CHAPTER XXX.

INTEGRATION. CAUCHY'S THEOREM ON CONTOUR INTEGRATION. TAYLOR'S THEOREM.

1266. Definition of Integration for a Function of a Complex Variable.

Let $f(z)$ be any single-valued function of z, and let any path of z on the z-plane be selected which does not pass through a point which makes $f(z)$ infinite, and along which the change in $f(z)$ is continuous.

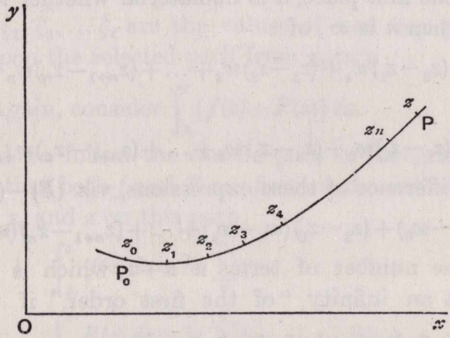

Fig. 377.

Let $z_0, z_1, z_2, \dots z_n, z_{n+1} (=z)$ be an infinitesimally close array of points on this path from an initial point P_0, (z_0), to another point P, (z).

Then the limit (provided a limit exists) of the sum when n is infinite of the series

$$(z_1-z_0)f(z_0)+(z_2-z_1)f(z_1)+(z_3-z_2)f(z_2)+\dots+(z-z_n)f(z_n),$$

when the moduli

$$|z_1-z_0|, \quad |z_2-z_1|, \quad |z_3-z_2| \dots |z-z_n|$$

are each indefinitely decreased, so that the successive elements of the z-path are all infinitesimally small, is called the integral of $f(z)\,dz$ for the selected path, and is denoted by

$$\int_{z_0}^{z} f(z)\,dz.$$

1267. Obviously, the last term of the series, having an infinitesimal modulus, the series may, if desired, be supposed to stop at the term $(z_n - z_{n-1})f(z_{n-1})$, as in the case of a function of a real variable (Arts. 11 and 12).

1268. This definition clearly includes that of functions of a real variable (Art. 11) as a particular case, the "selected path" for the variation of x in that case lying upon the x-axis.

1269. **General Properties of an Integral.**

Properties of the integral, corresponding to those of Articles 322, etc., for a real variable, may be established. Let $w_r \equiv f(z_r)$.

Then, in the first place, it is immaterial whether we consider the limit, when n is ∞, of

$$(z_1-z_0)w_0+(z_2-z_1)w_1+(z_3-z_2)w_2+\ldots+(z_{n+1}-z_n)w_n \ldots \quad \equiv (A),$$

or of

$$(z_1-z_0)w_1+(z_2-z_1)w_2+(z_3-z_2)w_3+\ldots+(z_{n+1}-z_n)w_{n+1} \ldots \quad \equiv (B).$$

For the difference of these expressions, viz. $(B)-(A)$, is

$$(z_1-z_0)(w_1-w_0)+(z_2-z_1)(w_2-w_1)+\ldots+(z_{n+1}-z_n)(w_{n+1}-w_n),$$

in which the number of terms is $n+1$, which is ultimately infinite, but an infinity "of the first order," if we regard $\frac{1}{n+1}$ as an infinitesimal of the first order.

Let the greatest of the moduli of the several terms be

$$|z_r-z_{r-1}| \times |w_r-w_{r-1}|,$$

which is finite, as the path of z has been chosen so as not to pass through a point for which w becomes infinite. Then, since the z-points are taken infinitely close to each other, and the function w is continuous for variations of z along the path, $|z_r-z_{r-1}|$ is an infinitesimal of at least the first order, and $|w_r-w_{r-1}|$ is also an infinitesimal of at least the first order.

Hence the difference of the (*A*) and (*B*) series cannot exceed the value of the product of

(an infinity of the first order) × (an infinitesimal of the first order) × (an infinitesimal of the first order),

i.e. a finite quantity multiplied by an infinitesimal, and must therefore vanish in the limit.

