CHAPTER XXXI.

ELLIPTIC INTEGRALS AND FUNCTIONS.

1329. **The Legendrian Standard Integrals and the Jacobian Functions.**

In proceeding to the further consideration of the Jacobian Elliptic Functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ already introduced in Chapter XI., we shall adopt the same order of discussion as that followed in the description of the ordinary circular functions and of their inverses in Trigonometry; viz.

(1) The nature of their Periodicity; (2) The establishment of their Addition Formulae; (3) The examination of formulae arising therefrom.

We have defined $\operatorname{sn}(u, k)$ as the value of z, which makes $u=\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$, where $k<1$, and $\operatorname{cn}(u, k)$, $\operatorname{dn}(u, k)$ are defined as $\sqrt{1-z^2}$ and $\sqrt{1-k^2z^2}$ respectively.

1330. **Periodicity of the Extended Circular Functions.**

Let us examine first the simpler integral $u=\int_0^z \frac{dz}{\sqrt{1-z^2}}$, the function $\sin u$ being considered as not hitherto known, but now defined by the equation $z=\sin u$, so that the inverse function $\sin^{-1}z$ is $\int_0^z \frac{dz}{\sqrt{1-z^2}}$, and z is not restricted to real values, but may be a complex variable.

1331. If we write $w^2=\frac{1}{1-z^2}$, w is a two-branched function, its two branches being $w_1=+\frac{1}{\sqrt{1-z^2}}$ and $w_2=-\frac{1}{\sqrt{1-z^2}}$, and individually characterised as assuming the respective values $+1$ and -1 at the origin.

The branch-points are at $z=1$ and at $z=-1$. These points are also poles of the function. There are no other singularities.

The region between an infinite circle whose centre is the origin O, and a double loop enclosing the two branch-points, is synectic, and the infinite circle is therefore deformable into and reconcilable with the double loop. Hence, considering either branch, say w_1, $\int w_1\,dz$ taken round the infinite circle has the same value as $\int w_1\,dz$ taken in the same sense round the double loop.

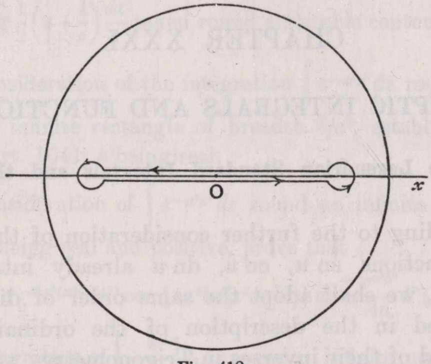


Fig. 418.

Now round the infinite circle, along which we may put $z=Re^{\iota\theta}$ and $dz/z=\iota\,d\theta$, where R is infinite, we have

$$\int w_1\,dz=\int\frac{dz}{\sqrt{1-z^2}}=\frac{1}{\iota}\int\frac{dz}{z},\ |z|\text{ being very large,}$$

$$=\frac{1}{\iota}\int_0^{2\pi}\iota\,d\theta=2\pi.$$

Hence $\int w_1\,dz$, taken round the double loop, is also $=2\pi$.

Again, in integrating round an infinitesimal circle whose centre is at the branch-point $z=1$, put $z=1+re^{\iota\theta}$.

Then $$\int w_1\,dz=\int_0^{2\pi}\frac{\iota re^{\iota\theta}\,d\theta}{\sqrt{2+re^{\iota\theta}}\sqrt{-re^{\iota\theta}}}=\sqrt{r}\int_0^{2\pi}\frac{e^{\iota\frac{\theta}{2}}d\theta}{\sqrt{2+re^{\iota\theta}}}=0,$$

when r is indefinitely diminished. Similarly the integral round the infinitesimal circle with centre at $z=-1$ also vanishes.

Hence the integral for the loop round $z=1$ is in the limit

$$=\int_0^1 w_1\,dz+\int_C w_1\,dz+\int_1^0 w_2\,dz,$$

where $\int_C w_1\,dz$ indicates the integration for the circuit round $z=1$; and w_1 has changed into w_2 after performing the circuit once (Fig. 419); and since $w_2=-w_1$, this reduces to

$$=2\int_0^1 w_1\,dz\equiv 2\int_0^1\frac{dz}{\sqrt{1-z^2}}=L_1,\text{ say.}$$

Similarly, the value of the integral $\int w_1\,dz$ for the loop round $z=-1$ is

$$=\int_0^{-1} w_1\,dz+\int_{c'} w_1\,dz+\int_{-1}^{0} w_2\,dz,$$

where c' refers to the circuit of the infinitesimal circle round $z=-1$ and $\int_{c'} w_1\,dz$ vanishes. Hence, for this loop, we have

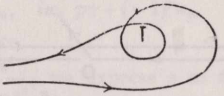
$$\int_0^{-1} w_1\,dz+\int_0^{-1} w_1\,dz=2\int_0^{-1} w_1\,dz=2\int_0^{-1}\frac{dz}{\sqrt{1-z^2}}$$

$$=-2\int_0^{1}\frac{dz}{\sqrt{1-z^2}}=L_{-1},\text{ say.}$$

$$\left.\begin{array}{ll}\text{Thus} & L_1+L_{-1}=0\\ \text{and} & L_1-L_{-1}=\text{integral for the whole loop}=2\pi\,;\end{array}\right\}$$

$$\therefore\ L_1=\pi,\quad L_{-1}=-\pi\,;\quad i.e.\ \int_0^{1}\frac{dz}{\sqrt{1-z^2}}=\frac{\pi}{2}\ \text{ and }\ \int_0^{-1}\frac{dz}{\sqrt{1-z^2}}=-\frac{\pi}{2},$$

the direction of travel in each case being the "positive" direction as defined earlier.



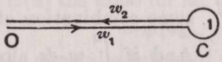


Fig. 419.

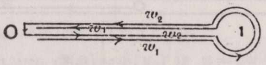


Fig. 420.

Now, if one of the branch-points, say $z=1$, be encircled *twice*, the path starting from the origin and returning to it after two encirclings, may be deformed into two loops round the point, and the integral, leaving out the integrals for the two infinitesimal circuits about the branch-point, which vanish, is $=\int_0^1 w_1\,dz+\int_1^0 w_2\,dz+\int_0^1 w_2\,dz+\int_1^0 w_1\,dz$, which is zero, and w_1 has changed to w_2 and back to w_1 in the double circuit, *i.e.* to its original value at the origin.

Thus, for a loop with an *even* number of circuits round one pole, we have a zero contribution with no aggregate change of branch, but for a loop with an *odd* number of circuits round one pole, the equivalent is obviously a single loop, $=2\int_0^1 w_1\,dz=\pi$, accompanied by a change of branch from w_1 to w_2 on arriving back at the origin.

The same thing happens for several encirclements of $z=-1$, starting from the origin with value w_1, except that for an *odd* number we have a contribution $2\int_0^{-1} w_1\,dz\equiv-\pi$; and w_1 has become w_2 or w_1 according as there have been an *odd* or an *even* number of encirclings of the branch-point.

When *both* branch-points are encircled n times in the positive direction, the integral will be $n \,.\, 2\pi$ with no change of branch, or if the pair be encircled p times in the positive direction and q times in the negative direction, the contribution will be $(p \sim q)\, 2\pi = 2n \,.\, \pi$, where n is the excess of the number of positive encirclements over the number of negative ones. And such an encircling of both points will result in w_1 being restored as the final branch of the function when z has returned to the starting point.

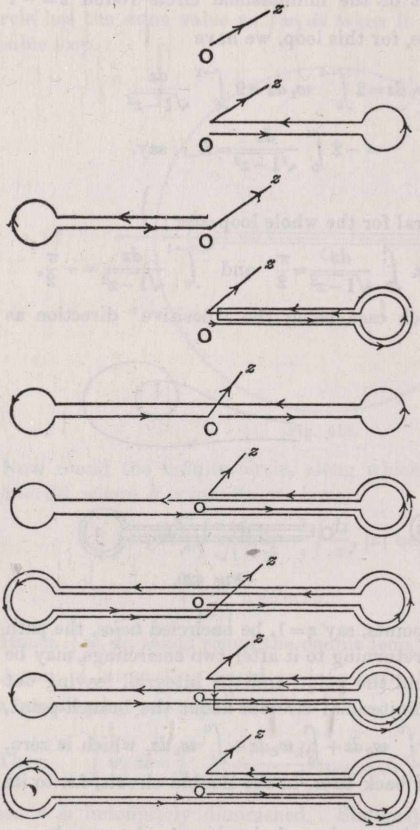


Fig. 421.

Now any path from O to z is reconcilable with a linear direct path, together with such loops as have been described above or some combination of them. And if $\int_0^z w_1\, dz$ along the straight path be called u_0, the contribution to the total integral from O to z by any other path deformable into the straight line OP with a system of loops will be $+u_0$ or $-u_0$, according as z, after having described its loop system and before commencing the portion OP, has returned to the origin with a value w_1 or a value w_2 for the function, and the total for any path will be u_0 or $-u_0$, as the case may be, together with whatever may accrue from the several encirclings of the branch-points.

Thus the total values of the integral $\int_0^z w_1\, dz$ are:

(1) for the direct path alone, $\int_0^z w_1\, dz = u_0$;

(2) for an *odd* number of circuits of one loop + a direct path, $\quad = L_1 - u_0$ or $= L_{-1} - u_0$;

(3) for an *even* number of encirclements of one branch-point + a direct path, $\quad = u_0$;

(4) for n encirclements of both branch-points + a direct path, $\quad = n(L_1 - L_{-1}) + u_0$;

(5) for n complete encirclements of both branch-points combined with an *odd* number of encirclements of one of them + a direct path, $\quad = n(L_1 - L_{-1}) + L_1 - u_0$ or $= n(L_1 - L_{-1}) + L_{-1} - u_0$;

(6) for n complete encirclements of both branch-points with an *even* number of encirclements of one + a direct path, $\quad = n(L_1 - L_{-1}) + u_0$;

and seeing that $L_1 - L_{-1}$ would be replaced by $-L_1 + L_{-1}$ if the description were in the opposite direction, these results are all of one or other of the forms

$$2p\pi + u_0 \quad \text{or} \quad (2p+1)\pi - u_0, \quad \textit{i.e.}\ p\pi + (-1)^p u_0,$$

p being some integer positive or negative.

If then, in the equation $u = \int_0^z \frac{dz}{\sqrt{1-z^2}}$, we express z as $z = \phi(u)$, it appears that as all these paths lead finally to the same point z, we must have $\phi(u)$ the same for all the paths

$$= \phi(u_0), \quad \textit{i.e.}\ \phi(u_0) = \phi\{p\pi + (-1)^p u_0\},$$

and the general solution of the equation $\phi(u) = \phi(u_0)$ is $u = p\pi + (-1)^p u_0$.

This is the ordinary result of trigonometry, and for a real variable it is a well-known theorem that $\sin u = \sin\{p\pi + (-1)^p u\}$.

1332. Let us next put $\sqrt{1-z^2} = \chi(u)$, and enquire which of the above values of u lead to the same value of $\sqrt{1-z^2}$.

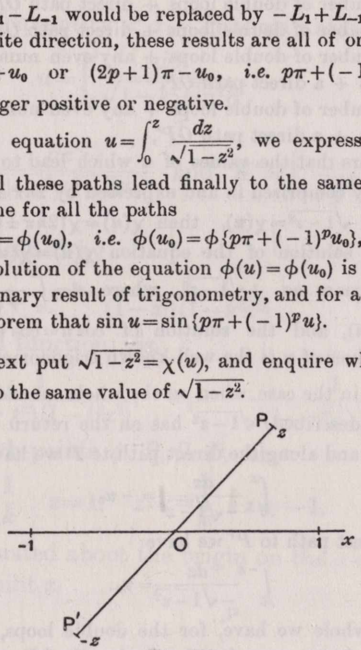


Fig. 422.

Clearly the function $\sqrt{1-z^2}$ has the same value at P', $(-z)$, as it has at P, (z) (Fig. 422).

Hence, besides the various paths which lead from O to P must be considered those which lead from O to P'. And it is not all the paths which have been considered from O to P thus restoring the value z at P, which *also restore the value of* $\sqrt{1-z^2}$. For after a description of an odd number of single loops, $\sqrt{1-z^2}$ has become $-\sqrt{1-z^2}$. Hence, in order to arrive at P or at P' with the value $+\sqrt{1-z^2}$, we can only take the cases of description of an even number of single loops; also a double loop traversed any number of times will restore the value $+\sqrt{1-z^2}$.

We therefore have the following cases:

(1) for a direct path from O to P, u_0;

(2) for a direct path from O to P',

$$\int_0^{-z}\sqrt{1-z^2}\,dz=-\int_0^{z}\sqrt{1-z^2}\,dz=-u_0;$$

(3) for an even number of loops round either branch-point + a direct path OP, $\Big\}\ u_0$;

(4) for an even number of loops round either branch-point + a direct path OP', $\Big\}\ -u_0$;

(5) for any number of double loops + direct path OP, $2n\pi+u_0$;

(6) for any number of double loops + direct path OP', $2n\pi-u_0$;

(7) for any number of double loops + any even number of single loops + a direct path OP, $\Big\}\ 2n\pi+u_0$;

(8) for any number of double loops + any even number of single loops + a direct path OP', $\Big\}\ 2n\pi-u_0$.

Hence it appears that the values of u which lead to the same value of $\sqrt{1-z^2}$ are exactly comprised in and expressed by $2n\pi\pm u_0$, *i.e.*

$$\text{if } \sqrt{1-z^2}=\chi(u),\quad \text{then } \chi(u)=\chi(2n\pi\pm u),$$

and the general solution of the equation $\chi(u)=\chi(u_0)$ is $u=2n\pi\pm u_0$. Thus, defining $\cos u$ as $+\sqrt{1-z^2}$, where $u=\int_0^z\frac{dz}{+\sqrt{1-z^2}}$, we have $\cos u=\cos(2n\pi\pm u)$, and the solution of $\cos u=\cos u_0$ is $u=2n\pi\pm u_0$, which for real values of u is the well-known trigonometrical result.

1333. Further, in the case when on the whole an *odd* number of single loops have been described, $\sqrt{1-z^2}$ has on the return of z to the origin become $-\sqrt{1-z^2}$, and along the direct path to P we have

$$\int_0^z\frac{dz}{-\sqrt{1-z^2}}=-u_0,$$

and along the direct path to P' we have

$$\int_0^{-z}\frac{dz}{-\sqrt{1-z^2}}=u_0.$$

So that on the whole we have, for the double loops, $2n\pi$; for an odd number of single loops, $\pm\pi$; for the final path OP or OP', $\pm u_0$, giving the general value of u as $(2n\pm1)\pi\pm u_0$ *i.e.* $(2\lambda+1)\pi\pm u_0$. And these values will give $-\sqrt{1-z^2}$ at the final position, *i.e.* $\chi(u)=-\chi\{(2\lambda+1)\pi\pm u\}$, which is the same as the corresponding result of trigonometry, viz., λ being an integer, $\cos u=-\cos\{(2\lambda+1)\pi\pm u\}$

1334. From the integral $u=\int_0^z \frac{dz}{\sqrt{1-z^2}}$ it is also *directly* obvious by expansion and integration that u is an odd function of z, in which the first term of the expansion in powers of z is z; and, therefore, by reversion of series, that z is an odd function of u, in which the first term of the expansion in powers of u is u. Hence it appears, from this consideration also, that if $z=\phi(u)$, then $\phi(u)=-\phi(-u)$. And further, since $\sqrt{1-z^2}$ is an *even* function of u, we have $\chi(u)=\chi(-u)$. Also $Lt_{u=0}\frac{z}{u}=1$, *i.e.* $Lt_{u=0}\frac{\sin u}{u}=1$.

1335. Periodicity of the Elliptic Functions.

We now turn to the consideration on similar lines of

$$u=\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

where k is a real quantity <1. This may also be written as

$$u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

where $z=\sin\theta$.

Let $K=\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$ and $K'=\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}}$, where $k^2+k'^2=1$.

The function defined by

$$w^2=\frac{1}{(1-z^2)(1-k^2z^2)}$$

is a two-branched function, viz.

$$w_1=+\frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}},\quad w_2=-\frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

having four branch-points A, B, C, D, viz.

$$z=\frac{1}{k},\quad z=1,\quad z=-\frac{1}{k},\quad z=-1,$$

symmetrically situated about the origin on the x-axis.

Let P be the point z.

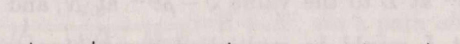


Fig. 423.

There are no branch-points other than A, B, C, D (Art. 1296). These branch-points are also poles of the function, and there are no other singularities of any kind. We shall first consider the integration $\int_0^{\frac{1}{k}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$, the path of the integration being:

(1) along the x-axis from $x=0$ to $x=1-\rho$, viz. O to L in Fig. 424;

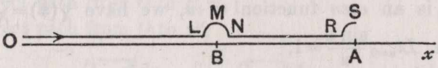


Fig. 424.

(2) round the small semicircle LMN, centre at $z=1$ and radius ρ;

(3) along the x-axis from $x=1+\rho$ to $\frac{1}{k}-\rho$, viz. NR in the figure;

(4) along a quadrantal arc, centre at $z=\frac{1}{k}$ and radius ρ, viz. RS.

In this integration which passes the point B, where $z=1$, the sign of $1-z$ changes at B and the integrand becomes imaginary. We have then to examine the behaviour of the factor $\sqrt{1-z}$ as we pass round the semicircle LMN, but do not complete the circuit, about the branch-point. Put

$$z=1+\rho e^{\iota\theta}.$$

Then $\sqrt{1-z}=\sqrt{-\rho e^{\iota\theta}}$, and in passing round the semicircle LMN *above* B, θ decreases from $\theta=\pi$ to $\theta=0$, and $\sqrt{1-z}$ changes from the value $\sqrt{-\rho e^{\iota\pi}}$ at L to the value $\sqrt{-\rho e^{\iota 0}}$ at N; that is, its value has been multiplied by $e^{-\frac{\iota\pi}{2}}$ or $-\iota$ in passing round the semicircle.

Therefore w_1 becomes ιw_1 in passing *over* B.

If we pass *under* B, we have a change in $\sqrt{1-z}$ from the value $\sqrt{-\rho e^{\iota\pi}}$ at L to the value $\sqrt{-\rho e^{\iota 2\pi}}$ at N, and therefore the value at L would be multiplied by $e^{\frac{\iota\pi}{2}}$ in passing to N; that is, w_1 would become $-\iota w_1$.

Since the value of $\sqrt{1-z}$ at L may be written as $\sqrt{\rho}$, where ρ is $1-x$, x being the abscissa of L, it becomes $-\iota\sqrt{\rho}$ at N, where $\rho=x-1$, x being now the abscissa of N, and along NR there is no further change of amplitude. Hence

From O to L $\quad \sqrt{1-z}=\sqrt{1-x}$, x increasing from 0 to $1-\rho$.

From L to N round LMN $\Big\}$ $\sqrt{1-z}=\sqrt{-\rho e^{\iota\theta}}$, θ decreasing from π to 0.

From N to A $\quad \sqrt{1-z}=-\iota\sqrt{x-1}$, x increasing from $1+\rho$ to $\frac{1}{k}$.

The factor $\sqrt{1-kz}=\sqrt{1-kx}$ from O to R. But A being in this case a branch-point, we take a quadrantal arc with centre A and small radius ρ, avoiding the branch-point.

Put $z=\frac{1}{k}+\rho e^{\iota\theta}$ Then $\sqrt{1-kz}=\sqrt{-k\rho e^{\iota\theta}}$, in which θ decreases from $\theta=\pi$ to $\theta=\frac{\pi}{2}$. We thus have as the contributions from OL, LMN, NR and RS respectively,

$$\int_0^{1-\rho}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\quad \int_\pi^0\frac{\iota\rho e^{\iota\theta}\,d\theta}{\sqrt{-\rho e^{\iota\theta}(2+\rho e^{\iota\theta})[1-k^2(1+\rho e^{\iota\theta})^2]}},$$

$$\int_{1+\rho}^{\frac{1}{k}-\rho}\frac{dx}{-\iota\sqrt{(x^2-1)(1-k^2x^2)}}\ \text{and}\ \int_\pi^{\frac{\pi}{2}}\frac{\iota\rho e^{\iota\theta}\,d\theta}{-\iota\sqrt{\left\{\left(\frac{1}{k}+\rho e^{\iota\theta}\right)^2-1\right\}(-k\rho e^{\iota\theta})(2+k\rho e^{\iota\theta})}}$$

and when ρ is indefinitely small the second and fourth vanish and the first is ultimately K. Transform the third by writing $k^2x^2+k'^2x'^2=1$; whence

$$dx=-\frac{1}{k}\frac{k'^2x'\,dx'}{\sqrt{1-k'^2x'^2}}\ \text{and}\ \sqrt{x^2-1}=\sqrt{\frac{1-k'^2x'^2}{k^2}-1}=\frac{k'}{k}\sqrt{1-x'^2}.$$

Hence the third becomes ultimately

$$\iota\int_1^{\frac{1}{k}}\frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}=\iota\int_1^0\left(-\frac{1}{k}\right)\frac{k'^2x'\,dx'}{\sqrt{1-k'^2x'^2}}\frac{k}{k'\sqrt{1-x'^2}}\frac{1}{k'x'}$$

$$=\iota\int_0^1\frac{dx'}{\sqrt{(1-x'^2)(1-k'^2x'^2)}}=\iota K';$$

that is, $\displaystyle\int_0^{\frac{1}{k}}\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=K+\iota K'$, *via* a path *above* B,

and $\qquad\qquad\qquad\qquad =K-\iota K'$, *via* a path *below* B.

