CHAPTER XXXII.

ELLIPTIC INTEGRALS (*continued*). THE WEIERSTRASSIAN FORMS.

1380. It was stated in Chapter XI. that the integration of $\int\frac{dx}{\sqrt{Q}}$, where Q is a rational quartic function of x, could be made to depend by a suitable homographic substitution upon the integration $u=\int_0^z\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$, where k is real and <1, and the properties of z when expressed as a function of u, as also those of $\sqrt{1-z^2}$ and $\sqrt{1-k^2z^2}$, have been discussed in the last chapter. This is the **Legendrian** and **Jacobian** mode of procedure.

A more modern method is due to Weierstrass. In this method the same integral, viz. $\int\frac{dx}{\sqrt{Q}}$, is shown to be also reducible by a suitable homographic transformation to the form $u=\int_z^\infty\frac{dz}{\sqrt{4z^3-Iz-J}}$, where I, J are certain constants, viz. functions of the coefficients of Q, and of the constants of the homographic transformation formulae. The function u, regarded as dependent upon z, is considered as the inverse function, and z expressed as a function of u as the direct function. It is usual to write $z=\wp(u)$, or $\wp(u, I, J)$ if it be desired to put into evidence the values of I and J. $\wp(u)$ is called the **Weierstrassian** Function.

The letters g_2, g_3 are very commonly used instead of I and J, but as powers of these letters occur very frequently there appears to be less risk of error in practice if we use the I, J notation.

1381. The modes of reduction of the general integral $\int \frac{dx}{\sqrt{Q}}$ to the respective Legendrian and Weierstrassian forms will be discussed at length in the next chapter. For the present we shall be occupied with an examination of the nature and properties of the function $\wp(u)$ and the allied functions $\zeta(u)$ and $\sigma(u)$, respectively defined by the equations

$$\zeta(u) = -\int \wp(u)\,du = \frac{d}{du}\log\sigma(u).$$

These are respectively referred to as the **Weierstrassian Zeta** and **Sigma** functions.

1382. **Preliminary Remarks.**

The general binary quartic

$$Q \equiv a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$$

possesses two invariants for a linear transformation

$$x = l_1X + m_1Y, \quad y = l_2X + m_2Y,$$

viz. $$I \equiv a_0a_4 - 4a_1a_3 + 3a_2^2,$$

the quadratic invariant, or quadrinvariant,

$$J \equiv a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_4a_1^2 - a_2^3$$

$$\equiv \begin{vmatrix} a_0, & a_1, & a_2 \\ a_1, & a_2, & a_3 \\ a_2, & a_3, & a_4 \end{vmatrix}, \text{ the cubic invariant, or cubinvariant.}$$

If a transformation of this kind has reduced the original quartic to the form

$$0 \, . \, X^4 + 4X^3Y + 6 \, . \, 0X^2Y^2 + 4a_3'XY^3 + a_4'Y^4,$$

then for this new form

$$I' = 0 \, . \, a_4' - 4 \, . \, 1a_3' + 3 \, . \, 0^2 = -4a_3' \text{ and } J' = \begin{vmatrix} 0, & 1, & 0 \\ 1, & 0, & a_3' \\ 0, & a_3', & a_4' \end{vmatrix} = -a_4',$$

and the form has become

$$Y(4X^3 - I'XY^2 - J'Y^3),$$

or if Y be unity, $4X^3 - IX - J$, the accents being dropped as the meanings of I and J will be obvious.

1383. If e_1, e_2, e_3 be the roots of the equation $4z^3 - Iz - J = 0$, so that $4z^3 - Iz - J \equiv 4(z-e_1)(z-e_2)(z-e_3)$, we shall lose no generality in assuming for the present that e_1, e_2, e_3 are all real. For it will be shown that if two of these quantities be complementary imaginaries, say e_2, e_3, then a substitution of the form $\zeta-\eta_1=(z-e_2)(z-e_3)/(z-e_1)$ will reduce the integration

$$\int_z^\infty \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}$$

to the similar form

$$\int_\zeta^\infty \frac{d\zeta}{\sqrt{4(\zeta-\eta_1)(\zeta-\eta_2)(\zeta-\eta_3)}},$$

where η_1, η_2, η_3 are all real constants such that $\eta_1+\eta_2+\eta_3=0$ (Art. 1456). We therefore assume for the present that e_1, e_2, e_3 are all real, $e_1+e_2+e_3=0$ and $e_1>e_2>e_3$. We also have

$$\frac{I}{4}=-(e_2e_3+e_3e_1+e_1e_2)$$
$$=\frac{e_1{}^2+e_2{}^2+e_3{}^2}{2}=e_1{}^2-e_2e_3=e_2{}^2-e_3e_1=e_3{}^2-e_1e_2,$$
$$\frac{J}{4}=e_1e_2e_3.$$

1384. **The Differential Coefficients of $\wp(u)$.**

The integral $\wp^{-1}(z)=u\equiv\int_z^\infty \frac{dz}{\sqrt{4z^3-Iz-J}}$ is made definite at the upper limit, the integrand vanishing when z is infinite.

Differentiating, $\frac{dz}{du}=-\sqrt{4z^3-Iz-J}$, *i.e.* $\wp'(u)=-\sqrt{4\wp^3(u)-I\wp(u)-J}$, *i.e.* $\wp'^2(u)=4\wp^3(u)-I\wp(u)-J$. Hence also

$$\wp''(u)=6\wp^2(u)-\tfrac{1}{2}I=6z^2-\tfrac{1}{2}I,\quad \wp'''(u)=12\wp(u)\wp'(u)=12zz',$$
$$\wp^{\text{iv}}(u)=12[\wp'^2(u)+\wp(u)\wp''(u)]=12\left[10z^3-\frac{3I}{2}z-J\right],$$
$$\wp^{\text{v}}(u)=[360\wp^2(u)-18I]\wp'(u)=(360z^2-18I)z', \text{ etc.};$$

whence it appears that the successive differential coefficients of $\wp(u)$ with regard to u are alternately irrational and rational functions of $\wp(u)$.

1385. **Periodicity of $\wp(u)$.**

It has already been seen that the function w defined by $w^2=1/4(z-e_1)(z-e_2)(z-e_3)$ is a two-branched function having branch-points at $z=e_1$, $z=e_2$, $z=e_3$, and at $z=\infty$ (Art. 1295), and that in consequence $\int_z^\infty \frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}$ has three periods $2\omega_1$, $2\omega_2$, $2\omega_3$, where

$$\omega_1=\int_{e_1}^\infty w\,dz, \quad \omega_2=\int_{e_2}^\infty w\,dz, \quad \omega_3=\int_{e_3}^\infty w\,dz,$$

these periods being not independent but connected by a linear relation, viz. $\omega_1-\omega_2+\omega_3=0$. Of the three we shall consider $2\omega_1$ and $2\omega_3$ to be the independent periods.

We have also shown that if u_0 be any definite value of the integral $\int_z^\infty w\,dz$, say that obtained by integrating along any straight-line path extending from z to ∞, which does not pass through any of the points $z=e_1$, $z=e_2$, $z=e_3$, then all other values are comprised in the system,

$$\left.\begin{array}{l} u=2\lambda\omega_1+2\mu\omega_3+u_0, \\ u=2\lambda'\omega_1+2\mu'\omega_3+2\omega_1-u_0, \end{array}\right\} \text{ where } \lambda, \mu, \lambda', \mu' \text{ are integers.}$$

In consequence we have $\wp(2m\omega_1+2n\omega_3\pm u)=\wp(u)$, where m, n are integers, an equation which expresses the double periodicity of the function. And this is equivalent to the statement that the most general solution of the equation

$$\wp(u)=\wp(u_0) \text{ is } u=2m\omega_1+2n\omega_3\pm u_0,\ m, n \text{ being integers.}$$

Further, it follows that

$$\wp'\,(2m\omega_1+2n\omega_3+u)=\wp'\,(u), \quad \wp'\,(2m\omega_1+2n\omega_3-u)=-\wp'\,(u),$$

$$\wp''\,(2m\omega_1+2n\omega_2\pm u)=\wp''\,(u),$$

$$\wp'''(2m\omega_1+2n\omega_3+u)=\wp'''(u), \quad \wp'''(2m\omega_1+2n\omega_3-u)=-\wp'''(u),$$

and so on.

And in the special cases when $m=n=0$, we get

$$\wp\,(-u)=\wp\,(u), \quad \wp'\,(-u)=-\wp'\,(u),$$

$$\wp''(-u)=\wp''(u), \quad \wp'''(-u)=-\wp'''(u), \text{ etc.}$$

1386. These results are obvious from another consideration; viz. if we consider $(4z^3-Iz-J)^{-\frac{1}{2}}$ as expanded in a convergent series of negative powers of z, that expansion will begin with the term $\frac{1}{2z^{\frac{3}{2}}}+\dots$. Integrating between z and ∞, we have $u=\frac{1}{z^{\frac{1}{2}}}+\dots$; and squaring, $u^2=\frac{1}{z}+\dots$, and therefore by reversion of series $z=\frac{1}{u^2}+$even powers of u, *i.e.* $\wp(u)$ is an even function of u. [This expansion will be found carried out in Art. 1416.]

Thus $\wp'(u)$, $\wp''(u)$, $\wp'''(u)$... are alternately odd and even functions of u, whence $\wp(-u)=\wp(u)$, $\wp'(-u)=-\wp'(u)$, $\wp''(-u)=\wp''(u)$, etc., as stated.

Further, since these series for $\wp(u)$, $\wp'(u)$, $\wp''(u)$, ... all start with a negative power of u, it will be clear that $\wp(0)$, $\wp'(0)$, $\wp''(0)$, ... are all infinite, and the orders of these infinities are respectively those of $\frac{1}{u^2}, \frac{1}{u^3}, \frac{1}{u^4}, \ldots$, so that, for instance,

$$Lt_{u\to 0}\frac{\wp^3(u)}{\wp'^2(u)}=Lt_{u\to 0}\frac{\left(\frac{1}{u^2}\right)^3}{\left(-\frac{2}{u^3}\right)^2}=\frac{1}{4}.$$

1387. The Addition Formula for the Function $\wp(u)$.

Consider the solution of the Eulerian Equation $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$ for the case when

$$X\equiv 4x^3-Ix-J, \quad Y=4y^3-Iy-J.$$

Let $u=\int_x^\infty \frac{dx}{\sqrt{X}}$, $v=\int_y^\infty \frac{dy}{\sqrt{Y}}$, *i.e.* $x=\wp(u)$, $y=\wp(v)$. Then

$$\frac{dx}{du}=-\sqrt{X}, \quad \frac{dy}{dv}=-\sqrt{Y} \quad \text{and} \quad du+dv=-\left(\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}\right)=0.$$

Thus, one form of the integral is $u+v=C$, a constant. ...(1)

We can obtain another form of the integral as follows:

Introduce another variable t such that

$$\frac{dx}{\sqrt{X}}=-\frac{dy}{\sqrt{Y}}=-\frac{dt}{x-y},$$

and let $x+y=P$.

Then $$\frac{dP}{\sqrt{X}-\sqrt{Y}}=-\frac{dt}{x-y}, \quad \textit{i.e.}\ \frac{dP}{dt}=-\frac{\sqrt{X}-\sqrt{Y}}{x-y}.$$

Differentiating with regard to t,

$$\frac{d^2P}{dt^2}=-\frac{1}{x-y}\left[\frac{1}{2\sqrt{X}}\frac{dX}{dx}\frac{-\sqrt{X}}{x-y}-\frac{1}{2\sqrt{Y}}\frac{dY}{dy}\frac{\sqrt{Y}}{x-y}\right]$$
$$+\frac{\sqrt{X}-\sqrt{Y}}{(x-y)^2}\left[\frac{-\sqrt{X}}{x-y}-\frac{\sqrt{Y}}{x-y}\right]$$
$$=\frac{1}{(x-y)^2}\left[\frac{1}{2}\left(\frac{dX}{dx}+\frac{dY}{dy}\right)-\frac{X-Y}{x-y}\right].$$

Now

$$\frac{dX}{dx}+\frac{dY}{dy}=12(x^2+y^2)-2I, \quad \text{and} \quad \frac{X-Y}{x-y}=4(x^2+xy+y^2)-I;$$

$$\therefore \frac{d^2P}{dt^2}=\frac{2(x^2-2xy+y^2)}{(x-y)^2}=2, \quad \textit{i.e.}\ 2\frac{dP}{dt}\cdot\frac{d^2P}{dt^2}=4\frac{dP}{dt},$$

i.e. $$\left(\frac{dP}{dt}\right)^2=4(P+C') \quad \text{or} \quad P=\frac{1}{4}\left(\frac{dP}{dt}\right)^2-C',$$

where C' is a constant. ....(2)

Now this equation having been obtained on the supposition that $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0$, *i.e.* that $u+v=$ a constant C, it appears that C' is a constant, provided that C is a constant; *i.e.* C' is a function of C, say $\phi(C)$. We thus have the equation

$$P=\frac{1}{4}\left(\frac{dP}{dt}\right)^2-\phi(u+v),$$

and we have to identify the *form* of the function ϕ.

Now

$$P=x+y=\wp(u)+\wp(v), \quad \text{and} \quad \frac{dP}{dt}=-\frac{\sqrt{X}-\sqrt{Y}}{x-y}=\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)},$$

i.e. $$\phi(u+v)=\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2-x-y$$

$$=[\wp'^2(u)-2\wp'(u)\wp'(v)+\wp'^2(v)-4(x+y)(x-y)^2]/4(x-y)^2$$

$$=[\wp'^2(u)+2\wp'(u)\sqrt{4y^3-Iy-J}-Iy-J-4x^3$$
$$+4x^2y+4xy^2]/4(x-y)^2.$$

Now let v diminish indefinitely. Then $\wp(v)$ or y becomes infinitely great, and we have $\phi(u)=Lt\,\frac{4xy^2}{4y^2}=x=\wp(u)$, and the form of ϕ is now identified as that of the Weierstrassian function $\wp$.

Hence $$P=\frac{1}{4}\left(\frac{dP}{dt}\right)^2-\wp(u+v).$$

That is $$\wp(u+v)+\wp(u)+\wp(v)=\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2,$$

which, as it expresses $\wp(u+v)$ in terms of $\wp(u)$, $\wp(v)$ and their differential coefficients, forms the addition formula for this function.

