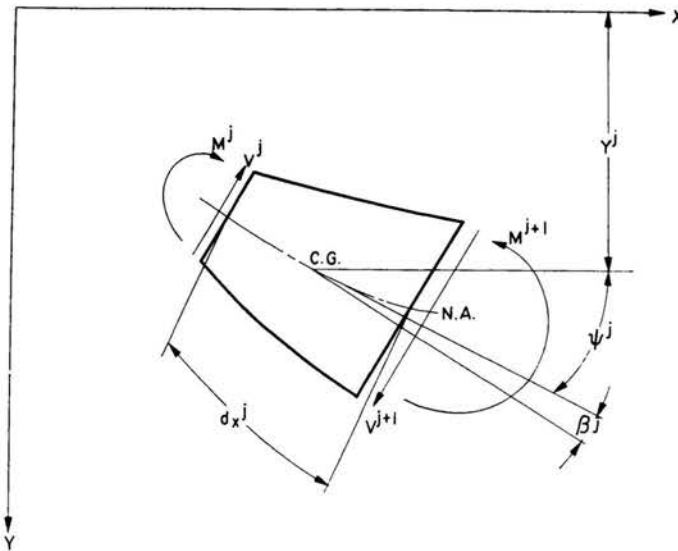


FIG. 1. Typical  $j$ th element.

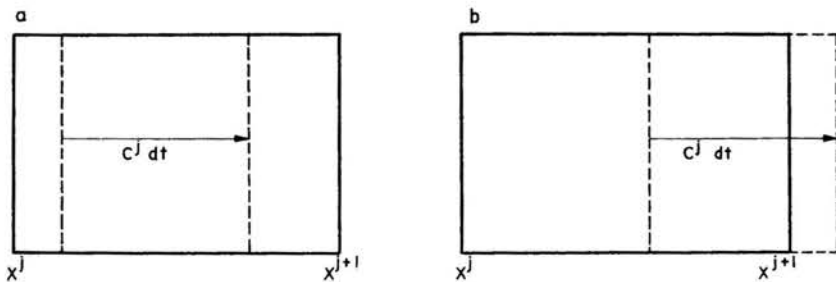
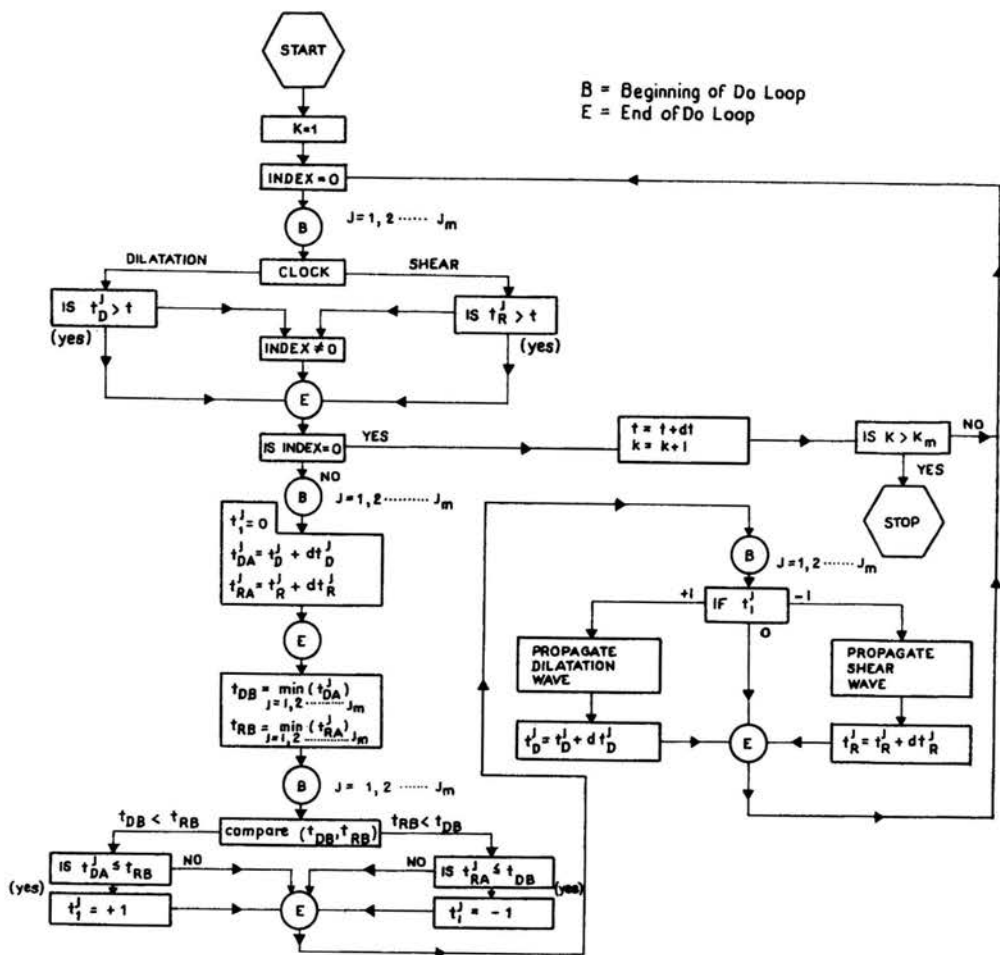


