CHAPTER XXXIII.

ELLIPTIC FUNCTIONS (*Continued*). REDUCTION TO STANDARD FORMS.

1446. Preliminary Considerations.

Taking the general integral $\int^x \frac{P\,dx}{\sqrt{Q}}$, where P is any rational algebraic function of x, and Q the quartic function

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4,$$

we now proceed to show how it may be reduced either to the Legendrian form or to the Weierstrassian form, as may be desired.

1447. We shall assume that the several coefficients occurring, viz. a_0, a_1, a_2, a_3, a_4, are all real constants.

The roots of a biquadratic $Q=0$ with real coefficients must be either (1) all real, (2) two real, two imaginary, or (3) all imaginary.

The roots of a cubic equation with real coefficients must be either (1) all real, or (2) one real, two imaginary.

Further imaginary roots occur "in pairs," and are conjugate, *i.e.* of form $\alpha\pm\iota\beta$, where α, β are real and $\iota=\sqrt{-1}$.

Hence when $a_0\neq 0$, Q must factorise, at the least, into two real quadratic factors, and it may further factorise into two linear factors and one irreducible quadratic factor, or into four linear factors, the coefficients of such factors being all real.

And when $a_0=0$, Q must factorise, at the least, into one real linear factor and one irreducible quadratic factor, or it may be into three real linear factors.

For the present we shall consider $a_0\neq 0$.

1448. **The Invariants.**

Now when any binary quartic

$$Q \equiv a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4 \equiv (a_0, a_1, a_2, a_3, a_4)(x, y)^4$$

is subjected to a linear transformation

$$x=l_1X+m_1Y, \quad y=l_2X+m_2Y,$$

so that the modulus of the transformation being

$$\Delta \equiv \begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} \equiv l_1m_2-l_2m_1,$$

Q takes the form

$$Q' \equiv a_0'X^4+4a_1'X^3Y+6a_2'X^2Y^2+4a_3'XY^3+a_4'Y^4$$
$$\equiv (a_0', a_1', a_2', a_3', a_4')(X, Y)^4,$$

the quadrinvariant $I \equiv a_0a_4-4a_1a_3+3a_2^2$ is of order 2 and weight 4;

the cubinvariant $J \equiv a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_4a_1^2-a_2^3$ is of order 3 and weight 6;

and if I', J' be the same functions of the new coefficients in Q', we have $I'=\Delta^4I$, $J'=\Delta^6J$, so that $I'^3/J'^2=I^3/J^2$; and this is an absolute invariant, being independent of the letters of the transformation formulae.

Now amongst the four letters l_1, m_1, l_2, m_2, there are three ratios at our choice, and sufficient, if they can be determined, to make either a_1' and a_3' both vanish, or a_0' and a_2' both vanish, and in either case we shall have a third choice between the three ratios still available for any other purpose of simplification which we may desire. The choice making a_1' and a_3' vanish is the Legendrian plan of attacking the problem of reduction. The choice making a_0' and a_2' vanish is the Weierstrassian method. The latter is the more modern and the simpler. We shall therefore consider it first.

1449. Reduction to the Weierstrassian Form.

If $a_0'=a_2'=0$, the invariants become

$$I' \equiv -4a_1'a_3', \quad J'=-a_1'^2a_4',$$

Q' becomes $\quad Y\left(4a_1'X^3-\dfrac{I'}{a_1'}XY^2-\dfrac{J'}{a_1'^2}Y^3\right),$

and a_1' still remains at our disposal.

We could make it unity by a proper final choice amongst the transformation letters. For the moment we reserve the choice. In any case we have seen that it is possible to transform Q to the form

$$Q' \equiv KY(4X^3 - g_2XY^2 - g_3Y^3),$$

where K, g_2, g_3 are certain constants which are functions of

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4;\ l_1,\ m_1,\ l_2,\ m_2.$$

1450. Now let

$$f(x) \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4,$$

and let the roots of $f(x)=0$ be α_0, α_1, α_2, α_3, so that

$$f(x) \equiv a_0(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3).$$

From what precedes it appears that by a proper choice amongst the letters l_1, m_1, l_2, m_2, in the homographic substitution $x=(l_1z+m_1)/(l_2z+m_2)$, $f(x)$ may be reduced to a form in which the term in z^4 is absent in the numerator.

Now $$x-\alpha_0 = \frac{(l_1-\alpha_0l_2)z+(m_1-\alpha_0m_2)}{l_2z+m_2},$$

and *if we make our first choice amongst the three disposable ratios* $l_1 : m_1 : l_2 : m_2$ *to be* $l_1=\alpha_0l_2$, we shall have

$$x-\alpha_0 = \frac{m_1 - \dfrac{l_1}{l_2}m_2}{l_2z+m_2} = \frac{-\Delta/l_2}{l_2z+m_2},\ \textit{i.e. } x=\alpha_0+\frac{\mu}{z-\eta}, \text{ say,}$$

and the two quantities μ, η are still at our disposal.

We now have

$$x-\alpha_1 = \alpha_0-\alpha_1+\frac{\mu}{z-\eta} = \frac{\alpha_0-\alpha_1}{z-\eta}\left(z-\eta+\frac{\mu}{\alpha_0-\alpha_1}\right),$$

$$x-\alpha_2 = \qquad\qquad = \frac{\alpha_0-\alpha_2}{z-\eta}\left(z-\eta+\frac{\mu}{\alpha_0-\alpha_2}\right),$$

$$x-\alpha_3 = \qquad\qquad = \frac{\alpha_0-\alpha_3}{z-\eta}\left(z-\eta+\frac{\mu}{\alpha_0-\alpha_3}\right),$$

and

$$f(x) = a_0\mu\frac{(\alpha_0-\alpha_1)(\alpha_0-\alpha_2)(\alpha_0-\alpha_3)}{(z-\eta)^4}$$
$$\times\left(z-\eta+\frac{\mu}{\alpha_0-\alpha_1}\right)\left(z-\eta+\frac{\mu}{\alpha_0-\alpha_2}\right)\left(z-\eta+\frac{\mu}{\alpha_0-\alpha_3}\right).$$

In order to arrange that the term in z^2 in this numerator shall be absent, we shall *make the choice of a relation between* η *and* μ, viz. that

$$3\eta=\mu\left(\frac{1}{a_0-a_1}+\frac{1}{a_0-a_2}+\frac{1}{a_0-a_3}\right),$$

and we *still have one choice left* amongst the constants at our disposal.

Moreover, since $dx=-\mu\, dz/(z-\eta)^2$, we have

$$\frac{dx}{\sqrt{f(x)}}=\frac{-\mu\, dz}{\sqrt{a_0\mu(a_0-a_1)(a_0-a_2)(a_0-a_3)}}$$

$$\times\frac{1}{\sqrt{\left(z-\eta+\dfrac{\mu}{a_0-a_1}\right)\left(z-\eta+\dfrac{\mu}{a_0-a_2}\right)\left(z-\eta+\dfrac{\mu}{a_0-a_3}\right)}}.$$

Let us *now make our final choice amongst the disposable transformation constants*, such that

$$\mu=\tfrac{1}{4}a_0(a_0-a_1)(a_0-a_2)(a_0-a_3).$$

Then, since $f(x)=a_0(x-a_0)(x-a_1)(x-a_2)(x-a_3)$, we have

$$\frac{1}{a_0}f'(x)=(x-a_1)(x-a_2)(x-a_3)+\text{terms containing }(x-a_0);$$

whence

$$\frac{1}{a_0}f'(a_0)=(a_0-a_1)(a_0-a_2)(a_0-a_3)=\frac{4\mu}{a_0};\ \therefore\ \mu=\tfrac{1}{4}f'(a_0).$$

Again,

$$\frac{1}{2a_0}f''(x)=(x-a_0)(x-a_1)+(x-a_0)(x-a_2)+(x-a_0)(x-a_3)$$
$$+(x-a_1)(x-a_2)+(x-a_1)(x-a_3)$$
$$+(x-a_2)(x-a_3);$$

whence

$$\frac{1}{2a_0}f''(a_0)=(a_0-a_2)(a_0-a_3)+(a_0-a_3)(a_0-a_1)+(a_0-a_1)(a_0-a_2);$$

and since $$\eta=\frac{1}{3}\left(\frac{\mu}{a_0-a_1}+\frac{\mu}{a_0-a_2}+\frac{\mu}{a_0-a_3}\right),$$

this gives $$\eta=\tfrac{1}{3}\cdot\tfrac{1}{4}f'(a_0)\frac{\dfrac{1}{2a_0}f''(a_0)}{\dfrac{1}{a_0}f'(a_0)},\quad \textit{i.e. } \eta=\tfrac{1}{24}f''(a_0).$$

Thus μ and η are now found, viz. $\mu=\frac{1}{4}f'(a_0)$, $\eta=\frac{1}{24}f''(a_0)$, and $\dfrac{dx}{\sqrt{f(x)}}=\dfrac{-dz}{\sqrt{4z^3-g_2z-g_3}}$, where g_2, g_3 remain to be expressed. And seeing that the relation $x=a_0+\dfrac{\mu}{z-\eta}$ gives an infinite value to z when $x=a_0$, we have

$$\int_{a_0}^{x}\frac{dx}{\sqrt{f(x)}}=\int_{z}^{\infty}\frac{dz}{\sqrt{4z^3-g_2z-g_3}}=\wp^{-1}(z, g_2, g_3);$$

and if this integral be called u, we have $z=\wp(u)$.

1451. If e_1, e_2, e_3 be the roots of $4z^3-g_2z-g_3=0$, we have

$$e_1+e_2+e_3=0,\quad e_1e_2+e_2e_3+e_3e_1=-\frac{g_2}{4},\quad e_1e_2e_3=\frac{g_3}{4}.$$

Moreover, regarding $4z^3-g_2z-g_3$ as the form assumed by the transformed quartic function $(a_0, a_1, a_2, a_3, a_4)(x, y)^4$, viz. $0\,.\,z^4+4a_1'z^3+6\,.\,0\,.\,z^2+4a_3'z+a_4'$, we have $a_1'=1$, $a_3'=-\frac{1}{4}g_2$, $a_4'=-g_3$; so that $I'=g_2$, $J'=g_3$.

Also we have

$$e_1=\eta-\frac{\mu}{a_0-a_1}=\frac{\mu}{3}\left(\frac{-2}{a_0-a_1}+\frac{1}{a_0-a_2}+\frac{1}{a_0-a_3}\right)$$

$$=\frac{1}{12}a_0[-2(a_0-a_2)(a_0-a_3)+(a_0-a_1)(a_0-a_3)+(a_0-a_1)(a_0-a_2)],$$

i.e. $$e_1=\frac{a_0}{12}[(a_0-a_2)(a_3-a_1)-(a_0-a_3)(a_1-a_2)].$$

Similarly

$$e_2=\frac{a_0}{12}[(a_0-a_3)(a_1-a_2)-(a_0-a_1)(a_2-a_3)],$$

$$e_3=\frac{a_0}{12}[(a_0-a_1)(a_2-a_3)-(a_0-a_2)(a_3-a_1)],$$

thus expressing the roots of the cubic $4z^3-g_2z-g_3=0$ in terms of the roots of the quartic $Q=0$; and therefore g_2, g_3 or what is the same thing, I' and J', are now known in terms of a_0, a_1, a_2, a_3 and a_0.

We shall now for convenience drop the accents from I and J as being no longer necessary, and these letters will therefore be for the future understood to refer to the new form of the quartic function $0\,.\,z^4+4z^3+6\,.\,0z^2-Iz-J$, and henceforth use I and J, as in the previous chapter, instead of the letters

g_2 and g_3 respectively as may be desirable, and the accents can be restored whenever we wish to institute a comparison with the corresponding symbols belonging to the original quartic Q.

1452. Our transformation is now complete, and we have

$$u=\int_{a_0}^{x}\frac{dx}{\sqrt{(a_0, a_1, a_2, a_3, a_4)(x, y)^4}}=\int_z^\infty\frac{dz}{\sqrt{4z^3-Iz-J}}$$
$$=\int_z^\infty\frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}=\wp^{-1}(z, I, J),$$

the transformation to effect the reduction being

$$x=a_0+\frac{\frac{1}{4}f'(a_0)}{z-\frac{1}{24}f''(a_0)}.$$

1453. **To find the Legendrian Moduli, the Roots of $Q=0$ being known.**

The transformation formula may be written

$$z=\eta+\frac{\mu}{x-\alpha_0};$$

we have also $\qquad e_1=\eta+\dfrac{\mu}{\alpha_1-\alpha_0},$

and $\therefore \qquad z-e_1=\dfrac{\mu}{x-\alpha_0}-\dfrac{\mu}{\alpha_1-\alpha_0}=\dfrac{\mu}{\alpha_0-\alpha_1}\dfrac{x-\alpha_1}{x-\alpha_0},$

i.e. $\qquad z-e_1=\dfrac{a_0}{4}\dfrac{f'(\alpha_0)}{\alpha_0-\alpha_1}\dfrac{x-\alpha_1}{x-\alpha_0};$

similarly $\quad z-e_2=\dfrac{a_0}{4}\dfrac{f'(\alpha_0)}{\alpha_0-\alpha_2}\dfrac{x-\alpha_2}{x-\alpha_0},\quad z-e_3=\dfrac{a_0}{4}\dfrac{f'(\alpha_0)}{\alpha_0-\alpha_3}\dfrac{x-\alpha_3}{x-\alpha_0}.$

Also the Legendrian moduli k, k' may be readily expressed in terms of α_0, α_1, α_2, α_3. For since (Art. 1414)

$$k^2=(e_2-e_3)/(e_1-e_3),\quad k'^2=(e_1-e_2)/(e_1-e_3),$$

we have

$$k^2=\frac{\dfrac{1}{\alpha_0-\alpha_3}-\dfrac{1}{\alpha_0-\alpha_2}}{\dfrac{1}{\alpha_0-\alpha_3}-\dfrac{1}{\alpha_0-\alpha_1}}=\frac{(\alpha_0-\alpha_1)(\alpha_3-\alpha_2)}{(\alpha_0-\alpha_2)(\alpha_3-\alpha_1)}=\{\alpha_0, \alpha_1, \alpha_3, \alpha_2\},$$

$$k'^2=\frac{\dfrac{1}{\alpha_0-\alpha_2}-\dfrac{1}{\alpha_0-\alpha_1}}{\dfrac{1}{\alpha_0-\alpha_3}-\dfrac{1}{\alpha_0-\alpha_1}}=\frac{(\alpha_0-\alpha_3)(\alpha_1-\alpha_2)}{(\alpha_0-\alpha_2)(\alpha_1-\alpha_3)}=\{\alpha_0, \alpha_3, \alpha_1, \alpha_2\}$$

1454. Cubic to find the Legendrian Moduli, available when the Roots of $Q=0$ are unknown.

We may obtain an equation for the determination of the moduli k and k' for the case in which none of the roots of $Q=0$ are known and are not readily obtainable.