It follows that $\operatorname{sn}(K+\iota K')=\frac{1}{k}$.

Now, noting that $\frac{1}{k}$ is the value of x when $x'=0$, and that $\sqrt{x^2-1}=\frac{k'}{k}\sqrt{1-x'^2}$, we have

$$\iota\sqrt{1-\frac{1}{k^2}}=\frac{k'}{k}, \quad i.e. \sqrt{1-\frac{1}{k^2}}=-\frac{\iota k'}{k};$$

$$\therefore \operatorname{cn}(K+\iota K')=-\frac{\iota k'}{k}; \text{ also } \operatorname{dn}(K+\iota K')=\sqrt{1-k^2\frac{1}{k^2}}=0.$$

1336. Remembering that when

$$u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

$$K=\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

and $x=\sin\theta=\operatorname{sn} u$, also observing that $x=0$ gives $u=0$, we have $\operatorname{sn} 0=0$, whence $\operatorname{cn} 0=1$ and $\operatorname{dn} 0=1$; also $\operatorname{sn} K=1$, whence $\operatorname{cn} K=0$ and $\operatorname{dn} K=\sqrt{1-k^2}=k'$.

1337. Again, if we write $-\theta$ for θ,

$$u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=-\int_0^{-\theta} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}};$$

$$\therefore -u=\int_0^{-\theta} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

Therefore $-\theta=\operatorname{am}(-u)$; $\operatorname{sn}(-u)=-\sin\theta=-\operatorname{sn} u$;
also $\operatorname{cn}(-u)=\operatorname{cn} u$; and $\operatorname{dn}(-u)=\operatorname{dn} u$.

1338. It also appears directly from the integral

$$u=\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

by expansion, that u is an *odd* function of z whose first term is z, and therefore, by reversion of series, that z is an *odd* function of u, the first term of the expansion being u, and therefore also that $Lt_{u=0}\frac{\operatorname{sn} u}{u}=1$.

Also that, since $\operatorname{cn} u=\sqrt{1-\operatorname{sn}^2 u}$ and $\operatorname{dn} u=\sqrt{1-k^2\operatorname{sn}^2 u}$, $\operatorname{cn} u$ and $\operatorname{dn} u$ are both *even* functions of z $(=\operatorname{sn} u)$, the first terms of the expansions being in each case unity. These facts also show that

$$\operatorname{sn}(-u)=-\operatorname{sn} u, \quad \operatorname{cn}(-u)=\operatorname{cn} u, \quad \operatorname{dn}(-u)=\operatorname{dn} u,$$

as seen before.

1339. The Elliptic Functions of 0, K, $K+\iota K'$. Collected Results.

We thus have

$$\operatorname{sn} 0=0, \qquad \operatorname{cn} 0=1, \qquad \operatorname{dn} 0=1,$$
$$\operatorname{sn} K=1, \qquad \operatorname{cn} K=0, \qquad \operatorname{dn} K=k',$$
$$\operatorname{sn}(K+\iota K')=\frac{1}{k}, \quad \operatorname{cn}(K+\iota K')=-\frac{\iota k'}{k}, \quad \operatorname{dn}(K+\iota K')=0.$$

1340. General Values.

We shall now consider the variety of values of u which will accrue from the integral

$$u=\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

in integrating from the origin to the point P, viz. z, along the different paths which may occur, as was done in Art. 1331, for

$$\int_0^z \frac{dz}{\sqrt{1-z^2}}.$$

There are four branch-points A, B, C, D, and four loops and it has been seen in Art. 1294 that for such a system any

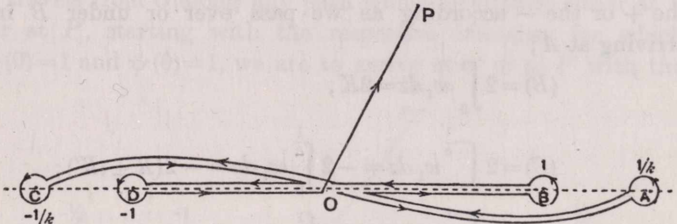


Fig. 425.

path starting from O and terminating at P is deformable into and reconcilable with

(1) a straight line from O to P

or (2) a straight-line path from O to P, together with a combination of loops,

and that in any system of loops about four branch-points there are two and only two groups which give different values to the integral taken from O to P, viz.

(i) those which consist of the integrations for sets of double loops + a direct path

or (ii) those which consist of the integrations for sets of double loops + a single loop + a direct-path.

Moreover, resuming the notation of Art. 1292, any two of the six possible double-loop systems may be selected as independent. This time we shall take these two double-loop systems as (AB) and (BD), and (B) as the principal single loop; and remembering that after every travel round a loop the branches of the function interchange, we have

$$u=\lambda(AB)+\mu(BD)+u_0 \quad \text{or} \quad u=\lambda'(AB)+\mu'(BD)+(B)-u_0$$

as the only possible forms of the result, where u_0 denotes, as before, integration along the straight-line path OP starting with the branch w_1, *i.e.* the same branch with which the whole integration was started from O.

Now $(A)=\int_0^{\frac{1}{k}} w_1\,dz+\int_a w_1\,dz+\int_{\frac{1}{k}}^0 w_2\,dz$, where $\int_a w_1\,dz$ refers to the integration round an infinitesimal circle with centre at A, which vanishes;

$$\therefore\ (A)=2\int_0^{\frac{1}{k}} w_1\,dz=2(K\pm\iota K'),$$

the + or the − according as we pass over or under B in arriving at A;

$$(B)=2\int_0^1 w_1\,dz=2K;$$

$$(C)=2\int_0^{-\frac{1}{k}} w_1\,dz=-2\int_0^{\frac{1}{k}} w_1\,dz=-2(K\pm\iota K');$$

$$(D)=2\int_0^{-1} w_1\,dz=-2\int_0^1 w_1\,dz=-2K;$$

and $\quad (AB)=(A)-(B)=\pm 2\iota K'$; $(BD)=(B)-(D)=4K$.

Hence the general values of the integral which accrue are

$$\left.\begin{aligned} &u=2\lambda\iota K'+4\mu K+u_0 \\ \text{or}\quad &u=2\lambda'\iota K'+4\mu' K+2K-u_0,\end{aligned}\right\}\ \text{where } \lambda,\ \mu,\ \lambda',\ \mu' \text{ are integers;}$$

that is, $\quad u=2p\iota K'+2qK+(-1)^q u_0$, where p, q are integers.

If we write $z=\phi(u)=\phi(u_0)$, it follows that

$$\phi(u_0)=\phi\{2p\iota K'+2qK+(-1)^q u_0\};$$

and taking q an even integer $=2r$,

$$\phi(u_0)=\phi(2p\iota K'+4rK+u_0),$$

so that $2\iota K'$ and $4K$ are independent periods of this function.

Conversely, it follows that the general solution of the equation $\phi(u)=\phi(u_0)$ is $u=2p\iota K'+2qK+(-1)^q u_0$, and $\phi(u)$ is the Jacobian function $\operatorname{sn} u$.

Hence $\qquad \operatorname{sn} u_0=\operatorname{sn}(2p\iota K'+2qK+(-1)^q u_0)$

or, which is the same thing, putting $(-1)^q u_0=v$,

$$\operatorname{sn}(2p\iota K'+2qK+v)=\operatorname{sn}(-1)^q v=(-1)^q \operatorname{sn} v.$$

As particular cases of this double periodicity, we have

$$\phi(u)=\phi(4K+u)=\phi(2K-u)=\phi(4K+2\iota K'+u)=\phi(6K-u)=\phi(2\iota K'+u)$$
$$=\phi[4(K+\iota K')+u]=\text{etc.}$$

1341. Having defined z as a function of u, $\equiv\phi(u)$, by the equation

$$u\equiv\int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

let us examine the periodicity of the expressions

$$\sqrt{1-z^2}\equiv\chi(u)\equiv\chi(u_0) \quad\text{and}\quad \sqrt{1-k^2z^2}\equiv\psi(u)\equiv\psi(u_0)$$

regarded as functions of u.

Let P and P' be the points z and $-z$ respectively. Then, as z travels from O along any path which terminates either at P or at P', starting with the respective branches for which $\chi(0)=1$ and $\psi(0)=1$, we are to arrive at P or at P' with the

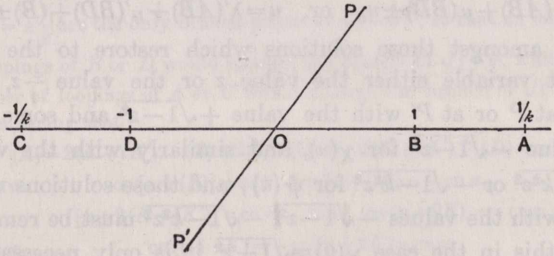


Fig. 426.

values $+\sqrt{1-z^2}$ and $+\sqrt{1-k^2z^2}$ respectively. And this will be effected, provided that either no change has occurred in the branches of the functions in the paths followed, or provided that in either case an even number of such changes have occurred. Such changes of branch occur

in $\chi(u)$ at each looping of B or of D, but not of A or C;

in $\psi(u)$ at each looping of A or of C, but not of B or D.

Hence in the case of $\chi(u)$ the number of times a single loop has been formed about B or about D must be even, but a double loop round B and D may occur any number of times. A double loop about A and B counts as a single loop about B.

In the case of $\psi(u)$ the number of times a single loop has been formed about A or about C must be even, but a double loop round A and C may occur any number of times. A double loop about A and B counts as a single loop about A.

Again, if the integral for the direct linear path OP be denoted as before by u_0, that for OP' is

$$\int_0^{-z}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}=-\int_0^z\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}=-u_0.$$

It has been seen that for the variety of paths from O to P the general value of the integral u is

$$u=\lambda(AB)+\mu(BD)+u_0 \quad\text{or}\quad u=\lambda'(AB)+\mu'(BD)+(B)-u_0.$$

It follows that the general value of the integral from O to P' will be expressed by

$$u=\lambda(AB)+\mu(BD)-u_0 \quad\text{or}\quad u=\lambda'(AB)+\mu'(BD)+(B)+u_0;$$

that is, for those which terminate at an unspecified one of the two points P or P',

$$u=\lambda(AB)+\mu(BD)\pm u_0 \quad\text{or}\quad u=\lambda'(AB)+\mu'(BD)+(B)\pm u_0.$$

Now amongst those solutions which restore to the independent variable either the value z or the value $-z$, some arrive at P or at P' with the value $+\sqrt{1-z^2}$ and some with the value $-\sqrt{1-z^2}$ for $\chi(u)$, and similarly with the values $+\sqrt{1-k^2z^2}$ or $-\sqrt{1-k^2z^2}$ for $\psi(u)$; and those solutions which arrive with the values $-\sqrt{1-z^2}$, $-\sqrt{1-k^2z^2}$ must be removed. To do this in the case $\chi(u)\equiv\sqrt{1-z^2}$ it is only necessary to select those cases in which the number of single loopings of B or of D must be even; that is, λ must be even and λ' must be odd. And in the case of $\psi(u)\equiv\sqrt{1-k^2z^2}$ we must select those cases in which the number of single loopings of A or of C must be even; that is, λ and λ' must both be even.

Thus for $\sqrt{1-z^2}$ the form of u is

$$u=2m(2\iota K')+\mu 4K\pm u_0 \text{ or } u=(2m'+1)(2\iota K')+\mu' 4K+2K\pm u_0,$$

in which the coefficients of $2\iota K'$ and $2K$ are both even or both odd, *i.e.* in one expression $u=p(2\iota K'+2K)+q4K\pm u_0$, where p and q are integers; and for $\sqrt{1-k^2z^2}$ the form of u is

$$u=2m(2\iota K')+\mu 4K\pm u_0 \quad \text{or} \quad u=2m'(2\iota K')+\mu' 4K+2K\pm u_0,$$

i.e., in one expression, $u=4p\iota K'+2qK\pm u_0$, where p and q are integers.

Thus $\qquad \sqrt{1-z^2}\equiv\chi(u)=\chi\{p(2\iota K'+2K)+q4K\pm u_0\}$

and $\qquad \sqrt{1-k^2z^2}\equiv\psi(u)=\psi(4p\iota K'+2qK\pm u_0)$.

The functions ϕ, χ, ψ are plainly sn, cn and dn respectively. Thus

$$\left.\begin{array}{ll} \operatorname{sn} v=\operatorname{sn}(2p\iota K'+2qK+(-1)^q v), & \text{with periods } 2\iota K', 4K,\\ \operatorname{cn} v=\operatorname{cn}(p(2\iota K'+2K)+q4K\pm v), & \text{with periods } 2\iota K'+2K,\ 4K,\\ \operatorname{dn} v=\operatorname{dn}(4p\iota K'+2qK\pm v), & \text{with periods } 4\iota K', 2K.\end{array}\right\}$$

Each function will have returned to its original value when the 'argument' has been increased by any multiple of $4\iota K'$ or of $4K$, which are therefore the whole periods for the group of functions, though individuals of the group will each have twice performed the whole cycle of their values in these intervals.

1342. We may examine this periodicity of $\operatorname{cn} u$ and $\operatorname{dn} u$ from a somewhat different point of view. Defining $\operatorname{cn} u$ as $+\sqrt{1-z^2}$ and $\operatorname{dn} u$ as $+\sqrt{1-k^2z^2}$, and noting that $z=\pm 1$ are the only branch-points of $\sqrt{1-z^2}$ and $z=\pm\frac{1}{k}$ are the only branch-points of $\sqrt{1-k^2z^2}$, so that an odd number of loopings of B or D would change the branch of $\sqrt{1-z^2}$, whilst an odd number of loopings of A or C would change the branch of $\sqrt{1-k^2z^2}$, and remembering that

$$(A)=2(K+\iota K'),\quad (B)=2K,\quad (C)=-2(K+\iota K'),\quad (D)=-2K,$$

we have $\qquad \operatorname{cn}[u+(A)]=\operatorname{cn} u,\quad \operatorname{cn}[u+(B)]=-\operatorname{cn} u,$

and $\therefore\quad \operatorname{cn}[u+2(K+\iota K')]=\operatorname{cn} u$; and $\operatorname{cn}(u+2K)=-\operatorname{cn} u$;

whence $\qquad \operatorname{cn}(u+4K)=-\operatorname{cn}(u+2K)=\operatorname{cn} u.$

Therefore $2(K+\iota K')$ and $4K$ are periods of $\operatorname{cn} u$, and

$$\operatorname{cn}[u+2\lambda(K+\iota K')+4\mu K]=\operatorname{cn} u,$$
$$\operatorname{cn}[u+2\lambda(K+\iota K')+2\mu K]=-\operatorname{cn} u\quad(\mu \text{ odd});$$

i.e. $\qquad \operatorname{cn}[u+2\lambda\iota K'+2(\lambda+\mu)K]=-\operatorname{cn} u\quad(\mu\text{ odd}),$

$$\operatorname{cn}[u+2\lambda\iota K'+2(\lambda+\mu)K]=\operatorname{cn} u\quad(\mu\text{ even}).$$

Similarly $\qquad \operatorname{dn}[u+(A)]=-\operatorname{dn} u,\quad \operatorname{dn}[u+(B)]=\operatorname{dn} u,$

i.e. $\qquad \operatorname{dn}(u+2K)=\operatorname{dn} u$; and $\operatorname{dn}[u+2(K+\iota K')]=-\operatorname{dn} u$;

whence $\qquad \operatorname{dn}[u+4(K+\iota K')]=-\operatorname{dn}[u+2(K+\iota K')]=\operatorname{dn} u.$

Further, $\quad \mathrm{dn}(u+2\iota K')= \quad \mathrm{dn}(u+2K+2\iota K')=-\mathrm{dn}\, u,$

$\mathrm{dn}(u+4\iota K')=-\mathrm{dn}(u+2\iota K') \quad = \quad \mathrm{dn}\, u$, etc.,

i.e. $\mathrm{dn}(u+2\lambda K+4\mu\iota K')=\mathrm{dn}\, u$; $\mathrm{dn}(u+2\lambda K+2\mu\iota K')=-\mathrm{dn}\, u$ if μ be odd.

We may sum up these results concisely thus :

$$\left.\begin{aligned}\mathrm{sn}(u+2p\iota K'+2qK)&=(-1)^q\,\mathrm{sn}\, u,\\ \mathrm{cn}(u+2p\iota K'+2qK)&=(-1)^{p+q}\,\mathrm{cn}\, u,\\ \mathrm{dn}(u+2p\iota K'+2qK)&=(-1)^p\,\mathrm{dn}\, u.\end{aligned}\right\}$$

1343. Values of sn ιu, cn ιu, dn ιu.

Let $\iota u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, and put $\sin\theta=\iota\tan\phi$, an imaginary transformation. Then $\cos\theta\, d\theta=\iota\sec^2\phi\, d\phi$ and $\cos\theta=\sec\phi$; then

$$\iota u=\int_0^\phi \frac{\iota\sec^2\phi\, d\phi}{\sec\phi\sqrt{1+k^2\tan^2\phi}}=\iota\int_0^\phi\frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}};$$

$$\therefore\ \phi=\mathrm{am}(u, k');\quad \therefore\ \mathrm{sn}(\iota u, k)=\iota\frac{\mathrm{sn}(u, k')}{\mathrm{cn}(u, k')};$$

whence $\mathrm{cn}(\iota u, k)=\dfrac{1}{\mathrm{cn}(u, k')}$; $\mathrm{dn}(\iota u, k)=\dfrac{\mathrm{dn}(u, k')}{\mathrm{cn}(u, k')}$.

These relations are true for all values of u real or complex.

1344. The Addition Formulae for Legendre's First Integral. Euler's Equation.

Let $u_1\equiv\int_0^{x_1}\frac{dz}{\sqrt{Z}}$, $u_2\equiv\int_0^{x_2}\frac{dz}{\sqrt{Z}}$, where $Z=(1-z^2)(1-k^2z^2)$.

Then $\quad x_1=\mathrm{sn}\, u_1, \quad x_2=\mathrm{sn}\, u_2.$

Consider the differential equation

$$\frac{dx_1}{\sqrt{X_1}}+\frac{dx_2}{\sqrt{X_2}}=0, \qquad\text{(A)}$$

where $\quad X_1\equiv(1-x_1^2)(1-k^2x_1^2), \quad X_2=(1-x_2^2)(1-k^2x_2^2).$

Let x_1 and x_2 be regarded as functions of a third variable t, such that $\dot{x}_1\equiv\dfrac{dx_1}{dt}=\sqrt{X_1}$; then $\dot{x}_2\equiv\dfrac{dx_2}{dt}=-\sqrt{X_2}$,

and $\dot{x}_1^2=1-(k^2+1)x_1^2+k^2x_1^4$; $\dot{x}_2^2=1-(k^2+1)x_2^2+k^2x_2^4$;

whence, differentiating and dividing by $2\dot{x}_1$ and $2\dot{x}_2$ respectively, $\ddot{x}_1=-(k^2+1)x_1+2k^2x_1^3$; $\ddot{x}_2=-(k^2+1)x_2+2k^2x_2^3$;

$$\left.\begin{aligned}&\text{Thus}\quad &\ddot{x}_1x_2-\ddot{x}_2x_1&=2k^2(x_1^2-x_2^2)x_1x_2,\\ &\text{whilst}\quad &\dot{x}_1^2x_2^2-\dot{x}_2^2x_1^2&=-(x_1^2-x_2^2)(1-k^2x_1^2x_2^2)\end{aligned}\right\}$$

Hence $\dfrac{\ddot{x}_1x_2-\ddot{x}_2x_1}{\dot{x}_1x_2-\dot{x}_2x_1}=-\dfrac{2k^2x_1x_2}{1-k^2x_1{}^2x_2{}^2}\dfrac{d}{dt}(x_1x_2)$;

whence $\log(\dot{x}_1x_2-\dot{x}_2x_1)=\log(1-k^2x_1{}^2x_2{}^2)+\text{const.}$,

i.e. $\dfrac{\dot{x}_1x_2-\dot{x}_2x_1}{1-k^2x_1{}^2x_2{}^2}=C$, and $\therefore\ \dfrac{x_2\sqrt{X_1}+x_1\sqrt{X_2}}{1-k^2x_1{}^2x_2{}^2}=C.$

Another form of the Integral of (A) is obviously

$$u_1+u_2=\int_0^{x_1}\frac{dx_1}{\sqrt{X_1}}+\int_0^{x_2}\frac{dx_2}{\sqrt{X_2}}=\text{const.}=C'.$$

It appears therefore that when u_1+u_2 is constant, so also is $\dfrac{x_2\sqrt{X_1}+x_1\sqrt{X_2}}{1-k^2x_1{}^2x_2{}^2}$ a constant.

One of these constants must therefore be a function of the other, say, $C=\phi(C')$.