1388. Symmetrical Form.

Taking a third function w, such that $u+v+w=0$, then

$$\wp(u+v)=\wp(-w)=\wp(w).$$

Therefore we have the symmetrical form

$$\wp(u)+\wp(v)+\wp(w)=\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2$$

$$=\frac{1}{4}\left[\frac{\wp'(v)-\wp'(w)}{\wp(v)-\wp(w)}\right]^2=\frac{1}{4}\left[\frac{\wp'(w)-\wp'(u)}{\wp(w)-\wp(u)}\right]^2,$$

by symmetry, and therefore

$$\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}=\frac{\wp'(v)-\wp'(w)}{\wp(v)-\wp(w)}=\frac{\wp'(w)-\wp'(u)}{\wp(w)-\wp(u)},$$

whence

$$\wp(u)\left[\wp'(v)-\wp'(w)\right]+\wp(v)\left[\wp'(w)-\wp'(u)\right]+\wp(w)\left[\wp'(u)-\wp'(v)\right]=0,$$

and we have the symmetrical relation

$$\begin{vmatrix} 1, & \wp(u), & \wp'(u) \\ 1, & \wp(v), & \wp'(v) \\ 1, & \wp(w), & \wp'(w) \end{vmatrix}=0.$$

1389. Various Results derived.

In the formula

$$\wp(u+v)+\wp(u)+\wp(v)=\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2$$

change the sign of v. Then, remembering that $\wp(-v)=\wp(v)$ and $\wp'(-v)=-\wp'(v)$ (Art. 1385), we have

$$\wp(u-v)+\wp(u)+\wp(v)=\frac{1}{4}\left[\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}\right]^2;$$

whence

$$\left.\begin{aligned} \wp(u+v)+\wp(u-v)+2\wp(u)+2\wp(v) &= \frac{1}{2}\frac{\wp'^2(u)+\wp'^2(v)}{\{\wp(u)-\wp(v)\}^2}, \\ \wp(u+v)-\wp(u-v) &= -\frac{\wp'(u)\wp'(v)}{\{\wp(u)-\wp(v)\}^2}. \end{aligned}\right\}$$

1390. Take a function of x, y, viz. $F(x, y)$, such that

$$F(x, y)\equiv 2xy(x+y)-I\frac{x+y}{2}-J,$$

so that $\qquad F(x, x)=4x^3-Ix-J=\wp'^2(u),$

and $\qquad F(y, y)=4y^3-Iy-J=\wp'^2(v).$

Then

$$\wp(u+v)+\wp(u-v)=\frac{1}{2}\,\frac{4x^3-Ix-J+4y^3-Iy-J}{(x-y)^2}-2(x+y)$$

$$=\{2xy(x+y)-\tfrac{1}{2}I(x+y)-J\}/(x-y)^2=F(x,y)/(x-y)^2;$$

whence $$\wp(u-v)+\wp(u+v)=\frac{F\{\wp(u),\,\wp(v)\}}{\{\wp(u)-\wp(v)\}^2};$$

also $$\wp(u-v)-\wp(u+v)=\frac{\wp'(u)\,\wp'(v)}{\{\wp(u)-\wp(v)\}^2},$$

$$\therefore\ \wp(u+v)=\frac{1}{2}\,\frac{F\{\wp(u),\,\wp(v)\}-\wp'(u)\,\wp'(v)}{\{\wp(u)-\wp(v)\}^2}.$$

1391. In the formula

$$\wp(u+v)+\wp(u)+\wp(v)=\frac{1}{4}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2,$$

let v approach to ultimate coincidence with u. Then

$$\wp(2u)+2\wp(u)=\frac{1}{4}\,Lt_{v\to u}\left[\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]^2=\frac{1}{4}\left\{\frac{\wp''(u)}{\wp'(u)}\right\}^2$$

$$=\frac{1}{4}\left\{\frac{d}{du}\log\wp(u)\right\}^2,$$

or $$=\frac{1}{4}\,\frac{\{6\wp^2(u)-\frac{1}{2}I\}^2}{4\wp^3(u)-I\wp(u)-J}.$$

1392. Hence

$$\wp(2u)=\frac{1}{4}\,\frac{\{6\wp^2(u)-\frac{1}{2}I\}^2}{4\wp^3(u)-I\wp(u)-J}-2\wp(u)=\frac{\{\wp^2(u)+\frac{1}{4}I\}^2+2J\wp(u)}{4\wp^3(u)-I\wp(u)-J},$$

which is a rational function of $\wp(u)$.

1393. Moreover

$$\frac{d^2}{du^2}\log\wp'(u)=\frac{d}{du}\,\frac{\wp''(u)}{\wp'(u)}=\frac{\wp'''(u)\,\wp'(u)-\wp''^2(u)}{\wp'^2(u)}$$

$$=[12\wp'^2(u)\,\wp(u)-4\wp'^2(u)\{\wp(2u)+2\wp(u)\}]/\wp'^2(u)=4\wp(u)-4\wp(2u);$$

$$\therefore\ \wp(2u)=\wp(u)-\frac{1}{4}\,\frac{d^2}{du^2}\log\wp'(u).$$

1394. Another form is

$$\wp(2u)-\wp(u)=-\frac{3\wp^4(u)-\frac{3}{2}I\wp^2(u)-3J\wp(u)-\frac{1}{16}I^2}{4\wp^3(u)-I\wp(u)-J}.$$

Since $\wp(2u)=\dfrac{\wp^4(u)+\frac{1}{2}I\wp^2(u)+2J\wp(u)+\frac{1}{16}I^2}{4\wp^3(u)-I\wp(u)-J}$, we have

$$4\wp(2u)-\wp(u)=\frac{3I\wp^2(u)+9J\wp(u)+\frac{1}{4}I^2}{4\{\wp(u)-e_1\}\{\wp(u)-e_2\}\{\wp(u)-e_3\}}$$

$$=\frac{A}{\wp(u)-e_1}+\frac{B}{\wp(u)-e_2}+\frac{C}{\wp(u)-e_3},$$

where $A=(3Ie_1^2+9Je_1+\frac{1}{4}I^2)/4(e_1-e_2)(e_1-e_3)$

$$=[-3(e_2e_3-e_1^2)e_1^2+9e_1^2e_2e_3+(e_2e_3-e_1^2)^2]/(e_1-e_2)(e_1-e_3)$$

$$=[(e_2e_3-e_1^2)(e_2e_3-4e_1^2)+9e_1^2e_2e_3]/(e_1-e_2)(e_1-e_3)$$

$$=(e_2e_3+2e_1^2)^2/(e_2e_3+2e_1^2)=e_2e_3+2e_1^2=(e_1-e_2)(e_1-e_3);$$

$$\therefore\ 4\wp(2u)-\wp(u)=\frac{(e_1-e_2)(e_1-e_3)}{\wp(u)-e_1}+\frac{(e_2-e_3)(e_2-e_1)}{\wp(u)-e_2}+\frac{(e_3-e_1)(e_3-e_2)}{\wp(u)-e_3}.$$

1395. Put $v=2u$ in the formula

$$\wp(v+u)+\wp(v-u)=\frac{F\{\wp(u),\wp(v)\}}{\{\wp(u)-\wp(v)\}^2}.$$

Then $\wp(3u)+\wp(u)=\dfrac{F\{\wp(2u),\wp(u)\}}{\{\wp(2u)-\wp(u)\}^2}$, so that $\wp(3u)$ can be expressed rationally in terms of $\wp(u)$.

1396. Now put $v=nu$. Then

$$\wp(n+1)u+\wp(n-1)u=\frac{F\{\wp(nu),\wp(u)\}}{\{\wp(nu)-\wp(u)\}^2},$$

which expresses $\wp(n+1)u$ in terms of $\wp(nu)$, $\wp(n-1)u$ and $\wp(u)$ in rational form, whence $\wp(n+1)u$ is a rational function of $\wp(u)$. Thus it appears that $\wp(2u)$, $\wp(3u)$, $\wp(4u)$, etc., can all be expressed as rational algebraic functions of $\wp(u)$. But the expressions for these successive forms rapidly increase in complexity.

1397. Again, using the formula

$$\wp(v+u)-\wp(v-u)=-\frac{\wp'(v)\wp'(u)}{\{\wp(v)-\wp(u)\}^2},$$

and putting $v=2u$, $3u$, etc.,

$$\wp(3u)-\wp(u)=-\frac{\wp'(2u)\wp'(u)}{\{\wp(2u)-\wp(u)\}^2},$$

$$\wp(4u)-\wp(2u)=-\frac{\wp'(3u)\wp'(u)}{\{\wp(3u)-\wp(u)\}^2},\ldots$$

$$\wp(n+1)u-\wp(n-1)u=-\frac{\wp'(nu)\wp'(u)}{\{\wp(nu)-\wp(u)\}^2},$$

from which $\wp(3u)$, $\wp(4u)$, ... may be successively calculated; and it is noticeable that

$$\wp'(2u)\wp'(u),\quad \wp'(3u)\wp'(u),\quad \wp'(4u)\wp'(u),\ldots$$

are all rational algebraic functions of $\wp(u)$.

1398. **General Value of $\wp(nu)-\wp(u)$.** SCHWARZ.

We shall show later that the general form of $\wp(nu)$ is given by the formula

$$\wp(nu)-\wp(u)=-\frac{\psi_{n-1}\psi_{n+1}}{\psi_n^{\,2}},$$

where ψ_n is expressed in terms of Sigma Functions.

Schwarz has shown that

$$\wp(nu)-\wp(u)=-\frac{1}{n^2}\frac{d^2}{du^2}\log\psi_n,$$

where $\psi_n=\dfrac{(-1)^{n-1}\Delta_n}{\{1!2!3!\dots(n-1)!\}^2}$ and Δ_n stands for the determinant

$$\begin{vmatrix} \wp'(u), & \wp''(u), & \wp'''(u), & \dots\wp^{(n-1)}(u) \\ \wp''(u), & \wp'''(u), & \wp^{\text{iv}}(u), & \dots\wp^{(n)}(u) \\ \dots & \dots & \dots & \dots \\ \wp^{(n-1)}(u), & \wp^{(n)}(u), & \wp^{(n+1)}(u), & \dots\wp^{(2n-3)}(u) \end{vmatrix}.$$

The method of establishing this result is pointed out by Greenhill (*E.F.*, p. 300, etc.), but the proof lies outside the scope of the present account.

For immediate purposes we may establish a difference equation which will suffice to give us the values of the function $\wp(nu)-\wp(u)$ in terms of $\wp(u)$ for low values of n, such as $n=3, 4, 5, 6$, etc., which is all that we shall require.

1399. **A Difference Equation.**

From the formula

$$\wp(v+u)+\wp(v-u)=\{2xy(x+y)-\tfrac{1}{2}I(x+y)-J\}/(x-y)^2,$$

where $x=\wp(u)$, $y=\wp(v)$, we have, by putting

$$v=nu \quad\text{and}\quad \wp(nu)-\wp(u)=R_n,$$

$$R_{n+1}+R_{n-1}=\frac{2x(x+R_n)(2x+R_n)-\frac{1}{2}I(2x+R_n)-J}{R_n^{\,2}}-2x$$

$$=\frac{(4x^3-Ix-J)+(6x^2-\frac{1}{2}I)R_n+2xR_n^{\,2}}{R_n^{\,2}}-2x$$

$$=\{\wp'^2(u)+R_n\wp''(u)\}/R_n^{\,2},$$

i.e.
$$R_{n+1}=\frac{\wp'^2(u)}{R_n^{\,2}}+\frac{\wp''(u)}{R_n}-R_{n-1}. \qquad\text{(I)}$$

Putting $\chi_2\equiv\wp''(u)=6x^2-\frac{1}{2}I$, $\chi_3\equiv\wp'^2(u)=4x^3-Ix-J$,

$$\chi_4\equiv 3x^4-\tfrac{3}{2}Ix^2-3Jx-\tfrac{1}{16}I^2=3\wp(u)\wp'^2(u)-\tfrac{1}{4}\wp''^2(u)$$
$$=\tfrac{1}{4}\{\wp'(u)\wp'''(u)-\wp''^2(u)\},$$

where the suffixes of χ denote the degree in x in each case, the difference equation is $R_{n+1}+R_{n-1}=\frac{\chi_3+\chi_2 R_n}{R_n^2}$, with the starting equations $R_1=0$, $R_2=-\frac{\chi_4}{\chi_3}$, whence $R_3=\frac{\chi_3(\chi_3^2-\chi_2\chi_4)}{\chi_4^2}=-\frac{\chi_3\chi_6}{\chi_4^2}$, say, where $\chi_6\equiv\chi_2\chi_4-\chi_3^2$.

The suffix notation will suffice until the case of R_5, when a second factor of degree 12 occurs after χ_{12} has been used. We may denote this second factor by ϕ_{12}.

1400. Other forms of the difference equation may be convenient, and may be used, now we have found R_3, for we may eliminate χ_2 or χ_3, or both of them.

Since

$$R_{n+1}R_n+R_nR_{n-1}=\chi_2+\frac{\chi_3}{R_n} \quad \text{and} \quad R_{n+2}R_{n+1}+R_{n+1}R_n=\chi_2+\frac{\chi_3}{R_{n+1}},$$

we have
$$R_{n+2}R_{n+1}-R_nR_{n-1}=-\chi_3\left(\frac{1}{R_n}-\frac{1}{R_{n+1}}\right),$$

i.e.
$$R_{n+2}=\frac{R_{n-1}}{R_{n+1}}R_n-\frac{\chi_3}{R_{n+1}}\left(\frac{1}{R_n}-\frac{1}{R_{n+1}}\right); \quad \text{.................(II)}$$

or again, $(R_{n+2}+R_n)R_{n+1}^2-(R_{n+1}+R_{n-1})R_n^2=\chi_2(R_{n+1}-R_n)$. ......(III)

From either of these equations or by another application of (*I*), R_4 can be found; after which we may eliminate both χ_2 and χ_3, and form an equation connecting the R's of any five consecutive suffixes, viz.

$$\begin{vmatrix} R_{n+1}^2(R_n+R_{n+2}), & R_{n+1}, & 1 \\ R_n^2(R_{n-1}+R_{n+1}), & R_n, & 1 \\ R_{n-1}^2(R_{n-2}+R_n), & R_{n-1}, & 1 \end{vmatrix}=0;$$

whence
$$\frac{(R_{n+1}-R_n)(R_{n+1}-R_{n-1})(R_{n+1}-R_{n-2})}{R_{n+1}^2}+\frac{(R_{n-1}-R_n)(R_{n-1}-R_{n+1})(R_{n-1}-R_{n+2})}{R_{n-1}^2}=0, \quad \text{...(IV)}$$

in which a factor has been inserted for symmetry.

Now, putting $n=2$ in (II), we may readily show that

$$R_4=-\frac{\chi_4\chi_{12}}{\chi_3\chi_6^2}, \text{ where } \chi_{12}\equiv\chi_3^2\chi_6-\chi_4^3;$$

putting $n=3$ in (IV), we similarly get

$$R_5=-\frac{\chi_3\chi_4\chi_6\phi_{12}}{\chi_{12}^2}, \text{ where } \phi_{12}\equiv\chi_{12}-\chi_6^2;$$

and putting $n=4$,

$$R_6=-\frac{\chi_{12}\phi_{24}}{\chi_3\chi_4^2\phi_{12}^2}, \text{ where } \phi_{24}\equiv\chi_3^2\chi_6\phi_{12}-\chi_{12}^2,$$

and so on.