Since $k^2=(e_2-e_3)/(e_1-e_3)$ and $k'^2=1-k^2$, we have

$$\left.\begin{aligned} k^2e_1-e_2+k'^2e_3&=0\\ e_1+e_2+e_3&=0;\end{aligned}\right\}$$

and

whence

$$\frac{e_1}{-(1+k'^2)}=\frac{e_2}{k'^2-k^2}=\frac{e_3}{1+k^2}=\frac{\sqrt{e_1e_3-e_2{}^2}}{\sqrt{3(k^2k'^2-1)}}$$

$$=\frac{\sqrt[3]{e_1e_2e_3}}{\sqrt[3]{-(1+k^2)(1+k'^2)(k'^2-k^2)}},$$

and $\qquad e_1e_3-e_2{}^2=-\frac{1}{4}I,\quad e_1e_2e_3=\frac{1}{4}J.$

Therefore $\displaystyle\sqrt{\frac{I}{12(1-k^2k'^2)}}=\sqrt[3]{\frac{J}{-4(2+k^2k'^2)(k'^2-k^2)}}.$

Writing $k^2k'^2=P$, $\displaystyle\frac{I^3}{4(1-P)^3}=27\frac{J^2}{(2+P)^2(1-4P)}=\frac{I^3-27J^2}{27P^2};$

whence $$\frac{P^2}{(1-P)^3}=\frac{4}{27}\left(1-27\frac{J^2}{I^3}\right),$$

and $\dfrac{J^2}{I^3}$ is an absolute invariant, free from the modulus of transformation, viz.

$$\begin{vmatrix} a_0, & a_1, & a_2\\ a_1, & a_2, & a_3\\ a_2, & a_3, & a_4\end{vmatrix}^2\Big/(a_0a_4-4a_1a_3+3a_2{}^2)^3.$$

when expressed in terms of the coefficients of the quartic Q.

This cubic for P may be solved by Cardan's method, and thus the product $k^2k'^2$ can be found; and as $k^2+k'^2=1$, both k and k' can be found.

1455. Illustrative Examples.

Ex. 1. Consider the integral $u\equiv\displaystyle\int_{-1}^{x}\frac{dx}{\sqrt{3x^4+17x^3+9x^2-5x}}.$

Here there are obvious roots of $f(x)=0$, viz. $x=0$ and $x=-1$,

$$f'(x)=12x^3+51x^2+18x-5,\quad f''(x)=36x^2+102x+18.$$

Taking the root $x=-1$ as α_0,

$$f'(-1)=16,\quad f''(-1)=-48,\quad \mu=\tfrac{1}{4}f'(-1)=4,\quad \eta=\tfrac{1}{24}f''(-1)=-2.$$

Hence the proper reduction formula is

$$x=a_0+\frac{\mu}{z-\eta}=-1+\frac{4}{z+2}=-\frac{z-2}{z+2}.$$

Then $f(x)=x(x+1)(3x^2+14x-5)=x(x+1)(x+5)(3x-1)$

$$=64(z-2)(z-1)(z+3)/(z+2)^4,$$

and $dx=-4dz/(z+2)^2$;

$$\therefore\ \frac{dx}{\sqrt{f(x)}}=-\frac{dz}{\sqrt{4(z-2)(z-1)(z+3)}}=-\frac{dz}{\sqrt{4z^3-28z+24}}.$$

Also $x=-1$ gives $z=\infty$;

$$\therefore\ u=\int_z^\infty\frac{dz}{\sqrt{4z^3-28z+24}}=\wp^{-1}(z,\,28,\,-24)\quad\text{and}\quad z=\wp(u).$$

In this case $e_1=2$, $e_2=1$, $e_3=-3$, $k^2=(e_2-e_3)/(e_1-e_3)=4/5$, $k'^2=1/5$,

$$\wp(u)=e_3+\frac{e_1-e_3}{\operatorname{sn}^2(u\sqrt{5})}=-3+\frac{5}{\operatorname{sn}^2(u\sqrt{5})};$$

$$\therefore\ \operatorname{sn}(u\sqrt{5})=\sqrt{5\frac{x+1}{x+5}},\quad u=\frac{1}{\sqrt{5}}\operatorname{sn}^{-1}\sqrt{5\frac{x+1}{x+5}}.$$

Ex. 2. Take the same example, and start with the root $x=0$.

Here $\quad a_0=0,\ f'(0)=-5,\ f''(0)=18,\ \mu=-5/4,\ \eta=3/4,$

$$x=-5/(4z-3),\quad dx=20\,dz/(4z-3)^2,$$

$$f(x)=1600(z-2)(z-1)(z+3)/(4z-3)^4,$$

$$\int_0^x\frac{dx}{\sqrt{f(x)}}=\int_{-\infty}^z\frac{dz}{\sqrt{4z^3-28z+24}},$$

$$u=\int_{-1}^x\frac{dx}{\sqrt{f(x)}}=\left[\int_{-1}^0+\int_0^x\right]\frac{dx}{\sqrt{f(x)}}=\left(\int_2^{-\infty}+\int_{-\infty}^z\right)\frac{dz}{\sqrt{4z^3-28z+24}}$$

$$=\int_2^z\frac{dz}{\sqrt{4z^3-28z+24}}=\left(\int_2^\infty-\int_z^\infty\right)\frac{dz}{\sqrt{4z^3-28z+24}}$$

$$=2\omega_1-\int_z^\infty\frac{dz}{\sqrt{4z^3-28z+24}}.$$

Hence $z=\wp(2\omega_1-u)=\wp(u)$, as before.

Ex. 3. Examine the same integral with the substitution $x=5\dfrac{s^2-1}{5-s^2}$.

Then $dx=\dfrac{40s\,ds}{(5-s^2)^2}$, $\quad x+1=\dfrac{4s^2}{5-s^2}$, $\quad x+5=\dfrac{20}{5-s^2}$, $\quad 3x-1=4\dfrac{4s^2-5}{5-s^2}$.

Hence $\quad u=\dfrac{1}{\sqrt{5}}\displaystyle\int_0^s\frac{ds}{\sqrt{(1-s^2)(1-\frac{4}{5}s^2)}}$; $\quad\therefore\ s=\operatorname{sn}(u\sqrt{5})$; mod. $\dfrac{2}{\sqrt{5}}$,

which agrees with the former result (Ex. 1), in which

$$\wp(u)=-3+\frac{5}{s^2}\quad\text{and}\quad x=-1+\frac{4}{\wp(u)+2}=-1+\frac{4s^2}{5-s^2}=5\frac{s^2-1}{5-s^2}.$$

1456. **Transformation for the Case of Unreal Values of the e's.**

So far e_1, e_2, e_3 have been considered real. Now suppose e_1 real and e_2, e_3 to be complementary imaginaries. Take the hyperbolic transformation $y-\eta_1=\dfrac{(x-e_2)(x-e_3)}{x-e_1}$, where η_1 is at our choice. Since $e_1+e_2+e_3=0$, we have

$$y-\eta_1=\frac{x^2+e_1x+e_2e_3}{x-e_1}=x+2e_1+\frac{e_2e_3+2e_1^2}{x-e_1}.$$

Let us choose $\eta_1=-2e_1$, *i.e.* choose the hyperbola so that the oblique asymptote passes through the origin. Then the graph of this transformation is a hyperbola with asymptotes $x=e_1$, $y=x$ and centre (e_1, e_1). Let (ξ_2, η_2), (ξ_3, η_3) be the points at which the tangent is parallel to the x-axis. These points are the ends of a diameter, and $\eta_2+\eta_3=2e_1=-\eta_1$; $\therefore\ \eta_1+\eta_2+\eta_3=0$. Moreover, ξ_1 and ξ_2, which are the roots of $\dfrac{dy}{dx}=0$, must be repeated roots of the equations $y=\eta_2$ and $y=\eta_3$ respectively, *i.e.*

$$y-\eta_2=\frac{(x-\xi_2)^2}{x-e_1} \quad \text{and} \quad y-\eta_3=\frac{(x-\xi_3)^2}{x-e_1},$$

whilst $\dfrac{dy}{dx}$, which is $1-\dfrac{e_2e_3+2e_1^2}{(x-e_1)^2}$, must take the form

$$\frac{dy}{dx}=\frac{(x-\xi_2)(x-\xi_3)}{(x-e_1)^2}.$$

Clearly the values of ξ_2, ξ_3 are $e_1\pm\sqrt{e_2e_3+2e_1^2}$.

$$\text{Thus} \int\frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}$$

$$=\int\frac{dy\,(x-e_1)^2}{(x-\xi_2)(x-\xi_3)}\cdot\frac{1}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}}$$

$$=\int\frac{(x-e_1)^2\,dy}{\sqrt{(x-e_1)(y-\eta_2)}\sqrt{(x-e_1)(y-\eta_3)}\sqrt{(4(x-e_1)^2(y-\eta_1)}}$$

$$=\int\frac{dy}{\sqrt{4(y-\eta_1)(y-\eta_2)(y-\eta_3)}},$$

in which $\eta_1+\eta_2+\eta_3=0$.

The nature of the transformation graph, in which the branches of the hyperbola cannot cut the line $y=\eta_1$, since e_2 and e_3 are imaginary, and which must therefore lie in the compartments between the asymptotes as shown in Fig. 427, establishes the fact that η_1, η_2, η_3 are essentially real quantities; $y=\eta_3$ and $y=\eta_2$ are the maximum and minimum ordinates of

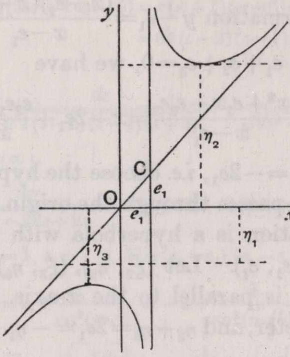

Fig. 427.

the graph, and the line $y=\eta_1=-2e_1$ is a line parallel to the x-axis at a distance twice as far below that axis as the centre is above it.

1457. **Analytical Examination of the same Transformation.**

If the roots of any cubic $a_0x^3+3a_1x^2+3a_2x+a_3=0$ be α_1, α_2, α_3, we have $a_0^6(\alpha_2-\alpha_3)^2(\alpha_3-\alpha_1)^2(\alpha_1-\alpha_2)^2=-27a_0^2\Delta$, where Δ is the discriminant, viz.

$$\Delta \equiv a_0^2a_3^2-6a_0a_1a_2a_3+4a_0a_2^3+4a_1^3a_3-3a_1^2a_2^2,$$

(Burnside and Panton, *Th. of Eq.*, p. 83.)

and the roots are all real or one real and two imaginary, according as Δ is $-^{\text{ve}}$ or $+^{\text{ve}}$.

In the case of the cubic $4x^3-Ix-J=0$, with roots e_1, e_2, e_3, we have $a_0=4$, $a_1=0$, $a_2=-\frac{1}{3}I$, $a_3=-J$; $\Delta=4^2J^2+4\,.\,4(-\frac{1}{3}I)^3=-\frac{16}{27}(I^3-27J^2)$, and
$$(e_2-e_3)^2(e_3-e_1)^2(e_1-e_2)^2=\tfrac{1}{16}(I^3-27J^2).$$

The roots are then all real or one real and two imaginary, according as I^3-27J^2 is $+^{\text{ve}}$ or $-^{\text{ve}}$. In the case we are considering, viz. one real, say e_1, and two imaginary, viz. $e_2=p+\iota q$, $e_3=p-\iota q$, p and q being real, and $e_1=-2p$, so that $e_1+e_2+e_3=0$, we have
$$I^3-27J^2=16(2\iota q)^2(9p^2+q^2)^2=-64q^2(9p^2+q^2)^2=-^{\text{ve}}.$$

But when we transform by the equation $y=x+\dfrac{R^2}{x-e_1}$, where
$$R^2=e_2e_3+2e_1^2=5p^2+q^2=+^{\text{ve}},$$
we have $\xi_2=e_1+R$, $\xi_3=e_1-R$, $\eta_2=e_1+2R$, $\eta_3=e_1-2R$, $\eta_1=-2e_1$; and in the new cubic, $4y^3-I'y-J'=0$, we have
$$I'^3-27J'^2=16(\eta_2-\eta_3)^2(\eta_3-\eta_1)^2(\eta_1-\eta_2)^2=16(4R)^2(3e_1-2R)^2(-3e_1-2R)^2$$
$$=256R^2(9e_1^2-4R^2)^2=256(5p^2+q^2)(16p^2-4q^2)^2=+^{\text{ve}}.$$

Hence all the roots of the new cubic are real.