Hence $\dfrac{x_2\sqrt{X_1}+x_1\sqrt{X_2}}{1-k^2x_1{}^2x_2{}^2}=\phi(u_1+u_2)$, and the form of ϕ may be readily identified. For, since $u_1=\displaystyle\int_0^{x_1}\frac{dz}{\sqrt{Z}}$ and $u_2=\displaystyle\int_0^{x_2}\frac{dz}{\sqrt{Z}}$, it is clear that,

if $x_1=0$ and therefore $X_1=1$, we have $u_1=0$,

and if $x_2=0$ and therefore $X_2=1$, we have $u_2=0$.

Putting $u_2=0$, we have $\phi(u_1)\equiv x_1=\operatorname{sn}u_1$. Hence the form of the function ϕ is identified as the elliptic function sn. Thus we have

$$\operatorname{sn}(u_1+u_2)=\frac{x_2\sqrt{1-x_1{}^2}\sqrt{1-k^2x_1{}^2}+x_1\sqrt{1-x_2{}^2}\sqrt{1-k^2x_2{}^2}}{1-k^2x_1{}^2x_2{}^2},$$

i.e. $$\mathbf{sn}(u_1+u_2)=\frac{\mathbf{sn}\,u_1\,\mathbf{cn}\,u_2\,\mathbf{dn}\,u_2+\mathbf{sn}\,u_2\,\mathbf{cn}\,u_1\,\mathbf{dn}\,u_1}{1-k^2\,\mathbf{sn}^2u_1\,\mathbf{sn}^2u_2}.$$

Remembering that

$\operatorname{sn}'u_1$, *i.e.* $\dfrac{d}{du_1}\operatorname{sn}u_1$, $=\operatorname{cn}u_1\operatorname{dn}u_1$ and $\operatorname{cn}'u_1=-\operatorname{sn}u_1\operatorname{dn}u_1$,

this formula may be written as

$$\mathbf{sn}(u_1+u_2)=\frac{\mathbf{sn}\,u_1\,\mathbf{sn}'u_2+\mathbf{sn}\,u_2\,\mathbf{sn}'u_1}{1-k^2\,\mathbf{sn}^2u_1\,\mathbf{sn}^2u_2}.$$

For shortness write $\operatorname{sn}u_1=s_1$, $\operatorname{sn}u_2=s_2$, $\operatorname{cn}u_1=c_1$, $\operatorname{cn}u_2=c_2$, $\operatorname{dn}u_1=d_1$, $\operatorname{dn}u_2=d_2$ and $1-k^2\operatorname{sn}^2u_1\operatorname{sn}^2u_2=D$.

Then $\mathbf{sn}(u_1+u_2)=(s_1c_2d_2+s_2c_1d_1)/D$ or $=(s_1s_2'+s_2s_1')/D$.

[Compare the ordinary addition formula of trigonometry, $\sin(u_1+u_2)=\sin u_1\cos u_2+\sin u_2\cos u_1$, which may be similarly written $=s_1c_2+s_2c_1$ or $=s_1s_2'+s_2s_1'$, viz. the case of the above elliptic function formula when $k=0$.]

1345. **To obtain $\mathbf{cn}(u_1+u_2)$**, we have

$$\begin{aligned}\mathrm{cn}^2(u_1+u_2)&=1-\mathrm{sn}^2(u_1+u_2)\\&=\{(1-k^2s_1^2s_2^2)^2-(s_1c_2d_2+s_2c_1d_1)^2\}/D^2\\&=(c_1^2c_2^2-2s_1s_2\,c_1c_2\,d_1d_2+s_1^2s_2^2\,d_1^2d_2^2)/D^2\,;\end{aligned}$$

$\therefore$ $\mathbf{cn}(u_1+u_2)=(c_1c_2-s_1s_2\,d_1d_2)/D$, the positive sign being taken because, when $u_2=0$, each side must become c_1. This may be also written

$$\mathbf{cn}(u_1+u_2)=(c_1c_2-c_1'c_2')/D.$$

[Compare with the trigonometrical formula for $\cos(u_1+u_2)$, which may be written $c_1c_2-s_1s_2$ or $c_1c_2-c_1'c_2'$, where $c_1=\cos u_1$, etc.]

1346. **To obtain $\mathbf{dn}(u_1+u_2)$**, we have

$$\begin{aligned}\mathrm{dn}^2(u_1+u_2)&=1-k^2\,\mathrm{sn}^2(u_1+u_2)\\&=\{(1-k^2s_1^2s_2^2)^2-k^2(s_1c_2d_2+s_2c_1d_1)^2\}/D^2\\&=(d_1^2d_2^2-2k^2s_1s_2\,c_1c_2\,d_1d_2+k^4\,s_1^2s_2^2\,c_1^2c_2^2)/D^2,\end{aligned}$$

and $\mathbf{dn}(u_1+u_2)=(d_1d_2-k^2s_1c_1\,s_2c_2)/D$, the positive sign being taken because, when $u_2=0$, each side must become d_1. This may be written as

$$\mathbf{dn}\,(u_1+u_2)=\left(d_1d_2-\frac{1}{k^2}d_1'd_2'\right)\Big/D.$$

1347. **Derived Results.**

From the three formulae

$$\left.\begin{aligned}\mathrm{sn}\,(u_1+u_2)&=(s_1c_2d_2+s_2c_1d_1)/D,\\\mathrm{cn}\,(u_1+u_2)&=(c_1c_2-s_1s_2d_1d_2)/D,\\\mathrm{dn}\,(u_1+u_2)&=(d_1d_2-k^2s_1s_2c_1c_2)/D,\end{aligned}\right\}\text{(I)},$$

we obtain, by changing the sign of u_2,

$$\left.\begin{aligned}\mathrm{sn}\,(u_1-u_2)&=(s_1c_2d_2-s_2c_1d_1)/D,\\\mathrm{cn}\,(u_1-u_2)&=(c_1c_2+s_1s_2d_1d_2)/D,\\\mathrm{dn}\,(u_1-u_2)&=(d_1d_2+k^2s_1s_2c_1c_2)/D,\end{aligned}\right\}\text{(II)}.$$

The addition and subtraction of formulae (I) and (II) in pairs gives

$$\left.\begin{aligned}
\operatorname{sn}(u_1+u_2)+\operatorname{sn}(u_1-u_2)&=2s_1c_2d_2/D,\\
\operatorname{sn}(u_1+u_2)-\operatorname{sn}(u_1-u_2)&=2s_2c_1d_1/D,\\
\operatorname{cn}(u_1+u_2)+\operatorname{cn}(u_1-u_2)&=2c_1c_2/D,\\
\operatorname{cn}(u_1+u_2)-\operatorname{cn}(u_1-u_2)&=-2s_1s_2d_1d_2/D,\\
\operatorname{dn}(u_1+u_2)+\operatorname{dn}(u_1-u_2)&=2d_1d_2/D,\\
\operatorname{dn}(u_1+u_2)-\operatorname{dn}(u_1-u_2)&=-2k^2s_1s_2c_1c_2/D,
\end{aligned}\right\}\text{(III).}$$

Replacing u_1+u_2 and u_1-u_2 by U_1, U_2 respectively and writing D' for $1-k^2\operatorname{sn}^2\dfrac{U_1+U_2}{2}\operatorname{sn}^2\dfrac{U_1-U_2}{2}$, we have

$$\operatorname{sn}U_1+\operatorname{sn}U_2=2\operatorname{sn}\frac{U_1+U_2}{2}\operatorname{cn}\frac{U_1-U_2}{2}\operatorname{dn}\frac{U_1-U_2}{2}\Big/D',$$

$$\operatorname{sn}U_1-\operatorname{sn}U_2=2\operatorname{sn}\frac{U_1-U_2}{2}\operatorname{cn}\frac{U_1+U_2}{2}\operatorname{dn}\frac{U_1+U_2}{2}\Big/D',$$

$$\operatorname{cn}U_1+\operatorname{cn}U_2=2\operatorname{cn}\frac{U_1+U_2}{2}\operatorname{cn}\frac{U_1-U_2}{2}\Big/D',$$

$$\operatorname{cn}U_1-\operatorname{cn}U_2=-2\operatorname{sn}\frac{U_1+U_2}{2}\operatorname{sn}\frac{U_1-U_2}{2}\operatorname{dn}\frac{U_1+U_2}{2}\operatorname{dn}\frac{U_1-U_2}{2}\Big/D',$$

$$\operatorname{dn}U_1+\operatorname{dn}U_2=2\operatorname{dn}\frac{U_1+U_2}{2}\operatorname{dn}\frac{U_1-U_2}{2}\Big/D'$$

$$\operatorname{dn}U_1-\operatorname{dn}U_2=-2k^2\operatorname{sn}\frac{U_1+U_2}{2}\operatorname{sn}\frac{U_1-U_2}{2}\operatorname{cn}\frac{U_1+U_2}{2}\operatorname{cn}\frac{U_1-U_2}{2}\Big/D'.$$

Again, by division of corresponding formulae from groups (I) and (II), and writing t_1 or $\operatorname{tn}u_1$ for $\tan\operatorname{am}u_1$ and $\operatorname{ctn}u_1$ for $\cot\operatorname{am}u_1$, etc.,

$$\left.\begin{aligned}
\operatorname{tn}(u_1\pm u_2)&=\frac{s_1c_2d_2\pm s_2c_1d_1}{c_1c_2\mp s_1s_2d_1d_2}=\frac{t_1d_2\pm t_2d_1}{1\mp t_1t_2d_1d_2},\\
\operatorname{ctn}(u_1\pm u_2)&=\frac{c_1c_2\mp s_1s_2d_1d_2}{s_1c_2d_2\pm s_2c_1d_1}=\frac{\operatorname{ctn}u_1\operatorname{ctn}u_2\mp\operatorname{dn}u_1\operatorname{dn}u_2}{\operatorname{ctn}u_2\operatorname{dn}u_2\pm\operatorname{ctn}u_1\operatorname{dn}u_1}.
\end{aligned}\right\}$$

1348. Following Cayley's notation (*Elliptic Functions*, p. 62), with a slight modification, let us write

$$s_1s_2'=A_1,\qquad c_1c_2=B_1,\qquad d_1d_2=C_1,$$

$$s_2s_1'=A_2,\quad -c_1'c_2'=B_2,\quad -\frac{1}{k^2}d_1'd_2'=C_2,\qquad 1-k^2s_1^2s_2^2=D,$$

$$P=s_1^2-s_2^2=c_2^2-c_1^2,$$

$$Q=1-s_1^2-s_2^2+k^2s_1^2s_2^2=c_1^2-s_2^2d_1^2=c_2^2-s_1^2d_2^2,$$

$$R=1-k^2s_1^2-k^2s_2^2+k^2s_1^2s_2^2=d_1^2-k^2s_2^2c_1^2=d_2^2-k^2s_1^2c_2^2,$$

$$s_1's_2'=S_1,\quad -c_2c_1'=T_1,\quad s_1c_1d_2=U_1,$$

$$-k'^2s_1s_2=S_2,\qquad c_1c_2'=T_2,\quad s_2c_2d_1=U_2.$$

A number of identical relations immediately arise amongst the capital letters. We have

(1) $A_1^2-A_2^2=s_1^2s_2'^2-s_2^2s_1'^2=s_1^2(1-s_2^2)(1-k^2s_2^2)-s_2^2(1-s_1^2)(1-k^2s_1^2)$
$$=(s_1^2-s_2^2)(1-k^2s_1^2s_2^2)=PD.$$

(2) $B_1^2-B_2^2=c_1^2c_2^2-c_1'^2c_2'^2=(1-s_1^2)(1-s_2^2)-s_1^2s_2^2(1-k^2s_1^2)(1-k^2s_2^2)$
$$=(1-s_1^2-s_2^2+k^2s_1^2s_2^2)(1-k^2s_1^2s_2^2)=QD.$$

(3) $C_1^2-C_2^2=d_1^2d_2^2-k^4s_1^2s_2^2c_1^2c_2^2=(1-k^2s_1^2)(1-k^2s_2^2)-k^4s_1^2s_2^2(1-s_1^2)(1-s_2^2)$
$$=(1-k^2s_1^2-k^2s_2^2+k^2s_1^2s_2^2)(1-k^2s_1^2s_2^2)=RD.$$

Hence
$$\frac{A_1^2-A_2^2}{P}=\frac{B_1^2-B_2^2}{Q}=\frac{C_1^2-C_2^2}{R}=D.$$

Again,

(4) $S_1^2-S_2^2=(1-s_1^2)(1-s_2^2)(1-k^2s_1^2)(1-k^2s_2^2)-(1-k^2)^2s_1^2s_2^2$
$$=(1-s_1^2-s_2^2+k^2s_1^2s_2^2)(1-k^2s_1^2-k^2s_2^2+k^2s_1^2s_2^2)=QR.$$

(5) $T_1^2-T_2^2=s_1^2(1-k^2s_1^2)(1-s_2^2)-s_2^2(1-k^2s_2^2)(1-s_1^2)$
$$=(s_1^2-s_2^2)(1-k^2s_1^2-k^2s_2^2+k^2s_1^2s_2^2)=RP.$$

(6) $U_1^2-U_2^2=s_1^2(1-s_1^2)(1-k^2s_2^2)-s_2^2(1-s_2^2)(1-k^2s_1^2)$
$$=(s_1^2-s_2^2)(1-s_1^2-s_2^2+k^2s_1^2s_2^2)=PQ.$$

Hence $\quad P(S_1^2-S_2^2)=Q(T_1^2-T_2^2)=R(U_1^2-U_2^2)=PQR.$

Also,

(7) $(B_1+B_2)(C_1-C_2)=(c_1c_2-s_1s_2d_1d_2)(d_1d_2+k^2s_1s_2c_1c_2)$
$$\begin{aligned}&=c_1c_2d_1d_2+k^2s_1s_2(1-s_1^2-s_2^2+s_1^2s_2^2)\\&\quad-s_1s_2(1-k^2s_1^2-k^2s_2^2+k^4s_1^2s_2^2)-k^2s_1^2s_2^2c_1c_2d_1d_2\\&=(c_1c_2d_1d_2-k'^2s_1s_2)D=(S_1+S_2)D,\end{aligned}$$

and similarly, or changing the sign of s_2,
$$(B_1-B_2)(C_1+C_2)=(S_1-S_2)D.$$

(8) $(C_1+C_2)(A_1-A_2)=(d_1d_2-k^2s_1s_2c_1c_2)(s_1c_2d_2-s_2c_1d_1)$
$$\begin{aligned}&=s_1c_2d_1(1-k^2s_2^2)-s_2c_1d_2(1-k^2s_1^2)\\&\quad-s_1^2s_2c_1k^2d_2(1-s_2^2)+s_1s_2^2c_2k^2d_1(1-s_1^2)\\&=(s_1c_2d_1-s_2c_1d_2)(1-k^2s_1^2s_2^2)=(T_1+T_2)D,\end{aligned}$$

and similarly, or changing the sign of s_2,
$$(C_1-C_2)(A_1+A_2)=(T_1-T_2)D.$$

(9) $(A_1+A_2)(B_1-B_2)=(s_1c_2d_2+s_2c_1d_1)(c_1c_2+s_1s_2d_1d_2)$
$$\begin{aligned}&=s_1c_1d_2(1-s_2^2)+s_2c_2d_1(1-s_1^2)\\&\quad+s_1^2s_2c_2d_1(1-k^2s_2^2)+s_1s_2^2c_1d_2(1-k^2s_1^2)\\&=(s_1c_1d_2+s_2c_2d_1)(1-k^2s_1^2s_2^3)=(U_1+U_2)D,\end{aligned}$$

and similarly, or changing the sign of s_2,
$$(A_1-A_2)(B_1+B_2)=(U_1-U_2)D.$$

Thus
$$\frac{(B_1\pm B_2)(C_1\mp C_2)}{S_1\pm S_2}=\frac{(C_1\pm C_2)(A_1\mp A_2)}{T_1\pm T_2}=\frac{(A_1\pm A_2)(B_1\mp B_2)}{U_1\pm U_2}=D.$$

With this notation, it follows at once that

$$\left.\begin{aligned}
\operatorname{sn}(u_1+u_2)&=\frac{A_1+A_2}{D}=\frac{P}{A_1-A_2}=\frac{U_1+U_2}{B_1-B_2}=\frac{T_1-T_2}{C_1-C_2},\\
\operatorname{sn}(u_1-u_2)&=\frac{A_1-A_2}{D}=\frac{P}{A_1+A_2}=\frac{U_1-U_2}{B_1+B_2}=\frac{T_1+T_2}{C_1+C_2},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\operatorname{cn}(u_1+u_2)&=\frac{B_1+B_2}{D}=\frac{Q}{B_1-B_2}=\frac{S_1+S_2}{C_1-C_2}=\frac{U_1-U_2}{A_1-A_2},\\
\operatorname{cn}(u_1-u_2)&=\frac{B_1-B_2}{D}=\frac{Q}{B_1+B_2}=\frac{S_1-S_2}{C_1+C_2}=\frac{U_1+U_2}{A_1+A_2},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\operatorname{dn}(u_1+u_2)&=\frac{C_1+C_2}{D}=\frac{R}{C_1-C_2}=\frac{T_1+T_2}{A_1-A_2}=\frac{S_1-S_2}{B_1-B_2},\\
\operatorname{dn}(u_1-u_2)&=\frac{C_1-C_2}{D}=\frac{R}{C_1+C_2}=\frac{T_1-T_2}{A_1+A_2}=\frac{S_1+S_2}{B_1+B_2}.
\end{aligned}\right\}$$

1349. A number of identities immediately appear.

For example, since

$$(B_1+B_2)(A_1-A_2)=D(U_1-U_2)$$

and

$$(B_1-B_2)(A_1+A_2)=D(U_1+U_2),$$

we have $B_1A_1-B_2A_2=DU_1$ and $B_1A_2-B_2A_1=DU_2$,

i.e. $s_1s_2'.\,c_1c_2+s_2s_1'.\,c_1'c_2'=s_1c_1d_2(1-k^2s_1{}^2s_1{}^2)$

and $s_2s_1'.\,c_1c_2+s_1s_2'.\,c_1'c_2'=s_2c_2d_1(1-k^2s_1{}^2s_2{}^2)$.

1350. More important however than such, are the following:

$$\left.\begin{aligned}
&\operatorname{sn}(u_1+u_2)+\operatorname{sn}(u_1-u_2)=\frac{2A_1}{D}, &&\operatorname{sn}(u_1+u_2)-\operatorname{sn}(u_1-u_2)=\frac{2A_2}{D};\\
&\operatorname{cn}(u_1+u_2)+\operatorname{cn}(u_1-u_2)=\frac{2B_1}{D}, &&\operatorname{cn}(u_1+u_2)-\operatorname{cn}(u_1-u_2)=\frac{2B_2}{D};\\
&\operatorname{dn}(u_1+u_2)+\operatorname{dn}(u_1-u_2)=\frac{2C_1}{D}, &&\operatorname{dn}(u_1+u_2)-\operatorname{dn}(u_1-u_2)=\frac{2C_2}{D},
\end{aligned}\right\}$$

which are the formulae of Group (III) in Cayley's notation.

$$\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)=\frac{A_1{}^2-A_2{}^2}{D^2}=\frac{PD}{D^2}=\frac{P}{D}=\frac{\operatorname{sn}^2u_1-\operatorname{sn}^2u_2}{1-k^2\operatorname{sn}^2u_1\operatorname{sn}^2u_2}\text{ or }(s_1{}^2-s_2{}^2)/D,$$

$$\operatorname{cn}(u_1+u_2)\operatorname{cn}(u_1-u_2)=\frac{B_1{}^2-B_2{}^2}{D^2}=\frac{QD}{D^2}=\frac{Q}{D}=\frac{\operatorname{cn}^2u_1-\operatorname{sn}^2u_2\operatorname{dn}^2u_1}{1-k^2\operatorname{sn}^2u_1\operatorname{sn}^2u_2}\text{ or }(c_1{}^2-s_2{}^2d_1{}^2)/D,$$

$$\operatorname{dn}(u_1+u_2)\operatorname{dn}(u_1-u_2)=\frac{C_1{}^2-C_2{}^2}{D^2}=\frac{RD}{D^2}=\frac{R}{D}=\frac{\operatorname{dn}^2u_1-k^2\operatorname{sn}^2u_2\operatorname{cn}^2u_1}{1-k^2\operatorname{sn}^2u_1\operatorname{sn}^2u_2}\text{ or }(d_1{}^2-k^2s_2{}^2c_1{}^2)/D,$$

$$1+\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)=1+\frac{s_1{}^2-s_2{}^2}{1-k^2s_1{}^2s_2{}^2}=(c_2{}^2+s_1{}^2d_2{}^2)/D,$$

$$1-\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)=1-\frac{s_1{}^2-s_2{}^2}{1-k^2s_1{}^2s_2{}^2}=(c_1{}^2+s_2{}^2d_1{}^2)/D,$$

$$1+k^2\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)=1+k^2\frac{s_1{}^2-s_2{}^2}{1-k^2s_1{}^2s_2{}^2}=(d_2{}^2+k^2s_1{}^2c_2{}^2)/D,$$

$$1-k^2\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)=1-k^2\frac{s_1{}^2-s_2{}^2}{1-k^2s_1{}^2s_2{}^2}=(d_1{}^2+k^2c_1{}^2s_2{}^2)/D,$$

$$1+\mathrm{cn}(u_1+u_2)\,\mathrm{cn}(u_1-u_2)=1+\frac{c_1^2-s_2^2d_1^2}{1-k^2s_1^2s_2^2}=(c_1^2+c_2^2)/D,$$

$$1+\mathrm{dn}(u_1+u_2)\,\mathrm{dn}(u_1-u_2)=1+\frac{d_1^2-k^2s_2^2c_1^2}{1-k^2s_1^2s_2^2}=(d_1^2+d_2^2)/D,$$

$$[1+\mathrm{sn}(u_1+u_2)][1+\mathrm{sn}(u_1-u_2)]=\mathrm{sn}(u_1+u_2)+\mathrm{sn}(u_1-u_2)+[1+\mathrm{sn}(u_1+u_2)\,\mathrm{sn}(u_1-u_2)]$$
$$=(2s_1c_2d_2+c_2^2+s_1^2d_2^2)/D=(c_2+s_1d_2)^2/D.$$

Again, $$\mathrm{cn}(u_1+u_2)\,\mathrm{dn}(u_1-u_2)=\frac{B_1+B_2}{D}\cdot\frac{C_1-C_2}{D}=\frac{S_1+S_2}{D}$$
$$=(s_1's_2'-k'^2s_1s_2)/D=(c_1c_2d_1d_2-k'^2s_1s_2)/D,$$

$$\mathrm{dn}(u_1+u_2)\,\mathrm{sn}(u_1-u_2)=\frac{C_1+C_2}{D}\cdot\frac{A_1-A_2}{D}=\frac{T_1+T_2}{D}=(c_1c_2'-c_2c_1')/D$$
$$=(c_2s_1d_1-c_1s_2d_2)/D,$$

$$\mathrm{sn}(u_1+u_2)\,\mathrm{cn}(u_1-u_2)=\frac{A_1+A_2}{D}\cdot\frac{B_1-B_2}{D}=\frac{U_1+U_2}{D}=(s_1c_1d_2+s_2c_2d_1)/D,$$

and so on for other cases.