FIG. 2. Wave index technique.

a — wave has not propagated out of cell,  
 b — wave has propagated out of cell.



B = Beginning of Do Loop  
 E = End of Do Loop

FIG. 3. Clock method.

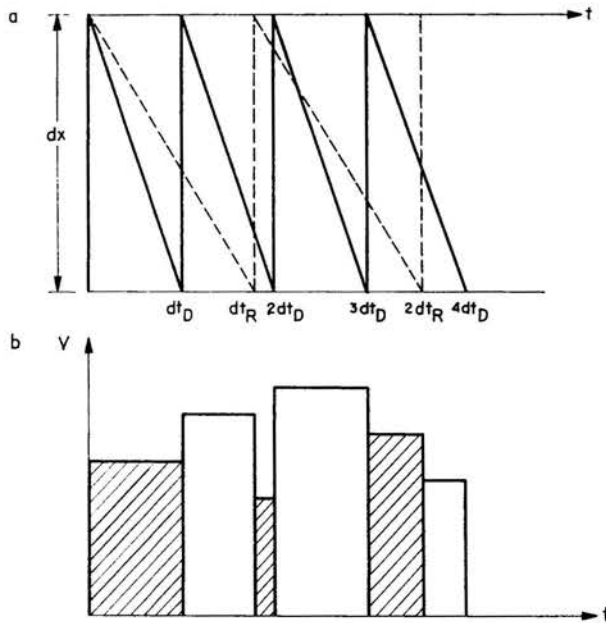


FIG. 4. Variation of shear force with respect to wave arrival.

## 5. Results

All of the parameters which were used to describe an elastic, visco-plastic beam in this paper are the same parameters which were used by BEJDA. Bejda's analysis of an elastic, visco-plastic beam was selected because he presents the most realistic yield criterion of

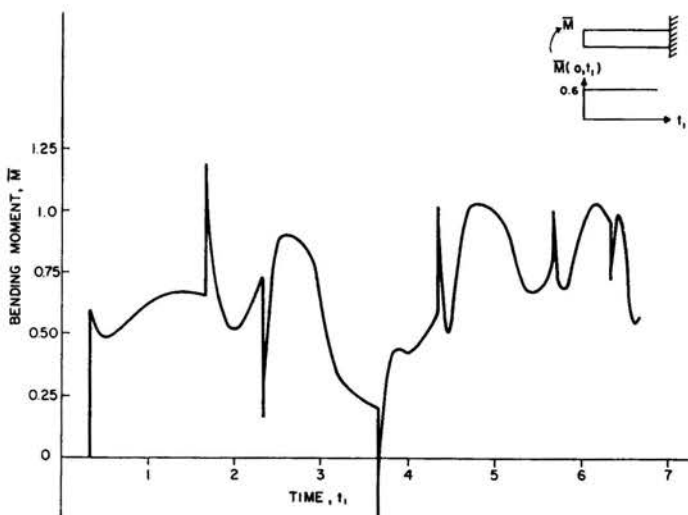


FIG. 5. Transient response of bending moment at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