1458. Illustrative Example.

Integrate $$u \equiv \int_x^1 \frac{dx}{\sqrt{x^4-12x^3+54x^2-100x+57}}.$$

Here $x=1$ is an obvious root of $f(x)=0$,

$$\left.\begin{aligned} f'(x) &= 4x^3-36x^2+108x-100, & f'(1)&=-24,\\ f''(x)&=12x^2-72x+108, & f''(1)&=48\,; \end{aligned}\right\}$$

$$\therefore\ \mu=\tfrac{1}{4}f'(1)=-6,\quad \eta=\tfrac{1}{24}f''(1)=2.$$

The transformation formula is $x=a_0+\dfrac{\mu}{z-\eta}=1-\dfrac{6}{z-2}$.

We also have

$$f(x)=(x-1)(x^3-11x^2+43x-57)=(x-1)(x-3)[(x-4)^2+3]\,;$$

hence two roots for x, and therefore also for z, in the transformed equation will be imaginary.

The transformation is

$$-\frac{6}{(z-2)^4}(-2)(z+1)(12)(z^2-z+1)=\frac{144}{(z-2)^4}(z^3+1)\,;$$

also $dx=\dfrac{6dz}{(z-2)^2}$; whence $\displaystyle\int_x^1\frac{dx}{\sqrt{f(x)}}=\int_z^\infty\frac{dz}{\sqrt{4z^3+4}}=\wp^{-1}(z,\,0,\,-4).$

Transform further by the rule of Art. 1456.

$$e_1=-1,\quad \eta_1=-2e_1=2,\quad y=\eta_1+\frac{z^2-z+1}{z+1}=\frac{z^2+z+3}{z+1}=z+\frac{3}{z+1},$$

and $\dfrac{dy}{dz}=1-\dfrac{3}{(z+1)^2}=0$ gives $z=\pm\sqrt{3}-1$.

Therefore $\eta_2=2\sqrt{3}-1,\quad \eta_3=-2\sqrt{3}-1\quad$ and $\quad\eta_1+\eta_2+\eta_3=0$,

$$y-\eta_2=\frac{(z-\sqrt{3}+1)^2}{z+1},\quad y-\eta_3=\frac{(z+\sqrt{3}+1)^2}{z+1}\,;$$

$$\begin{aligned}\therefore\ u\equiv\int_z^\infty\frac{dz}{\sqrt{4z^3+4}}&=\int_y^\infty\frac{dy}{(z+1)^2-3}\cdot\frac{(z+1)^2}{\sqrt{4(z+1)^2(y-\eta_1)}}\\ &=\int_y^\infty\frac{dy}{\sqrt{(z+1)(y-\eta_2)}\sqrt{(z+1)(y-\eta_3)}}\cdot\frac{z+1}{\sqrt{4(y-\eta_1)}}\\ &=\int_y^\infty\frac{dy}{\sqrt{4(y-\eta_1)(y-\eta_2)(y-\eta_3)}}=\int_y^\infty\frac{dy}{\sqrt{4(y-2)(y^2+2y-11)}}\\ &=\int_y^\infty\frac{dy}{\sqrt{4(y^3-15y+22)}}=\wp^{-1}(y,\,60,\,-88).\end{aligned}$$

In order of magnitude the values of the η's are

$$\eta_2=2\sqrt{3}-1,\quad \eta_1=2,\quad \eta_3=-2\sqrt{3}-1\,;$$

whence $$k^2=\frac{3+2\sqrt{3}}{4\sqrt{3}}=\frac{4+2\sqrt{3}}{8}=\sin^2 75^\circ.$$

Thus $y=\wp(u)=2+4\sqrt{3}\,\dfrac{\operatorname{dn}^2 2\sqrt[4]{3}u}{\operatorname{sn}^2 2\sqrt[4]{3}u}$, mod. sin 75°; whence we can express z and x in terms of u.

We have $$\operatorname{cn}^2 2\sqrt[4]{3}u=\frac{\wp(u)-2\sqrt{3}+1}{\wp(u)+2\sqrt{3}+1},$$

and $$u=\frac{1}{2\sqrt[4]{3}}\operatorname{cn}^{-1}\sqrt{\frac{y+1-2\sqrt{3}}{y+1+2\sqrt{3}}}$$

$$=\frac{1}{2\sqrt[4]{3}}\operatorname{cn}^{-1}\sqrt{\frac{2(7-5x+x^2)-\sqrt{3}(1-x)(3-x)}{2(7-5x+x^2)+\sqrt{3}(1-x)(3-x)}}, \text{ (mod. sin } 75^\circ).$$

1459. Reduction to the Legendrian Form.

We next turn to the other method of reduction referred to in Art. 1448, which endeavours to express $\int\frac{dx}{\sqrt{Q}}$ directly in the Legendrian form $\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$, $(k^2<1)$.

1460. **Preliminary Geometrical Considerations.**

It will be convenient to consider the expression Q made homogeneous by the introduction of the proper power of y where necessary, and written with binomial coefficients, as

$$Q\equiv a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4,$$

and to imagine it to have been factorised into two quadratic factors with real coefficients, as

$$Q\equiv(ax^2+2hxy+by^2)(a'x^2+2h'xy+b'y^2).$$

Consider the two concentric conics whose equations are

$$ax^2+2hxy+by^2=F, \quad a'x^2+2h'xy+b'y^2=G;$$

F and G being at our choice, we may select them so as to give real intersections P, Q, R, S, which will always be possible if one of the conics be an ellipse. Then it is plain that $PQRS$ is a parallelogram concentric with the conics, and that as PQ, QR form a pair of supplemental chords of both conics, the lines through the centre drawn parallel to the sides of the parallelogram form a common pair of conjugate diameters, viz. OX, OY It is therefore possible by a change of axes, to the axes OX, OY, to remove the term in XY in each of the two conics simultaneously by the same linear transformation, viz. $(x=\lambda X+\mu Y,\ y=\lambda' X+\mu' Y)$, say; λ, μ, λ', μ' being all *real* when one of the two conics is an ellipse, or when both of them are ellipses; and the conics becoming

$$AX^2+BY^2=F, \quad A'X^2+B'Y^2=G,$$

Q can thus be reduced to the form

$$Q' \equiv (AX^2+BY^2)(A'X^2+B'Y^2),$$

or, as we may write it,

$$Q' \equiv A_0X^4+6A_2X^2Y^2+A_4Y^4.$$

We may obviously make a further reduction by putting $X\sqrt[4]{A_0}=\xi$, $Y\sqrt[4]{A_4}=\eta$, thus reducing the quartic Q to the canonical form

$$Q \equiv \xi^4+6\lambda\xi^2\eta^2+\eta^4.$$

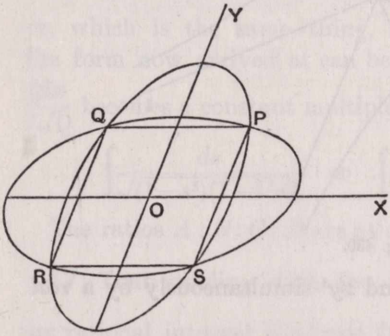

Fig. 428.

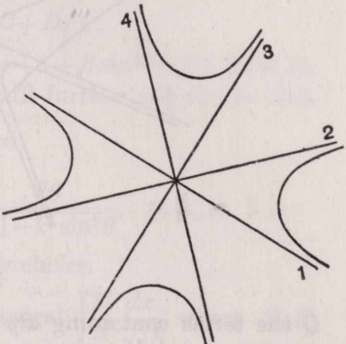

Fig. 429.

If both conics be hyperbolae, the common conjugate diameters may be imaginary lines. But in any case their equations are

$$\begin{vmatrix} x^2, & xy, & y^2 \\ b, & -h, & a \\ b', & -h', & a' \end{vmatrix}=0.$$

(Smith, *Conic Sections*, p. 196.)

We may, however, readily avoid an imaginary transformation. For, as has been seen, the only case in which it could occur would be that in which both conics are hyperbolae, as in the case shown in Fig. 429, where there are no real intersections. In this case the factors of Q are all linear. Call them (1), (2), (3), (4). Then, instead of taking the hyperbolae $(1)(2)=F$, $(3)(4)=G$, we might take the hyperbolae $(1)(4)=F$, $(2)(3)=G$ (Fig. 430), and with a proper choice of F and G we can ensure real intersections and real common conjugate axes to which we can refer the system. We infer therefore from these considerations that it is always possible to remove from

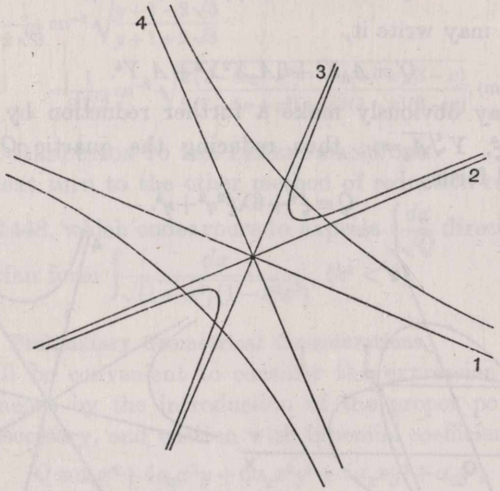

Fig. 430.

Q the terms containing x^3y and xy^3 simultaneously by a *real* linear transformation.

1461. If in the transformation formulae

$$x=\lambda X+\mu Y, \quad y=\lambda' X+\mu' Y,$$

we write $\lambda' X=\xi$, $\mu' Y=\eta$, the formulae take the simpler shape $x=\lambda_1\xi+\mu_1\eta$, $y=\xi+\eta$. It follows, therefore, that it is always possible, by a *real* substitution $x=(p+qz)/(1+z)$, to reduce Q from the general quartic form

$$Q\equiv a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4,$$

to the form $\quad Q\equiv(A_1z^2+B_1)(A_2z^2+B_2)/(1+z)^4;$

and since $dx=(q-p)\,dz/(1+z)^2$, we have

$$\frac{dx}{\sqrt{Q}}=(q-p)\frac{dz}{\sqrt{(A_1z^2+B_1)(A_2z^2+B_2)}},$$

and the values of p, q are in all cases real.

1462. **Outline of the Process of Transformation.**

As the whole discussion is necessarily somewhat lengthy, we may with advantage stop for a moment to outline what is to be done.

I. It has been shown that when $a_0 \neq 0$, we can always, by the transformation $x=(p+qz)/(1+z)$, remove odd powers of the variable from the radical, p and q being real.

It remains to show how the necessary values of p and q are to be found.

II. We shall show that the same transformation will also reduce the integral to the desired form in the case when $a_0=0$.

III. That by a further transformation

$$z^2=(A+Bs^2)/(C+Ds^2),$$

or, which is the same thing, $z^2=(A+B\sin^2\theta)/(C+D\sin^2\theta)$, the form now arrived at can be still further reduced so that $\int\frac{dx}{\sqrt{Q}}$ becomes a constant multiple of

$$\int\frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \quad\text{or}\quad \int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad (k<1).$$

The ratios $A:B:C:D$ are at our choice.

IV. That starting with the integral $\int\frac{M}{N}\frac{dx}{\sqrt{Q}}$, where M, N are rational integral algebraic functions of x, we obtain after the transformation $x=(p+qz)/(1+z)$ a result of form

$$\int\frac{[\phi(z^2)+z\psi(z^2)]\,dz}{\sqrt{(A_1z^2+B_1)(A_2z^2+B_2)}},$$

and that whilst $\int\frac{z\psi(z^2)\,dz}{\sqrt{(A_1z^2+B_1)(A_2z^2+B_2)}}$ can be reduced by earlier rules, the portion $\int\frac{\phi(z^2)\,dz}{\sqrt{(A_1z^2+B_1)(A_2z^2+B_2)}}$ can be expressed by means of Legendre's Integrals, and that therefore by these means $\int\frac{M\,dx}{N\sqrt{Q}}$ can in all cases be reduced to a system of algebraic, logarithmic, circular or hyperbolic functions together with one or more of the three standard Legendrian forms F, E or Π.

Hence, as in Art. 318, the integral $\int\frac{A+B\sqrt{Q}}{C+D\sqrt{Q}}dx$, where A, B, C, D are rational algebraic functions of x, and Q is now a rational quartic expression, can be reduced to the sum of a similar set of terms by aid of the elliptic functions now described.

1463. I. **First consider** $a_0 \neq 0$ and imagine Q to be factorised into two quadratic factors with real coefficients, as

$$Q = a_0(x^2+2\lambda x+\mu)(x^2+2\lambda' x+\mu').$$

Then putting $x=(p+qz)/(1+z)$,

$$x^2+2\lambda x+\mu=[(p+qz)^2+2\lambda(p+qz)(1+z)+\mu(1+z)^2]/(1+z)^2$$
$$=H(z^2+2fz+g)/(1+z)^2, \text{ where } H\equiv q^2+2\lambda q+\mu,$$

and $$\frac{1}{H}=\frac{f}{pq+\lambda(p+q)+\mu}=\frac{g}{p^2+2\lambda p+\mu}.$$

Similarly, $x^2+2\lambda' x+\mu'=H'(z^2+2f'z+g')/(1+z)^2$,

where H', f', g' are the same functions of p, q, λ', μ', as H, f, g are of p, q, λ, μ.

Hence $Q\equiv a_0 HH'(z^2+2fz+g)(z^2+2f'z+g')/(1+z)^4$.

We shall be able to make f and f' zero by taking p and q so that

$$pq+\lambda(p+q)+\mu=0 \quad\text{and}\quad pq+\lambda'(p+q)+\mu'=0,$$

i.e. $$\frac{pq}{\lambda\mu'-\lambda'\mu}=\frac{p+q}{\mu-\mu'}=\frac{1}{\lambda'-\lambda}=\frac{p-q}{\sqrt{(\mu-\mu')^2-4(\lambda'-\lambda)(\lambda\mu'-\lambda'\mu)}}.$$

Now $(\mu-\mu')^2-4(\lambda'-\lambda)(\lambda\mu'-\lambda'\mu)$

$$\equiv(\mu+\mu'-2\lambda\lambda')^2-4(\mu-\lambda^2)(\mu'-\lambda'^2)=K^2, \text{ say.}$$

So $p+q=(\mu-\mu')/(\lambda'-\lambda)$ and $p-q=K/(\lambda'-\lambda)$, whence p and q are found.