1270. It follows that if $w=f(z)$,

$$\int_{z_0}^{z} w\,dz \equiv \int_{z_0}^{z} f(z)\,dz = \sum_{r=1}^{r=n+1}(z_r - z_{r-1})f(z_{r-1}) = \sum_{1}^{n+1}(z_r - z_{r-1})f(z_r)$$

$$= -\sum_{1}^{n+1}(z_{r-1} - z_r)f(z_r) = -\int_{z}^{z_0} f(z)\,dz = -\int_{z}^{z_0} w\,dz.$$

1271. Again, since the sum of the series

$$(z_1 - z_0)f(z_0) + (z_2 - z_1)f(z_1) + (z_3 - z_2)f(z_2) + \dots + (z - z_n)f(z_n)$$

may be divided into any number of portions which together make up the whole series, we have

$$\int_{z_0}^{\zeta_1} f(z)\,dz + \int_{\zeta_1}^{\zeta_2} f(z)\,dz + \int_{\zeta_2}^{\zeta_3} f(z)\,dz + \dots + \int_{\zeta_r}^{z} f(z)\,dz = \int_{z_0}^{z} f(z)\,dz,$$

where $\zeta_1, \zeta_2, \zeta_3, \dots \zeta_r$ are the values of z at any points taken in order upon the selected path from z_0 to z.

1272. Again, consider $\int_{z_0}^{z}[f(z) \pm F(z)]\,dz$.

Provided we follow the same z-path of integration in both cases, and that both f and F are finite and continuous between the points z_0 and z on this path,

$$\int_{z_0}^{z} f(z)\,dz = Lt \sum_{0}^{n}(z_{r+1} - z_r)f(z_r),$$

$$\int_{z_0}^{z} F(z)\,dz = Lt \sum_{0}^{n}(z_{r+1} - z_r)F(z_r).$$

Hence

$$\int_{z_0}^{z} f(z)\,dz \pm \int_{z_0}^{z} F(z)\,dz = Lt \sum_{0}^{n}(z_{r+1} - z_r)[f(z_r) \pm F(z_r)]$$

$$= \int_{z_0}^{z}[f(z) \pm F(z)]\,dz.$$

And the same is true if there be any finite number of functions.

Also, somewhat more generally, if $\Sigma A_k f_k(z)$ stand for

$$A_1 f_1(z) + A_2 f_2(z) + \dots$$

for a finite number of functions, where $A_1, A_2, \dots$, are all independent of z, then

$$\int_{z_0}^{z} \Sigma A_k f_k(z)\,dz = \Sigma \int_{z_0}^{z} A_k f_k(z)\,dz,$$

so long as the same z-path is followed in each integration, and the conditions as to being finite and continuous from z_0 to z are satisfied by each of the functions.

The coefficients A_k may be any whatever, provided they are not functions of z, and the number of terms in the summation is finite.

And further, in these results each function has been supposed single-valued, or if not, that the same branch is adhered to throughout the integration in each case.

1273. So long as the path of integration from z_0 to z is finite, and passes through no critical points of $f(z)$, *i.e.* points for which $f(z)$ becomes infinite, and is a continuous path so far as variations of $f(z)$ are concerned, the integral $\int_{z_0}^{z} f(z)\,dz$ must be finite.

For this integral is, by definition,

$$Lt\,[(z_1-z_0)f(z_0)+(z_2-z_1)f(z_1)+(z_3-z_2)f(z_2)+\dots+(z-z_n)f(z_n)],$$

and, by supposition, none of the expressions $f(z_0)$, $f(z_1)$, … $f(z_n)$ have an infinite modulus.

If mod. $f(z_r) \equiv K$, say, be the greatest of their moduli, the modulus of the integral $\int_{z_0}^{z} f(z)\,dz$, which is

$$\not> Lt\,\Sigma\,\text{mod.}\,(z_{r+1}-z_r)\,\text{mod.}\,f(z_r),$$

is

$$\not> Lt\,K\Sigma\,\text{mod.}\,(z_{r+1}-z_r),$$

and $Lt\,\Sigma\,\text{mod.}\,(z_{r+1}-z_r)=$ the arc of the selected path from z_0 to z, $=S$, say, which, by supposition, is finite.

Hence the modulus of the integral is not greater than $K.S$, and is therefore finite. Hence the integral itself, $\int_{z_0}^{z} f(z)\,dz$, is finite.