Jacobi gives a list of 33 such results (*Fundamenta Nova*, pp. 32-34). These are quoted by Cayley (*Elliptic Functions*, pp. 65 and 66) and by Greenhill (*Elliptic Functions*, pp. 138, 139).

Several have been worked above as illustrative of the method to be followed. They are too numerous to remember, but any one of them may be readily obtained if wanted. This list we append as Examples.

EXAMPLES. (JACOBI.)

1351. In each case the denominator $D=1-k^2s_1^2s_2^2$, and the previous notation is adhered to, viz. $\mathrm{sn}\,u_1=s_1$, $\mathrm{sn}\,u_2=s_2$, etc.

Establish the results following:

1. $\mathrm{sn}(u_1+u_2)+\mathrm{sn}(u_1-u_2)=2s_1c_2d_2/D.$
2. $\mathrm{sn}(u_1+u_2)-\mathrm{sn}(u_1-u_2)=2s_2c_1d_1/D.$
3. $\mathrm{cn}(u_1+u_2)+\mathrm{cn}(u_1-u_2)=2c_1c_2/D.$
4. $\mathrm{cn}(u_1+u_2)-\mathrm{cn}(u_1-u_2)=-2s_1s_2d_1d_2/D.$
5. $\mathrm{dn}(u_1+u_2)+\mathrm{dn}(u_1-u_2)=2d_1d_2/D.$
6. $\mathrm{dn}(u_1+u_2)-\mathrm{dn}(u_1-u_2)=-2k^2s_1s_2c_1c_2/D.$
7. $\mathrm{sn}(u_1+u_2)\,\mathrm{sn}(u_1-u_2)=(s_1^2-s_2^2)/D.$
8. $1+\mathrm{sn}(u_1+u_2)\,\mathrm{sn}(u_1-u_2)=(c_2^2+s_1^2d_2^2)/D.$
9. $1-\mathrm{sn}(u_1+u_2)\,\mathrm{sn}(u_1-u_2)=(c_1^2+s_2^2d_1^2)/D.$
10. $1+k^2\mathrm{sn}(u_1+u_2)\,\mathrm{sn}(u_1-u_2)=(d_2^2+k^2s_1^2c_2^2)/D.$
11. $1-k^2\mathrm{sn}(u_1+u_2)\,\mathrm{sn}(u_1-u_2)=(d_1^2+k^2s_2^2c_1^2)/D.$
12. $1+\mathrm{cn}(u_1+u_2)\,\mathrm{cn}(u_1-u_2)=(c_1^2+c_2^2)/D.$
13. $1-\mathrm{cn}(u_1+u_2)\,\mathrm{cn}(u_1-u_2)=(s_1^2d_2^2+s_2^2d_1^2)/D.$

14. $1+\mathrm{dn}(u_1+u_2)\,\mathrm{dn}(u_1-u_2)=(d_1^2+d_2^2)/D.$

15. $1-\mathrm{dn}(u_1+u_2)\,\mathrm{dn}(u_1-u_2)=k^2(s_1^2c_2^2+s_2^2c_1^2)/D.$

16. $\{1\pm\mathrm{sn}(u_1+u_2)\}\{1\pm\mathrm{sn}(u_1-u_2)\}\quad=(c_2\pm s_1d_2)^2/D.$

17. $\{1\pm\mathrm{sn}(u_1+u_2)\}\{1\mp\mathrm{sn}(u_1-u_2)\}\quad=(c_1\pm s_2d_1)^2/D.$

18. $\{1\pm k\,\mathrm{sn}(u_1+u_2)\}\{1\pm k\,\mathrm{sn}(u_1-u_2)\}=(d_2\pm ks_1c_2)^2/D.$

19. $\{1\pm k\,\mathrm{sn}(u_1+u_2)\}\{1\mp k\,\mathrm{sn}(u_1-u_2)\}=(d_1\pm ks_2c_1)^2/D.$

20. $\{1\pm\mathrm{cn}(u_1+u_2)\}\{1\pm\mathrm{cn}(u_1-u_2)\}\quad=(c_1\pm c_2)^2/D.$

21. $\{1\pm\mathrm{cn}(u_1+u_2)\}\{1\mp\mathrm{cn}(u_1-u_2)\}\quad=(s_1d_2\mp s_2d_1)^2/D.$

22. $\{1\pm\mathrm{dn}(u_1+u_2)\}\{1\pm\mathrm{dn}(u_1-u_2)\}\quad=(d_1\pm d_2)^2/D.$

23. $\{1\pm\mathrm{dn}(u_1+u_2)\}\{1\mp\mathrm{dn}(u_1-u_2)\}\quad=k^2(s_1c_2\mp s_2c_1)^2/D.$

24. $\mathrm{sn}(u_1+u_2)\,\mathrm{cn}(u_1-u_2)=(s_1c_1d_2+s_2c_2d_1)/D.$

25. $\mathrm{sn}(u_1-u_2)\,\mathrm{cn}(u_1+u_2)=(s_1c_1d_2-s_2c_2d_1)/D.$

26. $\mathrm{sn}(u_1+u_2)\,\mathrm{dn}(u_1-u_2)=(s_1d_1c_2+s_2d_2c_1)/D.$

27. $\mathrm{sn}(u_1-u_2)\,\mathrm{dn}(u_1+u_2)=(s_1d_1c_2-s_2d_2c_1)/D.$

28. $\mathrm{cn}(u_1+u_2)\,\mathrm{dn}(u_1-u_2)=(c_1c_2d_1d_2-k'^2s_1s_2)/D.$

29. $\mathrm{cn}(u_1-u_2)\,\mathrm{dn}(u_1+u_2)=(c_1c_2d_1d_2+k'^2s_1s_2)/D.$

30. $\sin\{\mathrm{am}(u_1+u_2)+\mathrm{am}(u_1-u_2)\}=2s_1c_1d_2/D.$

31. $\sin\{\mathrm{am}(u_1+u_2)-\mathrm{am}(u_1-u_2)\}=2s_2c_2d_1/D.$

32. $\cos\{\mathrm{am}(u_1+u_2)+\mathrm{am}(u_1-u_2)\}=(c_1^2-s_1^2d_2^2)/D.$

33. $\cos\{\mathrm{am}(u_1+u_2)-\mathrm{am}(u_1-u_2)\}=(c_2^2-s_2^2d_1^2)/D.$

To the above list it is convenient to add for reference:

(*a*) $\mathrm{cn}(u_1+u_2)\,\mathrm{cn}(u_1-u_2)=(c_2^2-s_1^2d_2^2)/D=(c_1^2-s_2^2d_1^2)/D.$

(*b*) $\mathrm{dn}(u_1+u_2)\,\mathrm{dn}(u_1-u_2)=(d_1^2-k^2s_2^2c_1^2)/D=(d_2^2-k^2s_1^2c_2^2)/D.$

(*c*) $\{\mathrm{dn}(u_1+u_2)\pm\mathrm{cn}(u_1+u_2)\}\{\mathrm{dn}(u_1-u_2)\pm\mathrm{cn}(u_1-u_2)\}=(c_1d_2\pm c_2d_1)^2/D.$

(*d*) $\{\mathrm{dn}(u_1+u_2)\pm\mathrm{cn}(u_1+u_2)\}\{\mathrm{dn}(u_1-u_2)\mp\mathrm{cn}(u_1-u_2)\}=k'^2(s_1\mp s_2)^2/D.$

[(*c*) and (*d*) are given by Greenhill, *E.F.*, p. 262.]

1352. **Periodicity of the Functions considered by aid of the Addition Theorem.**

Starting with the addition formulae in which $D\equiv 1-k^2s_1^2s_2^2$,

$$\mathrm{sn}(u_1\pm u_2)=(s_1c_2d_2\pm s_2c_1d_1)/D;\quad \mathrm{cn}(u_1\pm u_2)=(c_1c_2\mp s_1s_2d_1d_2)/D;$$
$$\mathrm{dn}(u_1\pm u_2)=(d_1d_2\mp k^2s_1s_2c_1c_2)/D;$$

and putting $u_1=u$, $u_2=K$, we have, since $\mathrm{sn}\,K=1$, $\mathrm{cn}\,K=0$, $\mathrm{dn}\,K=k'$,

$$\mathrm{sn}(u+K)=(\mathrm{sn}\,u\,\mathrm{cn}\,K\,\mathrm{dn}\,K+\mathrm{sn}\,K\,\mathrm{cn}\,u\,\mathrm{dn}\,u)/D,$$

where $D=1-k^2\mathrm{sn}^2u=\mathrm{dn}^2u=d^2$,

i.e.
$$\left.\begin{aligned}\mathrm{sn}(u+K)&=\frac{c}{d},\quad \mathrm{cn}(u+K)=-\frac{k's}{d},\quad \mathrm{dn}(u+K)=\frac{k'}{d},\\ \mathrm{sn}(u-K)&=-\frac{c}{d},\quad \mathrm{cn}(u-K)=\frac{k's}{d},\quad \mathrm{dn}(u-K)=\frac{k'}{d}.\end{aligned}\right\}$$

Putting $u+K$ in these formulae in place of u,

$$\left.\begin{aligned}
&\operatorname{sn}(u+2K)=\frac{\operatorname{cn}(u+K)}{\operatorname{dn}(u+K)}=-s, \quad \operatorname{cn}(u+2K)=-c, \quad \operatorname{dn}(u+2K)=d,\\
&\operatorname{sn}(u+3K)=\frac{\operatorname{cn}(u+2K)}{\operatorname{dn}(u+2K)}=-\frac{c}{d}, \quad \operatorname{cn}(u+3K)=\frac{k's}{d}, \quad \operatorname{dn}(u+3K)=\frac{k'}{d},\\
&\operatorname{sn}(u+4K)=\frac{\operatorname{cn}(u+3K)}{\operatorname{dn}(u+3K)}=\ s, \quad \operatorname{cn}(u+4K)=\ c, \quad \operatorname{dn}(u+4K)=d.
\end{aligned}\right\}$$

Hence the functions have all returned to their original values with period $4K$. It will be noted that $\operatorname{dn} u$ was restored with two additions of K, and that $\operatorname{sn} u$ and $\operatorname{cn} u$ took the same value but the opposite sign after two additions of K.

In the same way, since

$$\operatorname{sn}(K+\iota K')=\frac{1}{k}, \quad \operatorname{cn}(K+\iota K')=-\frac{\iota k'}{k}, \quad \operatorname{dn}(K+\iota K')=0,$$

we have $\operatorname{sn}(u+K+\iota K')=\dfrac{1}{k}.\,cd/D$, where $D=1-k^2s^2.\dfrac{1}{k^2}=c^2$;

$$\begin{aligned}
\therefore\ &\operatorname{sn}(u+K+\iota K')=\frac{d}{kc}, &&\operatorname{cn}(u+K+\iota K')=-\frac{\iota k'}{kc}, &&\operatorname{dn}(u+K+\iota K')=\frac{\iota k's}{c}\\
&\operatorname{sn}(u+2K+2\iota K')=-s, &&\operatorname{cn}(u+2K+2\iota K')=\ c, &&\operatorname{dn}(u+2K+2\iota K')=-d,\\
&\operatorname{sn}(u+3K+3\iota K')=-\frac{d}{kc}, &&\operatorname{cn}(u+3K+3\iota K')=-\frac{\iota k'}{kc}, &&\operatorname{dn}(u+3K+3\iota K')=-\frac{\iota k's}{c}\\
&\operatorname{sn}(u+4K+4\iota K')=\ s, &&\operatorname{cn}(u+4K+4\iota K')=\ c, &&\operatorname{dn}(u+4K+4\iota K')=\ d,
\end{aligned}$$

and all the original values are again acquired after an addition of $4(K+\iota K')$, and it will be noted that after two additions of $K+\iota K'$, $\operatorname{cn} u$ resumed its original value, but $\operatorname{sn} u$ and $\operatorname{dn} u$ resumed their original values with the opposite sign.

Writing $u-K$ for u in the several cases of the last form,

$$\begin{aligned}
&\operatorname{sn}(u+\iota K')=\frac{\operatorname{dn}(u-K)}{k\operatorname{cn}(u-K)}=\frac{1}{ks}, &&\operatorname{cn}(u+\iota K')=-\frac{\iota d}{ks}, &&\operatorname{dn}(u+\iota K')=-\frac{\iota c}{s}\\
&\operatorname{sn}(u+K+2\iota K')=-\operatorname{sn}(u-K)=\frac{c}{d}, &&\operatorname{cn}(u+K+2\iota K')=\frac{k's}{d}, &&\operatorname{dn}(u+K+2\iota K')=-\frac{k'}{d}\\
&\operatorname{sn}(u+2K+3\iota K')=-\frac{1}{ks}, &&\operatorname{cn}(u+2K+3\iota K')=-\frac{\iota d}{ks}, &&\operatorname{dn}(u+2K+3\iota K')=\frac{\iota c}{s}\\
&\operatorname{sn}(u+3K+4\iota K')=\operatorname{sn}(u-K)=-\frac{c}{d}, &&\operatorname{cn}(u+3K+4\iota K')=\frac{k's}{d}, &&\operatorname{dn}(u+3K+4\iota K')=\frac{k'}{d}
\end{aligned}$$

the last three being the same results as for the functions of $u+3K$.

Again, writing $u-K$ for u,

$$\begin{aligned}
&\operatorname{sn}(u+2\iota K')=\frac{\operatorname{cn}(u-K)}{\operatorname{dn}(u-K)}=s, &&\operatorname{cn}(u+2\iota K')=-c, &&\operatorname{dn}(u+2\iota K')=-d,\\
&\operatorname{sn}(u+K+3\iota K')=\frac{d}{kc}, &&\operatorname{cn}(u+K+3\iota K')=\frac{\iota k'}{kc}, &&\operatorname{dn}(u+K+3\iota K')=-\frac{\iota k's}{c},\\
&\operatorname{sn}(u+3\iota K')=\frac{1}{k}\frac{\operatorname{dn}(u-K)}{\operatorname{cn}(u-K)}=\frac{1}{ks}, &&\operatorname{cn}(u+3\iota K')=\frac{\iota d}{ks}, &&\operatorname{dn}(u+3\iota K')=\frac{\iota c}{s}.
\end{aligned}$$

Writing $u+K$ for u in the functions of $u+K+\iota K'$,

$$\operatorname{sn}(u+2K+\iota K') = \frac{1}{k}\frac{\operatorname{dn}(u+K)}{\operatorname{cn}(u+K)} = -\frac{1}{ks},\quad \operatorname{cn}(u+2K+\iota K') = \frac{\iota d}{ks},\quad \operatorname{dn}(u+2K+\iota K') = -\frac{\iota c}{s}$$

$$\operatorname{sn}(u+3K+\iota K') = -\frac{1}{k}\frac{1}{\operatorname{sn}(u+K)} = -\frac{1}{k}\frac{d}{c},\quad \operatorname{cn}(u+3K+\iota K') = \frac{\iota k'}{kc},\quad \operatorname{dn}(u+3K+\iota K') = \frac{\iota k'}{c}$$

Writing $u+K$ for u in the functions of $u+2K+2\iota K'$,

$$\operatorname{sn}(u+3K+2\iota K') = -\operatorname{sn}(u+K) = -\frac{c}{d},\quad \operatorname{cn}(u+3K+2\iota K') = -\frac{k's}{d},\quad \operatorname{dn}(u+3K+2\iota K') = -\frac{k'}{d}$$

1353. We exhibit these results for arguments of form $u+pK+q\iota K'$, in tabular form for reference.

If Δ stand for the word denominator we have, tabulating the numerators only and indicating the several denominators,

	$+0.K$	$+K$	$+2K$	$+3K$	$+4K$
$+0.\iota K'$	s c d $\Delta=1$	c $-k's$ k' $\Delta=d$	$-s$ $-c$ d $\Delta=1$	$-c$ $k's$ k' $\Delta=d$	s c d $\Delta=1$
$+\iota K'$	1 $-\iota d$ $-\iota kc$ $\Delta=ks$	d $-\iota k'$ $\iota kk's$ $\Delta=kc$	-1 ιd $-\iota kc$ $\Delta=ks$	$-d$ $\iota k'$ $\iota kk's$ $\Delta=kc$	1 $-\iota d$ $-\iota kc$ $\Delta=ks$
$+2\iota K'$	s $-c$ $-d$ $\Delta=1$	c $k's$ $-k'$ $\Delta=d$	$-s$ c $-d$ $\Delta=1$	$-c$ $-k's$ $-k'$ $\Delta=d$	s $-c$ $-d$ $\Delta=1$
$+3\iota K'$	1 ιd ιkc $\Delta=ks$	d $\iota k'$ $-\iota kk's$ $\Delta=kc$	-1 $-\iota d$ ιkc $\Delta=ks$	$-d$ $-\iota k'$ $-\iota kk's$ $\Delta=kc$	1 ιd ιkc $\Delta=ks$
$+4\iota K'$	s c d $\Delta=1$	c $-k's$ k' $\Delta=d$	$-s$ $-c$ d $\Delta=1$	$-c$ $k's$ k' $\Delta=d$	s c d $\Delta=1$

If, for instance, $\operatorname{dn}(u+2K+3\iota K')$ be required, we look in the group of the third column and fourth row and find numerator $=\iota kc$, denominator $=ks$, and the result is $\iota \operatorname{cn} u/\operatorname{sn} u$.

The vertical order in each square is sn( ), cn( ), dn( ), Δ.

The fifth column and fifth row exhibit the fact, that after an addition of $4K$ or of $4\iota K'$ to the argument, each of the functions returns to its original value, and shows their double periodicity. The value of any function of the forms

$$\operatorname{sn}(u+pK+q\iota K'),\quad \operatorname{cn}(u+pK+q\iota K'),\quad \operatorname{dn}(u+pK+q\iota K'),$$

where p and q are integral, can now be written down; *e.g.*

$$\operatorname{cn}(u+5K+11\iota K')=\operatorname{cn}(u+K+3\iota K')=\iota k'/kc.$$

The tabulation is given by Cayley (*E.F.*, p. 77) with a slightly different notation.

1354. Putting $u=0$, all the functions in the table for which $\Delta=ks$ become infinite.

There are four such groups, *i.e.* twelve of the functions. Cayley points out the importance of their *ratios* even when themselves infinite, and writing I for the infinite factor $1/k\operatorname{sn}0$ we have, remembering that $c=1$ and $d=1$, in this case

$$\frac{\operatorname{sn}\iota K'}{1}=\frac{\operatorname{cn}\iota K'}{-\iota}=\frac{\operatorname{dn}\iota K'}{-\iota k}=\frac{\operatorname{sn}(2K+\iota K')}{-1}=\frac{\operatorname{cn}(2K+\iota K')}{\iota}=\frac{\operatorname{dn}(2K+\iota K')}{-\iota k}$$

$$=\frac{\operatorname{sn}3\iota K'}{1}=\frac{\operatorname{cn}3\iota K'}{\iota}=\frac{\operatorname{dn}3\iota K'}{\iota k}=\frac{\operatorname{sn}(2K+3\iota K')}{-1}=\frac{\operatorname{cn}(2K+3\iota K')}{-\iota}=\frac{\operatorname{dn}(2K+3\iota K')}{\iota k}=I.$$

1355. **Formula for sin $2u$, etc. Duplication Formulae.**

Putting $u_1=u_2=u$ in the addition formulae and writing s, c, d, D respectively for $\sin u$, $\cos u$, $\operatorname{dn}u$ and $1-k^2\operatorname{sn}^4u$,

(1) $\operatorname{sn}2u=2scd/D$, (2) $\operatorname{cn}2u=(c^2-s^2d^2)/D=(1-2s^2+k^2s^4)/D$,

(3) $\operatorname{dn}2u=(d^2-k^2s^2c^2)/D=(1-2k^2s^2+k^2s^4)/D$.