From the several connecting equations,

$$\chi_6=\chi_2\chi_4-\chi_3^2, \quad \chi_{12}=\chi_3^2\chi_6-\chi_4^3, \quad \phi_{12}=\chi_{12}-\chi_6^2,$$
$$\phi_{24}=\chi_3^2\chi_6\phi_{12}-\chi_{12}^2, \text{ etc.},$$

we can readily express χ_6, χ_{12}, ϕ_{12}, etc., in terms of the original quantities χ_2, χ_3, χ_4, so that the successive values of $\wp(nu)-\wp(u)$ may be obtained in terms of x. Collecting the results, we have

$$\wp(2u)-\wp(u)=-\frac{\chi_4}{\chi_3},\quad \wp(3u)-\wp(u)=-\frac{\chi_3\chi_6}{\chi_4^2},\quad \wp(4u)-\wp(u)=-\frac{\chi_4\chi_{12}}{\chi_3\chi_6^2},$$

$$\wp(5u)-\wp(u)=-\frac{\chi_3\chi_4\chi_6\phi_{12}}{\chi_{12}^2},\quad \wp(6u)-\wp(u)=-\frac{\chi_{12}\phi_{24}}{\chi_3\chi_4^2\phi_{12}^2},\ \text{etc.},$$

and the notation shows the nature of the factorisation of the several numerators and denominators.

If we change the notation, and write

$$\chi_3\equiv\psi_2^2,\quad \chi_4=\psi_3,\quad \chi_6=\psi_4/\psi_2,\quad \chi_{12}=\psi_5,\quad \phi_{12}=\psi_6/\psi_2\psi_3,\quad \phi_{24}=\psi_7,$$

etc., with $\psi_1=1$, we get

$$\wp(2u)-\wp(u)=-\frac{\psi_1\psi_3}{\psi_2^2},\quad \wp(3u)-\wp(u)=-\frac{\psi_2\psi_4}{\psi_3^2},$$

$$\wp(4u)-\wp(u)=-\frac{\psi_3\psi_5}{\psi_4^2},\quad \wp(5u)-\wp(u)=-\frac{\psi_4\psi_6}{\psi_5^2},$$

$$\wp(6u)-\wp(u)=-\frac{\psi_5\psi_7}{\psi_6^2},\ \text{etc.}$$

1401. Factorisation of ψ_3, etc.

If we consider the solution of $\wp(2u)=\wp(u)$, we may infer the factorisation of χ_4, *i.e.* ψ_3.

The equation gives $2u=2m\omega_1+2n\omega_3\pm u$. Therefore

$$u=\frac{2m}{3}\omega_1+\frac{2n}{3}\omega_3\quad\text{or}\quad 2m\omega_1+2n\omega_3.$$

The principal solutions are

$$\frac{2\omega_1}{3},\ \frac{2\omega_3}{3},\ \frac{2\omega_1}{3}+\frac{2\omega_3}{3},\ \frac{2\omega_1}{3}-\frac{2\omega_3}{3};$$

and any other solutions, such for instance as

$$\frac{4\omega_1}{3}+\frac{2\omega_3}{3},\ \frac{4\omega_1}{3}\pm\frac{6\omega_3}{3},\ \text{etc.},$$

are merely such that when added to one or other of the four principal solutions we obtain a complete period. Hence the factors of χ_4 are

$$\chi_4\equiv\psi_3\equiv3\left[\wp(u)-\wp\left(\frac{2\omega_1}{3}\right)\right]\left[\wp(u)-\wp\left(\frac{2\omega_3}{3}\right)\right]$$
$$\times\left[\wp(u)-\wp\left(\frac{2\omega_1+2\omega_2}{3}\right)\right]\left[\wp(u)-\wp\left(\frac{2\omega_1-2\omega_3}{3}\right)\right],$$

and since $\chi_4\equiv3\wp^4(u)-\frac{3}{2}I\wp^2(u)-3J\wp(u)-\frac{1}{16}I^2$, we have various results from the consideration of various symmetrical functions of the roots of the quartic $\chi_4=0$; for instance

$$\wp\left(\frac{2\omega_1}{3}\right)+\wp\left(\frac{2\omega_3}{3}\right)+\wp\left(\frac{2\omega_1+2\omega_3}{3}\right)+\wp\left(\frac{2\omega_1-2\omega_3}{3}\right)=0,$$

$$\wp\left(\frac{2\omega_1}{3}\right)\cdot\wp\left(\frac{2\omega_3}{3}\right)\cdot\wp\left(\frac{2\omega_1+2\omega_3}{3}\right)\cdot\wp\left(\frac{2\omega_1-2\omega_3}{3}\right)=-\tfrac{1}{48}I^2, \text{etc.},$$

and similar results will follow from a consideration of the equations $\wp(3u)=\wp(u)$, $\wp(4u)=\wp(u)$, etc.

1402. Let $Q_x\equiv 4(x-e_1)(x-e_2)(x-e_3)$, $x=\wp(u)$, $y=\wp(v)$, $z=\wp(w)$. Then

$$[\sqrt{y-e_1}\sqrt{(z-e_2)(z-e_3)}-\sqrt{z-e_1}\sqrt{(y-e_2)(y-e_3)}]^2$$
$$=(y-e_1)(z^2+e_1z+e_2e_3)+(z-e_1)(y^2+e_1y+e_2e_3)-\tfrac{1}{2}\sqrt{Q_y}\sqrt{Q_z}$$
$$=yz(y+z)-\tfrac{1}{4}I(y+z)-\tfrac{1}{2}J-e_1(y-z)^2-\tfrac{1}{2}\sqrt{Q_y}\sqrt{Q_z}$$
$$=\tfrac{1}{2}[F(y,z)-\sqrt{Q_y}\sqrt{Q_z}]-e_1(y-z)^2=(y-z)^2\left\{\frac{1}{2}\,\frac{F(y,z)-\sqrt{Q_y}\sqrt{Q_z}}{(y-z)^2}-e_1\right\}$$
$$=\{\wp(v)-\wp(w)\}^2\{\wp(v+w)-e_1\}.\quad \text{That is}$$

$$\sqrt{\wp(v+w)-e_1}\{\wp(v)-\wp(w)\}=\sqrt{y-e_1}\sqrt{(z-e_2)(z-e_3)}-\sqrt{z-e_1}\sqrt{(y-e_2)(y-e_3)}$$

with two similar equations.

1403. It will be noted that $\wp(v+w)-e_1$, $\wp(w+u)-e_2$, $\wp(u+v)-e_3$ are perfect squares.

1404. In the same way

$$\sqrt{\wp(v-w)-e_1}\{\wp(v)-\wp(w)\}=\sqrt{y-e_1}\sqrt{(z-e_2)(z-e_3)}+\sqrt{z-e_1}\sqrt{(y-e_2)(y-e_3)}$$

with two similar equations.

1405. If $2\omega_1$, $2\omega_2$, $2\omega_3$ be the three periods, then

$$\omega_1-\omega_2+\omega_3=0 \quad\text{and}\quad \wp(\omega_1)=e_1,\ \wp(\omega_2)=e_2,\ \wp(\omega_3)=e_3,$$

and since $e_1+e_2+e_3=0$, we have $\wp(\omega_1)+\wp(\omega_2)+\wp(\omega_3)=0$.

Also

$$\wp(2u)-\wp(\omega_1)=\frac{\wp^4(u)+\frac{1}{2}I\wp^2(u)+2J\wp(u)+\frac{1}{16}I^2}{\wp'^2(u)}-e_1\equiv\frac{Q}{\wp'^2(u)}, \text{ say,}$$

where

$$Q\equiv\wp^4(u)-4e_1\wp^3(u)+\tfrac{1}{2}I\wp^2(u)+(2J+e_1I)\wp(u)+(\tfrac{1}{16}I^2+e_1J).$$

Then this quartic function Q *is a perfect square*. For the solutions of $\wp(2u)=\wp(\omega_1)$ are given by $2u=2\lambda\omega_1+2\mu\omega_3\pm\omega_1$. That is $u=$ an odd multiple of $\frac{1}{2}\omega_1+$ a multiple of ω_3.

Now $\dfrac{\omega_1}{2}$ and $\dfrac{\omega_1}{2}+\omega_3$ are the only *independent* solutions, for any others are merely such that, with one or other of these, they make a complete period. Therefore the only different factors of Q are the two

$$\wp(u)-\wp\left(\frac{\omega_1}{2}\right) \text{ and } \wp(u)-\wp\left(\frac{\omega_1}{2}+\omega_3\right),$$

which must therefore be repeated. It is therefore indicated that

$$\wp(2u)-\wp(\omega_1)=\left[\wp(u)-\wp\left(\frac{\omega_1}{2}\right)\right]^2\left[\wp(u)-\wp\left(\frac{\omega_1}{2}+\omega_3\right)\right]^2\Big/\wp'^2(u),$$

no coefficient being required, because in $\wp(2u)$ the coefficient of $\wp^4(u)$ is to be $1/\wp'^2(u)$, which is so.

The actual factorisation is given in the next article, which will show that the repetition could not be such that one factor is repeated thrice.

1406. Since

$$I=-4(e_2e_3-e_1^2)\,;\; 2J+e_1I=4e_1(e_2e_3+e_1^2)\,;\; \tfrac{1}{16}I^2+e_1J=(e_2e_3+e_1^2)^2,$$

$$\begin{aligned}\wp(2u)-e_1 &=[\wp^4(u)-4e_1\wp^3(u)-2(e_2e_3-e_1^2)\wp^2(u)+4e_1(e_2e_3+e_1^2)\wp(u)+(e_2e_3+e_1^2)^2]/\wp'^2(u)\\ &=[\wp^2(u)-2e_1\wp(u)-(e_2e_3+e_1^2)]^2/\wp'^2(u)\\ &=[\{\wp(u)-e_1\}^2-(e_2e_3+2e_1^2)]^2/\wp'^2(u),\end{aligned}$$

which shows the actual factorisation of Q.

1407. The values of $\wp\left(\frac{\omega_1}{2}\right)$, $\wp\left(\frac{\omega_1}{2}+\omega_3\right)$ are therefore

$$e_1\pm\sqrt{e_2e_3+2e_1^2}, \quad \textit{i.e. } e_1\pm\sqrt{3e_1^2-\tfrac{1}{4}I},$$

and since $\wp\left(\frac{\omega_1}{2}\right)$ lies between e_1 and ∞ we take the positive sign for $\wp\left(\frac{\omega_1}{2}\right)$. [See Art. 1410.]

1408. We have also the relations

$$\wp\left(\frac{\omega_1}{2}\right)+\wp\left(\frac{\omega_1}{2}+\omega_3\right)=2e_1=2\wp(\omega_1)\,;\; \wp\left(\frac{\omega_1}{2}\right)\cdot\wp\left(\frac{\omega_1}{2}+\omega_3\right)=\frac{I}{4}-2\wp^2(\omega_1)$$

with other results. For instance

$$\sqrt{\wp(2u)-e_1}=-\left[\wp(u)-\wp\left(\frac{\omega_1}{2}\right)\right]\left[\wp(u)-\wp\left(\frac{\omega_1}{2}+\omega_3\right)\right]\Big/\wp'(u),$$

where the negative sign is chosen, because when u is very small

$$\wp(2u)=\frac{1}{4u^2}, \quad \wp(u)=\frac{1}{u^2}, \quad \wp'(u)=-\frac{2}{u^3}.$$

1409. Putting $z=e_1$, e_2 or e_3 in

$$\wp'^2(u)=4\wp^3(u)-I\wp(u)-J=4(z-e_1)(z-e_2)(z-e_3),$$

$$\wp'(\omega_1)=\wp'(\omega_2)=\wp'(\omega_3)=0.$$

Then $$\wp(u+\omega_1)=\frac{1}{4}\frac{\wp'^2(u)}{\{\wp(u)-\wp(\omega_1)\}^2}-\wp(u)-\wp(\omega_1);$$

$$\therefore\ \wp(u+\omega_1)-\wp(\omega_1)=\frac{1}{4}\frac{\wp'^2(u)-4\{\wp(u)+2\wp(\omega_1)\}\{\wp(u)-\wp(\omega_1)\}^2}{\{\wp(u)-\wp(\omega_1)\}^2}$$
$$=\{4z^3-Iz-J-4(z+2e_1)(z-e_1)^2\}/4(z-e_1)^2$$
$$=\{(12e_1{}^2-I)z-(J+8e_1{}^3)\}/4(z-e_1)^2,$$

and $$12e_1{}^2-I=4(e_1-e_2)(e_1-e_3),\quad J+8e_1{}^3=4e_1(e_1-e_2)(e_1-e_3).$$

Hence $$\wp(u+\omega_1)-\wp(\omega_1)=(e_1-e_2)(e_1-e_3)/(z-e_1),\ \ldots\ldots\ldots\ldots(1)$$

i.e. $$\{\wp(u+\omega_1)-\wp(\omega_1)\}\{\wp(u)-\wp(\omega_1)\}=\{\wp(\omega_1)-\wp(\omega_2)\}\{\wp(\omega_1)-\wp(\omega_3)\},\ \ldots(2)$$

with two similar results by a cyclical change of suffixes.