1274. When the number of functions $f_1(z), f_2(z), f_3(z), \dots f_n(z)$ is infinite, the functions being each single valued, or if multiple valued, the same branch being adhered to throughout the integration, the same theorem as that of Art. 1272 is true for their sum, provided that the sum forms a series which is uniformly and unconditionally convergent,* and provided the z-path of the integrations lies entirely within the circle of convergence and is finite; for if we write $u_1, u_2, u_3, \ldots$ for these functions, let $f(z)=u_1+u_2+u_3+\ldots+u_n+R_n$, where R_n is the remainder after n terms; and let the series

$$u_1+u_2+u_3+\ldots \text{ to } \infty$$

be uniformly and unconditionally convergent for all points within the region bounded by a circle of radius ρ, then, when n is indefinitely increased, $|R_n|$ vanishes.

But $$\int_{z_0}^{z}\left[f(z)-\sum_1^n u_r\right]dz=\int_{z_0}^{z} R_n\,dz,$$

and if $|R'|$ be the greatest value of $|R_n|$ along the path of integration, which is finite, and which lies within and does not cut the circle of convergence, then

$$\left|\int_{z_0}^{z} R_n\,dz\right| \text{ is } \ngtr \int_{z_0}^{z}|R'\,dz|, \quad \textit{i.e.} \ngtr |R'|\int_{z_0}^{z}|dz|,$$

$$\ngtr |R'| \times \text{the length of the path of integration}$$

$$\ngtr |R'| \times \text{a finite quantity},$$

and $|R'|$ is zero, by supposition, when n is made infinite;

$$\therefore\ Lt\left|\int_{z_0}^{z} R_n\,dz\right|=0, \quad \text{and therefore } \int_{z_0}^{z} R_n\,dz=0,$$

whence $$\int_{z_0}^{z} f(z)\,dz=\sum_1^\infty \int_{z_0}^{z} u_r\,dz,$$

where the path of integration is the same for each term of the series and the conditions of the series are as stated.

1275. Cauchy's Theorem.

It was shown in Chapter XV. that if ϕ and ψ be any two functions of x and y which are single valued, finite, and continuous at all points x, y which lie within or upon a given closed contour Γ of the x-y plane, then

$$\iint\left(\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right)dx\,dy=\int\left(\phi\frac{dx}{ds}+\psi\frac{dy}{ds}\right)ds,$$

* A knowledge of the general theory of infinite series and tests for convergency will be assumed here. The necessary information will be found in Professor Hobson's *Plane Trigonometry*, Chapter XIV., or in the *Treatise on the Theory of Functions*, by Harkness and Morley, Chapter III.

the surface integral being taken over the area bounded by the contour and the line integral being taken round the perimeter, the direction of the integration being such that in travelling along the arc in the direction of increase of s, the area bounded by the contour is always on the left-hand side.

Consider the function $w=f(z)=f(x+\iota y)=u+\iota v$, say.

Then u and v being conjugate functions of x and y (*Diff. Calc.*, Art. 190), we have

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\ \text{ and }\ \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}.$$

Now, from the above theorem, we have, by two applications,

$$\int(u\,dx-v\,dy)=-\iint\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dx\,dy=0$$

and $$\int(v\,dx+u\,dy)=\iint\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dx\,dy=0.$$

Hence $$\int f(z)\,dz=\int(u+\iota v)\,d(x+\iota y)$$

$$=\int(u\,dx-v\,dy)+\iota\int(v\,dx+u\,dy)$$

$$=0,$$

and the assumption in this theorem is that $f(z)$ is synectic within and upon the boundary of Γ along which the integration is conducted. That is, that $f(z)$ is a single-valued, continuous function which has no infinities, whether pole or essential singularity, within or upon the boundary of the contour. This extremely important theorem is due to Cauchy (*Comptes Rendus de l'Acad. des Sciences*, 1846).

1276. **Deformation of a Path.**

When w is a synectic function for a definite region Γ of the z-plane, let ACB, ADB be two z-paths which lie entirely within that region. Then it follows from Cauchy's theorem that

$$\int_A^B w\,dz\,(\text{along } ADB)+\int_B^A w\,dz\,(\text{along } BCA)=0,$$

as there are no singularities in the region between the two paths.