all papers which were read. Bejda's method of analysis was modified before employing the Direct Analysis method to solve the following problems: a step moment input, a step shear input, and a step linear velocity input. Finally, the resonant solution for a harmonic shear input is discussed.

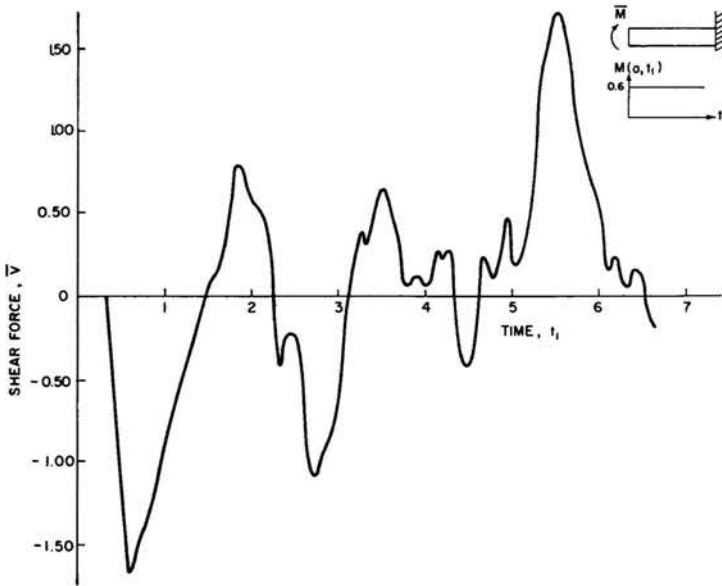


FIG. 6. Transient response of shear force at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

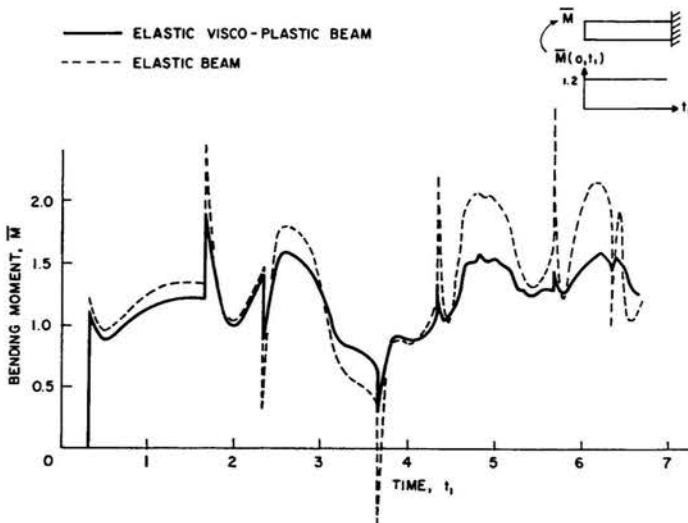


FIG. 7. Transient response of bending moment at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

Figures 5 and 6 illustrate the transient response of the bending moment and shear force respectively, due to a step moment input. In this particular example, the step moment input is in the elastic range. There is no visco-plastic effect at the test point until the reflected discontinuity wave arrives. The visco-plastic effect is essentially an attenuation condition as evidenced by the diminishing amplitude of the discontinuity. Because the visco-plastic term is only included in the plastic range, there is no mechanism that permits the energy to be dissipated in the elastic region. Therefore, a steady state solution is unobtainable. Instead, a solution is obtained that fluctuates around the static solution.