1410. We may therefore write the result of Art. 1394 as

$$4\wp(2u)=\wp(u)+\wp(u+\omega_1)+\wp(u+\omega_2)+\wp(u+\omega_3).\quad \text{[M. Trip., 1888.]}\ \ldots(3)$$

Other identities may be established. Thus, since

$$\wp(u+\omega_1)=e_1+\frac{(e_1-e_2)(e_1-e_3)}{z-e_1},$$

we have $$\wp'(u+\omega_1)=\frac{(e_3-e_1)(e_1-e_2)}{(z-e_1)^2}\wp'(u),$$

i.e. $$\wp'(u+\omega_1)=\frac{\{\wp(\omega_3)-\wp(\omega_1)\}\{\wp(\omega_1)-\wp(\omega_2)\}}{\{\wp(u)-\wp(\omega_1)\}^2}\wp'(u).$$

If in (1) we put $u=-\frac{1}{2}\omega_1$,

$$z=\wp\left(\frac{\omega_1}{2}\right)\quad\text{and}\quad \wp\left(\frac{\omega_1}{2}\right)-e_1=\pm\sqrt{(e_1-e_2)(e_1-e_3)}.\quad \text{(See Art. 1407.)}$$

Now $2\omega_1=2\int_{e_1}^{\infty}\frac{dz}{\sqrt{4z^3-Iz-J}}$ and is real; and as z increases from e_1 to ∞, u decreases from ω_1 to 0 and passes the value $\omega_1/2$ in the interval. Hence the value of z corresponding to $\frac{\omega_1}{2}$, that is $\wp\left(\frac{\omega_1}{2}\right)$, lies between e_1 and ∞, and is therefore $> e_1$. Hence we take the positive sign, and

$$\wp\left(\frac{\omega_1}{2}\right)=e_1+\sqrt{(e_1-e_2)(e_1-e_3)}.$$

Also, since $\wp'(u)=-\sqrt{4(z-e_1)(z-e_2)(z-e_3)}$, we have

$$\wp'\left(\frac{\omega_1}{2}\right)=-\sqrt{4\{\sqrt{(e_1-e_2)(e_1-e_3)}\}\{e_1-e_2+\sqrt{(e_1-e_2)(e_1-e_3)}\}\{e_1-e_3+\sqrt{(e_1-e_2)(e_1-e_3)}\}}$$
$$=-2\sqrt{(e_1-e_2)(e_1-e_3)}[\sqrt{e_1-e_2}+\sqrt{e_1-e_3}].$$

1411. It may also be shown that

$$\wp\left(\frac{\omega_3}{2}\right)=e_3-\sqrt{(e_1-e_3)(e_2-e_3)},\quad \wp\left(\frac{\omega_2}{2}\right)=e_2-\iota\sqrt{(e_1-e_2)(e_2-e_3)},$$

$$\wp'\left(\frac{\omega_3}{2}\right)=-2\iota\sqrt{(e_1-e_3)(e_2-e_3)}[\sqrt{e_1-e_3}+\sqrt{e_2-e_3}],$$

$$\wp'\left(\frac{\omega_2}{2}\right)=2\sqrt{(e_1-e_2)(e_2-e_3)}[\sqrt{e_1-e_2}+\iota\sqrt{e_2-e_3}].$$

1412. Again

$$\wp'(u+\omega_2)=\frac{(e_1-e_2)(e_2-e_3)}{(z-e_2)^2}\wp'(u),\quad \wp'(u+\omega_3)=\frac{(e_2-e_3)(e_3-e_1)}{(z-e_3)^2}\wp'(u).$$

Therefore

$$\wp'(u)\wp'(u+\omega_1)\wp'(u+\omega_2)\wp'(u+\omega_3)=16(e_2-e_3)^2(e_3-e_1)^2(e_1-e_2)^2,$$

and $$\therefore\ \frac{\wp''(u)}{\wp'(u)}+\frac{\wp''(u+\omega_1)}{\wp'(u+\omega_1)}+\frac{\wp''(u+\omega_2)}{\wp'(u+\omega_2)}+\frac{\wp''(u+\omega_3)}{\wp'(u+\omega_3)}=0.$$

1413. Also $\wp'(u)\dfrac{\wp(u+\omega_1)}{\wp'(u+\omega_1)}=\dfrac{e_1(e_2-e_3)}{(e_2-e_3)(e_3-e_1)(e_1-e_2)}(z-e_1)^2-(z-e_1)$, with two similar results.

$\therefore$ adding $$\wp'(u)\left\{\frac{\wp(u+\omega_1)}{\wp'(u+\omega_1)}++\right\}=-z=-\wp(u);$$

whence $$\frac{\wp(u)}{\wp'(u)}+\frac{\wp(u+\omega_1)}{\wp'(u+\omega_1)}+\frac{\wp(u+\omega_2)}{\wp'(u+\omega_2)}+\frac{\wp(u+\omega_3)}{\wp'(u+\omega_3)}=0.$$

1414. Weierstrassian Periods in Terms of Legendrian.

We have now to examine the relationship between the Legendrian and Weierstrassian systems. Taking e_1, e_2, e_3 as the roots of $4z^3-Iz-J=0$, and supposing them all real and $e_1>e_2>e_3$, the period $2\omega_1$ is defined as

$$2\int_{e_1}^{\infty}\frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}},$$

and is a *real* period ($z>e_1>e_2>e_3$).

Let $\quad z-e_1=(e_1-e_3)\cot^2\theta \quad$ and $\quad k^2=\dfrac{e_2-e_3}{e_1-e_3},$

which is positive and <1.

Then $\quad z-e_2=e_1-e_2+(e_1-e_3)\cot^2\theta=(e_1-e_3)\operatorname{cosec}^2\theta-(e_2-e_3)$
$\qquad=(e_1-e_3)(1-k^2\sin^2\theta)/\sin^2\theta,$

and $z-e_3=(e_1-e_3)/\sin^2\theta$; also $dz=-2(e_1-e_3)\operatorname{cosec}^2\theta\cot\theta\,d\theta$.

Again $z=e_1$ gives $\theta=\pi/2$ and $z=\infty$ gives $\theta=0$;

$$\therefore\ 2\omega_1=2\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{2(e_1-e_3)\operatorname{cosec}^2\theta\cot\theta\sin^2\theta\,d\theta}{(e_1-e_3)^{\frac{3}{2}}\cot\theta\sqrt{1-k^2\sin^2\theta}}$$

$$=\frac{2}{\sqrt{e_1-e_3}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{2K}{\sqrt{e_1-e_3}}.$$

Again (z real, and passing below $z=e_1$, see Art. 1335),

$$2\omega_2=2\int_{e_2}^{\infty}\frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}$$

$$=2\left\{\int_{e_2}^{e_1}+\int_{e_1}^{\infty}\right\}\frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}};$$

$$\therefore\ 2(\omega_2-\omega_1)=\frac{1}{\iota}\int_{e_2}^{e_1}\frac{dz}{\sqrt{(e_1-z)(z-e_2)(z-e_3)}}\quad (e_1>z>e_2>e_3).$$

Let $$z=e_1\cos^2\theta+e_2\sin^2\theta.$$

Then $e_1-z=(e_1-e_2)\sin^2\theta,\quad z-e_2=(e_1-e_2)\cos^2\theta,$

and $$z-e_3=(e_1-e_3)(1-k'^2\sin^2\theta),$$

where $$k'^2=\frac{e_1-e_2}{e_1-e_3}=1-\frac{e_2-e_3}{e_1-e_3}=1-k^2,$$

k' being positive and <1. Also $dz=-2(e_1-e_2)\sin\theta\cos\theta\,d\theta$.

Again $z=e_2$ gives $\theta=\frac{\pi}{2}$; $z=e_1$ gives $\theta=0$;

$$\therefore\ 2(\omega_2-\omega_1)=\frac{2}{\iota}\frac{1}{\sqrt{e_1-e_3}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k'^2\sin^2\theta}}=\frac{2K'}{\iota\sqrt{e_1-e_3}}.$$

Finally $$2\omega_3=2\int_{e_3}^{\infty}\frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}$$

$$=2\left\{\int_{e_3}^{e_2}+\int_{e_2}^{\infty}\right\}\frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}};$$

$$\therefore\ 2(\omega_3-\omega_2)=2\int_{e_3}^{e_2}\frac{dz}{\iota^2\sqrt{4(e_1-z)(e_2-z)(z-e_3)}}\quad(e_1>e_2>z>e_3).$$

Let $$z=e_2\sin^2\theta+e_3\cos^2\theta;$$

$$\therefore\ e_1-z=e_1-e_2\sin^2\theta-e_3(1-\sin^2\theta)=(e_1-e_3)(1-k^2\sin^2\theta),$$

$$e_2-z=(e_2-e_3)\cos^2\theta,\quad z-e_3=(e_2-e_3)\sin^2\theta,$$

$$dz=2(e_2-e_3)\sin\theta\cos\theta\,d\theta;$$

$z=e_3$ gives $\theta=0$, $z=e_2$ gives $\theta=\frac{\pi}{2}$;

$$\therefore\ 2(\omega_3-\omega_2)=\frac{2}{\iota^2\sqrt{e_1-e_3}}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=-\frac{2K}{\sqrt{e_1-e_3}}.$$

Hence $$\omega_1=\frac{K}{\sqrt{e_1-e_3}},\quad \omega_2=\frac{K-\iota K'}{\sqrt{e_1-e_3}},\quad \omega_3=\frac{-\iota K'}{\sqrt{e_1-e_3}},$$

and $\omega_1-\omega_2+\omega_3=0$, as it should be.

1415. Connection between the Jacobian and Weierstrassian Elliptic Functions.

In general, taking

$$u=\int_z^{\infty}\frac{dz}{\sqrt{4(z-e_1)(z-e_2)(z-e_3)}}\quad(e_1>e_2>e_3).$$

Put $z=e_1+(e_1-e_3)\cot^2\theta$, and we have

$$u=\frac{1}{\sqrt{e_1-e_3}}\int_0^{\theta}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\quad\text{where } k^2=\frac{e_2-e_3}{e_1-e_3}.$$

Then $\theta = \operatorname{am}\sqrt{e_1 - e_3}\,u,$

$$\wp(u) = e_1 + (e_1 - e_3)\cot^2\theta = e_3 + \frac{e_1 - e_3}{\sin^2\theta} = e_2 + \frac{e_1 - e_3}{\sin^2\theta}\left(1 - \frac{e_2 - e_3}{e_1 - e_3}\sin^2\theta\right),$$

i.e.
$$\wp(u) = e_1 + (e_1 - e_3)\frac{\operatorname{cn}^2\sqrt{e_1 - e_3}\,u}{\operatorname{sn}^2\sqrt{e_1 - e_3}\,u},$$
$$\wp(u) = e_2 + (e_1 - e_3)\frac{\operatorname{dn}^2\sqrt{e_1 - e_3}\,u}{\operatorname{sn}^2\sqrt{e_1 - e_3}\,u},$$
$$\wp(u) = e_3 + (e_1 - e_3)\frac{1}{\operatorname{sn}^2\sqrt{e_1 - e_3}\,u}, \quad \text{...............(A)}$$

which may also be written as

$$\operatorname{sn}^2\sqrt{e_1 - e_3}\,u = \frac{e_1 - e_3}{\wp(u) - e_3}, \quad \operatorname{cn}^2\sqrt{e_1 - e_3}\,u = \frac{\wp(u) - e_1}{\wp(u) - e_3},$$
$$\operatorname{dn}^2\sqrt{e_1 - e_3}\,u = \frac{\wp(u) - e_2}{\wp(u) - e_3}, \quad \text{.................(B)}$$

which show the connection between the Jacobian and Weierstrassian systems.

1416. Expansion of $\wp(u)$ in Powers of u.

Taking $u = \int_z^\infty \frac{dz}{\sqrt{4z^3 - Iz - J}}$, and $z > e_1 > e_2 > e_3$, we have

$$u = \int_z^\infty \frac{dz}{2z^{\frac{3}{2}}}\left[1 - \frac{1}{4}\left(\frac{I}{z^2} + \frac{J}{z^3}\right)\right]^{-\frac{1}{2}} dz, \text{ and a convergent expansion,}$$

$$u = \int_z^\infty \frac{dz}{2z^{\frac{3}{2}}}\left[1 + \frac{1}{2}\cdot\frac{1}{4}\left(\frac{I}{z^2} + \frac{J}{z^3}\right) + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{4^2}\left(\frac{I}{z^2} + \frac{J}{z^3}\right)^2 + \dots\right]$$
$$= \int_z^\infty dz\left[\frac{1}{2z^{\frac{3}{2}}} + \frac{I}{2^2\,.\,4}\frac{1}{z^{\frac{7}{2}}} + \frac{J}{2^2\,.\,4}\frac{1}{z^{\frac{9}{2}}} + \frac{1\,.\,3}{2^2\,.\,4^3}\frac{I^2}{z^{\frac{11}{2}}} + \dots\right]$$
$$= \frac{1}{z^{\frac{1}{2}}} + 0 + \frac{I}{2\,.\,4\,.\,5}\frac{1}{z^{\frac{5}{2}}} + \frac{J}{2\,.\,4\,.\,7}\frac{1}{z^{\frac{7}{2}}} + \frac{1\,.\,3}{2\,.\,4^3\,.\,9}\frac{I^2}{z^{\frac{9}{2}}} + \dots.$$

We have to reverse this series, and expand z in powers of u. Squaring, we notice that u^2 is a rational function of z, viz.

$$u^2 = \frac{1}{z} + 0 + \frac{I}{4\,.\,5}\frac{1}{z^3} + \frac{J}{4\,.\,7}\frac{1}{z^4} + \dots.$$

Then
$$z = \frac{1}{u^2} + 0 + \frac{I}{20}\frac{1}{u^2z^2} + \frac{J}{28}\frac{1}{u^2z^3} + \dots$$
$$= \frac{1}{u^2} + 0 + \frac{I}{20}u^2 + \frac{J}{28}u^4 + \dots \text{ to the first three terms.}$$

As z is obviously an even function of u, we may conclude that the expansion is of the form

$$z = \frac{1}{u^2} + 0 + \frac{I}{20}u^2 + \frac{J}{28}u^4 + \frac{A_6}{6!}u^6 + \frac{A_8}{8!}u^8 + \frac{A_{10}}{10!}u^{10} + \dots,$$

where A_6, A_8, ... remain to be found. As the work of reversion of series is somewhat laborious, we may now use the differential equation $z'''=12zz'$ (Art. 1384) to determine the coefficients from this point.

Now $$z'=\frac{-1.2}{u^3}+0+\frac{I}{10}u+\frac{J}{7}u^3+\frac{A_6}{5!}u^5+\frac{A_8}{7!}u^7+\frac{A_{10}}{9!}u^9+\dots,$$

$$z'''=\frac{-1.2.3.4}{u^5}+0+0+\frac{2.3J}{7}u+\frac{A_6}{3!}u^3+\frac{A_8}{5!}u^5+\frac{A_{10}}{7!}u^7+\dots;$$

whence $$\frac{A_6}{3!}=12\left(\frac{4A_6}{6!}+\frac{I^2}{200}\right),\quad \frac{A_8}{5!}=12\left(\frac{6A_8}{8!}+\frac{3IJ}{280}\right),$$

$$\frac{A_{10}}{7!}=12\left(\frac{8A_{10}}{10!}+\frac{2}{5}I\frac{A_6}{6!}+\frac{J^2}{4.7^2}\right),\text{ etc.,}$$

giving $$\frac{A_6}{6!}=\frac{I^2}{2^4.3.5^2},\quad \frac{A_8}{8!}=\frac{3IJ}{2^4.5.7.11},\quad \frac{A_{10}}{10!}=\frac{1}{2^4.3.13}\left(\frac{I^3}{2.5^3}+\frac{3J^2}{7^2}\right),\text{ etc.}$$

Hence
$$\wp(u)=\frac{1}{u^2}+0+\frac{I}{20}u^2+\frac{J}{28}u^4+\frac{I^2}{2^4.3.5^2}u^6+\frac{3IJ}{2^4.5.7.11}u^8+\frac{1}{2^4.3.13}\cdot\left(\frac{I^3}{2.5^3}+\frac{3J^2}{7^2}\right)u^{10}+\dots.$$

1417. It appears that $\wp(u)-\dfrac{1}{u^2}$ vanishes with u. That is, for very small values of u, $\wp(u)=\dfrac{1}{u^2}$. Also $Lt_{u\to 0}\dfrac{\wp(u)-\dfrac{1}{u^2}}{u^2}=\dfrac{I}{20}$, etc.

Again $\wp(u)\wp'(u)+\dfrac{2}{u^5}$ vanishes with u.

Moreover the expansions of $\wp'(u)$, $\wp''(u)$, $\wp'''(u)$, etc., are now all known to several terms.