This completely determines the necessary transformation, and we shall show that K is real; so that in all cases p and q are real.

The form of Q is now reduced to

$$Q\equiv a_0 HH'(z^2+g)(z^2+g')/(1+z)^4.$$

Also $dz=(q-p)\,dz/(1+z)^2$.

Therefore $$\frac{dx}{\sqrt{Q}}=\frac{q-p}{\sqrt{a_0 HH'}}\cdot\frac{dz}{\sqrt{(z^2+g)(z^2+g')}}.$$

1464. Next, to examine the Reality of K.

(i) When the roots of $Q=0$ are all imaginary, $\lambda^2<\mu$ and $\lambda'^2<\mu'$.

Let $\mu=\lambda^2+\rho^2$, $\mu'=\lambda'^2+\rho'^2$. Then

$$K^2=(\mu+\mu'-2\lambda\lambda')^2-4(\mu-\lambda^2)(\mu'-\lambda'^2)$$
$$=(\lambda^2+\rho^2+\lambda'^2+\rho'^2-2\lambda\lambda')^2-4\rho^2\rho'^2$$
$$=[(\lambda-\lambda')^2+(\rho-\rho')^2]\cdot[(\lambda-\lambda')^2+(\rho+\rho')^2]$$

and is essentially positive. Hence K is real and p, q both real.

(ii) When $Q=0$ has two real roots and two imaginary, $\lambda^2-\mu$ and $\lambda'^2-\mu'$ have opposite signs, and

$$K^2\equiv(\mu+\mu'-2\lambda\lambda')^2-4(\mu-\lambda^2)(\mu'-\lambda'^2)$$
$$=(\mu+\mu'-2\lambda\lambda')^2+\text{ a positive quantity }=+^{\text{ve}}.$$

Hence K is real, and therefore also p, q are both real.

(iii) When the roots of $Q=0$ are all real, say a_1, a_2, a_3, a_4 arranged in descending order of magnitude, we may take

$$2\lambda=-(a_1+a_2),\ \mu=a_1a_2,\ 2\lambda'=-(a_3+a_4),\ \mu'=a_3a_4;$$

$$\therefore\ K^2=(\mu+\mu'-2\lambda\lambda')^2-4(\mu-\lambda^2)(\mu'-\lambda'^2)$$
$$=[a_1a_2+a_3a_4-\tfrac{1}{2}(a_1+a_2)(a_3+a_4)]^2$$
$$-\tfrac{1}{4}[4a_1a_2-(a_1+a_2)^2]\cdot[4a_3a_4-(a_3+a_4)^2]$$
$$=(a_1-a_4)(a_2-a_3)(a_1-a_3)(a_2-a_4),$$

which is again positive, and therefore K, p, q are all real.

In the case $f=f'$, we may put $z+f=u$.

Then $Q\equiv a_0HH'(u^2+g-f^2)(u^2+g'-f^2)$, and the required form is taken without further reduction.

1465. II. **Case when $a_0=0$.**

In this case $\quad Q\equiv 4a_1x^3+6a_2x^2+4a_3x+a_4$.

The case $a_1=0$ need not be considered, as the integral would then reduce to a standard form.

One factor of Q must now be real. Let ϵ be the real root of $Q=0$.

Then $Q\equiv 4a_1(x-\epsilon)(x^2+2\lambda x+\mu)$, say. Then, putting

$$x=(p+qz)/(1+z),\text{ as before,}$$

$$x-\epsilon=[(p-\epsilon)+(q-\epsilon)z](1+z)/(1+z)^2=H'(z^2+2f'z+g')/(1+z)^2$$

say, and $x^2+2\lambda x+\mu=H(z^2+2fz+g)/(1+z)^2$, as before. Then proceeding as in Art. 1463,

$$H'=q-\epsilon, \quad 2H'f'=p+q-2\epsilon, \quad H'g'=p-\epsilon;$$

and making $f=f'=0$, $p+q=2\epsilon$ and $pq+\lambda(p+q)+\mu=0$

Therefore $p+q=2\epsilon$, $pq=-2\epsilon\lambda-\mu$, whence

$$p-q=2\sqrt{(\epsilon+\lambda)^2+\mu-\lambda^2}.$$

Thus, (i) if the factors of $x^2+2\lambda x+\mu$ be imaginary, $\lambda^2<\mu$, $p-q$ is real, and therefore p, q are both real;

(ii) if the factors of $x^2+2\lambda x+\mu$ be real, let the roots of $Q=0$ be e_1, e_2, e_3, arranged in descending order of magnitude. Then we may take $\epsilon=e_1$, $\lambda=-\dfrac{e_2+e_3}{2}$, $\mu=e_2e_3$, and

$$p-q=2\sqrt{[\{e_1-\tfrac{1}{2}(e_2+e_3)\}^2+e_2e_3-\tfrac{1}{4}(e_2+e_3)^2]}=2\sqrt{(e_1-e_2)(e_1-e_3)},$$

which is real, since $e_1>e_2>e_3$; and p, q are real in this case also. And the rest of Art. 1463 still applies, and the reduction to the Legendrian form is effected as before, Q becoming

$$4a_1HH'(z^2+g)(z^2+g')/(1+z)^4$$

and
$$\frac{dx}{\sqrt{Q}}=\frac{q-p}{\sqrt{4a_1HH'}}\frac{dz}{\sqrt{(z^2+g)(z^2+g')}}.$$

1466. We have therefore in all cases reduced the differential $\dfrac{dx}{\sqrt{Q}}$ to one of the forms $C\dfrac{dz}{\sqrt{\pm(z^2\pm\alpha^2)(z^2\pm\beta^2)}}$, where C may be taken a *real* constant function of a_0, a_1, a_2, a_3, a_4 of known value and α, β both real. For if $\sqrt{a_0HH'}$ or $\sqrt{4a_1HH'}$ be of unreal form, we may replace them by $\sqrt{-a_0HH'}$ or $\sqrt{-4a_1HH'}$ carrying the negative sign into the other radical.

The case $\sqrt{-(z^2+\alpha^2)(z^2+\beta^2)}$ is obviously unreal and need not be discussed, as we are now dealing with real functions.

1467. III. We have therefore only to consider the reduction of the five cases:

(1) $\sqrt{+(z^2-\alpha^2)(z^2-\beta^2)}$; (2) $\sqrt{-(z^2-\alpha^2)(z^2-\beta^2)}$;

(3) $\sqrt{+(z^2+\alpha^2)(z^2-\beta^2)}$; (4) $\sqrt{-(z^2+\alpha^2)(z^2-\beta^2)}$;

(5) $\sqrt{+(z^2+\alpha^2)(z^2+\beta^2)}$.

The final substitutions to reduce these five cases are all of the form $z^2=(A+B\sin^2\theta)/(C+D\sin^2\theta)$, where the values of the ratios $A:B:C:D$ are to be suitably chosen. We consider each case in detail.

1468. Case (1), $\sqrt{(z^2-\alpha^2)(z^2-\beta^2)}$; $\alpha^2>\beta^2$. This is unreal if z^2 lies between α^2 and β^2.

(i) $\alpha>\beta>z$. Put $z=\beta\sin\theta$, $k=\beta/\alpha$.

$$u=\int_0^z\frac{dz}{\sqrt{(z^2-\alpha^2)(z^2-\beta^2)}}=\frac{1}{\beta}\int_0^\theta\frac{\beta\cos\theta\,d\theta}{\sqrt{(\alpha^2-\beta^2\sin^2\theta)\cos^2\theta}}$$

$$=\frac{1}{\alpha}\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{1}{\alpha}\,\mathrm{am}^{-1}\theta.$$

Hence $z=\beta\,\mathrm{sn}\,\alpha u$; mod. β/α.

(ii) $z>\alpha>\beta$. Put $z=\alpha\,\mathrm{cosec}\,\theta$, $k=\beta/\alpha$.

$$u=\int_z^\infty\frac{dz}{\sqrt{(z^2-\alpha^2)(z^2-\beta^2)}}=-\frac{1}{\alpha}\int_0^\theta\frac{-\alpha\,\mathrm{cosec}\,\theta\cot\theta\,d\theta}{\sqrt{\cot^2\theta(\alpha^2\mathrm{cosec}^2\theta-\beta^2)}}$$

$$=\quad\frac{1}{\alpha}\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{1}{\alpha}\,\mathrm{am}^{-1}\theta.$$

Hence $z=\alpha/\mathrm{sn}\,\alpha u$; mod. β/α.

Also $$u'=\int_\alpha^z\frac{dz}{\sqrt{(z^2-\alpha^2)(z^2-\beta^2)}}=-\frac{1}{\alpha}\int_{\frac{\pi}{2}}^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$=\frac{1}{\alpha}\left(\int_0^{\frac{\pi}{2}}-\int_0^\theta\right)\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{1}{\alpha}\,(K-\mathrm{am}^{-1}\theta),$$

where K is the complete elliptic integral.

Hence $z=\alpha/\mathrm{sn}(K-\alpha u')=\alpha\,\mathrm{dn}(\alpha u')/\mathrm{cn}(\alpha u')$.

1469. Case (2), $\sqrt{-(z^2-\alpha^2)(z^2-\beta^2)}$; $\alpha^2>\beta^2$. This is unreal if z^2 *does not* lie between α^2 and β^2.

Put $z^2=\alpha^2-(\alpha^2-\beta^2)\sin^2\theta$, *i.e.* $\alpha^2\cos^2\theta+\beta^2\sin^2\theta$.

Then $\alpha^2-z^2=(\alpha^2-\beta^2)\sin^2\theta$, $\quad z^2-\beta^2=(\alpha^2-\beta^2)\cos^2\theta$,

$$dz=-(\alpha^2-\beta^2)\frac{\sin\theta\cos\theta\,d\theta}{\sqrt{\alpha^2-(\alpha^2-\beta^2)\sin^2\theta}},$$

$$u=\int_z^\alpha\frac{dz}{\sqrt{-(z^2-\alpha^2)(z^2-\beta^2)}}=\frac{1}{\alpha}\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{1}{\alpha}\,\mathrm{am}^{-1}\theta,$$

where $k^2=\dfrac{\alpha^2-\beta^2}{\alpha^2}$, $k'^2=\dfrac{\beta^2}{\alpha^2}$.

Hence

$$z^2=\alpha^2\operatorname{cn}^2(\alpha u)+\beta^2\operatorname{sn}^2(\alpha u),\quad i.e.\ z=\alpha\operatorname{dn}(\alpha u),\quad \text{mod. } \sqrt{1-\frac{\beta^2}{\alpha^2}}.$$

1470. Case (3), $\sqrt{(z^2+\alpha^2)(z^2-\beta^2)}$. This is unreal unless $z^2>\beta^2$. Put $z=\beta\sec\theta$.

$$u=\int_\beta^z\frac{dz}{\sqrt{(z^2+\alpha^2)(z^2-\beta^2)}}=\int_0^\theta\frac{\beta\sec\theta\tan\theta\,d\theta}{\sqrt{\beta^2\tan^2\theta(\beta^2\sec^2\theta+\alpha^2)}}$$

$$=\int_0^\theta\frac{d\theta}{\sqrt{\beta^2+\alpha^2\cos^2\theta}}=\frac{1}{\sqrt{\alpha^2+\beta^2}}\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\ \left(k^2=\frac{\alpha^2}{\alpha^2+\beta^2}\right),$$

$$=\frac{k}{\alpha}\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{k}{\alpha}\operatorname{am}^{-1}\theta.$$

Hence $z=\beta/\operatorname{cn}\left(\frac{\alpha u}{k}\right)$.

1471. Case (4), $\sqrt{-(z^2+\alpha^2)(z^2-\beta^2)}$. This is unreal unless $z^2<\beta^2$. Put $z=\beta\cos\theta$.

$$u=\int_z^\beta\frac{dz}{\sqrt{-(z^2+\alpha^2)(z^2-\beta^2)}}=\int_\theta^0\frac{-\beta\sin\theta\,d\theta}{\sqrt{\beta^2\sin^2\theta(\alpha^2+\beta^2\cos^2\theta)}}$$

$$=\frac{1}{\sqrt{\alpha^2+\beta^2}}\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{k}{\beta}\operatorname{am}^{-1}\theta,\ \left(k^2=\frac{\beta^2}{\alpha^2+\beta^2}\right).$$

Hence $z=\beta\operatorname{cn}\left(\frac{\beta u}{k}\right)$, mod. $\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$.

1472. Case (5), $\sqrt{(z^2+\alpha^2)(z^2+\beta^2)}$; $\alpha^2>\beta^2$. Put $z=\beta\tan\theta$.

$$u=\int_0^z\frac{dz}{\sqrt{(z^2+\alpha^2)(z^2+\beta^2)}}=\int_0^\theta\frac{d\theta}{\sqrt{\beta^2\sin^2\theta+\alpha^2\cos^2\theta}}$$

$$=\frac{1}{\alpha}\int_0^\theta\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{1}{\alpha}\operatorname{am}^{-1}\theta,\ \left(k^2=\frac{\alpha^2-\beta^2}{\alpha^2}\right).$$

Hence $z=\beta\operatorname{tn}(\alpha u)\left(\text{mod. }\sqrt{1-\frac{\beta^2}{\alpha^2}}\right)$.