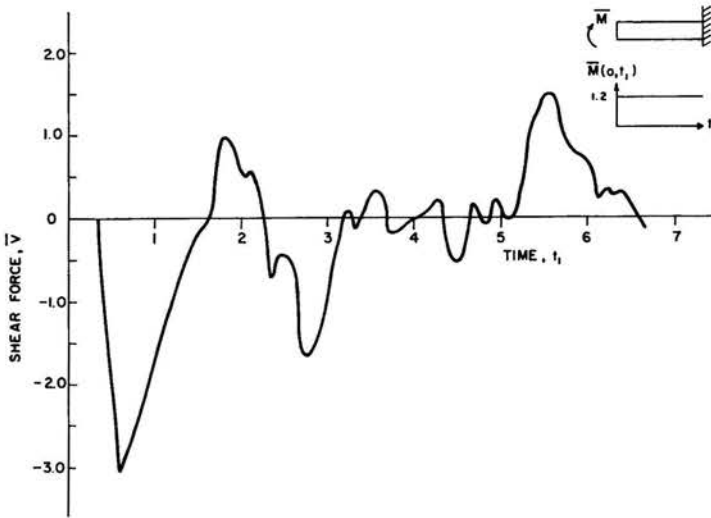


FIG. 8. Transient response of shear force at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

Figures 7 and 8 depict the transient response of the bending moment and shear force respectively, due to a step moment input. In this example the input is in the plastic range. When the discontinuity reaches the test point it has already been influenced by the visco-plasticity effect. The effect of the visco-plasticity is clearly demonstrated in figure 7, where

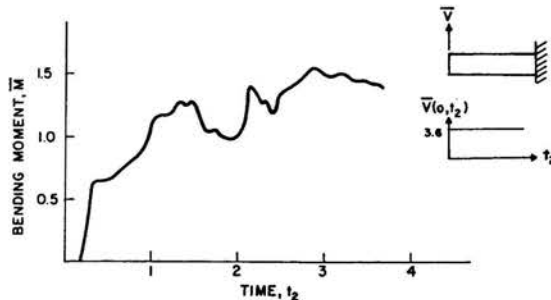


FIG. 9. Transient response of bending moment at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

the solution is compared to the solution for an elastic beam under identical loading conditions. In this problem the steady state solution is reached.

Figures 9 and 10 depict the transient response of the bending moment and shear force respectively, due to a step shear input. The steady state solution is approached more quickly

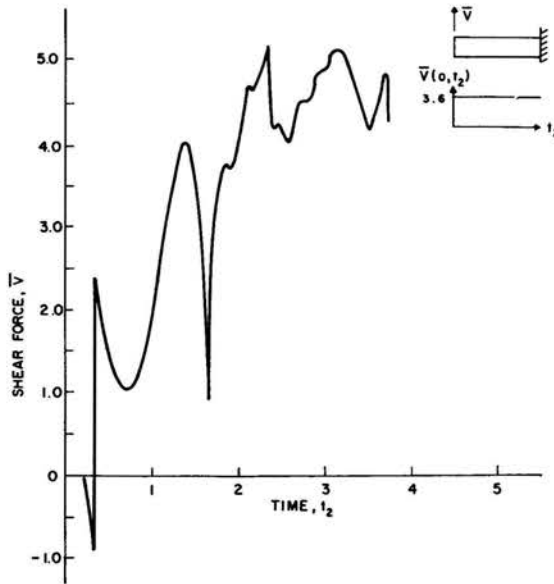


FIG. 10. Transient response of shear force at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

than in the previous example, because the input loading is further into the plastic range. The lack of significant fluctuations in the bending moment supports a contention that fluctuations in the shear force do not significantly influence the bending moment.