1418. **The Expansions of the Weierstrassian Zeta and Sigma Functions.**

Since $\zeta(u)=-\int\wp(u)\,du=\dfrac{d}{du}\log\sigma(u)$, we have

$$\zeta(u)=\frac{1}{u}+0-\frac{I}{2^2.3.5}u^3-\frac{J}{2^2.5.7}u^5-\frac{I^2}{2^4.3.5^2.7}u^7-\frac{I.J}{2^4.3.5.7.11}u^9-\frac{1}{2^4.3.11.13}\left(\frac{I^3}{2.5^3}+\frac{3J^2}{7^2}\right)u^{11}-\dots. \quad \text{(A)}$$

Also $$\int\zeta(u)\,du=\log u+0-\frac{I}{2^4.3.5}u^4-\frac{J}{2^3.3.5.7}u^6-\frac{I^2}{2^7.3.5^2.7}u^8-\frac{IJ}{2^5.3.5^2.7.11}u^{10}-\dots;$$

whence

$$\sigma(u)=e^{\int\zeta(u)\,du}=u\,.\,e^{-\frac{Iu^4}{2^4.3.5}}\,.\,e^{-\frac{Ju^6}{2^3.3.5.7}}\,.\,e^{-\frac{I^2u^8}{2^7.3.5^2.7}}\,.\,e^{-\frac{IJu^{10}}{2^5.3.5^2.7.11}}\dots$$

$$=u\left[1-\frac{Iu^4}{2^4.3.5}+\frac{I^2u^8}{2^9.3^2.5^2}-\dots\right]\left[1-\frac{Ju^6}{2^3.3.5.7}\dots\right]\times\left[1-\frac{I^2u^8}{2^7.3.5^2.7}\dots\right]\left[1-\frac{IJu^{10}}{2^5.3.5^2.7.11}\dots\right],$$

i.e. $$\sigma(u)=u+0-\frac{Iu^5}{2^4.3.5}-\frac{Ju^7}{2^3.3.5.7}-\frac{I^2u^9}{2^9.3^2.5.7}-\frac{IJu^{11}}{2^7.3^2.5^2.7.11}-\dots. \quad \text{(B)}$$

Equations (A) and (B) give the expansions of the Zeta and Sigma functions.

The constants of integration are in both cases taken zero. That is, $\zeta(u)-\frac{1}{u}$ and $\log\frac{\sigma(u)}{u}$ are taken as vanishing with u.

1419. We note that both $\zeta(u)$ and $\sigma(u)$ are odd functions of u, and that in consequence $\zeta(-u)=-\zeta(u)$, $\sigma(-u)=-\sigma(u)$.

Also that $\zeta(0)=\infty$, $\zeta'(0)=\infty$, $\zeta''(0)=\infty$, etc.,

$$\sigma(0)=0,\quad \sigma'(0)=1,\quad \sigma''(0)=0,\quad \sigma'''(0)=0,\quad \sigma^{\text{iv}}(0)=0,$$
$$\sigma^{\text{v}}(0)=-\tfrac{1}{2}I, \text{ etc.},$$

and for small values of u, $\zeta(u)=\frac{1}{u}$, $\sigma(u)=u$.

1420. Addition Formula for the Zeta Function.

Integrating the equation

$$\wp(u-v)-\wp(u+v)=\frac{\wp'(u)\wp'(v)}{\{\wp(u)-\wp(v)\}^2}$$

with respect to v, $\quad \zeta(u-v)+\zeta(u+v)=\frac{\wp'(u)}{\wp(u)-\wp(v)}+C$;

and putting $v=0$, $\wp(v)=\infty$; $\therefore\ 2\zeta(u)=C$;

$$\therefore\ \zeta(u-v)+\zeta(u+v)-2\zeta(u)=\frac{\wp'(u)}{\wp(u)-\wp(v)}. \quad\text{.........(1)}$$

Also $\zeta(u)$ being an odd function, $\zeta(u-v)=-\zeta(v-u)$.

Hence, interchanging u and v in equation (1),

$$-\zeta(u-v)+\zeta(u+v)-2\zeta(v)=-\frac{\wp'(v)}{\wp(u)-\wp(v)}. \quad\text{.........(2)}$$

Hence adding,

$$\zeta(u+v)-\zeta(u)-\zeta(v)=\frac{1}{2}\,\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)} \quad\text{............(3)}$$
$$=\{\wp(u+v)+\wp(u)+\wp(v)\}^{\frac{1}{2}},$$

or writing $u+v=-w$ and remembering that

$$\wp(-w)=\wp(w),\quad \zeta(-w)=-\zeta(w),$$
$$\zeta(u)+\zeta(v)+\zeta(w)+\sqrt{\wp(u)+\wp(v)+\wp(w)}=0,$$

where $u+v+w=0$. [See Greenhill, *E.F.*, p. 205.]

Changing the sign of v in (3),

$$\zeta(u-v)-\zeta(u)+\zeta(v)=\frac{1}{2}\,\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}. \quad\text{............(4)}$$

1421. By differentiating (3) and (4) with regard to u,

$$\frac{d}{du}\zeta(u+v)-\frac{d}{du}\zeta(u)=\frac{1}{2}\frac{d}{du}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}$$

and $$\frac{d}{du}\zeta(u-v)-\frac{d}{du}\zeta(u)=\frac{1}{2}\frac{d}{du}\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)};$$

whence $$\wp(u)-\wp(u+v)=\frac{1}{2}\frac{d}{du}\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)},$$

$$\wp(u)-\wp(u-v)=\frac{1}{2}\frac{d}{du}\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}.$$

1422. Addition Formula for the Sigma Function.

Integrating $\zeta(u-v)+\zeta(u+v)-2\zeta(u)=\dfrac{\wp'(u)}{\wp(u)-\wp(v)}$ with regard to u,

$$\log\sigma(u-v)+\log\sigma(u+v)-2\log\sigma(u)=\log\{\wp(u)-\wp(v)\}+C;$$

and since, when u is indefinitely small,

$$\sigma(u)=u \quad\text{and}\quad \wp(u)=\frac{1}{u^2},$$

$$\log\sigma(-v)+\log\sigma(v)=Lt_{u\to 0}\log u^2\left\{\frac{1}{u^2}-\wp(v)\right\}+C=C;$$

whence

$$\log\frac{\sigma(v-u)}{\sigma(v)}+\log\frac{\sigma(v+u)}{\sigma(v)}-2\log\sigma(u)=\log\{\wp(u)-\wp(v)\}, \quad (1)$$

i.e. $$\frac{\sigma(v-u)\,\sigma(v+u)}{\sigma^2(u)\,\sigma^2(v)}=\wp(u)-\wp(v)$$

and $$\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)}=\wp(v)-\wp(u). \quad\ldots\ldots\ldots\ldots\ldots(2)$$

Putting $v=nu$, we have

$$\wp(nu)-\wp(u)=-\frac{\sigma(n-1)u\,\sigma(n+1)u}{\sigma^2(nu)\,\sigma^2(u)}.$$

1423. If we integrate with regard to v instead of with regard to u, we have

$$-\log\sigma(u-v)+\log\sigma(u+v)-2v\zeta(u)=\int_0^v\frac{\wp'(u)}{\wp(u)-\wp(v)}dv;$$

whence $$\log e^{-2v\zeta(u)}\frac{\sigma(u+v)}{\sigma(u-v)}=\int_0^v\frac{\wp'(u)}{\wp(u)-\wp(v)}dv. \quad\ldots\ldots\ldots\ldots(3)$$

1424. Starting with

$$-\zeta(u-v)+\zeta(u+v)-2\zeta(v)=-\frac{\wp'(v)}{\wp(u)-\wp(v)},$$

and integrating with regard to u,

$$-\log\sigma(v-u)+\log\sigma(u+v)-2u\zeta(v)=-\int_0^u\frac{\wp'(v)}{\wp(u)-\wp(v)}\,du;$$

whence $$\log e^{-2u\zeta(v)}\frac{\sigma(v+u)}{\sigma(v-u)}=-\int_0^u\frac{\wp'(v)}{\wp(u)-\wp(v)}\,du. \quad\text{.........(4)}$$

1425. Since $\dfrac{d^2\log\sigma(u)}{du^2}=\dfrac{d\zeta(u)}{du}=-\wp(u)$, we have

$$\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)}=\frac{d^2}{du^2}\log\sigma(u)-\frac{d^2}{dv^2}\log\sigma(v). \quad\text{.......(5)}$$

1426. In the result

$$\frac{\sigma(u-v)\,\sigma(u+v)}{\sigma^2(u)\,\sigma^2(v)}=\wp(v)-\wp(u),$$

make v approach indefinitely closely to u. Then

$$\frac{\sigma(2u)}{\sigma^4(u)}=Lt_{v\to u}\frac{\wp(v)-\wp(u)}{\sigma(u-v)}=Lt_{v\to u}\frac{\wp'(v)}{-\sigma'(u-v)}=-\wp'(u),$$

for $\sigma'(0)=1$ (Art. 1419). Hence

$$\sigma(2u)=-\sigma^4(u)\,\wp'(u)=(-1)^1\sigma^{2^2}(u)\,\wp'(u).$$

1427. Differentiating $\wp(2u)-\wp(u)=-\frac{1}{4}\dfrac{d^2}{du^2}\log\wp'(u)$, we have

$$2\wp'(2u)-\wp'(u)=-\tfrac{1}{4}\frac{d^3}{du^3}\log\wp'(u),\ \text{etc.},$$

$$2^n\wp^{(n)}(2u)-\wp^{(n)}(u)=-\tfrac{1}{4}\frac{d^{n+2}}{du^{n+2}}\log\wp'(u).$$

Integrating the same equation,

$$-\tfrac{1}{2}\zeta(2u)+\zeta(u)+C=-\tfrac{1}{4}\frac{d}{du}\log\wp'(u)=-\tfrac{1}{4}\frac{\wp''(u)}{\wp'(u)},$$

and taking u indefinitely small, we have in the limit

$$-\frac{1}{2}\cdot\frac{1}{2u}+\frac{1}{u}+C=-\frac{1}{4}\cdot\frac{\dfrac{2\,.\,3}{u^4}+\dfrac{1}{10}I}{-\dfrac{2}{u^3}}=\frac{3}{4u};\quad\therefore\ C=0;$$

whence $$-\tfrac{1}{2}\zeta(2u)+\zeta(u)=-\tfrac{1}{4}\frac{\wp''(u)}{\wp'(u)}.$$

Again integrating $-\frac{1}{4}\log\sigma(2u)+\log\sigma(u)+C'=-\frac{1}{4}\log\wp'(u)$,
and diminishing u indefinitely,

$$-\tfrac{1}{4}\log 2u+\log u+C'=-\tfrac{1}{4}\log\left(-\frac{2}{u^3}\right)=\tfrac{3}{4}\log u-\tfrac{1}{4}\log 2-\tfrac{1}{4}\log(-1);$$

$$\therefore\ C'=-\tfrac{1}{4}\log(-1);$$

$$\therefore\ \log\frac{\sigma^4(u)}{\sigma(2u)}=\log\frac{-1}{\wp'(u)},\ \textit{i.e.}\ \sigma(2u)=-\sigma^4(u)\,\wp'(u),\ \text{as found before.}$$

1428. Putting $n=2$ in the formula

$$\wp(nu)-\wp(u)=-\frac{\sigma(n+1)u\,\sigma(n-1)u}{\sigma^2(nu)\,\sigma^2(u)},$$

we have $$\frac{\sigma(3u)\,\sigma(u)}{\sigma^2(2u)\,\sigma^2(u)}=\wp(u)-\wp(2u)=\frac{1}{4}\,\frac{d^2}{du^2}\log\wp'(u)\,;$$

$$\therefore\ \sigma(3u)=\tfrac{1}{4}\sigma^{3^2}(u)\,\wp'^2(u)\,\frac{d^2}{du^2}\log\wp'(u)=\frac{(-1)^2\,\sigma^{3^2}(u)}{(1\,!\,2\,!)^2}\begin{vmatrix}\wp'(u), & \wp''(u)\\ \wp''(u), & \wp'''(u)\end{vmatrix}.$$

1429. To find $\sigma(4u)$, we have

$$\sigma(4u)=-\sigma^4(2u)\,\wp'(2u)=-[\sigma^4(u)\,\wp'(u)]^4\,\wp'(2u)$$
$$=-\sigma^{4^2}(u)\,.\,\wp'^4(u)\,\wp'(2u),$$

and by aid of these results we might proceed to find $\sigma(5u)$, $\sigma(6u)$, etc.

1430. Corresponding to Euler's Theorem,

$$\cos\theta\cos 2\theta\cos 2^2\theta\ldots\cos 2^{n-1}\theta=\sin 2^n\theta/2^n\sin\theta,$$

we have $$\frac{\sigma(2^n u)}{\sigma^4(2^{n-1}u)}=-\wp'(2^{n-1}u),\quad \frac{\sigma(2^{n-1}u)}{\sigma^4(2^{n-2}u)}=-\wp'(2^{n-2}u),\ldots$$

$$\frac{\sigma(2^2u)}{\sigma^4(2u)}=-\wp'(2u),\quad \frac{\sigma(2u)}{\sigma^4(u)}=-\wp' u\,;$$

whence $$\frac{\sigma(2^n u)}{\sigma^{4^n}(u)}=-\wp'(2^{n-1}u)\,.\,\wp'^4(2^{n-2}u)\,.\,\wp'^{4^2}(2^{n-3}u)\ldots\wp'^{4^{n-1}}(u).$$

1431. Writing ψ_n for $\dfrac{\sigma(nu)}{(\sigma u)^{n^2}}$, we have

$$\frac{\psi_{n-1}\,\psi_{n+1}}{\psi_n{}^2}=\frac{\sigma(n-1)u}{(\sigma u)^{(n-1)^2}}\cdot\frac{\sigma(n+1)u}{(\sigma u)^{(n+1)^2}}\cdot\left\{\frac{(\sigma u)^{n^2}}{\sigma(nu)}\right\}^2=\frac{\sigma(n-1)u\,\sigma(n+1)u}{\sigma^2(nu)\,\sigma^2(u)}$$
$$=\wp(u)-\wp(nu)\,;$$

$$\therefore\ \wp(nu)-\wp(u)=-\frac{\psi_{n-1}\,\psi_{n+1}}{\psi_n{}^2}.$$

The value of $\psi_n(u)$ found by Schwarz has been shown in Art. 1398, expressed in terms of differential coefficients of $\wp(u)$.