Hence $$\int_A^B w\,dz\,(\text{along } ADB)=\int_A^B w\,dz\,(\text{along } ACB).$$

Hence, as far as the value of the integral is concerned, either path from A to B is *deformable into* the other without altering the value of $\int w\,dz$ along it. When one of these paths is the straight line AB itself, the other path is said to be "re-

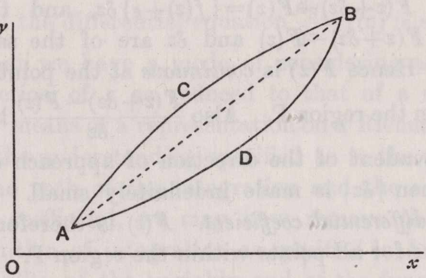

Fig. 378.

concilable with" a straight-line path of integration; and it will appear that such deformation of the path from A to B can be carried to any extent, provided that this deformation does not carry any part of the path of integration outside the boundary of the region Γ on the x-y plane, for which the function $f(z)$ is synectic.

1277. Differentiation of this Integral.

Writing ζ for z and taking $f(\zeta)$ as synectic throughout the singly connected region Γ of the z-plane, and starting from any selected point z_0, viz. A in Fig. 378, and travelling along any path to z, viz. the point B, both terminals and path lying entirely within the boundary of Γ, we see that the integral $\int_{z_0}^{z} f(\zeta)\,d\zeta$ is independent of the path of approach of ζ to the terminal z. Let $F(z)$ stand for this integral. Then it follows that $F(z)$ is a *single-valued* function of z; and it has been shown to be *finite* in Art. 1273. Let $z+\delta z$ be another point within the region Γ infinitesimally close to z. Then $F(z+\delta z)$, which is $\int_{z_0}^{z+\delta z} f(\zeta)\,d\zeta$, is also independent of the path of approach of ζ to $z+\delta z$. We may therefore select the same path as before from z_0 as far as the point z, together with any additional elementary path from z to $z+\delta z$ lying within the region Γ, and along this $f(\zeta)$ remains finite and continuous by supposition. The difference between $f(\zeta)$ and $f(z)$ for any point of this elementary path is therefore infinitesimal, and therefore we may write $\int_z^{z+\delta z} f(\zeta)\,d\zeta$ as $\{f(z)+\epsilon\}\,\delta z$, where the modulus of ϵ is infinitesimally small, ultimately vanishing with that of δz. Wherefore $F(z+\delta z)-F(z)=\{f(z)+\epsilon\}\,\delta z$, and therefore the moduli of $F(z+\delta z)-F(z)$ and δz are of the same order of smallness. Hence $F(z)$ is *continuous* at the point z, *i.e.* at any point within the region Γ. Also $\dfrac{F(z+\delta z)-F(z)}{\delta z}$ has a limiting value independent of the direction of approach of $z+\delta z$ to z, viz. $f(z)$, when $|\delta z|$ is made indefinitely small. That is $F(z)$ *possesses a differential coefficient.* $F(z)$ is therefore a synectic function of z for all points within the region Γ.

1278. **Definition of Integration regarded as a Solution of the Differential Equation $\dfrac{dy}{dz}=f(z)$.**

It now appears that the integral $\int_{z_0}^{z} f(\zeta)\,d\zeta$ defined in Art. 1266 as the limit of a summation from a definite starting point z_0 to a definite terminal point z along any selected path, both path and terminals lying within the region Γ, and the terminals being not within an infinitesimal distance of its boundary, throughout which region $f(z)$ is synectic, is a solution of the differential equation $\dfrac{dy}{dz}=f(z)$, whatever the starting point z_0 may be. And supposing z_0 to have been specifically selected, we may write the general solution of this equation as $y=C+\int_{z_0}^{z} f(\zeta)\,d\zeta$, where C is the integral from any *arbitrary* point of the region Γ along any path lying within Γ to the selected point z_0. In fact, we might regard the notation $y=C+\int_{z_0}^{z} f(\zeta)\,d\zeta$ as only another way of writing the differential equation, but one which emphasizes the interrogative character of the investigation it is proposed to conduct.

1279. **Extension of Former Definitions of Integration. Removal of Limitations.**

So long then as Γ is a singly connected region in the z-plane in which $f(z)$ has no singularities, whether poles, essential singularities or branch-points and the path of the integration lies entirely within the contour of Γ and the terminals do not lie within an infinitesimal distance of the boundary, the identity of the summation definition with that of a solution of the differential equation $\frac{dy}{dz}=f(z)$ is established.

Seeing that we have a mode of considering any multiple-valued function of z as reduced to that of a single-valued function by means of a representation on a Riemann's Surface, and under the understanding specified as to the nature of the function, the path of the integration and the existence of a differential coefficient, we may now remove the limitations of the definition of integration as specified in Art. 17, Vol. I., as to the reality of the variable, and of the function, and the stipulated condition as to the single-valued character of the functions dealt with. We may therefore regard the functions which have been subsequently treated as subjects of integration, as functions of a complex variable with such alterations in the several definitions of those functions as may be required in individual cases to give them intelligible meanings in consonance with such as they possess when functions of a real variable.