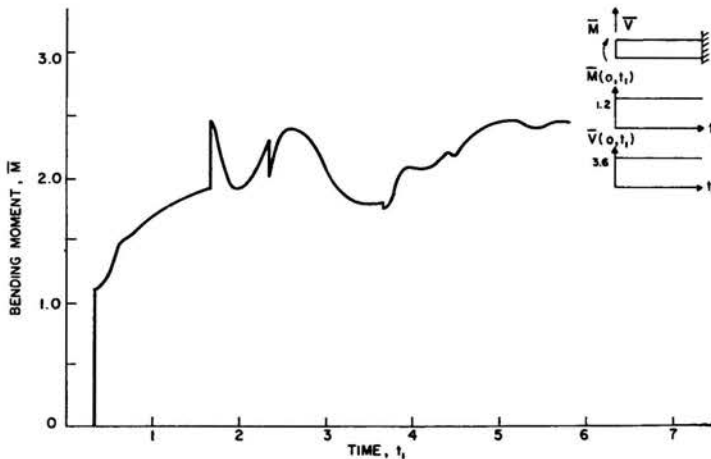


FIG. 11. Transient response of bending moment at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

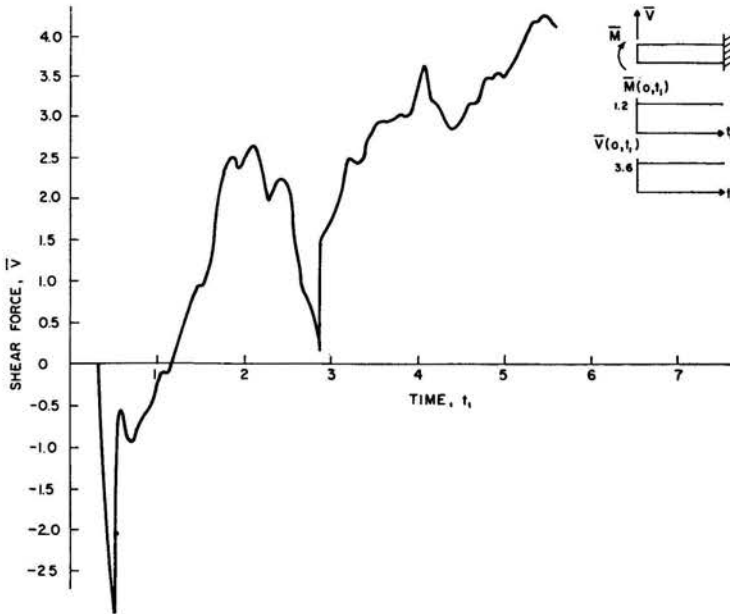


FIG. 12. Transient response of shear force at  $\bar{x} = 0.333$  for an elastic, visco-plastic.

Figures 11 and 12 illustrate the transient response of the bending moment and shear force, respectively, to a combined input (step moment and step shear). In this example the visco-plastic effect is more predominant than in the previous examples. This is because the values of the bending moment and shear force cause the dynamic response of the beam to be in the plastic range more frequently than in previous cases. The visco-plastic damping

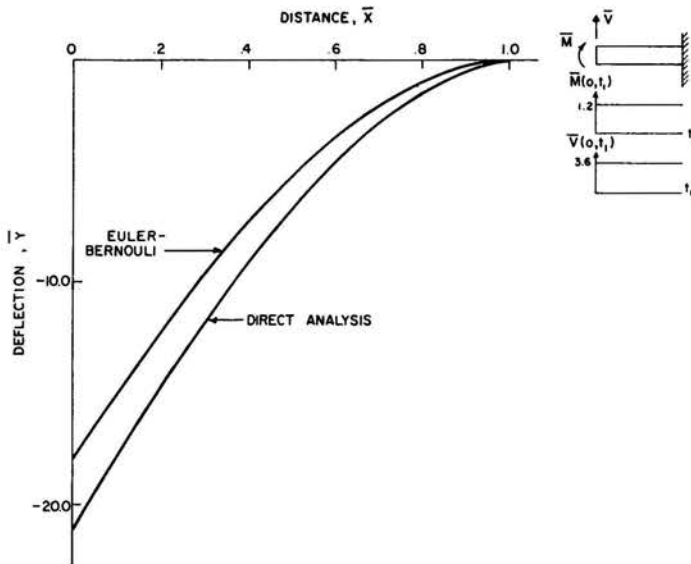


FIG. 13. Static deflection profile of an elastic, visco-plastic beam.

causes the discontinuity to diminish to a negligible value as the steady state solution is approached.