Supposing the functions R_n to have been found in terms of $\wp(u)$ as explained in Art. 1399, etc., ψ_n can also be expressed in the same manner. For

$$\frac{\psi_n\,\psi_{n-2}}{\psi_{n-1}^2}=-R_{n-1},\quad \frac{\psi_{n-1}\,\psi_{n-3}}{\psi_{n-2}^2}=-R_{n-2},\ldots\frac{\psi_4\,\psi_2}{\psi_3{}^2}=-R_3,\quad \frac{\psi_3\,\psi_1}{\psi_2{}^2}=-R_2\,;$$

$$\therefore\ \left(\frac{\psi_n\,\psi_{n-2}}{\psi_{n-1}^2}\right)^1\left(\frac{\psi_{n-1}\,\psi_{n-3}}{\psi_{n-2}^2}\right)^2\left(\frac{\psi_{n-2}\,\psi_{n-4}}{\psi_{n-3}^2}\right)^3\ldots\left(\frac{\psi_4\,\psi_2}{\psi_3{}^2}\right)^{n-3}\left(\frac{\psi_3\,\psi_1}{\psi_2{}^2}\right)^{n-2}$$
$$=(-1)^{\frac{(n-2)(n-1)}{2}}R_{n-1}\,R_{n-2}^2\,.\,R_{n-3}^3\ldots R_3^{n-3}\,R_2^{n-2},$$

and $\psi_2=\dfrac{\sigma(2u)}{\sigma^{2^2}(u)}=-\wp'(u)$, $\psi_1=1$; whence $(n>2)$

$$\frac{\psi_n}{\psi_2^{n-1}}=(-1)^{\frac{(n-2)(n-1)}{2}}R_{n-1}\,R_{n-2}^2\,R_{n-3}^3\ldots R_3^{n-3}\,R_2^{n-2}\,;$$

$$\therefore\ \psi_n = (-1)^{\frac{n(n-1)}{2}} \{\wp'(u)\}^{n-1} R_{n-1} R_{n-2}^2 R_{n-3}^3 \ldots R_3^{n-3} R_2^{n-2},$$

i.e. $$\frac{\sigma(nu)}{\{\sigma(u)\}^{n^2}} = (-1)^{n-1} (\wp' u)^{n-1} (\wp u - \wp 2u)^{n-2} (\wp u - \wp 3u)^{n-3} \ldots (\wp u - \wp \overline{n-1}\, u)^1.$$

1432. General Form of the Differential Coefficients of $\wp(u)$ with regard to u.

Writing P, P_1, P_2, etc., for $\wp(u)$, $\wp'(u)$, $\wp''(u)$, etc., for short, we have $$P_1{}^2 = 4P^3 - IP - J,$$

$$P_2 = 6P^2 - \tfrac{1}{2}I, \qquad P_3 = 12PP_1,$$

$$P_4 = 12P_1{}^2 + 12PP_2$$

$$= aP^3 + bP + c, \text{ say}, \qquad P_5 = (3aP^2 + b)P_1,$$

$$P_6 = 6aPP_1{}^2 + (3aP^2 + b)P_2$$

$$= a_1P^4 + b_1P^2 + c_1P + d_1, \text{ say}, \qquad P_7 = (4a_1P^3 + 2b_1P + c_1)P_1,$$

$$P_8 = (12a_1P^2 + 2b_1)P_1{}^2 + (4a_1P^3 + 2b_1P + c_1)P_2$$

$$= a_2P^5 + b_2P^3 + c_2P^2 + d_2P + e_2, \text{ say},$$

$$P_9 = (5a_2P^4 + 3b_2P^2 + 2c_2P + d_2)P_1,$$

etc.;

whence it appears

that P_2, P_4, P_6, ... are all rational functions of P

and that P_3, P_5, P_7, ... contain an irrational factor P_1.

If we suppose these equations solved to express the various powers of P in terms of P, P_1, P_2, ..., we have

$$P^2 = \tfrac{1}{6}(P_2 + \tfrac{1}{2}I), \qquad P^3 = \frac{1}{a}(P_4 - bP - c),$$

$$P^4 = \frac{1}{a_1}\{P_6 - \tfrac{1}{6}b_1(P_2 + \tfrac{1}{2}I) - c_1P - d_1\},$$

$$P^5 = \frac{1}{a_2}\left\{P_8 - \frac{b_2}{a}(P_4 - bP - c) - \frac{c_2}{6}(P_2 + \tfrac{1}{2}I) - d_2P - e_2\right\}, \text{ etc.};$$

whence it appears that any positive integral power of P can be expressed linearly in terms of P and its differential coefficients, and that the general result will be of the form

$$P^n = AP_{2n-2} + BP_{2n-6} + CP_{2n-8} + \ldots + KP_2 + LP + M,$$

in which no differential coefficient of an odd order occurs, and the coefficients are all functions of I and J not involving the variable and readily calculable in the early cases.

1433. Integration of Rational Integral Algebraic Functions of $\wp(u)$ with regard to u.

It follows from the last article that

$$\int P^n du = AP_{2n-3}+BP_{2n-7}+CP_{2n-9}+\dots$$
$$+KP_1+L\zeta(u)+Mu+\text{a const.},$$

in which the Zeta function appears from the integration of the term LP.

Any rational integral algebraic function of $\wp(u)$ and $\wp'(u)$, *i.e.* of P and P_1, can now be integrated. For if it be separated into two parts, the first containing all the even powers of $\wp'(u)$ and the second all the odd powers, then after substitution of $4P^3-IP-J$ for P_1^2, we have a result of the form $\phi(P)+\chi(P)P_1$, ϕ and χ being rational integral algebraic functions of P. And when $\phi(P)$ has been expressed as explained above as a linear function of P and its differential coefficients, each term is directly integrable. And if $\chi(P)$ be expressed in powers of P each term of $\chi(P)P_1$ is directly integrable, for $\int P^r P_1\,du = P^{r+1}/(r+1)$.

Moreover, since $P^rP_1=\dfrac{d}{du}\left(\dfrac{P^{r+1}}{r+1}\right)$, which is of form

$$\frac{d}{du}(AP_{2r}+\dots+M)=AP_{2r+1}+\dots,$$

it appears that P^rP_1 can be expressed as a linear function of P and its differential coefficients, and that the same is true of $\chi(P)P_1$, χ being rational and integral. Thus, whatever rational algebraic functions of P, ϕ and χ may be, the integral part of $\phi(P)+\chi(P)P_1$ is expressible in the form

$$A+A_0P+A_1P_1+A_2P_2+\dots,$$

and is integrable with respect to u and expressible in the form

$$C+Au+A_0\zeta(u)+A_1\wp(u)+A_2\wp'(u)+A_3\wp''(u)+\dots.$$

1434. Thus, for example, to integrate $\{\wp(u)+\wp'(u)\}^2$ with regard to u, we have

$$(P+P_1)^2=P^2+P_1^2+2PP_1=4P^3+P^2-IP-J+2PP_1$$
$$=\tfrac{4}{120}(P_4+18IP+12J)+\tfrac{1}{6}(P_2+\tfrac{1}{2}I)-IP-J+2PP_1$$
$$=\tfrac{1}{30}P_4+\tfrac{1}{6}P_2-\tfrac{2}{5}IP+(\tfrac{1}{12}I-\tfrac{3}{5}J)+2PP_1;$$

$$\therefore \int\{\wp(u)+\wp'(u)\}^2du=C+(\tfrac{1}{12}I-\tfrac{3}{5}J)u+\tfrac{2}{5}I\zeta(u)$$
$$+\tfrac{1}{6}\wp'(u)+\tfrac{1}{6}\wp''(u)+\tfrac{1}{30}\wp'''(u).$$

1435. If we differentiate equation (1) of Art. 1420 with regard to u,

$$\zeta'(u-v)+\zeta'(u+v)-2\zeta'(u)=\frac{\wp''(u)}{\wp(u)-\wp(v)}-\frac{\wp'^2(u)}{[\wp(u)-\wp(v)]^2},$$

and an interchange of u and v, or a differentiation of (2) of the same article with regard to v, gives

$$\zeta'(u-v)+\zeta'(u+v)-2\zeta'(v)=-\frac{\wp''(v)}{\wp(u)-\wp(v)}-\frac{\wp'^2(v)}{[\wp(u)-\wp(v)]^2};$$

a further differentiation with regard to v gives

$$-\zeta''(u-v)+\zeta''(u+v)-2\zeta''(v)$$
$$=-\frac{\wp'''(v)}{\wp(u)-\wp(v)}-\frac{3\wp'(v)\wp''(v)}{[\wp(u)-\wp(v)]^2}-\frac{2\wp'^3(v)}{[\wp(u)-\wp(v)]^3},$$

etc.

Thus we can form fractions containing $[\wp(u)-\wp(v)]^2$, $[\wp(u)-\wp(v)]^3$, etc., in the denominators with no functions of u in the numerators, and this will presently be found useful (Art. 1443); and since $\zeta'(u)=-\wp(u)$, we have

$$\frac{\wp'(v)}{\wp(u)-\wp(v)}=\zeta(u-v)-\zeta(u+v)+2\zeta(v),$$

$$\frac{\wp'^2(v)}{[\wp(u)-\wp(v)]^2}=\wp(u-v)+\wp(u+v)-2\wp(v)-\frac{\wp''(v)}{\wp(u)-\wp(v)},$$

$$\frac{2\wp'^3(v)}{[\wp(u)-\wp(v)]^3}=-\wp'(u-v)+\wp'(u+v)-2\wp'(v)-\frac{\wp'''(v)}{\wp(u)-\wp(v)}$$
$$-\frac{3\wp'(v)\wp''(v)}{[\wp(u)-\wp(v)]^2},$$

etc.

Integrating with regard to u,

$$\wp'(v)\int\frac{du}{\wp(u)-\wp(v)}=\log\sigma(u-v)-\log\sigma(u+v)+2u\zeta(v)+\text{const.},$$

$$\wp'^2(v)\int\frac{du}{[\wp(u)-\wp(v)]^2}=-\zeta(u-v)-\zeta(u+v)-2u\wp(v)$$
$$-\wp''(v)\int\frac{du}{\wp(u)-\wp(v)},$$

$$2\wp'^3(v)\int\frac{du}{[\wp(u)-\wp(v)]^3}=-\wp(u-v)+\wp(u+v)-2u\wp'(v)$$
$$-\wp'''(v)\int\frac{du}{\wp(u)-\wp(v)}-3\wp'(v)\wp''(v)\int\frac{du}{[\wp(u)-\wp(v)]^2},$$

etc.

Each such integral is therefore expressible by means of those which have preceded it, the first being completely integrated. So that all such functions as

$$\frac{1}{\wp(u)-a},\ \frac{1}{[\wp(u)-a]^2},\ \frac{1}{[\wp(u)-a]^3},\ \text{etc.,}$$

are integrable and expressible in terms of $\wp$, ζ or σ functions.

In the case where $\wp(v)=e_1$, e_2 or e_3, we have $v=\omega_1$, ω_2 or ω_3 and $\wp'(v)=0$.

We now have from the second result,

$$\wp''(\omega)\int\frac{du}{\wp(u)-e}=-\zeta(u-\omega)-\zeta(u+\omega)-2eu,$$

with corresponding suffixes for e and ω, replacing the first integration above, and so on for the other cases.

And $\wp''(\omega_1)=6e_1{}^2-\frac{1}{2}I=2e_2e_3+4e_1{}^2$, etc.

1436. As a particular case, if we put $\wp(v)=0$, v is a constant defined by $v=\int_0^\infty\frac{dz}{\sqrt{4z^3-Iz-J}}$. And

$$\wp'^2(v)=4\wp^3(v)-I\wp(v)-J=-J,\qquad \wp''(v)=6\wp^2(v)-\tfrac{1}{2}I=-\tfrac{1}{2}I,$$

$$\wp'''(v)=12\wp(v)\,\wp'(v)=0,\qquad \wp^{(\mathrm{iv})}(v)=-12J,\ \text{etc.};$$

whence the successive integrals $\int\frac{du}{\wp(u)}$, $\int\frac{du}{\wp^2(u)}$, $\int\frac{du}{\wp^3(u)}$, etc. may be at once expressed.

1437. **The integration of the function** $\frac{1}{\wp(u)-a}$ $(a\neq e_1,\ e_2$ or $e_3)$ may now be effected.

Let $a=\wp(v)$, which defines v as a certain constant, viz. $v\equiv\int_a^\infty\frac{dz}{\sqrt{4z^3-Iz-J}}$, and $\wp'(v)=-\sqrt{4a^3-Ia-J}$. Then

$$\frac{1}{\wp(u)-a}=\frac{1}{2\wp'(v)}\left[\frac{\wp'(u)+\wp'(v)}{\wp(u)-\wp(v)}-\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right]$$

$$=\frac{1}{\wp'(v)}\left[\{\zeta(u-v)-\zeta(u)+\zeta(v)\}-\{\zeta(u+v)-\zeta(u)-\zeta(v)\}\right]$$

$$=\frac{1}{\wp'(v)}[\zeta(u-v)-\zeta(u+v)+2\zeta(v)]\quad\text{(or by Art. 1435);}$$

whence

$$\int \frac{du}{\wp(u)-a} = \frac{1}{\wp'(v)}[\log\sigma(u-v)-\log\sigma(u+v)+2u\zeta(v)]+\text{const.}$$

$$= \frac{1}{\wp'(v)}\log e^{2u\zeta(v)}\frac{\sigma(u-v)}{\sigma(u+v)}+\text{const.}$$

1438. Art. 1435 shows that we also have

$$\wp'^2(v)\int\frac{du}{[\wp(u)-a]^2} = -\zeta(u-v)-\zeta(u+v)-2u\,\wp(v)$$
$$-\wp''(v)\int\frac{du}{\wp(u)-a},$$

$$2\wp'^3(v)\int\frac{du}{[\wp(u)-a]^3} = -\wp(u-v)+\wp(u+v)-2u\,\wp'(v)$$
$$-\wp'''(v)\int\frac{du}{\wp(u)-a}-3\wp'(v)\,\wp''(v)\int\frac{du}{\{\wp(u)-a\}^2},$$

and so on.

1439. **Integrals of form** $\int\frac{\wp'(u)}{\wp(u)-a}du$, $\int\frac{\wp'(u)}{\{\wp(u)-a\}^n}du$ are of course directly integrable as

$$\log[\wp(u)-a] \quad\text{and}\quad -\frac{1}{n-1}\frac{1}{[\wp(u)-a]^{n-1}}.$$

1440. **Integrals of form** $\int\frac{F[\wp(u)]}{\wp(u)-a}du$, where F is a rational integral algebraic function, can be integrated by expressing F in a series of form

$$A\wp^n(u)+B\wp^{n-1}(u)+\ldots+K\wp(u)+L,$$

and then dividing by $\wp(u)-a$, thus reducing the integrand to the form

$$A'\wp^{n-1}(u)+B'\wp^{n-2}(u)+\ldots+K'+\frac{L'}{\wp(u)-a},$$

and each of the terms of form $\lambda\wp^r(u)$ may be treated as in Art. 1433, whilst the integration of the last term is effected above.

1441. **Integrals of form**

$$\int\frac{F[\wp(u)]\,du}{[\wp(u)-a][\wp(u)-b]\ldots[\wp(u)-k]}$$

follow the ordinary rules of Partial Fractions in the first place with an integration of the several terms of the form $\Sigma\lambda\wp^r(u)+\Sigma\frac{\mu}{\wp(u)-a}$ which accrue, following the rules described above.