The proofs of general propositions as to integration given in Chapter IX. (Art. 321 onwards), which were there established under the understanding as to reality of the variable and single-valuedness of the function, are now superseded for the wider conception of the nature of the variable and the function by the general propositions of Arts. 1269 to 1274.

1280. **Loops.**

As the property presupposed for the function w may cease to hold and the function become meromorphic at certain points of the plane by virtue of the existence of Poles, Branch Points or other singularities, it is necessary to consider, in case the specific region Γ should include such points, what paths there are in this region which are deformable into a straight-line path from any one point O, which may be considered the origin, to any other point P of the region. Also we shall have to consider how the integral $\int_O^P w\,dz$ is affected when the path from O to P is not one which can be deformed into the straight path OP without passing through one of these singular points.

Imagine an infinitely extensible and contractible inelastic thread attached at the points O and P to the plane and lying in the plane. Imagine a pin stuck perpendicularly into the plane at a point A. It will be obvious that the thread might pass on either side the pin, or it might loop round it one or more times as in the paths in the diagram OXP, OSP (which is straight), OYP or OZP. In the case OXP the thread path can be deformed into the straight path OSP without moving the pin from the point A. But neither of the paths OYP, OZP can be so deformed whilst the thread lies in the plane without removing the pin. The path OXP is said to be "reconcilable with" a straight-line path. But the paths OYP, OZP are not so reconcilable.

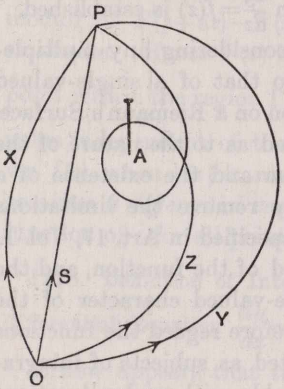

Fig. 379.

1281. The path OYP is "reconcilable with" a loop round A consisting of a straight line OB, a portion BCD of a small circle with centre at A, a straight line DO' parallel and equal to OB, and $O'P$, and the thread OYP may be deformed into this "loop and line" without crossing the pin at A.

The radius of the small circle may be regarded as any infinitesimal and the breadth of the canal BO an infinitesimal of higher order than the radius of the circle, so that the angle BAD is evanescent; the circle BCD may then be regarded as complete and the banks of the canal OB, $O'D$ as coincident. Thus B coincides with D and O' with O, and the figure will be as shown in diagram, No. 381. The portion of the deformation consisting of the small circle and the two banks of the narrow canal starting from O and terminating at O after passing once round the point A is technically known as a "Loop," and the integral $\int w\,dz$ taken round the circuit

OBCDO will be called (A), and if U_1 be the integral along *OP* the whole integral for the path will be $(A)+U_1$ the suffix in such cases denoting the number of loops that have been traversed before starting upon the portion of the path indicated by the letter to which the suffix is attached.

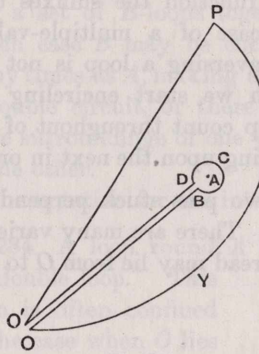

Fig. 380.

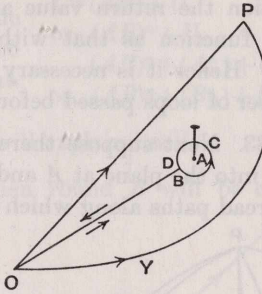

Fig. 381.

If A be an ordinary point of the plane the region within the small circle is synectic, as also along the canal, and $(A)=0$. The value of w on the return journey *DO* is the same as that of w on the outward path *OB*, and the integrations are of opposite sign and cancel; and the integral round the small circle separately vanishes.

No "loop" passes twice round the same point A without first returning to the starting point. The canal of the loop is usually but not necessarily taken straight (see Fig. 399, Art. 1294).