For convenience of reference we exhibit these cases in tabular form:

1473. Table of Substitutions, Etc.

Case.	$\sqrt{Q}$.	Limitation of z.	Substitution.	Mod. k.	Value of $u \equiv \int \frac{dz}{\sqrt{Q}}$.	Direct Form.
1	$\sqrt{(z^2-\alpha^2)(z^2-\beta^2)}$, $\alpha^2 > \beta^2$	$\alpha > \beta > z$	$z=\beta x$ $=\beta \sin\theta$	$\frac{\beta}{\alpha}$	$u=\int_0^z \frac{dz}{\sqrt{Q}}=\frac{1}{\alpha}\,\mathrm{am}^{-1}\theta$	$z=\beta\,\mathrm{sn}\,(\alpha u)$
		$z > \alpha > \beta$	$z=\alpha/x$ $=\alpha/\sin\theta$	$\frac{\beta}{\alpha}$	$u=\int_z^\infty \frac{dz}{\sqrt{Q}}=\frac{1}{\alpha}\,\mathrm{am}^{-1}\theta$	$z=\alpha/\mathrm{sn}\,(\alpha u)$
					$u'=\int_\alpha^z \frac{dz}{\sqrt{Q}}=\frac{1}{\alpha}(K-\mathrm{am}^{-1}\theta)$	$z=\alpha\,\mathrm{dn}\,(\alpha u')/\mathrm{cn}(\alpha u')$
2	$\sqrt{-(z^2-\alpha^2)(z^2-\beta^2)}$, $\alpha^2 > \beta^2$	$\alpha > z > \beta$	$z^2=\alpha^2-(\alpha^2-\beta^2)x^2$ $=\alpha^2\cos^2\theta+\beta^2\sin^2\theta$	$\sqrt{1-\frac{\beta^2}{\alpha^2}}$	$u=\int_z^\alpha \frac{dz}{\sqrt{Q}}=\frac{1}{\alpha}\,\mathrm{am}^{-1}\theta$	$z=\alpha\,\mathrm{dn}\,(\alpha u)$
3	$\sqrt{(z^2+\alpha^2)(z^2-\beta^2)}$	$z > \beta$	$z=\beta/\sqrt{1-x^2}$ $=\beta\sec\theta$	$\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$	$u=\int_\beta^z \frac{dz}{\sqrt{Q}}=\frac{k}{\alpha}\,\mathrm{am}^{-1}\theta$	$z=\beta/\mathrm{cn}\left(\frac{\alpha u}{k}\right)$
4	$\sqrt{-(z^2+\alpha^2)(z^2-\beta^2)}$	$z < \beta$	$z=\beta\sqrt{1-x^2}$ $=\beta\cos\theta$	$\frac{\beta}{\sqrt{\alpha^2+\beta^2}}$	$u=\int_z^\beta \frac{dz}{\sqrt{Q}}=\frac{k}{\beta}\,\mathrm{am}^{-1}\theta$	$z=\beta\,\mathrm{cn}\left(\frac{\beta u}{k}\right)$
5	$\sqrt{(z^2+\alpha^2)(z^2+\beta^2)}$	$\alpha > \beta$	$z=\beta x/\sqrt{1-x^2}$ $=\beta\tan\theta$	$\sqrt{1-\frac{\beta^2}{\alpha^2}}$	$u=\int_0^z \frac{dz}{\sqrt{Q}}=\frac{1}{\alpha}\,\mathrm{am}^{-1}\theta$	$z=\beta\,\mathrm{tn}\,(\alpha u)$

In all cases the substitutions are cases of $z^2=(A+B\sin^2\theta)/(C+D\sin^2\theta)$.

1474. The More General Case $\int \frac{M}{N}\frac{dx}{\sqrt{Q}}$.

Here M, N are any rational algebraic functions of x, and Q, as before, $=(a_0, a_1, a_2, a_3, a_4)(x, 1)^4$.

By a proper choice of p, q, the transformation

$$x=(p+qz)/(1+z)$$

has removed terms of odd degree from Q'. M/N becomes a rational algebraic function of z separable into two parts, the one an even, the other an odd function of z, expressible as

$$M/N \equiv \phi(z^2)+z\chi(z^2).$$

Hence $\int \frac{M}{N}\frac{dx}{\sqrt{Q}}$ is reducible to $\int \frac{\phi(z^2)}{\sqrt{Q'}}dz+\int \frac{z\chi(z^2)}{\sqrt{Q'}}dz$.

By putting $z^2=y$ the second integral is immediately reduced to a form integrable by earlier rules.

We have therefore only to consider the first integral.

Now $\phi(z^2)$ is itself separable into two parts, the first integral, the second fractional, and is expressible as

$$\phi(z^2) \equiv \Sigma\lambda z^{2r}+\Sigma\frac{\lambda'}{(\mu+\nu z^2)^s}.$$

But both $\int \frac{z^{2r}}{\sqrt{Q'}}dz$ and $\int \frac{dz}{(\mu+\nu z^2)^s\sqrt{Q'}}$ can, by integration by parts, or the use of reduction formulae, be connected with the integrals

$$\int \frac{dz}{\sqrt{Q'}},\quad \int \frac{z^2\,dz}{\sqrt{Q'}},\quad \int \frac{dz}{(\mu+\nu z^2)\sqrt{Q'}}\quad \text{(Arts. 271 to 274).}$$

Accordingly all functions of form $\int \frac{M}{N}\frac{dx}{\sqrt{Q}}$, where M, N, Q are of the forms specified, can be reduced to a series of known integrals, together with one or more of the integrals

$$\text{(i)}\ \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\quad \text{(ii)}\ \int_0^x \frac{x^2\,dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

$$\text{(iii)}\ \int_0^x \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}.$$

The second of these, viz.

$$=\frac{1}{k^2}\int_0^x \frac{1-(1-k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}}\,dx$$

$$=\frac{1}{k^2}\times(\text{first integral})-\frac{1}{k^2}\int_0^x\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

$$=\frac{1}{k^2}\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}-\frac{1}{k^2}\int_0^\theta\sqrt{1-k^2\sin^2\theta\,d\theta}.$$

Therefore any such integration as $\int\frac{M}{N}\frac{dx}{\sqrt{Q}}$ can be effected by aid of the three standard Legendrian forms

$$F(\theta, k),\quad E(\theta, k),\quad \Pi(\theta, k, n)\,;\ k<1.\quad \text{(See Art. 371.)}$$

The same is true of the more general form

$$\int\frac{A+B\sqrt{Q}}{C+D\sqrt{Q}}\,dx$$

discussed in Art. 1443.

1475. The Case when the Factorisation of Q is unknown.

To effect the foregoing reduction, a knowledge of the factorisation of the quartic Q has been presupposed. When there is a preliminary difficulty in this factorisation, we may still obtain the desired form by a use of the invariants I and J. Suppose the quartic made homogeneous by the introduction of a suitable power of y, and expressed as

$$Q\equiv a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4$$
$$\equiv(a_0, a_1, a_2, a_3, a_4)(x, y)^4,$$

and let it be reduced by the linear transformation

$$x=l_1X+m_1Y,\quad y=l_2X+m_2Y$$

to the form $\quad Q'\equiv(a_0', a_1', a_2', a_3', a_4')(X, Y)^4.$

Let $\Delta\equiv l_1m_2-l_2m_1$, viz. the modulus of the transformation.

Then $$x\,dy-y\,dx=\Delta(X\,dY-Y\,dX)$$

and $$\frac{x\,dy-y\,dx}{\sqrt{Q}}=\Delta\frac{X\,dY-Y\,dX}{\sqrt{Q'}},$$

i.e. writing $x/y=u,\ X/Y=U$,

$$\frac{du}{\sqrt{(a_0, a_1, a_2, a_3, a_4)(u, 1)^4}}=\Delta\frac{dU}{\sqrt{(a_0', a_1', a_2', a_2', a_4')(U, 1)^4}},$$

where $$u=\frac{l_1U+m_1}{l_2U+m_2}.$$

Also

$I \equiv a_0 a_4 - 4a_1 a_3 + 3a_2^2, \qquad J \equiv a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_4 a_1^2 - a_2^3$

are connected with I', J', the same functions of the accented letters, by the relations $I' = \Delta^4 I$, $J' = \Delta^6 J$, whence $I^3/J^2 = I'^3/J'^2$, in which we have an absolute invariant free from the coefficients of the transformation formulae.

Supposing the ratios $l_1 : m_1 : l_2 : m_2$ to have been so chosen as to make $a_1' = 0$ and $a_3' = 0$, as has been shown to be possible, with real values of these ratios, Q' takes the form

$$a_0' U^4 + 6a_2' U^2 + a_4',$$

which can now be supposed expressed as

$$a_0'(U^2 + p)(U^2 + q),$$

and we have to show that p, q can be found in terms of the original coefficients a_0, a_1, a_2, a_3, a_4.

We have

$$a_0' = a_0', \quad a_1' = 0, \quad 6a_2' = a_0'(p+q), \quad a_3' = 0, \quad a_4' = a_0' pq.$$

$$I' = a_0' \cdot a_0' pq + \frac{1}{12} a_0'^2 (p+q)^2 = \frac{a_0'^2}{12}[(p+q)^2 + 12pq],$$

$$J' = a_0' \cdot \frac{a_0'}{6}(p+q) \cdot a_0' pq - \frac{a_0'^3}{6^3}(p+q)^3 = \frac{a_0'^3}{216}(p+q)[36pq - (p+q)^2];$$

$$\therefore \frac{I^3}{J^2} = \frac{I'^3}{J'^2} = 27\frac{[(p+q)^2 + 12pq]^3}{(p+q)^2[36pq - (p+q)^2]^2};$$

whence $\dfrac{I^3 - 27J^2}{4 \cdot 27 \cdot I^3} = \dfrac{pq(p-q)^4}{[(p+q)^2 + 12pq]^3}$;

or putting $p = \rho q$,

$$\frac{\rho(\rho-1)^4}{(\rho^2 + 14\rho + 1)^3} = \frac{I^3 - 27J^2}{4 \cdot 27 I^3} = \frac{1}{16K}, \text{ say},$$

where $K = \dfrac{27}{4}\dfrac{I^3}{I^3 - 27J^2}$, and is a known function of the original coefficients. This is a sextic equation to find ρ, viz. the ratio of $p : q$.

1476. **Solution of the Sextic.**

The equation is obviously of the reciprocal class, and therefore its solution may be reduced to that of a cubic, and the cubic may be solved by Cardan's method.

Writing the equation as $\dfrac{(\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^4}{(\rho + \rho^{-1} + 14)^3} = \dfrac{1}{16K}$, put $(\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})^2 = \dfrac{16}{\theta - 1}$.

Then $\rho+\rho^{-1}+14=16\dfrac{\theta}{\theta-1}$, and the equation becomes

$$\left(\frac{16}{\theta-1}\right)^2\bigg/\left(\frac{16\theta}{\theta-1}\right)^3=\frac{1}{16K}\;;\;\textit{i.e.}\;\;\theta^3=K(\theta-1).$$

Now adopting Cardan's method, put $\theta=\eta+\zeta$; then

$$\eta^3+\zeta^3+(3\eta\zeta-K)(\eta+\zeta)+K=0\;;$$

and taking $\eta\zeta=\frac{1}{3}K$,

$$\eta^3+\frac{K^3}{3^3}\,\frac{1}{\eta^3}+K=0, \text{ a quadratic for } \eta.$$

Hence η and ζ can be found, and therefore also θ. Suppose θ_1 a real root of this equation, then $\rho^{\frac{1}{2}}-\rho^{-\frac{1}{2}}=4/\sqrt{\theta_1-1}$, and therefore

$$\rho^{\frac{1}{2}}+\rho^{-\frac{1}{2}}=2\sqrt{\theta_1+3}/\sqrt{\theta_1-1}.$$

Thus $\sqrt{\rho}=(2+\sqrt{\theta_1+3})/\sqrt{\theta_1-1}$ and $\rho=(7+\theta_1+4\sqrt{\theta_1+3})/(\theta_1-1)$.

Then a value of the ratio $p:q$ has been found, say $p_1:q_1$, where p_1, q_1 are specifically known numbers, so that $p/p_1=q/q_1=s$, say, *which remains to be found.*

Thus
$$\frac{du}{\sqrt{(a_0,\,a_1,\,a_2,\,a_3,\,a_4)(u,\,1)^4}}=\frac{\Delta}{\sqrt{a_0}}\,\frac{dU}{\sqrt{(U^2+p_1s)(U^2+q_1s)}}.$$

Putting $U=\sqrt{s}U'$, we have

$$\frac{du}{\sqrt{(a_0,\,a_1,\,a_2,\,a_3,\,a_4)(u,\,1)^4}}=\frac{\Delta}{s\sqrt{a_0}}\,\frac{\sqrt{s}\,dU'}{\sqrt{(U'^2+p_1)(U'^2+q_1)}}.$$

Finally, $\Delta=\sqrt[4]{\dfrac{I'}{I}}=\sqrt[4]{\dfrac{a_0'^2}{12I}}(p^2+14pq+q^2)=\sqrt[4]{\dfrac{a_0'^2s^2}{12I}(p_1^2+14p_1q_1+q_1^2)}$;

whence $\dfrac{\Delta}{\sqrt{a_0s}}=\sqrt[4]{\dfrac{p_1^2+14p_1q_1+q_1^2}{12I}}$, *and s is now known*, which completes the determination of p and q. We therefore have

$$\int\frac{du}{\sqrt{(a_0,\,a_1,\,a_2,\,a_3,\,a_4)(u,\,1)^4}}=\sqrt[4]{\frac{p_1^2+14p_1q_1+q_1^2}{12I}}\int\frac{dU'}{\sqrt{(U'^2+p_1)(U'^2+q_1)}}.$$

1477. Cayley points out that if one of the roots of the sextic for ρ be $\rho=\alpha=\beta^4$, the equation is of the form $\dfrac{(\rho^2+14\rho+1)^3}{\rho(\rho-1)^4}=\dfrac{(\alpha^2+14\alpha+1)^3}{\alpha(\alpha-1)^4}$, and that the solutions of the equation may be written

$$\beta^4,\;\;\frac{1}{\beta^4},\;\;\left(\frac{1-\beta}{1+\beta}\right)^4,\;\;\left(\frac{1+\beta}{1-\beta}\right)^4,\;\;\left(\frac{1-\iota\beta}{1+\iota\beta}\right)^4,\;\;\left(\frac{1+\iota\beta}{1-\iota\beta}\right)^4,$$

which the reader may verify. [*Elliptic Functions*, p. 320.]