1442. Ex. Thus

$$\int\frac{\wp^2(u)\,du}{[\wp(u)-a][\wp(u)-b][\wp(u)-c]}=\int\Sigma\frac{a^2}{(a-b)(a-c)}\frac{1}{\wp(u)-a}du$$

$$=\Sigma\frac{a^2}{(a-b)(a-c)}\frac{1}{\wp'(u_1)}\log e^{2u\zeta(u_1)}\frac{\sigma(u-u_1)}{\sigma(u+u_1)},$$

where $u_1=\int_a^\infty\frac{dz}{\sqrt{4z^3-Iz-J}}$, $u_2=$etc., $u_3=$etc., and

$$\wp'(u_1)=-\sqrt{4a^3-Ia-J},\text{ etc.}$$

1443. General Summing Up. Completion of the Method. We can now consider the general case of the integration of a function of form $(A+B\sqrt{Q})/(C+D\sqrt{Q})$, where A, B, C, D are rational algebraic functions of x and Q is a rational integral algebraic function of x of degree 3 or 4, thus extending the result of Art. 318. By exactly the same process as in Art. 318, the function may be thrown into the form $\frac{U}{V}+\frac{M}{N}\cdot\frac{1}{\sqrt{Q}}$, where U, V, M, N are rational integral algebraic functions of x. The transformation $x=a_0+\frac{\mu}{z-\eta}$ may be applied to both parts, or to the second part only, for $\int\frac{U}{V}dx$ is directly integrable in terms of x by the rules of the first seven chapters. But for the sake of uniformity in the result, let us suppose the same transformation is applied to both parts. Then, having determined μ and η so as to reduce $\frac{dx}{\sqrt{Q}}$ to the Weierstrassian form $\frac{-dz}{\sqrt{4z^3-Iz-J}}$, let us put, as in Art. 1432, $\wp(u)=P$, $\wp'(u)=P_1$, etc., where u is $\wp^{-1}(z)$. Then U/V and M/N, which are functions of x, take the forms U'/V' and M'/N' respectively, where U', V', M', N' are rational integral algebraic functions of P, or what is the same thing, z; and

$$\int\left(\frac{U}{V}+\frac{M}{N}\frac{1}{\sqrt{Q}}\right)dx=\int\frac{U'}{V'}\left[\frac{-\mu}{(z-\eta)^2}\right]dz+\int\frac{M'}{N'}\frac{dz}{P_1}$$

$$=\int\frac{U''}{V''}P_1\,du+\int\frac{M'}{N'}du,$$

where U''/V'' replaces $-U'\mu/V'(z-\eta)^2$, and U'', V'' are rational integral algebraic functions of z, *i.e.* of $\wp(u)$ or P, and M', N' are also rational integral algebraic functions of P.

Now U''/V'' and M'/N' can both be expressed partly as an algebraic series of powers of P and partly as a series of Partial Fractions.

Suppose

$$\frac{U''}{V''}\equiv\Sigma\lambda P^r+\Sigma\frac{\mu}{(P-\beta)^s} \quad\text{and}\quad \frac{M'}{N'}=\Sigma\lambda' P^{r'}+\Sigma\frac{\mu'}{(P-\beta')^{s'}},$$

which are the most general forms.

Then $\int P^r P_1\,du=\frac{P^{r+1}}{r+1}$; $\int\frac{P_1\,du}{(P-\beta)^s}=-\frac{1}{s-1}\frac{1}{(P-\beta)^{s-1}}$; and $\int\frac{P_1\,du}{P-\beta}=\log(P-\beta)$, so that all the terms of $\int\frac{U''}{V''}P_1\,du$ can be integrated in terms of P, *i.e.* of $\wp(u)$.

Also $\int P^{r'}du$ has been shown in Art. 1432 capable of integration, and the method to be followed has been there described.

Finally, the integration of terms of the form $\int\frac{du}{P-\beta'}$ or $\int\frac{du}{(P-\beta')^{s'}}$ has been discussed in Art. 1435. The total result is therefore expressible by aid of the Weierstrassian function $\wp(u)$ and its associated Zeta and Sigma functions, and the addition formula for each has been established.

This therefore completes the theory of the integration of the most general algebraic function of nature $(A+B\sqrt{Q})/(C+D\sqrt{Q})$, where Q is of degree 3 or 4, the cases of Q being of degree 1 or 2 having been completed in Art. 318.

1444. ILLUSTRATIVE EXAMPLE.

Consider the integration

$$U\equiv\int_z^\infty\frac{z^3\,dz}{(z-1)^2(z-2)\sqrt{4(z^3+1)}}\quad(2<z<\infty).$$

Let $z=\wp(u,\,0,\,-4)$, *i.e.* $\frac{dz}{\sqrt{4(z^3+1)}}=-du$; and let α, β be two constants defined by $\wp(\alpha)=2$, $\wp(\beta)=1$.

Then $\wp'^2(\alpha)=36, \wp'^2(\beta)=8, \wp''(\alpha)=6\,.\,2^2=24, \wp''(\beta)=6\,.\,1^2=6$, and we have

$$U=\int_0^u\left\{1+\frac{8}{z-2}-\frac{4}{z-1}-\frac{1}{(z-1)^2}\right\}du.$$

Hence, by Art. 1437,

$$U=u+8\,.\,\frac{1}{6}\log e^{2u\zeta(\alpha)}\frac{\sigma(u-\alpha)}{\sigma(u+\alpha)}-4\,.\,\frac{1}{\sqrt{8}}\log e^{2u\zeta(\beta)}\frac{\sigma(u-\beta)}{\sigma(u+\beta)}$$

$$-\frac{1}{8}\left\{-\zeta(u-\beta)-\zeta(u+\beta)-2u-\frac{6}{\sqrt{8}}\log e^{2u\zeta(\beta)}\frac{\sigma(u-\beta)}{\sigma(u+\beta)}\right\}+C,$$

and C is to be determined so that $U=0$ if $u=0$. Simplifying,

$$U=u+\frac{4}{3}\log e^{2u\zeta(\alpha)}\frac{\sigma(u-\alpha)}{\sigma(u+\alpha)}-\frac{13}{8\sqrt{2}}\log e^{2u\zeta(\beta)}\frac{\sigma(u-\beta)}{\sigma(u+\beta)}$$

$$+\frac{1}{8}\left\{2\zeta(u)+\frac{\wp'(u)}{\wp(u)-1}+2u\right\}+C;$$

and when u is diminished indefinitely,

$$0=\frac{4}{3}\log(-1)-\frac{13}{8\sqrt{2}}\log(-1)+\frac{1}{8}\,Lt\left\{\frac{2}{u}-\frac{\dfrac{2}{u^3}}{\dfrac{1}{u^2}-1}\right\}+C$$

$$=\frac{4}{3}\log(-1)-\frac{13}{8\sqrt{2}}\log(-1)+C.$$

Therefore subtracting,

$$U=\frac{5}{4}u+\frac{4}{3}\log e^{2u\zeta(\alpha)}\frac{\sigma(\alpha-u)}{\sigma(\alpha+u)}-\frac{13}{8\sqrt{2}}\log e^{2u\zeta(\beta)}\frac{\sigma(\beta-u)}{\sigma(\beta+u)}$$

$$+\frac{1}{4}\zeta(u)+\frac{1}{8}\frac{\wp'(u)}{\wp(u)-1},$$

where $\quad u=\wp^{-1}(z, 0, -4), \quad \alpha=\wp^{-1}(2), \quad \beta=\wp^{-1}(1).$

1445. For further development of this part of the Theory of Elliptic Functions, the reader must be referred to some book expressly dealing with this section of the subject, such as Professor Sir George Greenhill's treatise, where he will find a large number of very elegant applications of their use to the problems of higher Applied Mathematics, and a much more extensive account of them than space admits here.

PROBLEMS.

1. Reduce the integral

$$u \equiv \int_2^x \frac{dx}{\sqrt{4(x-2)(x-3)(2x-5)(3x-5)}} \quad (2 < x < 2.5)$$

to the Weierstrassian form, by putting $x = 2 + \frac{1}{y}$. Show that the moduli of the integral are $2/\sqrt{5}$ and $1/\sqrt{5}$, and that $u = \wp^{-1}\{1/(x-2)\}$.

Show also that $u = \frac{1}{\sqrt{5}} \operatorname{dn}^{-1} \sqrt{\frac{3-x}{3x-5}}$, mod. $\frac{2}{\sqrt{5}}$.

2. In the integral $u = \int_z^\infty \frac{dz}{\sqrt{4z^3 - 20z - 28}}$, show that if

$$z > e_1 > e_2 > e_3,$$

(i) $\wp(u) = \frac{1}{u^2} + u^2 + u^4 + \frac{1}{3}u^6 + \dots;$

(ii) $\zeta(u) = \frac{1}{u} - \frac{1}{3}u^3 - \frac{1}{5}u^5 - \frac{1}{21}u^7 - \dots;$

(iii) $\sigma(u) = u - \frac{1}{12}u^5 - \frac{1}{30}u^7 - \dots.$

3. If $2u \equiv \int_1^x \frac{dx}{\sqrt{(4x^2+17x+4)(2x^2-3x+1)}}$, show by putting

$$x = y/(y-5)$$

that the integral is reduced to Weierstrassian form. Prove also that

$$u = \frac{1}{\sqrt{5}} \wp^{-1}\left(\frac{5x}{x-1}, \ 84, \ -80\right) = \frac{1}{3\sqrt{5}} \operatorname{dn}^{-1}\left(\sqrt{\frac{1}{5}\,\frac{4x+1}{2x-1}}, \ \sqrt{\frac{2}{3}}\right).$$

4. Show that

$$32\wp'^3(u)\wp'(2u) = 64\wp^6(u) - 80I\wp^4(u) - 320J\wp^3(u) - 20I^2\wp^2(u) - 16IJ\wp(u) + (I^3 - 32J^2).$$

Also show that if $2u = \int_z^\infty \frac{dz}{\sqrt{z^3 - 2z - 1}}$, $\wp'(2u)$ contains $\wp(u)$ as a factor.

5. Show that for the integral $2u = \int_z^\infty \frac{dz}{\sqrt{z^3 - a^3}}$, the roots of the equation $\wp'(2u) = 0$ are given by $\wp(u) = a(\sqrt{3} \pm 1)$, $a\omega(\sqrt{3} \pm 1)$, $a\omega^2(\sqrt{3} \pm 1)$, where ω is one of the unreal cube roots of unity.

Show also that $\wp(2u) - \wp(u) = -\frac{3}{4} z \frac{z^3 - 4a^3}{z^3 - a^3}$, and that

$$\wp'''(u) = 24\{5\wp^3(u) - 2a^3\}.$$

6. If $2u=\int_z^\infty \frac{dz}{\sqrt{z^3-a^3}}$, show that $u=\frac{1}{2\sqrt[4]{3a^2}}\text{cn}^{-1}\left\{\frac{z-2a\sqrt{2}\cos 15^\circ}{z+2a\sqrt{2}\cos 15^\circ}\right\}$. Mod. sin 15°.

7. For any Weierstrassian Integral, show that

(i) $Lt_{u\to 0}\left\{\frac{u^4\wp''(u)-6}{u^2\wp(u)-1}\right\}=2$; (ii) $Lt_{u\to 0}\left\{\frac{u^2\zeta(u)-u}{\sigma(u)-u}\right\}=4$.

8. If $u=\wp^{-1}(z, 84, -80)$, show that the values of $\wp\left(\frac{\omega_1}{2}\right)$ and $\wp\left(\frac{\omega_1}{2}+\omega_3\right)$ are $4\pm 3\sqrt{3}$, and that

$$\wp'(u)\sqrt{\wp 2u-4}+\wp^2(u)-8\wp(u)-11=0.$$

Show also that

$$\left.\begin{aligned}\wp'(u+\omega_1)&=-27\wp'(u)/\{\wp(u)-4\}^2,\\ \wp'(u+\omega_2)&=18\wp'(u)/\{\wp(u)-1\}^2,\\ \wp'(u+\omega_3)&=-54\wp'(u)/\{\wp(u)+5\}^2.\end{aligned}\right\}$$

9. If $u\equiv\int_{e_1}^x \frac{dx}{\{(x-e_1)(x-e_2)(x-e_3)\}^{\frac{2}{3}}}$, transform the integral by the substitution $y^3=\frac{(x-e_2)(x-e_3)}{(x-e_1)^2}$, and show that

$$y=\wp\left\{\frac{u}{3}\sqrt{(e_1-e_2)(e_1-e_3)},\ 0,\ \frac{4e_2e_3-e_1{}^2}{(e_1-e_2)(e_1-e_3)}\right\}.$$

10. Prove the relations,

(i) $\sigma^2(u)\sigma(v+w)\sigma(v-w)+\sigma^2(v)\sigma(w+u)\sigma(w-u)$
$+\sigma^2(w)\sigma(u+v)\sigma(u-v)=0.$

(ii) $\wp(u)\sigma^2(u)\sigma(v+w)\sigma(v-w)+\wp(v)\sigma^2(v)\sigma(w+u)\sigma(w-u)$
$+\wp(w)\sigma^2(w)\sigma(u+v)\sigma(u-v)=0.$

(iii) $\wp^2(u)\sigma^2(u)\sigma(v+w)\sigma(v-w)+\wp^2(v)\sigma^2(v)\sigma(w+u)\sigma(w-u)$
$+\wp^2(w)\sigma^2(w)\sigma(u+v)\sigma(u-v)$
$=\sigma^2(u)\sigma^2(v)\sigma^2(w)\{\wp(v)-\wp(w)\}\{\wp(w)-\wp(u)\}\{\wp(u)-\wp(v)\}.$

(iv) $\sigma(v+w)\sigma(v-w)\sigma(u+x)\sigma(u-x)$
$+\sigma(w+u)\sigma(w-u)\sigma(v+x)\sigma(v-x)$
$+\sigma(u+v)\sigma(u-v)\sigma(w+x)\sigma(w-x)=0.$

[GREENHILL, *E.F.*, p. 208.]

(v) $\sigma^6(u)\sigma^3(v+w)\sigma^3(v-w)+\sigma^6(v)\sigma^3(w+u)\sigma^3(w-u)$
$+\sigma^6(w)\sigma^3(u+v)\sigma^3(u-v)$
$=3\sigma^2(u)\sigma^2(v)\sigma^2(w)\sigma(v+w)\sigma(v-w)\sigma(w+u)\sigma(w-u)\sigma(u+v)\sigma(u-v).$

11. If $u\equiv\wp^{-1}(z, I, J)$, find the values of

$$\int\wp^n(u)\,dz,\quad \int\frac{1}{\wp(u)}dz,\quad \int e^{\wp(u)}dz,\quad \int\frac{12\wp^2(u)-I}{\sqrt{4\wp^3(u)-I\wp(u)-J}}dz.$$

12. Find the values of

$$\int \wp^2(u)\,du, \quad \int \wp^3(u)\,du, \quad \int \wp^4(u)\,du, \quad \int \frac{du}{\wp(u)}, \quad \int \frac{du}{\wp^2(u)}, \quad \int \frac{du}{\wp^3(u)}.$$

13. Prove that

$$\Sigma(\wp u - e)(\wp v - \wp w)^2[\wp(v+w) - e]^{\frac{1}{2}}[\wp(v-w) - e]^{\frac{1}{2}} = 0,$$

where the sign of summation refers to any three arguments u, v, w, and e is any one of the usual quantities e_1, e_2, e_3.