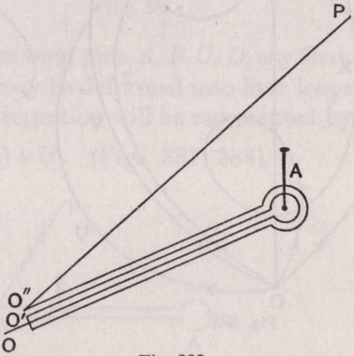

Fig. 382.

1282. If the thread initially lies as in the path Z of Fig. 379, passing round the pin twice before arriving at P, a deformation is possible into two loops + a straight path *OP*, as shown in Fig. 382, the points O, O', O'' being ultimately coincident. The value of the integration round this path we shall denote by $I\equiv(AA)+U_2$ or $(A^2)+U_2$.

If the thread passes round the pin n times before reaching P, the thread-path will in the same way be reconcilable with n A-loops + a linear path, and the value of the integral $\int w\,dz$ along it will be denoted by $I \equiv (A^n) + U_n$.

In the case of a single-valued function the suffixes used are of no account. But in the case of a multiple-valued function the return value after traversing a loop is not the same function as that with which we start encircling the loop. Hence it is necessary to keep count throughout of the number of loops passed before starting upon the next in order.

1283. Next suppose there are two pins stuck perpendicularly into the plane at A and at B. There are many varieties of thread paths along which the thread may lie from O to P.

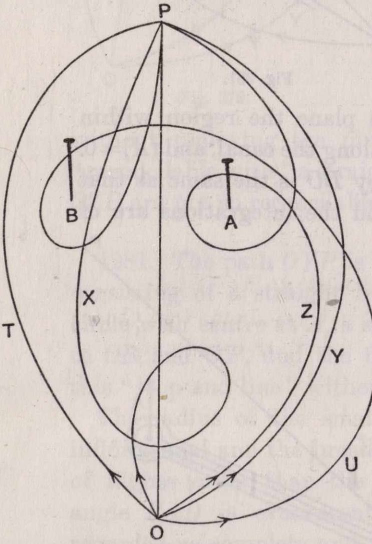

Fig. 383.

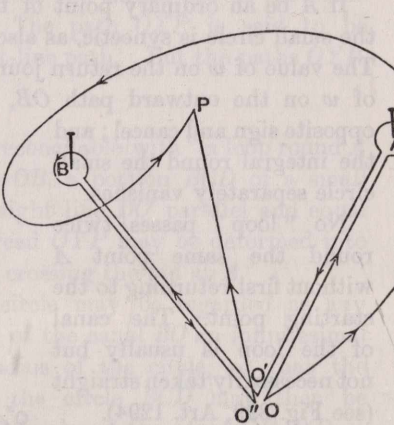

Fig. 384.

(1) It may be deformable without crossing a pin (as OXP) into the straight line OP.

(2) It may, if in position such as OYP, be deformable as before into an A-loop + a straight-line path OP. $I = (A) + U_1$.

(3) It may, if in a position such as OZP, be deformable into several A-loops + a straight-line path OP. $I = (A^n) + U_n$.

(4) It may, if in such a position as OTP, be deformable into a B-loop or into several B-loops + a straight-line path OP.

$$I=(B^n)+U_n.$$

(5) It may be that the thread path surrounds both pins several times, and then the system is deformable into a set of A-loops and a set of B-loops together with a straight path OP, in which case B may be encircled as many times as A, making each time a double circuit, or there may be more surroundings of one pin than of the other.

$$I=(AB)+U_2$$
$$\text{or } (AB)^n+U_{2n},$$
$$(AB)^n+(A^p_{2n})+U_{2n+p}$$
$$\text{or } (AB)^n+(B^q_{2n})+U_{2n+q}.$$

The notation for the integrals will explain itself.

1284. A loop round A and then round B will be called a "double loop." This term is often confined to the case when O lies between the points in question.

A double loop is deformable as shown in Figs. 385, 386, and

$$I=(AB)+U_2.$$

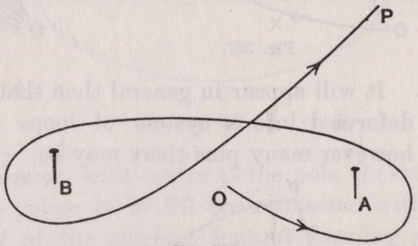

Fig. 385.