1478. When a reduction to the form

$$\int\frac{dU}{\sqrt{a_0'U^4+6a_2'U^2+a_4'}}\equiv\int\frac{dU}{\sqrt{a_0'(U^2+p)(U^2+q)}}$$

has been effected, then in case p and q are both real, *i.e.* $9a_2'^2>a_0'a_4'$, this factorisation will suffice. But in a case when p and q are imaginary, *i.e.* $9a_2'^2<a_0'a_4'$, we put $U=\lambda\sqrt{(1+T)/(1-T)}$, and we observe that a_0', a_4' could not be opposite signs, for if so $9a_2'^2>a_0'a_4'$.

We shall choose $\lambda=\sqrt[4]{\dfrac{a_4'}{a_0'}}$, which will be real. We have

$$dU=\lambda\frac{dT}{(1+T)^{\frac{1}{2}}(1-T)^{\frac{3}{2}}},$$

and

$$\begin{aligned}a_0'U^4+6a_2'U^2+a_4'&=[a_0'\lambda^4(1+T)^2+6a_2'\lambda^2(1-T^2)+a_4'(1-T)^2]/(1-T)^2\\&=2[(a_4'-3a_2'\lambda^2)T^2+(a_4'+3a_2'\lambda^2)]/(1-T)^2\\&=2\left[\sqrt{\frac{a_4'}{a_0'}}(\sqrt{a_0'a_4'}-3a_2')\right]\left[T^2+\frac{\sqrt{a_0'a_4'}+3a_2'}{\sqrt{a_0'a_4'}-3a_2'}\right]\Big/(1-T)^2,\end{aligned}$$

and

$$\frac{dU}{\sqrt{a_0'U^4+6a_2'U^2+a_4'}}=\frac{1}{\sqrt{2}[\sqrt{a_0'a_4'}-3a_2']^{\frac{1}{2}}}\frac{dT}{\sqrt{(1-T^2)\left(T^2+\dfrac{\sqrt{a_0'a_4'}+3a_2'}{\sqrt{a_0'a_4'}-3a_2'}\right)}}$$

which is now of real form, since $a_0'a_4'>9a_2'^2$ for the case considered.

1479. Illustrative Example.

It will be instructive to consider one case from several points of view.

Take $$u\equiv\int_3^x\frac{dx}{\sqrt{x^3-5x^2+4x+6}}$$

(*a*) First let us reduce it to the Legendrian form.

$$x^3-5x^2+4x+6=(x-3)(x^2-2x-2).$$

Put $$x=(p+qz)/(1+z),\quad dx=(q-p)\,dz/(1+z)^2.$$

$$x-3=[(p-3)+(q-3)z](1+z)/(1+z)^2.\quad \text{(See Art. 1465.)}$$

$$x^2-2x-2=[(p+qz)^2-2(p+qz)(1+z)-2(1+z)^2]/(1+z)^2.$$

Put $p-3+q-3=0$, $pq-(p+q)-2=0$, *i.e.* $p+q=6$, $pq=8$.

Take the solution $p=4$, $q=2$.

Then

$$x-3=(1-z^2)/(1+z)^2,\quad x^2-2x-2=2(3-z^2)/(1+z)^2,\quad dz=-2dz/(1+z)^2.$$

Also $x=3$ gives $z=1$;

$$\therefore\ u=-\sqrt{\frac{2}{3}}\int_1^z\frac{dz}{\sqrt{(1-z^2)(1-\frac{1}{3}z^2)}}=-\sqrt{\frac{2}{3}}\int_{\frac{\pi}{2}}^{\theta}\frac{d\theta}{\sqrt{1-\frac{1}{3}\sin^2\theta}}\quad(z=\sin\theta)$$

$=\sqrt{\frac{2}{3}}(K-\text{sn}^{-1}z)$, K being the real quarter-period, mod. $1/\sqrt{3}$;

$$\therefore\ z=\text{sn}(K-u\sqrt{\tfrac{3}{2}})=\text{cn}(u\sqrt{\tfrac{3}{2}})/\text{dn}(u\sqrt{\tfrac{3}{2}}),$$

i.e. $$x-3\equiv\frac{1-z}{1+z}=\frac{\text{dn}\,u\sqrt{3/2}-\text{cn}\,u\sqrt{3/2}}{\text{dn}\,u\sqrt{3/2}+\text{cn}\,u\sqrt{3/2}},\ \text{mod. }1/\sqrt{3}.$$

(*b*) Next let us reduce to the Weierstrassian form.

x^3-5x^2+4x+6 being already a cubic expression, it is only necessary to remove the term involving the square of the variable. Put $x=z+\frac{5}{3}$; $x=3$ gives $z=\frac{4}{3}$.

$$(x-3)[(x-1)^2-3]=\tfrac{1}{4}(4z^3-\tfrac{52}{3}z+\tfrac{368}{27}),\quad I=\tfrac{52}{3},\quad J=-\tfrac{368}{27};$$

$$\therefore\ u=\int_{\frac{4}{3}}^{z}\frac{2dz}{\sqrt{4z^3-\frac{52}{3}z+\frac{368}{27}}}=\left(\int_{\frac{4}{3}}^{\infty}-\int_z^{\infty}\right)\frac{2dz}{\sqrt{4z^3-\frac{52}{3}z+\frac{368}{27}}}=2\omega_1-2\wp^{-1}(z),$$

and

$$e_1=\tfrac{4}{3},\quad e_2=\sqrt{3}-\tfrac{2}{3},\quad e_3=-\sqrt{3}-\tfrac{2}{3},\quad k^2=\frac{e_2-e_3}{e_1-e_3}=\frac{4\sqrt{3}}{(\sqrt{3}+1)^2},\quad k'^2=\tan^2 15^\circ,$$

$$\omega_1=\frac{\overline{K}}{\sqrt{e_1-e_3}}=\frac{\overline{K}}{\sqrt{2+\sqrt{3}}}\quad \text{(Art. 1414)},$$

$\overline{K}$ *not* being the same as K in solution (a), the modulus being a different one.

$$\therefore\ \operatorname{sn}^2\left(\overline{K}-\sqrt{2+\sqrt{3}}\,\frac{u}{2}\right)=\frac{2+\sqrt{3}}{x-\frac{5}{3}+\sqrt{3}+\frac{2}{3}}=\frac{2+\sqrt{3}}{(x-3)+(2+\sqrt{3})};$$

$$\therefore\ \frac{\operatorname{cn}^2(u\cos 15^\circ)}{\operatorname{dn}^2(u\cos 15^\circ)}=\frac{1}{(x-3)\tan 15^\circ+1}\quad \text{(Art. 1352)},$$

and
$$(x-3)\tan 15^\circ=\frac{\operatorname{dn}^2(u\cos 15^\circ)}{\operatorname{cn}^2(u\cos 15^\circ)}-1=k'^2\frac{\operatorname{sn}^2(u\cos 15^\circ)}{\operatorname{cn}^2(u\cos 15^\circ)},$$

i.e.
$$x-3=\tan 15^\circ\operatorname{tn}^2(u\cos 15^\circ)\ ;\ \text{mod. } \sqrt[4]{3}(\sqrt{3}-1).$$

(c) The results arrived at by these two processes are of different form, the moduli being different.

Take the integral $\displaystyle\int_{\frac{\pi}{2}}^{\theta}\frac{d\theta}{\sqrt{1-\frac{1}{3}\sin^2\theta}}$ occurring in the Legendrian reduction.

Put $\dfrac{1-\sin\theta}{1+\sin\theta}=(2+\sqrt{3})\cot^2\phi$, so that when $\theta=\dfrac{\pi}{2}$, $\phi=\dfrac{\pi}{2}$.

Then
$$\sin\theta=\frac{1-\cot 15^\circ\cot^2\phi}{1+\cot 15^\circ\cot^2\phi},\quad \cos\theta=\frac{2\sqrt{\cot 15^\circ}\cot\phi}{1+\cot 15^\circ\cot^2\phi},$$

$$d\theta=\frac{2\sqrt{\cot 15^\circ}\operatorname{cosec}^2\phi\,d\phi}{1+\cot 15^\circ\cot^2\phi},$$

and
$$1-\frac{1}{3}\sin^2\theta=\frac{2}{3}\,\frac{1+4\cot 15^\circ\cot^2\phi+\cot^2 15^\circ\cot^4\phi}{(1+\cot 15^\circ\cot^2\phi)^2}$$
$$=\frac{2}{3}\cdot\frac{\cot^2 15^\circ\,.\operatorname{cosec}^4\phi}{(1+\cot 15^\circ\cot^2\phi)^2}\left(1-\frac{\cos 30^\circ}{\cos^2 15^\circ}\sin^2\phi\right).$$

Hence

$$u=-\sqrt{\frac{2}{3}}\int_{\frac{\pi}{2}}^{\theta}\frac{d\theta}{\sqrt{1-\frac{1}{3}\sin^2\theta}}=-\sqrt{\frac{2}{3}}\int_{\frac{\pi}{2}}^{\phi}\frac{\sqrt{6}}{\sqrt{\cot 15^\circ}}\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}},\left(\lambda=\frac{\sqrt{\cos 30^\circ}}{\cos 15^\circ}\right)$$
$$=\frac{1}{\cos 15}\int_{\phi}^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}}=\frac{1}{\cos 15}[\overline{K}-\operatorname{am}^{-1}\phi].$$

Thus
$$\phi=\operatorname{am}(\overline{K}-u\cos 15^\circ),\ \left(\text{mod. }\frac{\sqrt{\cos 30^\circ}}{\cos 15^\circ}\right);$$

whence
$$\sin\phi=\operatorname{sn}(K-u\cos 15^\circ)=\frac{\operatorname{cn}(u\cos 15^\circ)}{\operatorname{dn}(u\cos 15^\circ)},\quad \text{mod. }\frac{\sqrt{\cos 30^\circ}}{\cos 15^\circ},$$

$$\cos\phi=\operatorname{cn}(\overline{K}-u\cos 15^\circ)=\tan 15^\circ\frac{\operatorname{sn}(u\cos 15^\circ)}{\operatorname{dn}(u\cos 15^\circ)}\quad \text{(Art. 1352)}.$$

Hence $\cot\phi=\tan 15^\circ\operatorname{tn}(u\cos 15^\circ)$,

and
$$x-3=\cot 15^\circ\cot^2\phi=\tan 15^\circ\operatorname{tn}^2(u\cos 15^\circ),$$

which is the same result as that obtained in solution (b).

1480. LANDEN'S TRANSFORMATION.

From the above example it appears that the reduction of an elliptic integral to the Legendrian form is not unique.

The transformations

$$x=3+\frac{1-\sin\theta}{1+\sin\theta} \quad \text{and} \quad x=3+\cot 15^\circ \cot^2\phi$$

both succeeded in such a reduction, but the moduli in the two cases were different.

For the general theory of such transformations the reader is referred to Cayley (*E. Functions*) or Greenhill (*E. Functions*).

One well-known transformation, however, must be noticed before leaving the matter, viz. that due to Landen.

Taking two variables θ_1, θ_2 connected by the equation $\sin(2\theta_1-\theta_2)=\mu\sin\theta_2$, so that θ_1 and θ_2 vanish together, we have $\cot(2\theta_1-\theta_2)(2d\theta_1-d\theta_2)=\cot\theta_2\, d\theta_2$; whence

$$2d\theta_1 . \cot(2\theta_1-\theta_2)=d\theta_2\{\cot(2\theta_1-\theta_2)+\cot\theta_2\}=\frac{\sin 2\theta_1\, d\theta_2}{\sin\theta_2\sin(2\theta_1-\theta_2)};$$

$$\therefore \frac{2\sin\theta_2\, d\theta_1}{\sin 2\theta_1}=\frac{d\theta_2}{\cos(2\theta_1-\theta_2)}=\frac{d\theta_2}{\sqrt{1-\mu^2\sin^2\theta_2}}.$$

Also $\sin 2\theta_1 . \cot\theta_2-\cos 2\theta_1=\mu$, $\cot\theta_2=(\mu+\cos 2\theta_1)/\sin 2\theta_1$;

$$\therefore \operatorname{cosec}^2\theta_2=(1+\mu^2+2\mu\cos 2\theta_1)/\sin^2 2\theta_1$$

and
$$\frac{\sin^2 2\theta_1}{\sin^2\theta_2}=(1+\mu)^2\left[1-\frac{4\mu}{(1+\mu)^2}\sin^2\theta_1\right];$$

$$\therefore \frac{2}{1+\mu}\int_0^{\theta_1}\frac{d\theta_1}{\sqrt{1-\frac{4\mu}{(1+\mu)^2}\sin^2\theta_1}}=\int_0^{\theta_2}\frac{d\theta_2}{\sqrt{1-\mu^2\sin^2\theta_2}}=u, \text{ say};$$

$$\therefore u=\operatorname{am}^{-1}(\theta_2,\mu)=\frac{2}{1+\mu}\operatorname{am}^{-1}\left(\theta_1, \frac{2\sqrt{\mu}}{1+\mu}\right);$$

or, what is the same thing,

$$\sin\theta_1=\operatorname{sn}\frac{1+\mu}{2}u, \left(\text{mod.}\frac{2\sqrt{\mu}}{1+\mu}\right); \quad \sin\theta_2=\operatorname{sn}u, \ (\text{mod.}\ \mu),$$

or putting $\qquad x_1\equiv\sin\theta_1, \quad x_2=\sin\theta_2,$

$$u=\int_0^{x_2}\frac{dx_2}{\sqrt{(1-x_2^2)(1-\mu^2x_2^2)}}=\frac{2}{1+\mu}\int_0^{x_1}\frac{dx_1}{\sqrt{(1-x_1^2)\left\{1-\frac{4\mu x_1^2}{(1+\mu)^2}\right\}}},$$

so that $u = \operatorname{sn}^{-1}(x_2, \mu) = \frac{2}{1+\mu}\operatorname{sn}^{-1}\left(x_1, \frac{2\sqrt{\mu}}{1+\mu}\right)$, and therefore u is expressible in either of these ways as an inverse elliptic function.