[MATH. TRIP., 1896.]

14. Prove that

$$8\wp'(u)\wp'(2u) = \wp'^2(u) - 3I\wp(u) - 18J - 4\Sigma\frac{(e_1-e_2)^2(e_1-e_3)^2}{\wp(u)-e_1}.$$

15. Prove that

$$\sqrt{\wp(2u)-e_1} + \sqrt{\wp(2u)-e_2} + \sqrt{\wp(2u)-e_3} = \{12\wp^2(u) - I\}/4\wp'(u).$$

16. Show that

$$4\int \wp(2u)\wp'(u)\,du = \tfrac{1}{2}\wp^2(u) + \log(\wp u - e_1)^{\alpha_1}(\wp u - e_2)^{\alpha_2}(\wp u - e_3)^{\alpha_3},$$

where $\alpha_1 = (e_1 - e_2)(e_1 - e_3), \quad \alpha_2 = \text{etc.}, \quad \alpha_3 = \text{etc.}$

17. If $\phi(u, v) = \dfrac{\sigma(u+v)}{\sigma(u)\sigma(v)} e^{-u\zeta(v)}$, show that

(i) $\phi(u, v)\phi(u, -v) = \wp(u) - \wp(v)$;

(ii) $\phi(u, \omega_1) = \phi(u, -\omega_1) = \sqrt{\wp(u) - e_1}$.

18. Putting $\dfrac{\sigma(u+\omega_1)}{\sigma(\omega_1)} e^{-u\zeta(\omega_1)} = \sigma_1(u)$, etc., etc., show that

$$\sigma(2u) = 2\sigma(u)\sigma_1(u)\sigma_2(u)\sigma_3(u).$$

[GREENHILL, *E.F.*, p. 208.]

19. If the function $\phi(u, v)$ be defined by the equation

$$\log\phi(u, v) = \tfrac{1}{2}\int_0^u \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\,du,$$

show that (i) $\phi(u, v)\,\phi(u, -v) = \wp(u) - \wp(v)$;

(ii) $\dfrac{1}{\phi}\dfrac{\partial\phi}{\partial u} = \zeta(u+v) - \zeta(u) - \zeta(v)$;

(iii) $\dfrac{1}{\phi}\dfrac{\partial^2\phi}{\partial u^2} = 2\wp(u) + \wp(v)$.

Hence give the general solution of the following case of Lamé's Equation, viz.

$$\frac{1}{y}\frac{d^2y}{du^2} = 2\wp(u) + \wp(v).$$

[GREENHILL, *E.F.*, p. 210.]

20. Prove the results

(i) $$-2\frac{\wp'(u)\wp'^2(v)}{\{\wp(u)-\wp(v)\}^3}=\wp'(u+v)+\wp'(u-v)+\frac{\wp''(v)}{\wp'(v)}\{\wp(u-v)+\wp(u+v)\};$$

(ii) $$\frac{\wp''(u)\wp'(v)+\wp''(v)\wp'(u)}{\{\wp(u)-\wp(v)\}^2}-2\wp'(u)\wp'(v)\frac{\wp'(u)-\wp'(v)}{\{\wp(u)-\wp(v)\}^3}=-2\wp'(u+v);$$

(iii) $$\frac{\wp''(u)\wp'(v)-\wp''(v)\wp'(u)}{\{\wp(u)-\wp(v)\}^2}-2\wp'(u)\wp'(v)\frac{\wp'(u)+\wp'(v)}{\{\wp(u)-\wp(v)\}^3}=2\wp'(u-v);$$

(iv) $$\frac{\{\wp'(v)\}^2\wp''(u)+\{\wp'(u)\}^2\wp''(v)}{\{\wp(u)-\wp(v)\}^2}=\{\wp'(v)-\wp'(u)\}\wp'(u-v)$$
$$-\{\wp'(v)+\wp'(u)\}\wp'(u+v).$$

21. Obtain from the definition of the function $\wp(u)$ the formulae

$$(a)\ \wp(u+v)+\wp(u)+\wp(v)=m^2; \qquad (b)\ \wp(u)-\wp(u+v)=\frac{\partial m}{\partial u},$$

where $2m=\{\wp'(u)-\wp'(v)\}/\{\wp(u)-\wp(v)\}$. [MATH. TRIP. II., 1918.]

22. Prove that

$$\int\frac{du}{\wp(u)-e_1}=-\frac{1}{e_2e_3+2e_1^2}\left[e_1u+\zeta(u)+\frac{1}{2}\frac{\wp'(u)}{\wp(u)-e_1}\right].$$

23. Prove that $\sigma_\lambda(2u)+\sigma_\mu(2u)=2\sigma_\lambda^2(u)\sigma_\mu^2(u)$, where λ, μ are any two of the integers 1, 2, 3. [MATH. TRIP., 1890.]

24. If $\vartheta(u)=\wp(u+\omega)+\wp(u)-e$, $\sigma=e'-e''$, prove that

$$\frac{\sigma}{\vartheta(u)+2e}+\frac{e-e'}{\wp(u)-e'}-\frac{e-e''}{\wp(u)-e''}=0$$

and $$[\vartheta'(u)]^2=4(\vartheta u-E_1)(\vartheta u-E_2)(\vartheta u-E_3),$$

where E_1, E_2, E_3 are respectively $e\pm(9e^2-\sigma^2)^{\frac{1}{2}}$ and $-2e$.

[MATH. TRIP. II., 1919.]

25. Show that the function $\{\wp(u)-e_1\}^{\frac{1}{2}}$ is a single-valued function of u, and obtain its periods and its addition equation.

[MATH. TRIP. II., 1918.]

26. If $u\equiv\int_\alpha^\phi\frac{d\phi}{\{(\sin\phi-\sin\alpha)(1-\sin\beta\sin\phi)\}^{\frac{1}{2}}}$, verify that $\sin\phi$ is expressible as a single-valued function of u in the form

$$(\sin\phi-\sin\alpha)/(\sin\phi+1)=\tfrac{1}{2}(1-\sin\alpha)\sin^2(pu,k),$$

where

$$p^2=\tfrac{1}{2}(1-\sin\alpha\sin\beta),\ k^2=\tfrac{1}{2}(1-\sin\alpha)(1+\sin\beta)/(1-\sin\alpha\sin\beta).$$

[MATH. TRIP. II., 1918.]

27. State the properties of the elliptic function $\wp(u)$, which prove that there is a single-valued function $a(u)$, such that $a^2(u)=\wp(u)-e_1$ and $ua(u)=1$ when $u=0$.

Defining similarly $b(u)=\{\wp(u)-e_2\}^{\frac{1}{2}}$, $c(u)=\{\wp(u)-e_3\}^{\frac{1}{2}}$, prove that

$$a(u+v)=\frac{a(u)\,b(v)\,c(v)-a(v)\,b(u)\,c(u)}{a^2(v)-a^2(u)}.$$

[MATH. TRIP. II., 1916.]

28. With the notation of the last question, show that if

$$a'(u) = \frac{da(u)}{du},$$

(i) $a(u+\omega)\,a(u) = a'(\omega) = -a^2(\tfrac{1}{2}\omega)$;

(ii) $2a(u)\,b(u)\,c(u)\,a(2u) = a^4(u) - a^4(\tfrac{1}{2}\omega)$;

(iii) $\displaystyle\int_0^u \left\{\frac{1}{u} - a(u)\right\} du = \log\left[\tfrac{1}{2}u\{b(u)+c(u)\}\right]$.

[MATH. TRIP. II., 1916.]

29. Prove that

(i) $\wp(\tfrac{1}{2}\omega) + \wp(\tfrac{1}{2}\omega+\omega') = 2e_1$;

(ii) $\wp(\tfrac{1}{2}\omega) - \wp(\tfrac{1}{2}\omega+\omega') = 2\{(e_1-e_2)(e_1-e_3)\}^{\frac{1}{2}}$;

(iii) $\wp'(\tfrac{1}{2}\omega) = -2\{(e_1-e_2)(e_1-e_3)\}^{\frac{1}{2}}\{(e_1-e_2)^{\frac{1}{2}} + (e_1-e_3)^{\frac{1}{2}}\}$.

[MATH. TRIP. II., 1913.]

30. Prove the formulae

$$\operatorname{sn}\alpha\operatorname{sn}\beta = \frac{\operatorname{cn}\alpha\operatorname{cn}\beta - \operatorname{cn}(\alpha+\beta)}{\operatorname{dn}(\alpha+\beta)} = \frac{\operatorname{dn}\alpha\operatorname{dn}\beta - \operatorname{dn}(\alpha+\beta)}{k^2\operatorname{cn}(\alpha+\beta)},$$

and hence verify Cayley's theorem, that if $\alpha+\beta+\gamma+\delta=0$, then

$$k'^2 - k^2k'^2\operatorname{sn}\alpha\operatorname{sn}\beta\operatorname{sn}\gamma\operatorname{sn}\delta + k^2\operatorname{cn}\alpha\operatorname{cn}\beta\operatorname{cn}\gamma\operatorname{cn}\delta - \operatorname{dn}\alpha\operatorname{dn}\beta\operatorname{dn}\gamma\operatorname{dn}\delta = 0.$$

Prove independently that with Weierstrass' notation the addition theorem may be expressed in the form

$$(e_2-e_3)\sigma_1\alpha\,\sigma_1\beta\,\sigma_1\gamma + (e_3-e_1)\sigma_2\alpha\,\sigma_2\beta\,\sigma_2\gamma + (e_1-e_2)\sigma_3\alpha\,\sigma_3\beta\,\sigma_3\gamma = 0,$$

where $\alpha+\beta+\gamma=0$; and show that the equivalent of Cayley's Theorem is

$$(e_2-e_3)\sigma_1\alpha\,\sigma_1\beta\,\sigma_1\gamma\,\sigma_1\delta + (e_3-e_1)\sigma_2\alpha\,\sigma_2\beta\,\sigma_2\gamma\,\sigma_2\delta + (e_1-e_2)\sigma_3\alpha\sigma_3\beta\sigma_3\gamma\,\sigma_3\delta + (e_2-e_3)(e_3-e_1)(e_1-e_2)\,\sigma\alpha\,\sigma\beta\,\sigma\gamma\,\sigma\delta = 0,$$

where $\alpha+\beta+\gamma+\delta=0$. [MATH. TRIP. II., 1890.]

31. Show that $\dfrac{\sigma(3u)}{\sigma^9(u)} = \tfrac{1}{4}\{\wp'(u)\wp'''(u) - \wp''^2(u)\}$

[MATH. TRIP. II., 1889.]

Show further that this result when expressed as a function of $\wp(u)$ is

$$3\wp^4(u) - \tfrac{3}{2}I\wp^2(u) - 3J\wp(u) - \frac{I^2}{16}.$$

32. Evaluate (i) $\displaystyle\int\{\wp(u)-\wp(v)\}^2du$; (ii) $\displaystyle\int\{\wp(u)-\wp(v)\}^{-2}du$.

[MATH. TRIP. II., 1889.]

33. If one straight line cut the cubic curve $y^2 = ax^3 + bx + c$ in (x_1, y_1), (x_2, y_2), (x_3, y_3), and a consecutive straight line cut the curve in (x_1+dx_1, y_1+dy_1), etc., prove that

$dx_1/y_1 + dx_2/y_2 + dx_3/y_3 = 0$. [MATH. TRIP. I., 1914.]

34. If a variable straight line cut the cubic $y^3 = ax^3 + bx^2 + cx + d$ at the points (x_1, y_1), (x_2, y_2), (x_3, y_3), and a contiguous straight line cut the curve in $(x_1 + dx_1, y_1 + dy_1)$, etc., prove that

(i) $y_1 y_2 y_3 = a x_1 x_2 x_3 + b(x_2 x_3 + x_3 x_1 + x_1 x_2) + c(x_1 + x_2 + x_3) + d$;

(ii) $dx_1/y_1^2 + dx_2/y_2^2 + dx_3/y_3^2 = 0$. [GREENHILL, *E.F.*, p. 170.]

35. Show that $[\wp(\omega_1 - u) - e_1][\wp u - e_1] = (e_1 - e_2)(e_1 - e_3)$.

36. If $u = \int_0^x (x^2 + a^2)^{-\frac{1}{2}} (x^2 + b^2)^{-\frac{1}{2}} dx$, express x as a single-valued function of u. [MATH. TRIP. II., 1919.]

37. Prove that $\dfrac{1}{\wp u - e_l} = \dfrac{\wp(u - \omega_l) - e_l}{(e_l - e_m)(e_l - e_n)}$, where l, m, n are the numbers 1, 2, 3, taken in some order. [MATH. TRIP. II., 1913.]

38. Develop a proof that if $u \equiv \int_0^x \dfrac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$, then x and $\sqrt{1-x^2}$ are single-valued functions of u. Explain clearly what conditions the path of integration must satisfy and how you fix the value of the integrand at every point of the path.

Express x as a single-valued function of u when

$$u = \int_0^x \frac{dt}{\sqrt{(1-2t)(1+t^2)}}.$$ [MATH. TRIP. II., 1916.]

39. If $2\omega_1$ and $2\omega_3$ be a pair of primitive periods of the elliptic functions,

(i) Show that $\dfrac{\wp'(u+\omega_1)}{\wp'(u)} = -\left\{\dfrac{\wp\left(\frac{\omega_1}{2}\right) - \wp(\omega_1)}{\wp(u) - \wp(\omega_1)}\right\}^2$.

(ii) If $x = \dfrac{\wp\left(\frac{\omega_1}{2}\right) - \wp(\omega_1)}{\wp\left(\frac{\omega_3}{2}\right) - \wp(\omega_1)}$, then

$$x^2 = -\frac{\wp'\left(\frac{\omega_3}{2} + \omega_1\right)}{\wp'\left(\frac{\omega_3}{2}\right)} \quad \text{and} \quad x^4 = \frac{\wp(\omega_3) + 2\wp\left(\frac{\omega_3}{2} + \omega_1\right)}{\wp(\omega_3) + 2\wp\left(\frac{\omega_3}{2}\right)}.$$

Hence show how to express the coordinates of a point on the quintic $y = x(x^4 - 1)$ as elliptic functions of a single parameter.

[BURNSIDE, *Proc. L.M. Soc.*, 1892.]

40. Show that

$$E(3u) - 3E(u) = \frac{8k^2 s^3 c^3 d^3}{1 - 6k^2 s^4 + 4(k^2 + k^4) s^6 - 3k^4 s^8}.$$

[MATH. TRIP. II., 1913.]