Hence we deduce, writing $t\equiv\operatorname{tn}u\equiv\operatorname{sn}u/\operatorname{cn}u$,

(4) $1+\operatorname{cn}2u=2c^2/D$, (5) $1-\operatorname{cn}2u=2s^2d^2/D$,

(6) $\dfrac{1-\operatorname{cn}2u}{1+\operatorname{cn}2u}=t^2d^2$, (7) $\operatorname{cn}2u=\dfrac{1-t^2d^2}{1+t^2d^2}$,

(8) $1+\operatorname{dn}2u=2d^2/D$, (9) $1-\operatorname{dn}2u=2k^2s^2c^2/D$,

(10) $\dfrac{1-\operatorname{dn}2u}{1+\operatorname{dn}2u}=\dfrac{k^2s^2c^2}{d^2}$, (11) $\operatorname{dn}2u=\dfrac{d^2-k^2s^2c^2}{d^2+k^2s^2c^2}$,

(12) $\dfrac{1-\operatorname{dn}2u}{1+\operatorname{cn}2u}=k^2s^2$, $\therefore\ d^2=1-k^2s^2=\dfrac{\operatorname{cn}2u+\operatorname{dn}2u}{1+\operatorname{cn}2u}$,

(13) $\dfrac{1+\operatorname{cn}2u}{1+\operatorname{dn}2u}=\dfrac{c^2}{d^2}$, (14) $1-k^2\dfrac{1+\operatorname{cn}2u}{1+\operatorname{dn}2u}=1-\dfrac{k^2c^2}{d^2}=\dfrac{k'^2}{d^2}$,

i.e. $\dfrac{k'^2+\operatorname{dn}2u-k^2\operatorname{cn}2u}{1+\operatorname{dn}2u}=\dfrac{k'^2}{d^2}$,

(15) $1-k^2\dfrac{1-\operatorname{cn}2u}{1+\operatorname{dn}2u}=1-k^2s^2=d^2$, *i.e.* $\dfrac{k'^2+\operatorname{dn}2u+k^2\operatorname{cn}2u}{1+\operatorname{dn}2u}=d^2$,

(16) $\operatorname{cn}2u+\operatorname{dn}2u=2c^2d^2/D$ and $\dfrac{\operatorname{cn}2u+\operatorname{dn}2u}{1+\operatorname{dn}2u}=c^2$.

(17) From (15) and (16),

$$\frac{\operatorname{sn}^2u}{1-\operatorname{cn}2u}=\frac{\operatorname{cn}^2u}{\operatorname{cn}2u+\operatorname{dn}2u}=\frac{\operatorname{dn}^2u}{k'^2+\operatorname{dn}2u+k^2\operatorname{cn}2u}=\frac{1}{1+\operatorname{dn}2u}.$$

1356. **Dimidiation Formulae.**

By writing $\frac{u}{2}$ for u, we have

$$\operatorname{sn}^2\frac{u}{2}=\frac{1-\operatorname{cn}u}{1+\operatorname{dn}u},\quad \operatorname{cn}^2\frac{u}{2}=\frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{dn}u},\quad \operatorname{dn}^2\frac{u}{2}=\frac{k'^2+\operatorname{dn}u+k^2\operatorname{cn}u}{1+\operatorname{dn}u}.$$

1357. Again, since

$$\operatorname{dn}2u-\operatorname{cn}2u=2k'^2s^2/D,\quad 1+\operatorname{cn}2u=2c^2/D,\quad 1+\operatorname{dn}2u=2d^2/D,$$
$$k'^2+\operatorname{dn}2u-k^2\operatorname{cn}2u=2k'^2/D,$$

we have $$\frac{k'^2s^2}{\operatorname{dn}2u-\operatorname{cn}2u}=\frac{c^2}{1+\operatorname{cn}2u}=\frac{d^2}{1+\operatorname{dn}2u}=\frac{k'^2}{k'^2+\operatorname{dn}2u-k^2\operatorname{cn}2u};$$

and putting $\frac{u}{2}$ for u, we obtain further formulae for $\operatorname{sn}\frac{u}{2}$, $\operatorname{cn}\frac{u}{2}$, $\operatorname{dn}\frac{u}{2}$, viz.

$$\operatorname{sn}^2\frac{u}{2}=\frac{\operatorname{dn}u-\operatorname{cn}u}{k'^2+\operatorname{dn}u-k^2\operatorname{cn}u},\quad \operatorname{cn}^2\frac{u}{2}=\frac{k'^2(1+\operatorname{cn}u)}{k'^2+\operatorname{dn}u-k^2\operatorname{cn}u},\quad \operatorname{dn}^2\frac{u}{2}=\frac{k'^2(1+\operatorname{dn}u)}{k'^2+\operatorname{dn}u-k^2\operatorname{cn}u}.$$

1358. **Triplication Formulae.**

Writing $u_1=u$, $u_2=2u$ in the addition formula for $\operatorname{sn}(u_1+u_2)$,

$$\operatorname{sn}3u=(\operatorname{sn}u\operatorname{cn}2u\operatorname{dn}2u+\operatorname{sn}2u\operatorname{cn}u\operatorname{dn}u)(1-k^2\operatorname{sn}^2u\operatorname{sn}^22u),$$

and substituting for $\operatorname{sn}2u$, $\operatorname{cn}2u$, $\operatorname{dn}2u$ their values from (1), (2), (3) of Art. 1355, we obtain, after a little reduction,

$$\operatorname{sn}3u/\operatorname{sn}u=\{3-4(1+k^2)s^2+6k^2s^4-k^4s^8\}/D',$$

and similarly $$\operatorname{cn}3u/\operatorname{cn}u=\{1-4s^2+6k^2s^4-4k^4s^6+k^4s^8\}/D',$$

$$\operatorname{dn}3u/\operatorname{dn}u=\{1-4k^2s^2+6k^2s^4-4k^2s^6+k^4s^8\}/D',$$

where $$D'=1-6k^2s^4+4k^2(1+k^2)s^6-3k^4s^8.$$

Cayley gives also the following results, which may be verified without difficulty:

$$\frac{1-\operatorname{sn}3u}{1+\operatorname{sn}u}\cdot D'=(1-2s+2k^2s^3-k^2s^4)^2;\quad \frac{1+\operatorname{sn}3u}{1-\operatorname{sn}u}\cdot D'=(1+2s-2k^2s^3-k^2s^4)^2;$$

$$\frac{1-k\operatorname{sn}3u}{1+k\operatorname{sn}u}\cdot D'=(1-2ks+2ks^3-k^2s^4)^2;\quad \frac{1+k\operatorname{sn}3u}{1-k\operatorname{sn}u}\cdot D'=(1+2ks-2ks^3-k^2s^4)^2.$$

The formulae for $\operatorname{sn}\lambda u$, $\operatorname{cn}\lambda u$, $\operatorname{dn}\lambda u$ for the cases $\lambda=4$, 5, 6 and 7 are also given by Cayley (*Ell. F.*, pp 78 and 81 onwards), but these formulae rapidly become more and more complicated. According to Cayley the cases $\lambda=6$ and $\lambda=7$ are due to Baehr (*Grunert's Archiv*, xxxvi. pp. 125 to 176).

1359. **Dimidiation Formulae for the Periods.**

$$\operatorname{sn}^2\frac{u}{2}=\frac{1-\operatorname{cn}u}{1+\operatorname{dn}u},\quad \operatorname{cn}^2\frac{u}{2}=\frac{\operatorname{cn}u+\operatorname{dn}u}{1+\operatorname{dn}u},\quad \operatorname{dn}^2\frac{u}{2}=\frac{k'^2+\operatorname{dn}u+k^2\operatorname{cn}u}{1+\operatorname{dn}u},$$

give many results for the functions of $u+p\frac{K}{2}+q\frac{\iota K'}{2}$, p and q being integers.

Putting $u=0$ in the formulae of the table, and therefore $s=0$, $c=1$, $d=1$,

$$\operatorname{sn}\frac{K}{2}=\sqrt{\frac{1-\operatorname{cn}K}{1+\operatorname{dn}K}}=\frac{1}{\sqrt{1+k'}};\quad \operatorname{cn}\frac{K}{2}=\sqrt{\frac{\operatorname{cn}K+\operatorname{dn}K}{1+\operatorname{dn}K}}=\frac{\sqrt{k'}}{\sqrt{1+k'}};$$

$$\operatorname{dn}\frac{K}{2}=\sqrt{\frac{k'^2+\operatorname{dn}K+k^2\operatorname{cn}K}{1+\operatorname{dn}K}}=\sqrt{k'}$$

$$\left.\begin{aligned}
\operatorname{sn}\frac{\iota K'}{2}&=\sqrt{\frac{1-\operatorname{cn}\iota K'}{1+\operatorname{dn}\iota K'}}=\sqrt{\frac{1+\iota I}{1-\iota kI}}\,(I=\infty)=\sqrt{-\frac{1}{k}}=\frac{\iota}{\sqrt{k}};\\
\operatorname{cn}\frac{\iota K'}{2}&=\sqrt{\frac{\operatorname{cn}\iota K'+\operatorname{dn}\iota K'}{1+\operatorname{dn}\iota K'}}=\sqrt{\frac{-\iota I-\iota kI}{1-\iota kI}}\,(I=\infty)=\frac{\sqrt{1+k}}{\sqrt{k}};\\
\operatorname{dn}\frac{\iota K'}{2}&=\sqrt{\frac{k'^2+\operatorname{dn}\iota K'+k^2\operatorname{cn}\iota K'}{1+\operatorname{dn}\iota K'}}=\sqrt{\frac{k'^2-\iota kI-k^2\iota I}{1-\iota kI}}\,(I=\infty)=\sqrt{1+k}.
\end{aligned}\right\}$$

$$\operatorname{sn}\frac{K+\iota K'}{2}=\sqrt{\frac{1-\operatorname{cn}(K+\iota K')}{1+\operatorname{dn}(K+\iota K')}}=\frac{\sqrt{k+\iota k'}}{\sqrt{k}}=\frac{1}{\sqrt{2k}}(\sqrt{1+k}+\iota\sqrt{1-k});$$

$$\begin{aligned}\operatorname{cn}\frac{K+\iota K'}{2}&=\sqrt{\frac{\operatorname{cn}(K+\iota K')+\operatorname{dn}(K+\iota K')}{1+\operatorname{dn}(K+\iota K')}}\\
&=\sqrt{-\iota\frac{k'}{k}}=\sqrt{\frac{k'}{k}}\left(\cos\frac{3\pi}{2}+\iota\sin\frac{3\pi}{2}\right)^{\frac{1}{2}}=\sqrt{\frac{k'}{2k}}(-1+\iota);\end{aligned}$$

$$\begin{aligned}\operatorname{dn}\frac{K+\iota K'}{2}&=\sqrt{\frac{k'^2+\operatorname{dn}(K+\iota K')+k^2\operatorname{cn}(K+\iota K')}{1+\operatorname{dn}(K+\iota K')}}\\
&=\sqrt{k'}\sqrt{k'-\iota k}=\sqrt{\frac{k'}{2}}\,[\sqrt{1+k'}-\iota\sqrt{1-k'}].\end{aligned}$$

The reader will find no difficulty in completing for himself and tabulating the various results for the cases $p=0, 1, 2, 3$; $q=0, 1, 2, 3$. Such a table is given by Cayley (*E.F.*, p. 74).

1360. We now have

$$\operatorname{sn}\left(u+\frac{K}{2}\right)=\frac{s\sqrt{\dfrac{k'}{1+k'}}\cdot\sqrt{k'}+\dfrac{1}{\sqrt{1+k'}}cd}{1-k^2s^2\dfrac{1}{1+k'}}=\frac{1}{\sqrt{1+k'}}\,\frac{k's+cd}{c^2+k's^2};$$

$$\operatorname{cn}\left(u+\frac{K}{2}\right)=\frac{c\sqrt{\dfrac{k'}{1+k'}}-sd\dfrac{1}{\sqrt{1+k'}}\sqrt{k'}}{1-k^2s^2\dfrac{1}{1+k'}}=\sqrt{\frac{k'}{1+k'}}\,\frac{c-sd}{c^2+k's^2};$$

$$\operatorname{dn}\left(u+\frac{K}{2}\right)=\frac{d\sqrt{k'}-k^2s\dfrac{1}{\sqrt{1+k'}}c\dfrac{\sqrt{k'}}{\sqrt{1+k'}}}{1-k^2s^2\dfrac{1}{1+k'}}=\sqrt{k'}\,\frac{d-(1-k')sc}{c^2+k's^2},$$

with many similar results, and such results may be thrown into other forms. For example, we may show that

$$\operatorname{sn}\left(u+\frac{K}{2}\right)=\frac{1}{\sqrt{1+k'}}\,\frac{d+sc(1+k')}{c+sd},\quad \operatorname{cn}\left(u+\frac{K}{2}\right)=\sqrt{\frac{k'}{1+k'}}\,\frac{c^2-k's^2}{c+sd}.$$

1361. Other formulae may be obtained by direct application of the dimidiary formulae to the results for $2u+pK+q\iota K'$, *e.g.*

$$\operatorname{sn}(2u+K)=\frac{\operatorname{cn}2u}{\operatorname{dn}2u},\quad \operatorname{cn}(2u+K)=-k'\frac{\operatorname{sn}2u}{\operatorname{dn}2u},\quad \operatorname{dn}(2u+K)=\frac{k'}{\operatorname{dn}2u};$$

whence $$\operatorname{sn}^2\left(u+\frac{K}{2}\right)=\frac{1-\operatorname{cn}(2u+K)}{1+\operatorname{dn}(2u+K)}=\frac{\operatorname{dn}2u+k'\operatorname{sn}2u}{\operatorname{dn}2u+k'},\text{ etc.},$$

and many other formulae are similarly obtainable.

1362. **A General Proposition.**

Let U be a function of three variables ϕ_1, ϕ_2, ϕ_3, between which there is a connecting relation, viz.

$$d\phi_1/\Delta\phi_1+d\phi_2/\Delta\phi_2+d\phi_3/\Delta\phi_3=0,$$

and suppose the function U to be such that when any one of the three, say ϕ_3, is regarded as a constant, then U vanishes in one of the two cases ($\phi_1=\phi_3$, $\phi_2=0$) or ($\phi_2=\phi_3$, $\phi_1=0$), and provided also that $\dfrac{\partial U}{\partial\phi_1}\Delta\phi_1=\dfrac{\partial U}{\partial\phi_2}\Delta\phi_2$, then U must be zero always.

For if $\phi_3=$const., $d\phi_3=0$ and $d\phi_1/\Delta\phi_1+d\phi_2/\Delta\phi_2=0$, *i.e.* $d\phi_1/\Delta\phi_1=-d\phi_2/\Delta\phi_2=\lambda$, say, and this would have been equally true if the connecting equation were

$$d\phi_1/\Delta\phi_1+d\phi_2/\Delta\phi_2-d\phi_3/\Delta\phi_3=0.$$

But

$$dU=\frac{\partial U}{\partial\phi_1}d\phi_1+\frac{\partial U}{\partial\phi_2}d\phi_2+\frac{\partial U}{\partial\phi_3}d\phi_3=\lambda\left[\frac{\partial U}{\partial\phi_1}\Delta\phi_1-\frac{\partial U}{\partial\phi_2}\Delta\phi_2\right]=0;$$

$\therefore$ $U=$const.$=C$, say. But in the case ($\phi_1=\phi_3$, $\phi_2=0$), $U=0$; $\therefore$ $C=0$. Therefore U vanishes.

1363. Case I. Let

$$u_1=\int_0^{\phi_1}\frac{d\theta}{\Delta\theta},\quad u_2=\int_0^{\phi_2}\frac{d\theta}{\Delta\theta},\quad u_3=\int_0^{\phi_3}\frac{d\theta}{\Delta\theta}\quad\text{and}\quad U\equiv u_1+u_2-u_3.$$

Then $$\frac{\partial U}{\partial u_1}=1,\quad \frac{\partial U}{\partial u_2}=1,\quad \frac{\partial u_1}{\partial\phi_1}=\frac{1}{\Delta\phi_1},\quad \frac{\partial u_2}{\partial\phi_2}=\frac{1}{\Delta\phi_2},$$

and $$\frac{\partial U}{\partial\phi_1}\Delta\phi_1-\frac{\partial U}{\partial\phi_2}\Delta\phi_2=1-1=0.$$

Also, if $\phi_1=\phi_3$ and $\phi_2=0$, we have $u_1=u_3$ and $u_2=0$, *i.e.* $u_1+u_2-u_3=0$. Hence the conditions of the general theorem are satisfied, and $u_1+u_2-u_3=0$ always, *i.e.* according to

Legendre's notation $F\phi_1+F\phi_2=F\phi_3$, which is the addition formula for the first Legendrian Integral.

That is, $\quad F(\operatorname{am} u_1)+F(\operatorname{am} u_2)=F(\operatorname{am} u_3).$

Another mode of treatment (Art. 1342) of the equation $d\phi_1/\Delta\phi_1+d\phi_2/\Delta\phi_2=0$ led to the result that

$$\frac{\operatorname{sn} u_1 \operatorname{cn} u_2 \operatorname{dn} u_2+\operatorname{sn} u_2 \operatorname{cn} u_1 \operatorname{dn} u_1}{1-k^2 \operatorname{sn}^2 u_1 \operatorname{sn}^2 u_2}=\text{const.}$$

when $\phi_3=$const., so that $u_3=$const.; and as $(u_1=u_3,\ u_2=0)$ satisfies this, the constant is $\operatorname{sn} u_3$, so that

$$u_1+u_2=\operatorname{sn}^{-1}\frac{s_1c_2d_2+s_2c_1d_1}{1-k^2s_1{}^2s_2{}^2}, \text{ as before.}$$

1364. Case II. With the same definition of u_1, u_2, u_3, and taking

$$v_1=\int_0^{\phi_1}\Delta\theta\,d\theta,\quad v_2=\int_0^{\phi_2}\Delta\theta\,d\theta,\quad v_3=\int_0^{\phi_3}\Delta\theta\,d\theta,$$

and $U\equiv v_1+v_2-v_3-k^2\sin\phi_1\sin\phi_2\sin\phi_3$, then, proceeding as before,

$$\begin{aligned}\frac{\partial U}{\partial\phi_1}\Delta\phi_1-\frac{\partial U}{\partial\phi_2}\Delta\phi_2&=\Delta\phi_1[\Delta\phi_1-k^2\cos\phi_1\sin\phi_2\sin\phi_3]\\&\qquad-\Delta\phi_2[\Delta\phi_2-k^2\cos\phi_2\sin\phi_1\sin\phi_3]\\&=(\Delta\phi_1)^2-(\Delta\phi_2)^2-k^2\sin\phi_3[\Delta\phi_1\cos\phi_1\sin\phi_2-\Delta\phi_2\cos\phi_2\sin\phi_1]\\&=(1-k^2s_1{}^2)-(1-k^2s_2{}^2)-k^2\frac{s_1c_2d_2+s_2c_1d_1}{1-k^2s_1{}^2s_2{}^2}(s_2c_1d_1-s_1c_2d_2)\\&=k^2[(s_2{}^2-s_1{}^2)(1-k^2s_1{}^2s_2{}^2)+s_1{}^2(1-s_2{}^2)(1-k^2s_2{}^2)\\&\qquad-s_2{}^2(1-s_1{}^2)(1-k^2s_1{}^2)]/(1-k^2s_1{}^2s_2{}^2)\\&=0.\end{aligned}$$

Also, if $\phi_2=0$, $v_2=0$ and if $\phi_1=\phi_3$, $v_1=v_3$, and $\therefore U=0$ in this case; $\therefore U=0$ always, and

$$\therefore\ v_1+v_2-v_3=k^2\sin\phi_1\sin\phi_2\sin\phi_3;$$

and writing $v_1=E\phi_1$, $v_2=E\phi_2$, $v_3=E\phi_3$, viz. the Legendrian notation,

$$E\phi_1+E\phi_2-E\phi_3=k^2\sin\phi_1\sin\phi_2\sin\phi_3;$$

and since $\phi_1=\operatorname{am} u_1$, $\phi_2=\operatorname{am} u_2$, $\phi_3=\operatorname{am} u_3=\operatorname{am}(u_1+u_2)$, we have

$$E\operatorname{am} u_1+E\operatorname{am} u_2-E\operatorname{am}(u_1+u_2)=k^2\operatorname{sn} u_1\operatorname{sn} u_2\operatorname{sn}(u_1+u_2),$$

which constitutes the addition formula for the second class of Legendrian Elliptic Integrals.