Writing λ for $\frac{2\sqrt{\mu}}{1+\mu}$ and $\lambda' = \frac{1-\mu}{1+\mu}$, *i.e.* $\lambda^2 + \lambda'^2 = 1$, we have $\frac{2}{1+\mu} = 1 + \lambda'$, $\mu = \frac{1-\lambda'}{1+\lambda'}$, and the connection between x_1 and x_2 is obtained from the initial formula

$\sin(2\theta_1 - \theta_2) = \mu \sin\theta_2$, viz. $2x_1\sqrt{1-x_1^2}\sqrt{1-x_2^2} - (1-2x_1^2)x_2 = \mu x_2$,

i.e. $\frac{x_2}{\sqrt{1-x_2^2}} = \frac{2x_1\sqrt{1-x_1^2}}{1+\mu-2x_1^2}$; whence $x_2 = (1+\lambda')x_1\sqrt{\frac{1-x_1^2}{1-\lambda^2 x_1^2}}$.

Therefore

$$\operatorname{sn}^{-1}(x_1, \lambda) = \frac{1}{1+\lambda'}\operatorname{sn}^{-1}\left\{(1+\lambda')x_1\sqrt{\frac{1-x_1^2}{1-\lambda^2x_1^2}}, \frac{1-\lambda'}{1+\lambda'}\right\}.$$

This is known as Landen's Transformation.

For many such results and other transformations, see Greenhill, *E.F.*, pp. 55, 56, and Chapter X. Greenhill gives a very elegant interpretation of the above transformation with reference to the motion of a pendulum (pages 318, 319, *E.F.*).

In such transformations, when $F(\theta_1, k)$ becomes $MF(\theta_2, k')$, F representing the first Legendrian Integral, M is technically known as the "Multiplier," and the relation between k and k' is known as the "Modular Equation." Thus, in the foregoing case the multiplier is $\frac{1}{2}(1+\mu)$, and the modular equation is $\lambda(\mu+1) = 2\sqrt{\mu}$.

1481. Illustrative Examples.

Ex. 1. Reduce $v = \int_{\sqrt{11}-3}^{x} \frac{dx}{\sqrt{x^4 + 8x^3 + 20x^2 + 56x - 20}}$

to standard Legendrian form.

We have $U \equiv x^4 + 8x^3 + 20x^2 + 56x - 20 \equiv (x^2 + 2x + 10)(x^2 + 6x - 2)$.

Here, with the notation of Art. 1463, $\lambda = 1$, $\mu = 10$; $\lambda' = 3$, $\mu' = -2$,

$$\left.\begin{array}{l} pq + (p+q) + 10 = 0, \\ pq + 3(p+q) - 2 = 0, \end{array}\right\} \text{ giving } \left.\begin{array}{r} p+q = 6, \\ pq = -16, \end{array}\right\}$$

i.e. $$\left.\begin{array}{l} p = 8, \\ q = -2, \end{array}\right\} \text{ and } x = \frac{p+qz}{1+z} = \frac{8-2z}{1+z}.$$

$$x^2+2x+10=10(9+z^2)/(1+z)^2,\quad x^2+6x-2=10(11-z^2)/(1+z)^2,$$
$$dx=-10dz/(1+z)^2\,;$$
$$\therefore\ \frac{dx}{\sqrt{U}}=-\frac{dz}{\sqrt{-(z^2+9)(z^2-11)}},$$

which is Case 4, Art. 1473. Put $z=\sqrt{11}\cos\theta$.

Then $$\frac{dx}{\sqrt{U}}=\frac{\sqrt{11}\sin\theta\,d\theta}{\sqrt{11\sin^2\theta(20-11\sin^2\theta)}}=\frac{1}{2\sqrt{5}}\,\frac{d\theta}{\sqrt{1-\frac{11}{20}\sin^2\theta}},$$

and the limits for x corresponding to 0 and θ for θ, are $\sqrt{11}-3$ to x.

Therefore $$v=\frac{1}{2\sqrt{5}}\int_0^\theta\frac{d\theta}{\sqrt{1-\frac{11}{20}\sin^2\theta}}=\frac{1}{2\sqrt{5}}F\left(\theta,\frac{\sqrt{55}}{10}\right),$$

and $$2v\sqrt{5}=\mathrm{cn}^{-1}\frac{1}{\sqrt{11}}\cdot\frac{8-x}{x+2}\quad(\text{mod. }\tfrac{1}{10}\sqrt{55}).$$

Ex. 2. Examine the same integral without factorisation. With the notation of Art. 1475,

$$a_0=1,\quad a_1=2,\quad a_2=\tfrac{10}{3},\quad a_3=14,\quad a_4=-20,$$
$$I=a_0a_4-4a_1a_3+3a_2{}^2=-\tfrac{296}{3},$$
$$J=a_0a_2a_4+2a_1a_2a_3-a_0a_3{}^2-a_4a_1{}^2-a_2{}^3=-\tfrac{892}{27},$$
$$\frac{I^3-27J^2}{108I^3}=\frac{3^2.\,5^4.\,11}{2^7.\,37^3}.$$

Hence, following the notation of Arts. 1475, 1476, our equation for θ is

$$\theta^3=\frac{2^3.\,37^3}{3^2.\,5^4.\,11}\ (\theta-1).$$

To simplify, let $$\theta=\frac{2\,.\,37}{5^2}t\,;$$

$$\therefore\ t^3=\frac{5^2}{3^2.\,11}\left(\frac{2\,.\,37}{5^2}t-1\right),\quad \textit{i.e.}\ t^3=\frac{74}{99}t-\frac{25}{99},$$

of which an obvious root is $t=-1$.

Hence $\theta=-\dfrac{74}{25}$ and $\rho+\dfrac{1}{\rho}+14=\dfrac{16\times74}{99}$, *i.e.* $\rho=-\dfrac{9}{11}$ or $-\dfrac{11}{9}$.

Therefore $$\frac{p}{-9}=\frac{q}{11}=s,\ \text{say}\,;\quad p_1=-9,\quad q_1=11.$$

Then $$\Delta=\sqrt[4]{\frac{s^2}{12I}(9^2-14\,.\,9\,.\,11+11^2)}=\sqrt{s},$$

and $$v=\frac{\Delta}{s}\int\frac{\sqrt{s}\,dU'}{\sqrt{(U'^2-9)(U'^2+11)}}=\int\frac{dU'}{\sqrt{(U'^2-9)(U'^2+11)}}.$$

Let $U'=3\sec\theta'$. Then $x=\sqrt{11}-3$ gives $Q=0$, $U'=3$, $\theta'=0$;

$$\therefore\ v=\int_0^{\theta'}\frac{3\sec\theta'\tan\theta'\,d\theta'}{\sqrt{9\tan^2\theta'(9\sec^2\theta'+11)}}=\frac{1}{2\sqrt{5}}\int_0^{\theta'}\frac{d\theta'}{\sqrt{1-\frac{11}{20}\sin^2\theta'}}=\frac{1}{2\sqrt{5}}F(\theta',\tfrac{1}{10}\sqrt{55}),$$

which agrees with the result of Ex. 1.

Ex. 3. Consider the integral $u \equiv \int_0^x \frac{x^{-\frac{2}{3}}dx}{\sqrt{1-x^2}}$ [Legendre, *Exercices*, p. 56].

This does not become infinite in the vicinity of $x=0$ (Art. 348).

Put $x=(1+z^2)^{-\frac{3}{2}}$, $dx=-3z(1+z^2)^{-\frac{5}{2}}dz$, $1-x^2=(3+3z^2+z^4)z^2/(1+z^2)^3$;

$$\therefore\ u=3\int_z^\infty \frac{dz}{\sqrt{z^4+3z^2+3}}.$$

The factorisation *of the desired form* $(U^2+p)(U^2+q)$ is

$$\left(z^2+\frac{3+\iota\sqrt{3}}{2}\right)\left(z^2+\frac{3-\iota\sqrt{3}}{2}\right).$$

Therefore p and q are complex. Following Art. 1478, put

$$z=\sqrt[4]{3}\sqrt{\frac{1+T}{1-T}},\quad dz=\frac{\sqrt[4]{3}\,dT}{\sqrt{1-T^2}(1-T)},$$

and $z=\infty$ gives $T=1$, and

$$z^4+3z^2+3=[(6-3\sqrt{3})\,T^2+(6+3\sqrt{3})]/(1-T)^2;$$

$$\therefore\ u=3\int_T^1 \frac{\sqrt[4]{3}\,dT}{\sqrt{1-T^2}(1-T)}\cdot\frac{1-T}{\sqrt{\frac{3}{2}(\sqrt{3}-1)^2\left[T^2+\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2\right]}}$$

$$=\frac{3^{\frac{3}{4}}}{2\sin 15^\circ}\int_T^1 \frac{dT}{\sqrt{(1-T^2)(T^2+\cot^2 15^\circ)}}$$

$$=\frac{3^{\frac{3}{4}}}{2}\operatorname{cn}^{-1}T=\frac{3^{\frac{3}{4}}}{2}\operatorname{cn}^{-1}\frac{z^2-\sqrt{3}}{z^2+\sqrt{3}}=\frac{3^{\frac{3}{4}}}{2}\operatorname{cn}^{-1}\frac{x^{-\frac{2}{3}}-\sqrt{3}-1}{x^{-\frac{2}{3}}+\sqrt{3}-1},$$

i.e. $$u=\frac{3^{\frac{3}{4}}}{2}\operatorname{cn}^{-1}\frac{1-2\sqrt{2}x^{\frac{2}{3}}\cos 15^\circ}{1+2\sqrt{2}x^{\frac{2}{3}}\sin 15^\circ},\ (\text{mod.}\ \sin 15^\circ).$$

PROBLEMS.

1. Find the values of I and J for the quartic function
$$\phi \equiv x^4 - 6\lambda x^2 y^2 + y^4,$$
and show that $4\lambda^3 - I\lambda - J = 0$. Form also the Hessian of the quartic, and the discriminant.

2. Examine the modification in the reduction to Weierstrassian form which accrues from the quartic Q having one root α_0 zero, *i e.* $a_4 = 0$. Show that in this case
$$e_1 = a_0 \frac{\alpha_1\alpha_2\alpha_3}{12}\left(\frac{1}{\alpha_2} + \frac{1}{\alpha_3} - \frac{2}{\alpha_1}\right), \quad e_2 = a_0 \frac{\alpha_1\alpha_2\alpha_3}{12}\left(\frac{1}{\alpha_3} + \frac{1}{\alpha_1} - \frac{2}{\alpha_2}\right),$$
$$e_3 = a_0 \frac{\alpha_1\alpha_2\alpha_3}{12}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} - \frac{2}{\alpha_3}\right);$$
and that
$$k^2 = \frac{1/\alpha_2 - 1/\alpha_3}{1/\alpha_1 - 1/\alpha_3}.$$

3. If
$$\phi \equiv a_0 (x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)(x - \alpha_4 y),$$
and
$$P = \alpha_2 - \alpha_3, \quad Q = \alpha_3 - \alpha_1, \quad R = \alpha_1 - \alpha_2,$$
$$P' = \alpha_1 - \alpha_4, \quad Q' = \alpha_2 - \alpha_4, \quad R' = \alpha_3 - \alpha_4,$$
show that
$$I = \frac{a_0^2}{24}(P^2P'^2 + Q^2Q'^2 + R^2R'^2),$$
$$J = -\frac{a_0^3}{432}(QQ' - RR')(RR' - PP')(PP' - QQ'),$$
and
$$\Delta \equiv I^3 - 27J^2 = \frac{a_0^6}{256} P^2Q^2R^2P'^2Q'^2R'^2.$$

Also, if $S_1 = \Sigma\alpha_1$, $S_2 = \Sigma\alpha_1\alpha_2$, $S_3 = \Sigma\alpha_1\alpha_2\alpha_3$, $S_4 = \alpha_1\alpha_2\alpha_3\alpha_4$, show that
$$I = \frac{a_0^2}{12}(12S_4 - 3S_1S_3 + S_2^2), \quad J = \frac{a_0^3}{12^3}\begin{vmatrix} 12, & -3S_1, & 2S_2 \\ -3S_1, & 2S_2, & -3S_3 \\ 2S_2, & -3S_3, & 12S_4 \end{vmatrix}.$$

4. If $\phi \equiv x^4 + 6\lambda x^2y^2 + y^4$ and the Hessian $H = \dfrac{1}{12^2}\begin{vmatrix} \phi_{xx}, & \phi_{xy} \\ \phi_{xy}, & \phi_{yy} \end{vmatrix}$, show that $H - k\phi$ is a perfect square if $k = \lambda$, $-\frac{1}{2}(\lambda + 1)$ or $-\frac{1}{2}(\lambda - 1)$.

5. Show that $\wp^{-1}(z, 76, -120) = \dfrac{1}{2\sqrt{2}} \operatorname{sn}^{-1} \dfrac{2\sqrt{2}}{\sqrt{z+5}}$; mod. $\dfrac{\sqrt{7}}{2\sqrt{2}}$.

6. Show that $\wp^{-1}(z, 28, -24) = \dfrac{1}{\sqrt{5}} \operatorname{dn}^{-1} \sqrt{\dfrac{z-1}{z+3}}$; mod. $\dfrac{2}{\sqrt{5}}$.