The damped solution (Fig. 13) for an elastic, visco-plastic beam is identical to the damped solution for an elastic beam. This is a consequence of the dynamic characteristics of visco-plasticity and the inability of strain rate theory to account for the existence of permanent set. The input for this problem is in the plastic range rendering the damped solution physically meaningless. The absence of any permanent set is inconsistent with plastic deformation theory and is therefore, incorrect. Strain rate theory is limited to a dynamic problem, and in particular, for a short duration following impact. The transient response presented herein are consistent with the above limitations.

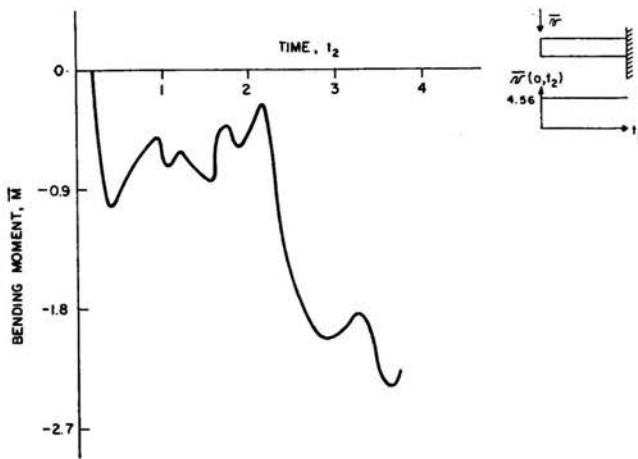


FIG. 14. Transient response of bending moment  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

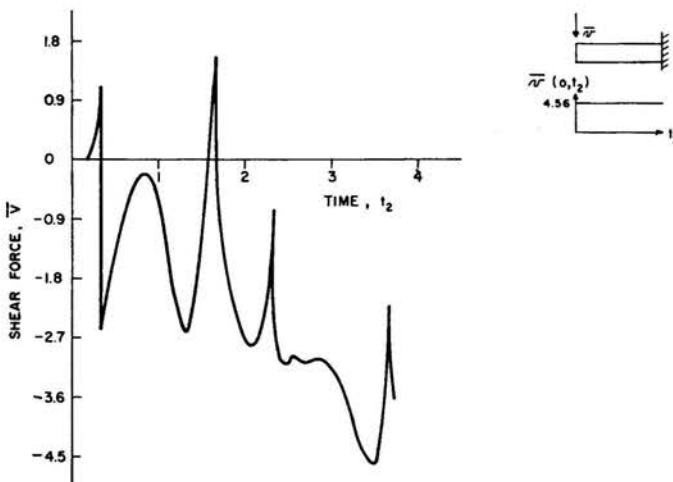


FIG. 15. Transient response of shear force at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.



Figures 14 and 15 illustrate the transient response of the bending moment and shear force respectively, due to a step linear velocity input in the plastic range. This example is presented for completeness because it is common practice to specify a shock input in the form of a velocity profile.