1365. Case III. Let

$$w_1=\int_0^{\phi_1}\frac{d\theta}{(1+n\sin^2\theta)\Delta\theta},\quad w_2=\text{etc.},\quad w_3=\text{etc.},$$

where $\phi_1=\operatorname{am}u_1$, etc. Then, putting

$$U=w_1+w_2-w_3-\int\frac{dR}{1+aR^2},\quad an=(n+1)(n+k^2),$$

$$R=\frac{n\sin\phi_1\sin\phi_2\sin\phi_3}{1+n-n\cos\phi_1\cos\phi_2\cos\phi_3},$$

we may verify as before by the general theorem that $U=0$, *i.e.*

$$\Pi\phi_1+\Pi\phi_2-\Pi\phi_3=\frac{1}{\sqrt{a}}\tan^{-1}R\sqrt{a}\quad\text{or}\quad\frac{1}{\sqrt{-a}}\tanh^{-1}R\sqrt{-a},$$

which is the addition formula for a Legendrian Integral of the third class (see Cayley, *E.F.*, pp. 104 to 106).

The work of this verification is necessarily somewhat cumbrous, and it is found best to proceed to discuss the Third Legendrian Integral $\Pi(\theta, n, k)$ after a modification of its form. Taking $\theta=\operatorname{am}u$ as before, $\frac{du}{d\theta}=\frac{1}{\Delta\theta}=\frac{1}{\operatorname{dn}u}$. Let $n=-k^2\operatorname{sn}^2a$, a being not necessarily real; then the transformed integral is

$$\Pi(\theta, n, k)=\int_0^u\frac{du}{1-k^2\operatorname{sn}^2a\operatorname{sn}^2u}.$$

But instead of considering the original function $\Pi(\theta, n, k)$, it is convenient to consider a somewhat different form $\Pi(u, a)$, defined as $\equiv\int_0^u\frac{k^2\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a\operatorname{sn}^2u\,du}{1-k^2\operatorname{sn}^2a\operatorname{sn}^2u}$.

The connexion between $\Pi(u, a)$ and $\Pi(\theta, n, k)$ is then

$$\Pi(u, a)=k^2\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a\int_0^\theta\frac{\sin^2\theta\,d\theta}{(1+n\sin^2\theta)\Delta\theta}$$

$$=\frac{k^2}{n}\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a\int_0^\theta\frac{(1+n\sin^2\theta)-1}{(1+n\sin^2\theta)\Delta\theta}\,d\theta$$

$$=\frac{k^2}{n}\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a\{F(\theta, k)-\Pi(\theta, n, k)\},$$

and the new function is proportional to the difference of the first and third Legendrian forms.

1366. Jacobian Zeta, Eta, Theta Functions. Introductory.

These functions, denoted respectively by $Z(u)$, $H(u)$, $\Theta(u)$, are defined as

$$Z(u)\equiv\int_0^u\left(\mathrm{dn}^2u-\frac{E_1}{K}\right)du,\quad \frac{\Theta'(u)}{\Theta(u)}\equiv Z(u),\quad \frac{H(u)}{\Theta(u)}=\sqrt{k}\,\mathrm{sn}\,u$$

with a constant of integration in the second case, such that $\Theta(0)=\sqrt{\frac{2k'K}{\pi}}$, and k being the modulus in each case. Also E_1 in the first of these Jacobian Elliptic Functions is the complete Legendrian Integral of the second kind with limits 0 and $\pi/2$ (Art. 375).

1367. Obvious Elementary Properties.

Clearly $\quad Z(0)=0$ and $Z(-u)=-Z(u)$.

Also $\quad Z(u)+\frac{E_1}{K}u=\int_0^u \mathrm{dn}^2u\,du=\int_0^\theta \Delta\theta\,d\theta=E(\theta)=E(\mathrm{am}\,u)$

in the Legendrian notation, *i.e.* $Z(u)=E(\mathrm{am}\,u)-\frac{E_1}{K}u$ in that notation.

Again

$$\Theta(u)=\sqrt{\frac{2k'K}{\pi}}\,e^{\int_0^u Z(u)\,du}\quad\text{and}\quad H(u)=\sqrt{\frac{2kk'K}{\pi}}\,\mathrm{sn}\,u\,e^{\int_0^u Z(u)\,du}$$

Also $\quad \Theta(-u)=\sqrt{\frac{2k'K}{\pi}}\,e^{\int_0^{-u}Z(t)dt}=\sqrt{\frac{2k'K}{\pi}}\,e^{-\int_0^u Z(-w)dw}\ (t=-w)$

$$=\sqrt{\frac{2k'K}{\pi}}\,e^{\int_0^u Z(w)dw}=\Theta(u),$$

$$H(-u)=\sqrt{k}\,\mathrm{sn}(-u)\Theta(-u)=-\sqrt{k}\,\mathrm{sn}\,u\,\Theta(u)=-H(u).$$

Also $\quad H(0)=0$ and $Lt_{u=0}\frac{H(u)}{u}=\sqrt{\frac{2kk'K}{\pi}}$.

Thus $Z(u)$ and $H(u)$ are odd functions of u, and $\Theta(u)$ is an even function of u.

1368. Properties of the Second Legendrian Integral.

(i) $\quad E(-\phi)=\int_0^{-\phi}\Delta\theta\,d\theta=-\int_0^\phi\Delta\chi\,d\chi,\ (\theta=-\chi),\ =-E(\phi).$

(ii) $\quad E(\pi\pm\phi)=\int_0^{\pi\pm\phi}\Delta\theta\,d\theta=\left(\int_0^\pi+\int_\pi^{\pi\pm\phi}\right)\Delta\theta\,d\theta$

$$=\left(\int_0^\pi+\int_0^{\pm\phi}\right)\Delta\chi\,d\chi,\ (\theta=\pi+\chi\text{ in second}),\ =2E_1\pm E\phi.$$

(iii) $E(2\pi\pm\phi)=2E_1+E(\pi\pm\phi)=4E_1\pm E(\phi)$, and generally $E(n\pi\pm\phi)=2nE_1\pm E(\phi)$, *i.e.* $E(n\pi\pm\text{am}\,u)=2nE_1\pm E(\text{am}\,u)$.

(iv) Again, with $u=\int_0^\theta \frac{d\phi}{\Delta\phi}$, $v=\int_0^\theta \Delta\phi\,d\phi$,

$$\theta=\text{am}\,u,\quad v=E(\text{am}\,u),$$

and if $\theta=0$, $u=0$ and $v=0$, *i.e.* $E(\text{am}\,0)=0$; whilst if $\theta=\frac{\pi}{2}$,

$$u=F_1\equiv K,\quad v=E_1,\quad i.e.\ E(\text{am}\,K)=E_1.$$

(v) Moreover $E(\text{am}\,u)+E(\text{am}\,K)-E\,\text{am}(u+K)$

$$=k^2\text{sn}\,u\sin\frac{\pi}{2}\,\text{sn}(u+K)=k^2\frac{\text{sn}\,u\,\text{cn}\,u}{\text{dn}\,u};$$

$$\therefore\ E\,\text{am}(u+K)=\ E(\text{am}\,u)+E_1-k^2\frac{\text{sn}\,u\,\text{cn}\,u}{\text{dn}\,u}.$$

Also $-E\,\text{am}(u-K)=-E(\text{am}\,u)+E_1+k^2\dfrac{\text{sn}\,u\,\text{cn}\,u}{\text{dn}\,u}.$

1369. Addition Formula for the Zeta Function, etc.

The formulae for $\text{dn}(u+v)$, $\text{dn}(u-v)$ of Art. 1347 give

$$\text{dn}^2(u+v)-\text{dn}^2(u-v)=-4k^2\frac{\text{sn}\,u\,\text{cn}\,u\,\text{dn}\,u\,\text{sn}\,v\,\text{cn}\,v\,\text{dn}\,v}{(1-k^2\text{sn}^2u\,\text{sn}^2v)^2};$$

and integrating with regard to v from $v=a$ to $v=u$,

$$\left[Z(u+v)+\frac{E_1}{K}(u+v)\right]_{v=a}^{v=u}+\left[Z(u-v)+\frac{E_1}{K}(u-v)\right]_{v=a}^{v=u}$$
$$=-\frac{2}{\text{sn}^2u}\left[\frac{\text{sn}\,u\,\text{cn}\,u\,\text{dn}\,u}{1-k^2\text{sn}^2u\,\text{sn}^2v}\right]_{v=a}^{v=u},$$

i.e.
$$\left\{Z(2u)+\frac{E_1}{K}2u-Z(u+a)-\frac{E_1}{K}(u+a)\right\}$$
$$+\left\{Z(0)+\frac{E_1}{K}.0-Z(u-a)-\frac{E_1}{K}(u-a)\right\}$$
$$=-\frac{2\,\text{sn}\,u\,\text{cn}\,u\,\text{dn}\,u}{\text{sn}^2u}\left(\frac{1}{1-k^2\text{sn}^4u}-\frac{1}{1-k^2\text{sn}^2a\,\text{sn}^2u}\right)$$
$$=-2k^2\frac{\text{sn}\,u\,\text{cn}\,u\,\text{dn}\,u}{1-k^2\text{sn}^2u}\cdot\frac{\text{sn}^2u-\text{sn}^2a}{1-k^2\text{sn}^2a\,\text{sn}^2u}$$
$$=-k^2\text{sn}\,2u\,\text{sn}(u+a)\,\text{sn}(u-a)\quad\text{(Arts. 1351 and 1355)};$$

$$\therefore\ Z(u+a)+Z(u-a)-Z(2u)=k^2\text{sn}\,2u\,\text{sn}(u+a)\,\text{sn}(u-a).\quad\text{(I)}$$

Putting $\alpha=0$, we have

$$Z(2u)-2Z(u)=-k^2\operatorname{sn}2u\operatorname{sn}^2u. \quad \text{..............(II)}$$

Adding

$$Z(u+\alpha)+Z(u-\alpha)-2Z(u)=k^2\operatorname{sn}2u\left\{\frac{\operatorname{sn}^2u-\operatorname{sn}^2\alpha}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2\alpha}-\operatorname{sn}^2u\right\}$$

$$=-k^2\operatorname{sn}2u\frac{\operatorname{sn}^2\alpha(1-k^2\operatorname{sn}^4u)}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2\alpha},$$

i.e. $$Z(u+\alpha)+Z(u-\alpha)-2Z(u)=-2k^2\frac{\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u\operatorname{sn}^2\alpha}{1-k^2\operatorname{sn}^2\alpha\operatorname{sn}^2u}; \quad \text{(III)}$$

and writing $u+\alpha=u_1$, $u-\alpha=u_2$, $2u=u_1+u_2$, Eq. (I) becomes

$$Z(u_1+u_2)=Z(u_1)+Z(u_2)-k^2\operatorname{sn}u_1\operatorname{sn}u_2\operatorname{sn}(u_1+u_2), \quad \text{(IV)}$$

which constitutes an addition formula for the Zeta Function.

1370. Substituting for $Z(u)$ its value $E(\operatorname{am}u)-\frac{E_1}{K}u$, we have

$$E(\operatorname{am}u_1)+E(\operatorname{am}u_2)-E(\operatorname{am}\overline{u_1+u_2})=k^2\operatorname{sn}u_1\operatorname{sn}u_2\operatorname{sn}(u_1+u_2),$$

viz. the addition formula of the Second Legendrian Integral.

If in (IV) we write $u_1+u_2+u_3=0$, we have the symmetrical form

$$Z(u_1)+Z(u_2)+Z(u_3)=-k^2\operatorname{sn}u_1\operatorname{sn}u_2\operatorname{sn}u_3.$$

1371. From (III), we have at once

$$\frac{\Theta'(u+\alpha)}{\Theta(u+\alpha)}+\frac{\Theta'(u-\alpha)}{\Theta(u-\alpha)}-2\frac{\Theta'(u)}{\Theta(u)}=-2k^2\frac{\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u\operatorname{sn}^2\alpha}{1-k^2\operatorname{sn}^2u\operatorname{sn}^2\alpha},$$

i.e. $$\left[\log\frac{\Theta(u+\alpha)\,\Theta(u-\alpha)}{\Theta^2(u)}\right]_{u=0}^{u=u}=\left[\log(1-k^2\operatorname{sn}^2\alpha\operatorname{sn}^2u)\right]_0^u,$$

i.e. $$\frac{\Theta(u+\alpha)\,\Theta(u-\alpha)}{\Theta^2(\alpha)\,\Theta^2(u)}\Theta^2(0)=1-k^2\sin^2\alpha\operatorname{sn}^2u. \quad \text{......(V)}$$

1372. If we integrate with regard to α, instead of with regard to u, from 0 to α,

$$\log\frac{\Theta(\alpha+u)}{\Theta(\alpha-u)}-2\alpha Z(u)=-2\Pi(\alpha,u), \quad \text{............(VI)}$$

and interchanging u and α,

$$\log\frac{\Theta(u+\alpha)}{\Theta(u-\alpha)}-2uZ(\alpha)=-2\Pi(u,\alpha), \quad \text{.........(VII)}$$

i.e. $$\Pi(u,\alpha)=\log e^{uZ(\alpha)}\left\{\frac{\Theta(u-\alpha)}{\Theta(u+\alpha)}\right\}^{\frac{1}{2}},$$

which expresses the Legendrian Integral of the Third Kind in terms of the Jacobian Zeta and Theta functions.

There are in this form two arguments only, viz. u and a, instead of the three, θ, k, u, in the Legendrian form (see Greenhill, *E.F.*, p. 192).

1373. From (VI) and (VII),

$$\Pi(u, a)-\Pi(a, u)=uZ(a)-aZ(u). \quad \text{.........(VIII)}$$

Since
$$\Pi(u_1, a)=u_1Z(a)+\tfrac{1}{2}\log\frac{\Theta(u_1-a)}{\Theta(u_1+a)},$$

$$\Pi(u_2, a)=u_2Z(a)+\tfrac{1}{2}\log\frac{\Theta(u_2-a)}{\Theta(u_2+a)},$$

and
$$\Pi(u_1+u_2, a)=(u_1+u_2)\,Z(a)+\tfrac{1}{2}\log\frac{\Theta(u_1+u_2-a)}{\Theta(u_1+u_2+a)},$$

we have
$$\left.\begin{aligned}&\Pi(u_1, a)+\Pi(u_2, a)-\Pi(u_1+u_2, a)=\tfrac{1}{2}\log\Omega,\\ &\text{where}\quad \Omega=\frac{\Theta(u_1-a)\,\Theta(u_2-a)\,\Theta(u_1+u_2+a)}{\Theta(u_1+a)\,\Theta(u_2+a)\,\Theta(u_1+u_2-a)},\end{aligned}\right\}\quad\text{(IX)}$$

which is a form of the addition formula for the Third Legendrian Integral. Various forms of the function Ω will be found in Cayley, *E.F.*, pages 157, etc., and *The Messenger of Math.*, vol. x. (Glaisher).

1374. In this brief notice of these important functions, we have in the main followed the course suggested by Dr. Glaisher in his note in the *Proceedings of the Lond. Math. Soc.*, vol. xvii.

1375. Integration of Expressions involving the Jacobian Functions.

[We shall write s, c, d for $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ respectively when desirable for abridgment.]

(1) $$\int \operatorname{sn} u\,du = -\int\frac{d\operatorname{cn} u}{\sqrt{1-k^2\operatorname{sn}^2 u}}=-\int\frac{d\operatorname{cn} u}{\sqrt{k'^2+k^2\operatorname{cn}^2 u}}=-\frac{1}{k}\sinh^{-1}\frac{k\operatorname{cn} u}{k'}$$
$$=-\frac{1}{k}\log\frac{\operatorname{dn} u+k\operatorname{cn} u}{k'}=\frac{1}{k}\log\sqrt{\frac{d-kc}{d+kc}},\ \text{or other forms.}$$

(2) $$\int \operatorname{cn} u\,du = \int\frac{d\operatorname{sn} u}{\sqrt{1-k^2\operatorname{sn}^2 u}}=\frac{1}{k}\sin^{-1}(k\operatorname{sn} u)=\frac{1}{k}\cos^{-1}(\operatorname{dn} u),\ \text{or other forms.}$$

(3) $$\int \operatorname{dn} u\,du = \int d\theta=\theta=\operatorname{am} u.$$

(4) $$\int \operatorname{sn}^2 u\,du = \frac{1}{k^2}\int(1-\operatorname{dn}^2 u)\,du=\frac{1}{k^2}(u-E\operatorname{am} u)=\frac{1}{k^2}\left\{u-\left(Zu+\frac{E_1}{K}\,u\right)\right\}.$$

(5) $\int \mathrm{cn}^2 u\, du = \frac{1}{k^2}\int (\mathrm{dn}^2 u - k'^2)\, du = \frac{1}{k^2}(E \,\mathrm{am}\, u - k'^2 u) = \frac{1}{k^2}\left\{\left(Zu + \frac{E_1}{K}u\right) - k'^2 u\right\}.$

(6) $\int \mathrm{dn}^2 u\, du = E\,\mathrm{am}\, u = Zu + \frac{E_1}{K}u.$

(7) $\int \mathrm{sn}^3 u\, du = -\int \mathrm{sn}^2 u \frac{d(\mathrm{cn}\, u)}{\mathrm{dn}\, u} = -\int \frac{(1-c^2)\,dc}{\sqrt{k'^2 + k^2c^2}}$

$$= -\frac{1}{k^2}\int \frac{dc}{\sqrt{k'^2+k^2c^2}} + \frac{1}{k^2}\int \sqrt{k'^2+k^2c^2}\,dc = -\frac{1+k^2}{2k^3}\sinh^{-1}\frac{kc}{k'} + \frac{1}{2k^2}cd.$$

(8) $\int \mathrm{cn}^3 u\, du = \int \frac{(1-s^2)\,ds}{\sqrt{1-k^2s^2}} = \frac{1}{k^2}\int\left(\sqrt{1-k^2s^2} - \frac{k'^2}{\sqrt{1-k^2s^2}}\right)ds$

$$= \frac{1}{2k^2}sd + \frac{2k^2-1}{2k^3}\sin^{-1}(ks).$$

(9) $\int \mathrm{dn}^3 u\, du = \int (1-k^2\sin^2\theta)\,d\theta = \frac{2-k^2}{2}\theta + \frac{k^2}{4}\sin 2\theta = \frac{2-k^2}{2}\,\mathrm{am}\, u + \frac{k^2}{2}\,\mathrm{sn}\, u\,\mathrm{cn}\, u,$

etc.

(10) $\int \frac{du}{\mathrm{sn}\, u} = -\int \frac{dc}{(1-c^2)\sqrt{k'^2+k^2c^2}}$, which suggests putting $y = \frac{d}{s}$; whence

$$dy = -\frac{c}{s^2}\,du,\ s^2 = 1/(k^2+y^2),\ c^2 = (y^2-k'^2)/(k^2+y^2);$$

$$\therefore \int \frac{du}{\mathrm{sn}\, u} = -\int \frac{s}{c}\,dy = -\int \frac{dy}{\sqrt{y^2-k'^2}} = -\cosh^{-1}\left(\frac{y}{k'}\right) = -\cosh^{-1}\left(\frac{\mathrm{dn}\, u}{k'\,\mathrm{sn}\, u}\right).$$

(11) $\int \frac{du}{\mathrm{cn}\, u}$. Putting $y = \frac{d}{c}$, $dy = k'^2\frac{s}{c^2}du$, $s^2 = \frac{y^2-1}{y^2-k^2}$, $c^2 = \frac{k'^2}{y^2-k^2}$;

$$\therefore \int \frac{du}{\mathrm{cn}\, u} = \frac{1}{k'^2}\int \frac{k'}{\sqrt{y^2-1}}\,dy = \frac{1}{k'}\cosh^{-1} y = \frac{1}{k'}\cosh^{-1}\left(\frac{\mathrm{dn}\, u}{\mathrm{cn}\, u}\right).$$

(12) $\int \frac{du}{\mathrm{dn}\, u} = \int \frac{d\theta}{1-k^2\sin^2\theta} = \int \frac{\mathrm{cosec}^2\theta\, d\theta}{\cot^2\theta + k'^2} = -\frac{1}{k'}\cot^{-1}\frac{\cot\theta}{k'} = -\frac{1}{k'}\cot^{-1}\frac{\mathrm{ctn}\, u}{k'}.$

1376. Again $\frac{d^2}{du^2}\log \mathrm{sn}\, u = \frac{d}{du}\frac{cd}{s} = -d^2 - k^2c^2 - \frac{c^2d^2}{s^2} = k^2s^2 - \frac{1}{s^2},$

$$\frac{d^2}{du^2}\log \mathrm{cn}\, u = -\frac{d}{du}\frac{sd}{c} = -d^2 + k^2s^2 - \frac{s^2d^2}{c^2} = -k^2c^2 - \frac{k'^2}{c^2},$$

$$\frac{d^2}{du^2}\log \mathrm{dn}\, u = -k^2\frac{d}{du}\frac{sc}{d} = -k^2c^2 + k^2s^2 - k^4\frac{s^2c^2}{d^2} = \frac{k'^2}{d^2} - d^2.$$

Hence (13) $\int \frac{du}{\mathrm{sn}^2 u} = -\frac{\mathrm{cn}\, u\,\mathrm{dn}\, u}{\mathrm{sn}\, u} + u - E\,\mathrm{am}\, u.$

(14) $\int \frac{du}{\mathrm{cn}^2 u} = \frac{1}{k'^2}\frac{\mathrm{sn}\, u\,\mathrm{dn}\, u}{\mathrm{cn}\, u} - \frac{1}{k'^2}(E\,\mathrm{am}\, u - k'^2 u).$

(15) $\int \frac{du}{\mathrm{dn}^2 u} = -\frac{k^2}{k'^2}\frac{\mathrm{sn}\, u\,\mathrm{cn}\, u}{\mathrm{dn}\, u} + \frac{1}{k'^2}E\,\mathrm{am}\, u.$

1377. Other positive or negative integral powers of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ may be integrated with regard to u by the reduction formulae of Examples 24, 25, 26 at the end of the chapter, which can be verified at once by putting respectively $P=s^{n-1}cd$, $c^{n-1}sd$, $d^{n-1}sc$ and differentiating.