7. Show that $\wp^{-1}(z, 36, 0) = \dfrac{1}{\sqrt{6}} \operatorname{cn}^{-1} \sqrt{\dfrac{z}{z+3}}$; mod. $\dfrac{1}{\sqrt{2}}$.

8. Reduce the integral $u=\int_1^x \frac{dx}{\sqrt{-70x^4+253x^3-327x^2+179x-35}}$ to Weierstrassian form, and show that $u=\wp^{-1}\left(\frac{x}{x-1}\right)$. Show also that it can be expressed in a Legendrian form with a modulus $\frac{1}{2}$, viz. $u=\frac{1}{\sqrt{6}}\operatorname{sn}^{-1}\sqrt{12\frac{x-1}{7x-5}}$.

9. Show that if $e_1>e_2>e_3$ and $e_1+e_2+e_3=0$, the substitution $z=e_3+\frac{e_1-e_3}{x^2}$ will convert the Weierstrassian Integral

$$\int_z^\infty \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}$$

into the Legendrian form

$$\frac{1}{\sqrt{e_1-e_3}}\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

where $k^2=\frac{e_2-e_3}{e_1-e_3}$, and conversely that the substitution $x=\sqrt{\frac{e_1-e_3}{z-e_3}}$ will convert the standard Legendrian form into the Weierstrassian.

10. Reduce $\int_z^\infty \frac{dz}{\sqrt{4z(z^2-9)}}$ to the Legendrian form

$$\frac{1}{\sqrt{6}}\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-\frac{1}{2}x^2)}},$$

and show with the usual notation that

$$K=\omega_1\sqrt{6},\quad K-\iota K'=\omega_2\sqrt{6},\quad -\iota K'=\omega_3\sqrt{6}.$$

11. Show that $\int_z^\infty \frac{dz}{\sqrt{z(z^2-4)}}=\operatorname{sn}^{-1}\frac{2}{\sqrt{z+2}}$.

12. In the standard Legendrian form $\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ discuss the degenerate forms assumed when $k=0$ and when $k=1$, and state to what forms $\operatorname{sn}^{-1}x$, $\operatorname{cn}^{-1}x$, $\operatorname{dn} x$ and $\operatorname{tn} x$ ultimately degenerate in these cases.

13. Discuss the integration of the degenerate cases of

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)}},$$

(i) when $\alpha=\beta$, (ii) when $\alpha=\beta=\gamma$, (iii) when $\alpha=\beta=\gamma=\delta$.

14. Discuss the integration of the degenerate cases of

$$\int_z^\infty \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}},\quad \left\{\begin{matrix} e_1>e_2>e_3, \\ e_1+e_2+e_3=0 \end{matrix}\right\},$$

(i) when $e_2=e_3$, (ii) when $e_1=e_2$, (iii) when $e_1=e_2=e_3$.

15. Express both in Weierstrassian and in Legendrian notation the integration
$$u=\int_t^\infty \frac{t\,dt}{\sqrt{t^6+3t^4-6t^2-8}}.$$

16. Make use of the substitution $x^3+x^{-3}=2t^{-\frac{3}{2}}$ to reduce the integral $u=\int_0^x \frac{du}{\sqrt[3]{1+x^6}}$ to the form of an elliptic integral, and reduce it to the standard Weierstrassian form.

17. Use the substitution $t^3=(1+x+x^2)/(1-x)^2$ in the integration $u=\int_1^x \frac{dx}{(1-x^3)^{\frac{2}{3}}}$; and show that $t=\wp\left(\frac{u}{\sqrt{3}}, 0, 1\right)$.

18. Show that if
$$2u=\int_2^x \frac{dx}{\sqrt{(x-2)(5x-11)(11x-21)(3x-7)}} \quad (2<x<2{\cdot}2),$$
$$u=\wp^{-1}\left(\frac{x-1}{x-2}, 304, -960\right)=\frac{1}{4}\,\mathrm{sn}^{-1}4\sqrt{\frac{x-2}{11x-21}} \left(\text{mod.}\ \sqrt{\frac{7}{8}}\right).$$

19. Show that the solutions of the sextic equation
$$\frac{(\rho^2+14\rho+1)^3}{\rho(\rho-1)^4}=\frac{(\beta^8+14\beta^4+1)^3}{\beta^4(\beta^4-1)^4}$$
are β^4, $\frac{1}{\beta^4}$, $\left(\frac{1-\beta}{1+\beta}\right)^4$, $\left(\frac{1+\beta}{1-\beta}\right)^4$, $\left(\frac{1-\iota\beta}{1+\iota\beta}\right)^4$ and $\left(\frac{1+\iota\beta}{1-\iota\beta}\right)^4$.

[Cayley.]

20. Transform the integral $u=\int_0^1 \frac{dx}{(1-x^6)^{\frac{5}{6}}}$ into one in which z is the variable by the relation $4x^6(1-x^6)=z^6$, and the result by putting $z^2=1/(1+y^2)$; and lastly, by the further transformation
$$y=\sqrt[4]{3}\tan\frac{\phi}{2};$$
showing that $\quad \mathrm{sn}\left(\frac{3^{\frac{1}{4}}}{4^{\frac{1}{3}}}u\right)=\frac{\pi}{2}, \quad$ (mod. sin 15°).

Hence show that $u=1{\cdot}927622\ldots$, and verify this otherwise.

[Bertrand, *I.C.*, p. 687.]

21. Show by Landen's Transformation $2\sin(2\phi-\theta)=\sin\theta$ that
$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}}=\frac{4}{3}\int_0^{\frac{\pi}{3}} \frac{d\phi}{\sqrt{1-\frac{8}{9}\sin^2\phi}}.$$

22. Express by means of the Weierstrassian elliptic functions $\wp(u)$, $\zeta(u)$, $\sigma(u)$ the results of the following integrations:

(i) $\displaystyle\int_z^a \frac{z\,dz}{\sqrt{z^3-1}}$, $(1<z)$; (ii) $\displaystyle\int_z^\infty \frac{dz}{(z-2)\sqrt{z^3-1}}$, $(2<z)$;

(iii) $\displaystyle\int_z^\infty \frac{z^3\,dz}{(z-2)^2\sqrt{z^3-1}}$, $(2<z)$;

(iv) $\displaystyle\int_x^\infty \frac{dx}{(x-1)(x-2)\sqrt{x^3-5x^2+4x+6}}$, $(3<x)$;

(v) $\displaystyle\int_x^1 \frac{x\,dx}{\sqrt{x^4-12x^3+54x^2-100x+57}}$, $(x<1)$.

23. Express by Weierstrassian functions the second Legendrian standard form $\displaystyle\int_0^\theta \sqrt{1-k^2\sin^2\theta}\,d\theta$.

24. Express by Weierstrassian functions the third Legendrian standard form $\displaystyle\int_0^x \frac{dx}{(1-a^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}}$.

25. If $u=\frac{1}{2}\sqrt{11}\displaystyle\int_0^x \frac{dx}{\sqrt{(x^2+x+1)(3x^2+x+1)}}$, prove that

$$x(\sqrt{11}\,\mathrm{cn}\,u-\mathrm{sn}\,u)=2\,\mathrm{sn}\,u,\quad (\mathrm{mod.}\ \sqrt{\tfrac{8}{11}}).$$ [Ox. II. P., 1913.]

26. If $u=15\displaystyle\int_1^x \frac{dx}{\sqrt{1105x^4-904x^3-210x^2+8x+1}}$, prove that

$$x(3\,\mathrm{cn}\,u-2\,\mathrm{dn}\,u)=\mathrm{dn}\,u,\quad (\mathrm{mod.}\ 1/5).$$ [Ox. II. P., 1915.]

27. If $u=\displaystyle\int_0^x \frac{dx}{(1+x^2-2x^4)^{\frac{1}{2}}}$ express x as a single-valued function of u by help of (i) Jacobi's functions, (ii) Weierstrass' functions.

[Math. Trip. II., 1914.]

Prove that $x\sqrt{3}\,\mathrm{dn}\,(u\sqrt{3})=\mathrm{sn}\,(u\sqrt{3})$, (mod. $\sqrt{2/3}$).

28. Show that the integral

$$\int_{a_1}^x \{(x-a_1)(x-a_2)(x-a_3)(x-a_4)\}^{-\frac{1}{2}}\,dx$$

is transformed to the integral

$$2\{(a_4-a_2)(a_1-a_3)\}^{-\frac{1}{2}}\int_0^y \{(1-y^2)(1-k^2y^2)\}^{-\frac{1}{2}}\,dy$$

by the relations $y^2=(a_2-a_4)(x-a_1)/(a_2-a_1)(x-a_4)$,

$k^2=(a_2-a_1)(a_3-a_4)/(a_3-a_1)(a_2-a_4)$,

and obtain an expression for the general value of the former integral.

[Math. Trip. II., 1913.]

29. A heavy particle attached to a fixed point by a light thread of length a oscillates under the action of gravity in a vertical plane. Show that the height of the particle above the lowest point of its path at time t from the lowest position is

$$2a \sin^2 \frac{\alpha}{2} \operatorname{sn}^2\left(\sqrt{\frac{g}{a}}\, t\right), \quad \left(\text{mod. } \sin \frac{\alpha}{2}\right),$$

where 2α is the whole angle of swing.

30. Show that the potential of a uniform thin ring at any point is

$$4\gamma m a \int_{r_1}^{r_2} \frac{dr}{\{(r^2 - r_1^2)(r_2^2 - r^2)\}^{\frac{1}{2}}},$$

where γ is the constant of gravitation, m the mass per unit length, a the radius of the ring, r the distance of the point from a point of the ring, r_1 and r_2 the least and greatest values of r. Prove also that the potential may be expressed in the form $8\gamma m \dfrac{a}{r_1 + r_2} K$, where K is the complete elliptic integral of the first kind with modulus $(r_2 - r_1)/(r_2 + r_1)$. [Ox. II. P., 1914.]

31. A heavy elastic string which is uniform when unstretched is passed through a smooth semicircular tube which is held in a vertical plane with its vertex upwards. The radius of the tube is r. The modulus of the elastic string is equal to the weight of a length r of the unstretched string. It is observed that the two equal portions which hang vertically outside the tube are each equal in length to the radius. Show that the unstretched length of the portion which lies within the tube is

$$\frac{4r}{\sqrt{5}} \operatorname{dn}^{-1} \sqrt{\frac{3}{5}}, \quad \left(\text{mod. } \frac{2}{\sqrt{5}}\right).$$ [Ox. II. P., 1915].

32. Assuming that the law of central attractive force under which an orbit $u = f(\theta)$ can be described is given by $P/h^2u^2 = u + \dfrac{d^2u}{d\theta^2}$, show that if a particle describes an orbit $r = a \operatorname{cn} \theta\sqrt{3}$ under the action of a central attraction μu^5, the modulus of the elliptic function is $3^{-\frac{1}{2}}$.
[Ox. II. P., 1913.]

33. A particle of unit mass is projected horizontally with velocity u, and moves under gravity in a resisting medium such that the path is a portion of a circle of radius a. Show that the motion will cease after a time $\sqrt{\dfrac{2a}{g}} \operatorname{dn}^{-1} 2^{-\frac{1}{2}}$, (mod. $2^{-\frac{1}{2}}$). [Ox. II. P., 1913.]

34. Show that the area A bounded by the y-axis, the asymptote $x=1$ and the curve $y^2(x-1)(x-3)\{(x-4)^2+3\}=1$ is

$$\frac{1}{\sqrt[4]{3}}\operatorname{cn}^{-1}\frac{14-3\sqrt{3}}{13}, \quad (\text{mod.}\sin 75^\circ).$$

35. If A be the area in the positive quadrant bounded by the curve $2y^2x(x^2+4x+1)=3$, the coordinate axes and an abscissa x, show that

$$(x+1)/(x-1)=\operatorname{dn} A/\operatorname{cn} A, \quad (\text{mod.}\tan \pi/6).$$

36. A ring is generated by the motion of a circle such that its plane passes through the centre of an ellipse and a perpendicular to the plane of the ellipse through the centre, and the centre of the circle lies on the ellipse. Show that the volume of the ring is $4\pi Kbc^2$, where b is the semi-axis minor of the ellipse, K the complete elliptic integral of the first kind with its modulus equal to the eccentricity of the ellipse and c $(<b)$ the radius of the circle.

[C.S., 1895.]

37. Prove that the equation of the osculating plane at any point of the curve $x=a\operatorname{sn}u$, $y=b\operatorname{cn}u$, $z=c\operatorname{dn}u$, (mod. k), is

$$\frac{x}{a}k^2(1-k^2)\operatorname{sn}^3u-\frac{y}{b}k^2\operatorname{cn}^3u+\frac{z}{c}\operatorname{dn}^3u=1-k^2.$$

[Ox. II. P., 1902.]

38. An elliptic wire of semi-axes a and b moves so that its plane is always parallel to a fixed plane while its centre describes in a perpendicular plane a circle of radius c which is greater than either a or b, and the minor axis is perpendicular to the latter plane. Prove that the ring surface formed by the circumference of the wire cuts itself in two hyperbolic edges, and that its volume is

$$\frac{16}{3}\frac{bc}{a}\{(c^2+a^2)E-(c^2-a^2)K\},$$

where K and E are the complete elliptic integrals of the first and second kinds with modulus a/c. [MATH. TRIP. 1886.]

39. If the modulus k and the amplitude ϕ of the elliptic integral $F(\phi, k)$ be given by $k=\cos\pi/12$, $\cos\phi=2-\sqrt{3}$, then will

$$F(\phi, k)=\{\sqrt{\pi}\,\Gamma(\tfrac{1}{3})\}/\{3^{\frac{3}{4}}\,.\,\Gamma(\tfrac{5}{6})\}.$$

[J. C. MALET, *E.T.*, 9677.]