Figures 16, 17, 18 and 19 represent the response of an elastic, visco-plastic beam to an elastic harmonic shear input. The period of the harmonic input is determined from the first modal period for a comparable elastic beam [10]. The dynamic response of an elastic beam is identical to the dynamic response of an elastic, visco-plastic beam until the solu-

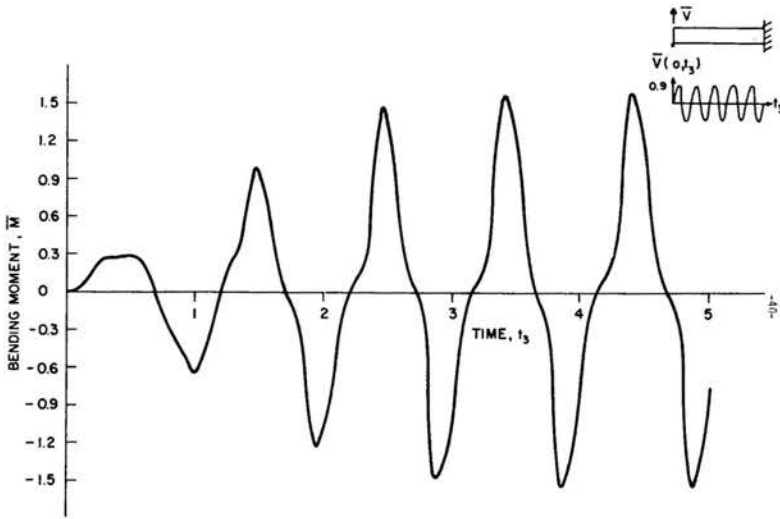


FIG. 16. Transient response of bending moment at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

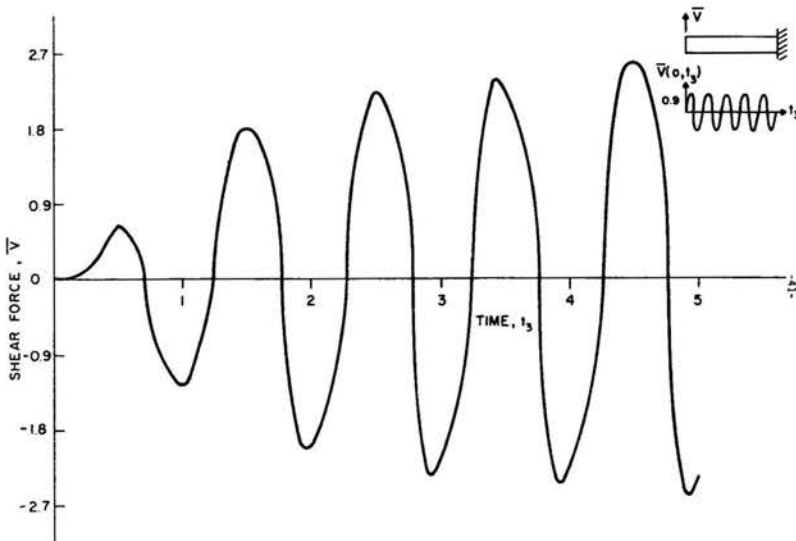


FIG. 17. Transient response of shear force at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

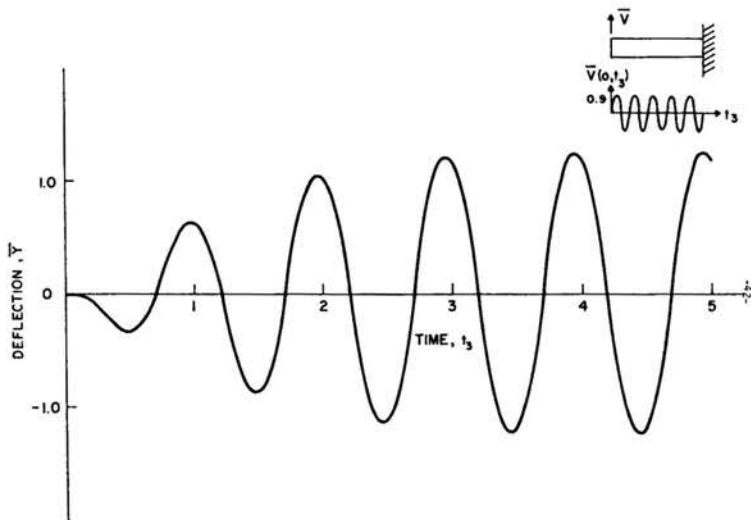


FIG. 18. Transient of deflection at  $\bar{x} = 0.333$  for an elastic, visco-plastic beam.