1378. Again, by aid of the Period formulae of Art. 1352, viz.

$$\frac{c}{d}=\operatorname{sn}(u+K),\qquad \frac{s}{d}=-\frac{1}{k'}\operatorname{cn}(u+K),\qquad \frac{1}{d}=\frac{1}{k'}\operatorname{dn}(u+K),$$

$$\frac{1}{s}=k\operatorname{sn}(u+\iota K'),\qquad \frac{d}{s}=-\frac{k}{\iota}\operatorname{cn}(u+\iota K'),\qquad \frac{c}{s}=-\frac{1}{\iota}\operatorname{dn}(u+\iota K'),$$

$$\frac{d}{c}=k\operatorname{sn}(u+K+\iota K'),\qquad \frac{1}{c}=-\frac{k}{\iota k'}\operatorname{cn}(u+K+\iota K'),\qquad \frac{s}{c}=\frac{1}{\iota k'}\operatorname{dn}(u+K+\iota K'),$$

we may readily deduce the integrals of integral powers of

$$\frac{c}{d},\quad \frac{d}{c},\quad \frac{s}{d},\quad \frac{d}{s},\quad \frac{c}{s},\quad \frac{s}{c}.$$

Thus, for example,

$$\int\frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}\,du=\int\operatorname{sn}^2(u+K)\,du=\frac{1}{k^2}\{(u+K)-E\operatorname{am}(u+K)\}$$

$$=\frac{1}{k^2}\left\{u-E\operatorname{am}u+k^2\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u}\right\}+\text{const.}$$

1379. Again, since

$$\Pi(u,\,a)=\int_0^u\frac{k^2\operatorname{sn}^2u\operatorname{sn}a\operatorname{cn}a\operatorname{dn}a}{1-k^2\operatorname{sn}^2a\operatorname{sn}^2u}\,du=\frac{\operatorname{cn}a\operatorname{dn}a}{\operatorname{sn}a}\int_0^u\left(\frac{1}{1-k^2\operatorname{sn}^2a\operatorname{sn}^2u}-1\right)du,$$

we have $$\int_0^u\frac{du}{1-k^2\operatorname{sn}^2a\operatorname{sn}^2u}=\frac{\operatorname{sn}a}{\operatorname{cn}a\operatorname{dn}a}\Pi(u,\,a)+u,$$

i.e. $$\int_0^u\frac{du}{1-k\operatorname{sn}a\operatorname{sn}u}+\int_0^u\frac{du}{1+k\operatorname{sn}a\operatorname{sn}u}=2\left[\frac{\operatorname{sn}a}{\operatorname{cn}a\operatorname{dn}a}\Pi(u,\,a)+u\right],$$

whilst

$$\int_0^u\frac{du}{1-k\operatorname{sn}a\operatorname{sn}u}-\int_0^u\frac{du}{1+k\operatorname{sn}a\operatorname{sn}u}=\int_0^u\frac{2k\operatorname{sn}a\operatorname{sn}u}{1-k^2\operatorname{sn}^2a\operatorname{sn}^2u}\,du$$

$$=\frac{k\operatorname{sn}a}{\operatorname{cn}a\operatorname{dn}a}\int_0^u(\operatorname{sn}\overline{u+a}+\operatorname{sn}\overline{u-a})\,du,$$

which is integrable by (1), Art. 1375; whence by addition and subtraction the two integrals

$$\int_0^u\frac{du}{1-k\operatorname{sn}a\operatorname{sn}u},\quad \int_0^u\frac{du}{1+k\operatorname{sn}a\operatorname{sn}u}\quad\text{are determined.}$$

PROBLEMS.

1. Show that $\dfrac{d}{du}\dfrac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u}=\dfrac{2}{\operatorname{cn} 2u+\operatorname{dn} 2u}$. [Ox. II. P., 1903.]

2. Prove that

(a) $\sqrt{(1-k^2\operatorname{sn}^4 u)(k'+\operatorname{dn} 2u)}/\sqrt{1+k'}=1-(1-k')\operatorname{sn}^2 u$;

(b) $\sqrt{(\operatorname{cn} 2u+\iota k'\operatorname{sn} 2u)(1-k^2\operatorname{sn}^4 u)}=\iota k'\operatorname{sn} u+\operatorname{cn} u\operatorname{dn} u$.

3. Prove that the equation of the osculating plane at the point u on the curve $x=a\operatorname{sn} u$, $y=b\operatorname{cn} u$, $z=c\operatorname{dn} u$ is

$$\frac{x}{a}k^2k'^2\operatorname{sn}^3 u-\frac{y}{b}k^2\operatorname{cn}^3 u+\frac{z}{c}\operatorname{dn}^3 u=k'^2.$$ [Ox. II. P., 1902.]

4. If $u=\int_0^x\{(a^2+x^2)(b^2+x^2)\}^{-\frac{1}{2}}a\,dx$, show that

$$x=b\operatorname{tn} u,\ (\text{mod. }\sqrt{a^2-b^2}/a),\ a>b.$$ [Ox. II. P., 1902.]

5. If the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ be *defined* by means of

$$\frac{d}{du}\operatorname{sn} u=\operatorname{cn} u\operatorname{dn} u,\quad \frac{d}{du}\operatorname{cn} u=-\operatorname{sn} u\operatorname{dn} u,\quad \frac{d}{du}\operatorname{dn} u=-k^2\operatorname{sn} u\operatorname{cn} u,$$

$$\operatorname{sn} 0=0,\quad \operatorname{cn} 0=1,\quad \operatorname{dn} 0=1,$$

prove that (i) $\operatorname{dn}^2 u=1-k^2\operatorname{sn}^2 u=1-k^2+k^2\operatorname{cn}^2 u$;

(ii) $\dfrac{\operatorname{sn} u\operatorname{cn} v+\operatorname{cn} u\operatorname{sn} v}{\operatorname{dn} u+\operatorname{dn} v}$ is a function of $u+v$.

[Ox. II. P., 1901.]

6. If $x\sqrt{2-\sqrt{3}}=\cos\phi$ and the differential $\dfrac{dx}{\sqrt{1+2x^2\sqrt{3}-x^4}}$ is transformed into $\dfrac{a\,d\phi}{\sqrt{1-\sin^2\alpha\sin^2\phi}}$, find the values of a and α.

[Caius, 1885.]

7. Prove the following results:

u	$\frac{K}{2}$	$\frac{3K}{2}$	$\frac{K+2\iota K'}{2}$	$\frac{3K+2\iota K'}{2}$	$\frac{\iota K'}{2}$	$\frac{2K+\iota K'}{2}$
$\operatorname{sn} u$	$\frac{1}{\sqrt{1+k'}}$	$\frac{1}{\sqrt{1+k'}}$	$\frac{1}{\sqrt{1-k'}}$	$\frac{1}{\sqrt{1-k'}}$	$\frac{\iota}{\sqrt{k}}$	$\frac{1}{\sqrt{k}}$
u	$\frac{3\iota K'}{2}$	$\frac{2K+3\iota K'}{2}$	$\frac{K+\iota K'}{2}$	$\frac{3K+\iota K'}{2}$	$\frac{K+3\iota K'}{2}$	$\frac{3K+3\iota K'}{2}$
$\operatorname{sn} u$	$\frac{-\iota}{\sqrt{k}}$	$\frac{1}{\sqrt{k}}$	$\sqrt{\frac{k+\iota k'}{k}}$	$\sqrt{\frac{k-\iota k'}{k}}$	$\sqrt{\frac{k-\iota k'}{k}}$	$\sqrt{\frac{k+\iota k'}{k}}$

and find the values of $\operatorname{cn} u$, $\operatorname{dn} u$ in each case.

[See Table in Cayley, *E.F.*, p. 74.]

8. If $\tan\frac{1}{8}\pi\sin\phi=\sin\psi=x\sqrt{1-x^2}/\sqrt{1+x^2}$, prove that
$$\int_0^x\frac{dx}{\sqrt{1-x^8}}=\frac{1}{2\sqrt{2}}\int_0^\phi\frac{d\phi}{\sqrt{1-\tan^2\frac{1}{8}\pi\sin^2\phi}}+\sin^2\tfrac{1}{8}\pi\int_0^\psi\frac{d\psi}{\sqrt{1-\tan^2\frac{1}{8}\pi\sin^2\psi}}.$$
[MATH TRIP., 1896.]

9. Prove that $\operatorname{cn}\frac{1}{4}\iota K'\operatorname{dn}\frac{1}{4}\iota K'\div\operatorname{sn}\frac{1}{4}\iota K'=-\iota(1+\sqrt{k})\sqrt{1+k}$
and $\operatorname{dn}\frac{1}{4}\iota K'\div\operatorname{sn}\frac{1}{4}\iota K'\operatorname{cn}\frac{1}{4}\iota K'=-\iota\{1+\sqrt{1+k}\}\sqrt{k}$.
[MATH. TRIP., 1896.]

10. If $\operatorname{tn}u_1=T_1\operatorname{dn}u_1$, $\operatorname{tn}u_2=T_2\operatorname{dn}u_2$, $\operatorname{dn}u_1=D_1^{-1}$, $\operatorname{dn}u_2=D_2^{-1}$, show that
$$\text{(i) }\operatorname{tn}(u_1+u_2)=\frac{T_1+T_2}{D_1D_2-T_1T_2},\quad\text{and}\quad\text{(ii) }\operatorname{tn}2u=\frac{2\operatorname{tn}u\operatorname{dn}u}{1-\operatorname{tn}^2u\operatorname{dn}^2u}.$$

11. Prove $\sin[\operatorname{am}(u+v)+\operatorname{am}(u-v)]=2\operatorname{sn}u\operatorname{cn}u\operatorname{dn}v/D$,
$$1+\operatorname{dn}(u+v)\operatorname{dn}(u-v)=(\operatorname{dn}^2u+\operatorname{dn}^2v)/D,$$
where $D=1-k^2\operatorname{sn}^2u\operatorname{sn}^2v$.

Prove that

12. $$\operatorname{sn}\left(u+\frac{K}{2}\right)=\frac{1}{\sqrt{1+k'}}\frac{k's+cd}{1-(1-k')s^2}=\frac{1}{\sqrt{1+k'}}\frac{d+(1+k')sc}{c+sd}$$
$$=\frac{1}{\sqrt{1+k'}}\sqrt{\frac{d+(1+k')sc}{d+(1-k')sc}}=\sqrt{\frac{\operatorname{dn}2u+k'\operatorname{sn}2u}{k'+\operatorname{dn}2u}}.$$
[CAYLEY.]

13. $$\operatorname{cn}\left(u+\frac{K}{2}\right)=\sqrt{\frac{k'}{1+k'}}\frac{c-sd}{1-(1-k')s^2}$$
$$=\sqrt{\frac{k'}{1+k'}}\frac{c^2-k's^2}{c+sd}=\sqrt{k'}\sqrt{\frac{1-\operatorname{sn}2u}{k'+\operatorname{dn}2u}}.$$

14. $$\operatorname{dn}\left(u+\frac{K}{2}\right)=\sqrt{k'}\frac{d-(1-k')sc}{1-(1-k')s^2}=\sqrt{k'}\frac{cd+k's}{c+sd}$$
$$=\sqrt{k'}\sqrt{\frac{1+k'\operatorname{dn}2u-k^2\operatorname{sn}2u}{k'+\operatorname{dn}2u}}.$$

15. $$\operatorname{sn}\left(u+\frac{\iota K'}{2}\right)=\frac{1}{\sqrt{k}}\frac{(1+k)s+\iota cd}{1+ks^2}=\frac{1}{\sqrt{k}}\sqrt{\frac{(1+k)s+\iota cd}{(1+k)s-\iota cd}}$$
$$=\frac{1}{\sqrt{k}}\sqrt{\frac{k\operatorname{sn}2u+\iota\operatorname{dn}2u}{\operatorname{sn}2u-\iota\operatorname{cn}2u}}.$$
[CAYLEY.]

16. $$\operatorname{cn}\left(u+\frac{\iota K'}{2}\right)=\sqrt{\frac{1+k}{k}}\frac{c-\iota sd}{1+ks^2}=\sqrt{\frac{1+k}{k}}\frac{1-ks^2}{c+\iota sd}$$
$$=\sqrt{\frac{1+k}{k}}\sqrt{\frac{1-ks^2}{1+ks^2}\cdot\frac{c-\iota sd}{c+\iota sd}}=\frac{1}{\sqrt{k}}\sqrt{\frac{\operatorname{dn}2u+k\operatorname{cn}2u}{\operatorname{cn}2u+\iota\operatorname{sn}2u}}.$$

17. $\operatorname{dn}\left(u+\frac{\iota K'}{2}\right)=\sqrt{1+k}\,\frac{d-\iota ksc}{1+ks^2}=\sqrt{1+k}\,\frac{1-ks^2}{d+\iota ksc}$

$$=\sqrt{\frac{k'^2\operatorname{sn}2u-\iota\operatorname{cn}2u-\iota k\operatorname{dn}2u}{\operatorname{sn}2u-\iota\operatorname{cn}2u}}.$$

18. $\operatorname{sn}\left(u+\frac{K+\iota K'}{2}\right)=\sqrt{\frac{k+\iota k'}{k}}\,\frac{-\iota k's+cd}{1-k(k+\iota k')s^2}$

$$=\sqrt{\frac{k+\iota k'}{k}}\,\frac{c+(k-\iota k')sd}{d+ksc}=\sqrt{\frac{k+\iota k'}{k}}\,\sqrt{\frac{c+(k-\iota k')sd}{c+(k+\iota k')sd}}$$

$$=\frac{1}{\sqrt{k}}\sqrt{\frac{k\operatorname{cn}2u+\iota k'}{\operatorname{cn}2u+\iota k'\operatorname{sn}2u}}.$$ [CAYLEY.]

19. Show that

(i) $s^2\frac{d}{du}\log s=-c^2\frac{d}{du}\log c=-\frac{d^2}{k^2}\frac{d}{du}\log d=scd,$

(ii) $c^2\frac{d}{du}td=c^2d^2-c^2+d^2,$

(iii) $s^2\frac{d}{du}\frac{cd}{s}=-c^2-s^2d^2,$

(iv) $\operatorname{dn}^2(u+\iota K')=d^2+\frac{d}{du}\left(\frac{cd}{s}\right).$

20. Show that $\operatorname{sn}^2(u_1+u_2)-\operatorname{sn}^2(u_1-u_2)=2\frac{\partial}{\partial u_1}\frac{s_1{}^2s_2c_2d_2}{1-k^2s_1{}^2s_2{}^2}.$

21. Show that

(i) $\int_0^u\sqrt{\frac{1-\operatorname{cn}2u}{1+\operatorname{cn}2u}}\,du=-\log\operatorname{cn}u,$

(ii) $\int_0^u\sqrt{\frac{1-\operatorname{dn}2u}{1+\operatorname{dn}2u}}\,du=-\frac{1}{k}\log\operatorname{dn}u,$

(iii) $\int_0^u\operatorname{sn}u\sqrt{\frac{1+\operatorname{cn}2u}{1+\operatorname{dn}2u}}\,du=-\frac{1}{k^2}\log\operatorname{dn}u,$

(iv) $\int_0^u\sqrt{\frac{1-\operatorname{sn}2u}{1+\operatorname{sn}2u}}\,du=\frac{1}{k'}\log\left[\sqrt{1+k'}\operatorname{sn}\left(u+\frac{K}{2}\right)\right].$

22. Find the values of

(i) $\int\operatorname{cn}u\,du,$ (ii) $\int\frac{\operatorname{sn}u}{\operatorname{cn}u}\,du,$ (iii) $\int\frac{\operatorname{sn}^2u\operatorname{dn}u}{\operatorname{cn}^2u}\,du.$

23. If $I_n=\int(\operatorname{sn}u)^n du$, show that

$$(n+1)k^2I_{n+2}-n(1+k^2)I_n+(n-1)I_{n-2}=s^{n-1}cd.$$

24. If $I_n=\int(\operatorname{cn}u)^n du$, show that

$$(n+1)k^2I_{n+2}-n(k-k'^2)I_n-(n-1)k'^2I_{n-2}=c^{n-1}sd.$$

25. If $I_n = \int (\operatorname{dn} u)^n du$, show that

$$(n+1) I_{n+2} - n(1+k'^2) I_n + (n-1) k'^2 I_{n-2} = k^2 d^{n-1} sc.$$

26. If $I_n = \int \left(\frac{\operatorname{sn} u}{\operatorname{dn} u}\right)^n du$, show that

$$(n+1)k^2 I_{n+2} - n(1+k^2) I_n + (n-1) I_{n-2} = -k'^2 \frac{sc^{n-1}}{d^{n+1}},$$

and obtain reduction formulae for $\int \left(\frac{\operatorname{cn} u}{\operatorname{dn} u}\right)^n du$ and $\int \frac{du}{(\operatorname{dn} u)^n}$ similarly.

27. Prove that

(i) $$\frac{1+\operatorname{dn}(u+v)}{\operatorname{sn}(u+v)} = k^2 \frac{\operatorname{sn} u \operatorname{cn} v - \operatorname{sn} v \operatorname{cn} u}{\operatorname{dn} v - \operatorname{dn} u},$$

[M. TRIP. II., 1915.]

(ii) $$\frac{\operatorname{dn}(u-v) - \operatorname{cn}(u-v)}{\operatorname{sn}(u-v)} = \frac{\operatorname{dn} u \operatorname{cn} v - \operatorname{cn} u \operatorname{dn} v}{\operatorname{sn} u + \operatorname{sn} v}.$$

[SIR J. J. THOMSON.]

28. Show that $\operatorname{sn}(u_1+u_2)$

$$= \frac{s_1c_2d_2 + s_2c_1d_1}{1 - k^2 s_1{}^2 s_2{}^2} = \frac{s_1c_1d_2 + s_2c_2d_1}{c_1c_2 + s_1s_2d_1d_2} = \frac{s_1c_2d_1 + s_2c_1d_2}{d_1d_2 + k^2 s_1s_2c_1c_2} = \frac{s_1{}^2 - s_2{}^2}{s_1c_2d_2 - s_2c_1d_1}.$$

[M. TRIP. II., 1889.]

29. If u_1, u_2, u_3, u_4 be any arguments, and x, y, z respectively denote

$$\frac{\operatorname{sn}(u_4-u_1)\operatorname{sn}(u_2-u_3)}{\operatorname{sn}(u_4+u_1)\operatorname{sn}(u_2+u_3)}, \quad \frac{\operatorname{sn}(u_4-u_2)\operatorname{sn}(u_3-u_1)}{\operatorname{sn}(u_4+u_2)\operatorname{sn}(u_3+u_1)}, \quad \frac{\operatorname{sn}(u_4-u_3)\operatorname{sn}(u_1-u_2)}{\operatorname{sn}(u_4+u_3)\operatorname{sn}(u_1+u_2)},$$

prove that $$x+y+z+xyz=0.$$ [M. TRIP. III., 1885.]

30. If $x_{\lambda\mu}$ denote the function

$$\operatorname{sn}(u_\lambda - u_\mu)\operatorname{cn}(u_\lambda + u_\mu)/\operatorname{cn}(u_\lambda - u_\mu)\operatorname{sn}(u_\lambda + u_\mu),$$

then $x_{41}x_{42}x_{43}x_{12}x_{23}x_{31} + x_{41}x_{23} + x_{42}x_{31} + x_{43}x_{12} = 0.$ [M. TRIP. II., 1889.]

31. Find the values of $\int \operatorname{dn} u \, du$, $\int \frac{du}{\operatorname{dn} u}$, $\int \frac{\operatorname{cn} u}{\operatorname{sn} u} du$.

[M. TRIP. II., 1888.]

32. Prove the formulae

(i) $$3\int \operatorname{dn}^4 u \, du = 2(1+k'^2) \operatorname{ezn} u + k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u - k'^2 u,$$

(ii) $$k'^2 \int \frac{\operatorname{sn} u \, du}{1+\operatorname{sn} u} = \operatorname{ezn}(u+K+\iota K') + \frac{\operatorname{dn} u}{\operatorname{cn} u},$$

(iii) $$k\int_0^K \operatorname{sn} u \, du = \frac{1}{2}\log\frac{1+k}{1-k},$$

where $\operatorname{ezn} u = \frac{E_1 u}{K} + \operatorname{zn} u$, and $\operatorname{zn} u$ is Jacobi's Zeta function $Z(u)$.

[M. TRIP. II., 1888.]

33. Show that $\operatorname{sn}(x+K)=\dfrac{c}{d}$, $\operatorname{sn}(x+2K)=-s$, $\operatorname{sn}(\iota x)=\iota\operatorname{tn}(x, k')$.

[M. Trip., 1876.]

Prove that, if $D=1-k^2s_1{}^2s_2{}^2$,

34. (i) $\operatorname{cn}(u_1+u_2)\operatorname{cn}(u_1-u_2)=(c_1{}^2-s_2{}^2d_1{}^2)/D=(c_2{}^2-s_1{}^2d_2{}^2)/D$;

(ii) $\operatorname{dn}(u_1+u_2)\operatorname{dn}(u_1-u_2)=(d_1{}^2-k^2s_2{}^2c_1{}^2)/D=(d_2{}^2-k^2s_1{}^2c_2{}^2)/D$.

35. (i) $\operatorname{cn}(u_1+u_2)\operatorname{cn}(u_1-u_2)+\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)$
$=(c_2{}^2-s_2{}^2d_1{}^2)/D$;

(ii) $\operatorname{cn}(u_1+u_2)\operatorname{cn}(u_1-u_2)-\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)$
$=(c_1{}^2-s_1{}^2d_2{}^2)/D$;

(iii) $\operatorname{dn}(u_1+u_2)\operatorname{dn}(u_1-u_2)+k^2\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)$
$=(d_2{}^2-k^2s_2{}^2c_1{}^2)/D$;

(iv) $\operatorname{dn}(u_1+u_2)\operatorname{dn}(u_1-u_2)-k^2\operatorname{sn}(u_1+u_2)\operatorname{sn}(u_1-u_2)$
$=(d_1{}^2-k^2s_1{}^2c_2{}^2)/D$.

36. (i) $\dfrac{1-\operatorname{sn}(u-\alpha)}{1+\operatorname{sn}(u-\alpha)}\cdot\dfrac{1-\operatorname{sn}(u+\alpha)}{1+\operatorname{sn}(u+\alpha)}=\left\{\dfrac{\operatorname{sn}(K-\alpha)-\operatorname{sn}u}{\operatorname{sn}(K-\alpha)+\operatorname{sn}u}\right\}^2$;

(ii) $\dfrac{1+k\operatorname{sn}(u-\alpha)}{1-k\operatorname{sn}(u-\alpha)}\cdot\dfrac{1-k\operatorname{sn}(u+\alpha)}{1+k\operatorname{sn}(u+\alpha)}=\left\{\dfrac{1-k\operatorname{sn}\alpha\operatorname{sn}(u+K)}{1+k\operatorname{sn}\alpha\operatorname{sn}(u+K)}\right\}^2$.

37. (i) $\operatorname{tn}(u+\alpha)+\operatorname{tn}(u-\alpha)=\dfrac{2\operatorname{sn}u\operatorname{cn}u\operatorname{dn}\alpha}{\operatorname{cn}^2\alpha-\operatorname{dn}^2\alpha\operatorname{sn}^2u}$;

(ii) $\operatorname{tn}(u+\alpha)-\operatorname{tn}(u-\alpha)=\dfrac{2\operatorname{sn}\alpha\operatorname{cn}\alpha\operatorname{dn}u}{\operatorname{cn}^2\alpha-\operatorname{dn}^2\alpha\operatorname{sn}^2u}$.

38. Verify the identity $k^2k'^2S-k^2C+D-k'^2=0$, where S denotes the product of the four sn functions with arguments $u\pm v$, $u\pm w$, C denotes the product of the four cn functions and D the product of the four dn functions with the same arguments. [M. Trip. II., 1914.]

39. Prove that the length of the curve of intersection of two right circular cylinders, whose axes are at right angles and radii a, b $(a<b)$, is $8a\displaystyle\int_0^{\frac{\pi}{2}}\left(\frac{1-k^2\sin^4\phi}{1-k^2\sin^2\phi}\right)^{\frac{1}{2}}d\phi$, where $k^2=a^2/b^2$; and verify the result when $a=b$. [St. John's, 1886.]

40. Prove that the relation

$$\frac{M\,dy}{\{(1-y^2)(1-\lambda^2y^2)\}^{\frac{1}{2}}}=\frac{dx}{\{(1-x^2)(1-k^2x^2)\}^{\frac{1}{2}}},$$

where M is a constant, can be satisfied by an equation of the form $yV=U$, in which U, V are integral polynomials.

41. Show that the envelope of

$$y(\operatorname{cn} u \operatorname{dn} u + k \operatorname{sn}^2 u) - x(\operatorname{dn} u - k \operatorname{cn} u) \operatorname{sn} u = ak \operatorname{sn} u$$

is $\quad kP + Q + \dfrac{k'^2}{ak}x = 0, \quad$ where $\quad P^{\frac{2}{3}} + \left(\dfrac{y}{ak^2}\right)^{\frac{2}{3}} = 1, \quad Q^{\frac{2}{3}} + \left(\dfrac{ky}{a}\right)^{\frac{2}{3}} = 1.$

[This is St. Laurent's result for the caustic by refraction for parallel rays falling upon a circle. See Heath's *Optics*, Art. 108.]

42. Show that the envelope of the straight line

$$k'^2 x \operatorname{sn} u + (\operatorname{cn} u + k \operatorname{dn} u)\, y = k \operatorname{sn} u (\operatorname{dn} u + k \operatorname{cn} u)$$

is $$\frac{k'^2}{k}x = k^2\left[1 - \left(\frac{y}{k^2}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}} + k\left[1 - (ky)^{\frac{2}{3}}\right]^{\frac{3}{2}}.$$

[Cayley on Caustics, *Ph. Tr.*, 1856.]

43. A particle under the action of a central attraction

$$\frac{\mu}{r^3}\left[1 - \frac{(l-r)^3}{e^2 l r^2}\right]$$

moves from an apse at distance $l/(1+e)$ with velocity $\sqrt{\mu}(1+e)/e$; show that the orbit described is $l/r = 1 + e \operatorname{cn} \theta$, mod. $1/\sqrt{2}$.

[Tait and Steele, *Dyn. of a Particle*, p. 393.]

44. Show that Euler's Equations of motion of a body about a fixed point under the action of no forces, viz. $A\dfrac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 = 0$, $B\dfrac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 = 0$, $\; C\dfrac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 = 0$, are satisfied by $\omega_1 = a \operatorname{sn} \lambda(t - \tau)$, $\omega_2 = b \operatorname{cn} \lambda(t - \tau)$, $\omega_3 = c \operatorname{dn} \lambda(t - \tau)$, provided the six constants a, b, c, λ, τ, k be suitably chosen

[Kirchoff. *See* Routh, *Rig. Dyn.*]

[For the treatment of these equations by aid of the Weierstrassian functions, the reader is referred to Greenhill, *Ell. F.*, Arts. 104-114.]

45. Prove that

$$-\iota k^{\frac{1}{2}} \operatorname{sn}(u + \tfrac{1}{2}\iota K') = \frac{cd - \iota(1+k)s}{1 + ks^2} = \frac{1 + ks^2}{cd + \iota(1+k)s} = \frac{d - \iota ksc}{c + \iota sd} = \frac{c - \iota sd}{d + \iota ksc}.$$

[M. Trip., 1888.]

46. Prove that

$$-k \operatorname{sn}^2(u + \tfrac{1}{2}\iota K') = \frac{D - \iota kS}{C + \iota S} = \frac{C - \iota S}{D + \iota kS} = \frac{C - kD - \iota k'^2 S}{D - kC} = \frac{D - kC}{C - kD + \iota k'^2 S},$$

where S, C, D denote sn $2u$, cn $2u$, dn $2u$ respectively.

[M. Trip., 1888.]

47. Prove that $\displaystyle\int_K^u \sqrt{\frac{\operatorname{dn} 2u + \operatorname{cn} 2u}{\operatorname{dn} 2u - \operatorname{cn} 2u}}\, du = \frac{1}{k'} \log \operatorname{sn} u.$

48. Show how $\operatorname{sn} mu$ may be expressed in terms of $\operatorname{sn} u$, where m is an integer; and if m be odd, prove that the numerator of $1-\operatorname{sn} mu$ when so expressed consists of a perfect square multiplied by the factor $1-(-1)^{\frac{1}{2}(m-1)}\operatorname{sn} u$. [CAYLEY, *E.F.*, p. 90.]

49. If $k^2=-\omega$, where ω is an imaginary cube root of unity, prove that
$$\frac{1-\operatorname{sn}(\omega-\omega^2)u}{1+\operatorname{sn}(\omega-\omega^2)u}=\frac{1-\operatorname{sn} u}{1+\operatorname{sn} u}\left(\frac{1-\omega\operatorname{sn} u}{1+\omega\operatorname{sn} u}\right)^2.$$

50. Prove that
$$\left\{\frac{1-k^2\dfrac{\operatorname{cn}^2(u+v)\operatorname{cn}^2(u-v)}{\operatorname{dn}^2(u+v)\operatorname{dn}^2(u-v)}}{1-k^2\operatorname{sn}^2(u+v)\operatorname{sn}^2(u-v)}\right\}^{\frac{1}{2}}=k'\frac{1-k^2\operatorname{sn}^2 u\operatorname{sn}^2 v}{1-k^2\operatorname{sn}^2 u-k^2\operatorname{sn}^2 v+k^2\operatorname{sn}^2 u\operatorname{sn}^2 v}.$$
[MATH. TRIP., 1878.]

51. Prove that
$$\frac{\operatorname{sn} u}{u}=\frac{\operatorname{cn}\frac{1}{2}u\operatorname{dn}\frac{1}{2}u\,.\operatorname{cn}\frac{1}{4}u\operatorname{dn}\frac{1}{4}u\,.\operatorname{cn}\frac{1}{8}u\operatorname{dn}\frac{1}{8}u\ldots}{(1-k^2 sn^4\frac{1}{2}u)(1-k^2\operatorname{sn}^4\frac{1}{4}u)(1-k^2\operatorname{sn}^4\frac{1}{8}u)\ldots}.$$
[MATH. TRIP., 1878.]

52. Prove that
$$\frac{1-\operatorname{sn} u}{1+\operatorname{sn} u}=\frac{1}{k'^2}\frac{\operatorname{cn}^2\frac{1}{2}(u+K)\operatorname{dn}^2\frac{1}{2}(u+K)}{\operatorname{sn}^2\frac{1}{2}(u+K)}.$$
[MATH. TRIP., 1878.]

53. Show that if $U=\operatorname{sn}(u+a_1)\operatorname{sn}(u+a_2)\operatorname{sn}(2u+a_1+a_2)$, then
$$\int U\,du=-\frac{1}{2k^2}\log\left[1-k^2\operatorname{sn}^2(u+a_1)\operatorname{sn}^2(u+a_2)\right].$$

54. Show that
$$\frac{\Theta^2(x+a)\,\Theta^2(y+a)\,\Theta(x+y-2a)}{\Theta^2(x-a)\,\Theta^2(y-a)\,\Theta(x+y+2a)}=\frac{1-k^2\operatorname{sn}^2(x-a)\operatorname{sn}^2(y-a)}{1-k^2\operatorname{sn}^2(x+a)\operatorname{sn}^2(y+a)}.$$
[GLAISHER.]

55. Show that
$$\int_0^u\frac{\operatorname{cn} u-\operatorname{sn} u\operatorname{dn} u}{\operatorname{cn} u+\operatorname{sn} u\operatorname{dn} u}du=\frac{1}{k'}\log\left\{\sqrt{1+k'}\operatorname{sn}\left(u+\frac{K}{2}\right)\right\}.$$

56. Prove that in a spherical triangle ABC, obtuse angled at C, we may replace $\cos a$, $\cos b$, $\cos c$, $\cos A$, $\cos B$, $\cos C$ respectively by $\operatorname{cn} u$, $\operatorname{cn} v$, $\operatorname{cn}(u+v)$, $\operatorname{dn} u$, $\operatorname{dn} v$, $-\operatorname{dn}(u+v)$, and then
$$\cos^2 p=1-k^2\operatorname{sn}^2 u\operatorname{sn}^2 v,$$
where p is the perpendicular arc from C on AB, and point out any other analogies between elliptic functions and spherical trigonometry. [MATH. TRIP. III., 1884.]

57. Prove that

$$\text{(i)}\ \Theta(2u) = \frac{\Theta^4(u)}{\Theta^2(0)}(1 - k^2 \operatorname{sn}^4 u);$$

$$\text{(ii)}\ \Theta(3u) = \frac{\Theta^2(2u)\,\Theta(u)}{\Theta^2(0)}(1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 2u).$$

58. Prove that $Z(u) = \dfrac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} - \dfrac{\pi u}{2KK'} + \iota Z(\iota u, k')$.

59. Solve completely the differential equations

$$\text{(i)}\ \frac{d^2u}{dt^2} + n^2u + \alpha u^2 = 0; \quad \text{(ii)}\ \frac{d^2u}{dt^2} + n^2u + \beta u^3 = 0.$$

[MATH. TRIP., 1878.]

Show that in case (i) u is of the form

$$u = a - b\frac{1 - \operatorname{cn}\frac{K}{T}(t-\tau)}{1 + \operatorname{cn}\frac{K}{T}(t-\tau)}, \quad \text{with} \begin{cases} b^2 = (a-m)^2 + n^2, \\ k^2 = \dfrac{1}{2}\left(1 + \dfrac{a-m}{b}\right), \\ \dfrac{K^2}{T^2} = \dfrac{2}{3}\alpha b, \end{cases}$$

$$\left.\begin{aligned} &\text{or}\quad u = -a - (a-b)\operatorname{tn}^2\frac{K}{T}(t-\tau), \\ &\text{or}\quad u = c\operatorname{cn}^2\frac{K}{T}(t-\tau) - b\operatorname{sn}^2\frac{K}{T}(t-\tau), \end{aligned}\right\} \text{with} \begin{cases} (a+c)\,k^2 = b + c, \\ \dfrac{K^2}{T^2} = \dfrac{1}{6}\alpha(a+c), \end{cases}$$

and in case (ii)

$$u = a\operatorname{cn}\frac{K}{T}(t-\tau), \text{ with } (a^2+b^2)k^2 = a^2, \quad \frac{K^2}{T^2} = \frac{1}{2}\beta(a^2+b^2).$$

[SOL. S.H. PROBLEMS, 1878.]

60. Prove that if a uniform chain fixed at two points rotate in relative equilibrium with constant angular velocity about an axis in the same plane with the line joining the two points and free from the action of gravity, the form of the curve assumed by the chain will be given by $y = b \operatorname{sn} K\dfrac{x}{a}$, the axis of rotation being the axis of x.

[GREENHILL, M. TRIP., 1878.]

61. Differentiations being denoted by accents, show that

$$\frac{\operatorname{cn}'' u}{\operatorname{cn} u} - \frac{\operatorname{sn}'' u}{\operatorname{sn} u} = k^2, \quad \frac{\operatorname{dn}'' u}{\operatorname{dn} u} - \frac{\operatorname{cn}'' u}{\operatorname{cn} u} = k'^2, \quad \frac{\operatorname{sn}'' u}{\operatorname{sn} u} - \frac{\operatorname{dn}'' u}{\operatorname{dn} u} = -1.$$

62. If $\dfrac{dx}{\sqrt{1-x^4}} + \dfrac{dy}{\sqrt{1-y^4}} = 0$, obtain the relation between x and y in an integral form.

[MATH. TRIP., 1876.]

63. Transform the differential $dx/\sqrt{(1-x^2)(1-k^2x^2)}$ into a like expression having, instead of k, the modulus $2\sqrt{k}/(1+k)$.

64. Accents denoting differentiations, prove that

(i) $\begin{vmatrix} \operatorname{sn} u, & \operatorname{sn}' u, & \operatorname{sn}'' u \\ \operatorname{cn} u, & \operatorname{cn}' u, & \operatorname{cn}'' u \\ \operatorname{dn} u, & \operatorname{dn}' u, & \operatorname{dn}'' u \end{vmatrix} = -k'^2$; (ii) $\begin{vmatrix} \operatorname{sn} u, & \operatorname{sn}' u, & \operatorname{sn}''' u \\ \operatorname{cn} u, & \operatorname{cn}' u, & \operatorname{cn}''' u \\ \operatorname{dn} u, & \operatorname{dn}' u, & \operatorname{dn}''' u \end{vmatrix} = 0$

65. Show that

(i) $\begin{vmatrix} s^2, & ss', & s'^2 \\ c^2, & cc', & c'^2 \\ d^2, & dd', & d'^2 \end{vmatrix} = k'^2 scd$; [MATHEWS. *See* GREENHILL, *E.F.*, p. 349.]

(ii) $$\begin{vmatrix} \operatorname{cn} u, & \operatorname{cn} u, & \operatorname{cn} u, & \operatorname{cn} u \\ \operatorname{cn} u, & \operatorname{dn} u, & \operatorname{cn} u, & \operatorname{cn} u \\ \operatorname{cn} u, & \operatorname{cn} u, & \operatorname{dn} u, & \operatorname{cn} u \\ \operatorname{cn} u, & \operatorname{cn} u, & \operatorname{cn} u, & \operatorname{dn} u \end{vmatrix} = \frac{8k'^6 \operatorname{cn} u \operatorname{sn}^6 \frac{u}{2}}{\left(1 - k^2 \operatorname{sn}^4 \frac{u}{2}\right)^3}.$$

66. Show that for four arguments u_1, u_2, v_1, v_2, if differentiations of the elliptic functions with regard to their respective arguments be denoted by accents,

$$\begin{vmatrix} \operatorname{dn} 2u_1, & \operatorname{dn} 2u_2, & \operatorname{cn} 2u_2, & \operatorname{cn} 2u_1 \\ \operatorname{cn} 2u_1, & \operatorname{cn} 2u_2, & \operatorname{dn} 2u_2, & \operatorname{dn} 2u_1 \\ \operatorname{dn} 2v_1, & \operatorname{dn} 2v_2, & \operatorname{cn} 2v_2, & \operatorname{cn} 2v_1 \\ \operatorname{cn} 2v_1, & \operatorname{cn} 2v_2, & \operatorname{dn} 2v_2, & \operatorname{dn} 2v_1 \end{vmatrix}$$

$$= \frac{16k'^4}{U_1^2 U_2^2 V_1^2 V_1^2}[U_1 V_2 \operatorname{sn}'^2 u_1 \operatorname{sn}'^2 v_2 - U_2 V_1 \operatorname{sn}'^2 u_2 \operatorname{sn}'^2 v_1]$$
$$\times [U_1 V_2 \operatorname{sn}^2 u_1 \operatorname{sn}^2 v_2 - U_2 V_1 \operatorname{sn}^2 u_1 \operatorname{sn}^2 v_2],$$

where $\dfrac{U_1}{1-k^2\operatorname{sn}^4 u_1} = \dfrac{U_2}{1-k^2\operatorname{sn}^4 u_2} = \dfrac{V_1}{1-k^2\operatorname{sn}^4 v_1} = \dfrac{V_2}{1-k^2\operatorname{sn}^4 v_2} = 1.$

67. Show that

$$\begin{vmatrix} 1, & \operatorname{cn} u, & \operatorname{dn} u \\ 1, & \operatorname{cn} v, & \operatorname{dn} v \\ 1, & \operatorname{cn} w, & \operatorname{dn} w \end{vmatrix} = -4k^2k'^2 \Pi \operatorname{sn}\frac{v+w}{2} \operatorname{sn}\frac{v-w}{2} \frac{1 - k^2 \operatorname{sn}^2 \frac{v}{2} \operatorname{sn}^2 \frac{w}{2}}{1 - k^2 \operatorname{sn}^4 \frac{u}{2}}.$$

[OX. II. P., 1914.]

68. Prove that

$$\begin{vmatrix} \operatorname{sn}^2(u+v), & \operatorname{sn}(u+v)\operatorname{sn}(u-v), & \operatorname{sn}^2(u-v) \\ \operatorname{cn}^2(u+v), & \operatorname{cn}(u+v)\operatorname{cn}(u-v), & \operatorname{cn}^2(u-v) \\ \operatorname{dn}^2(u+v), & \operatorname{dn}(u+v)\operatorname{dn}(u-v), & \operatorname{dn}^2(u-v) \end{vmatrix} = \frac{8k'^2 s_1 s_2^3 c_1 c_2 d_1 d_2}{(1 - k^2 s_1^2 s_2^2)^3}.$$

[MATH. TRIP. II., 1913.]

69. If $m^2+n^2=1$, prove that

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\sin^2\phi\,d\theta\,d\phi}{(1-m^2\sin^2\theta)^{\frac{3}{2}}(1-n^2\sin^2\phi)^{\frac{1}{2}}}$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\cos^2\phi\,d\theta\,d\phi}{(1-m^2\sin^2\theta)^{\frac{1}{2}}(1-n^2\sin^2\phi)^{\frac{3}{2}}}.$$

70. If $u=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{m^2\cos^2\theta+n^2\cos^2\phi}{\sqrt{1-m^2\sin^2\theta}\sqrt{1-n^2\sin^2\phi}}d\theta\,d\phi$, then $\frac{du}{dm}=0$.

[γ, 1891.]

71. P and Q are points one on each of two circles in parallel planes with a common axis through the centres C, C' at right angles to the planes; $CC'=b$ and the radii are A and a, $PQ=r$ and the angle between the planes $C'CP$ and $CC'Q$ is ϵ. Evaluate the integral $M\equiv\iint\frac{\cos\epsilon}{r}ds\,ds'$, the integrations extending round each circle, and throw the result into the form

$$M=4\pi\sqrt{Aa}\left[\left(c-\frac{1}{2c}\right)F_1-cE_1\right],$$

where F_1 and E_1 are complete Elliptic Integrals.