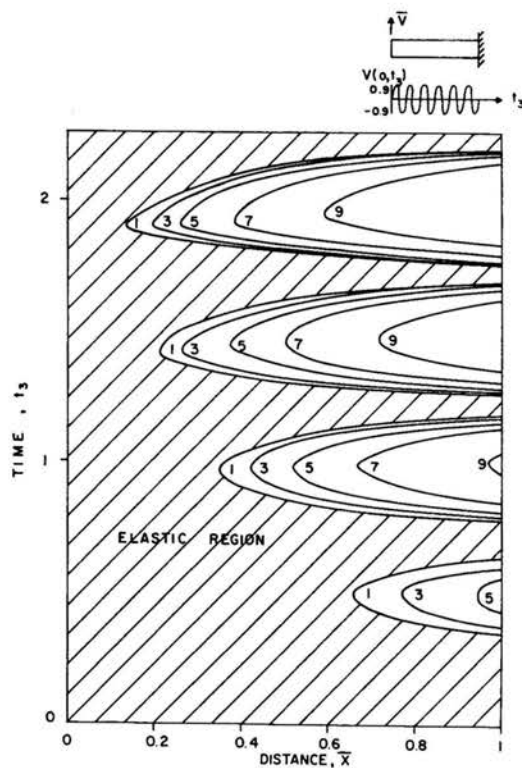


FIG. 19. Elastic-plastic map for a harmonic input.

tion enters the plastic region. At this point the visco-plastic effects influence the response. Two noticeable characteristics are evident, namely, the existence of harmonic distortion and the existence of a bounded solution.

Harmonic distortion is symbolic of visco-plastic damping. In Figs. 16 and 17 the effect of harmonic distortion on the bending moment and shear force respectively, is demonstrated. A particularly interesting aspect of harmonic distortion is the resulting unsymmetrical dynamic response. The nonsymmetry can best be observed by examining the shear force in Fig. 17. The ascent portion of the shear force (near the crest of the response) is steeper than the descent portion. The appearance of this nonsymmetry is caused by two separate phenomena. The visco-plastic term is a non-linear quantity because its value is dependent on the value of the bending moment and shear force. In addition, a symmetrical response is a characteristic of a steady state solution and not a transient characteristic. The response depicted in Figs. 16 and 17 is in actuality a pseudo steady state solution composed of a cyclic chain of transient solutions. Every time the response enters the plastic range from the elastic range (or vice versa) the equations representing the beam are altered by the addition (or removal) of the visco-plastic terms. Whenever this happens a new initial value problem must be solved.

Interestingly, there is not noticeable harmonic distortion present in Fig. 18, the deflection curve. The characteristic can be attributed to the fact that the deflection is the time integration of the velocity. The harmonic distortions that exist are not of sufficient magnitude to be evidenced in a numerical integration.

The region in which the solution is bounded, as well as the amount of harmonic distortion, is dependent upon the beam geometry, the loading functions and the visco-plastic terms. New numerical values used in the visco-plastic terms may become more appropriate than the values or terms now being used. To incorporate these changes into the Direct Analysis method requires very little reprogramming.

Figure 19 is a diagram depicting the degree of plasticity. The higher numbers indicate a response further into the plastic range than response indicated by the lower numbers. This figure is presented for a qualitative comparison rather than a quantitative comparison. The number "one" signifies the elastic-plastic boundary.

The diagram confirms the fact that the highest degree of plasticity exists at the fixed end of the cantilever beam. At the free end of the beam, and the surrounding neighbourhood, the response is always in the elastic range as the response must follow the elastic harmonic input.

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