In the same way, if there be several pins A, B, C, D, say four, any thread path such as OXP may be deformed into four loops and a straight path, and the integration will be represented by

$$I=(A)+(B_1)+(C_2)+(D_3)+U_4 \quad \text{(Figs. 387, 388)},$$

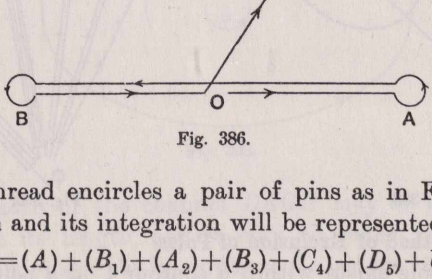

Fig. 386.

or if the thread encircles a pair of pins as in Fig. 389, the deformation and its integration will be represented by

$$I=(A)+(B_1)+(A_2)+(B_3)+(C_4)+(D_5)+U_6$$
$$\text{or} \quad (AB)+(AB)_2+(C_4)+(D_5)+U_6.$$

If the thread encircles three pins ABC, as shown in Fig. 391, the deformation and the integration will be indicated by

$$I=(A)+(B_1)+(C_2)+(A_3)+(B_4)+(C_5)+(D_6)+U_7,$$

and similarly in any other case.

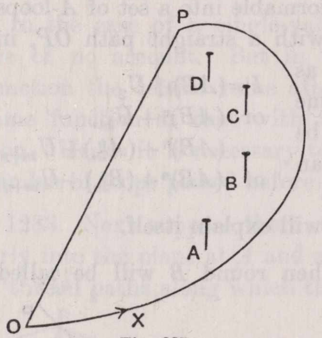

Fig. 387.

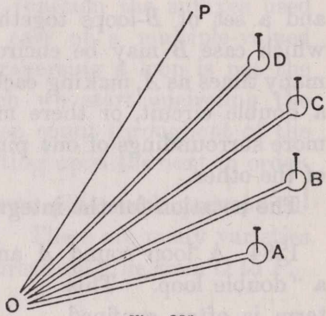

Fig. 388.

It will appear in general then that any thread path may be deformed into a system of loops + a straight-line path, however many pins there may be.

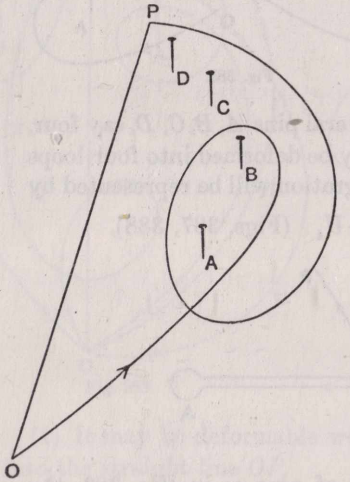

Fig. 389.

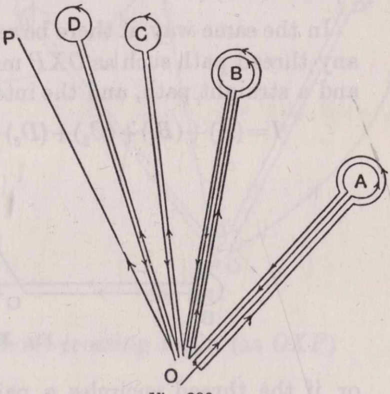

Fig. 390.

1285. Method of Exclusion of Poles.

When a pole exists within a contour Γ at a point $z=a$ and not within an infinitesimal distance of the boundary, it may be excluded from the integration by the artifice of altering the boundary, as indicated in Fig. 392, by the introduction of a loop so as to exclude the pole from the new contour Γ'.

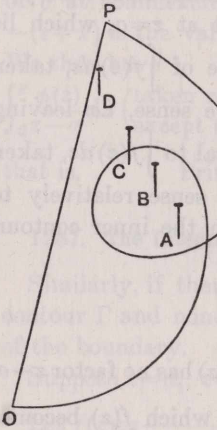

Fig. 391.

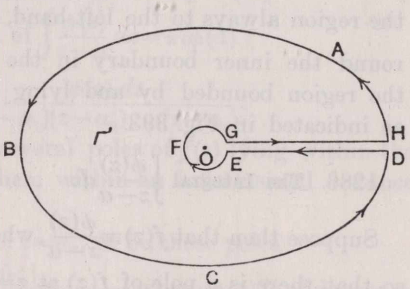

Fig. 392.

A small circle EFG is drawn with centre at the pole O (viz. $z=a$), and two adjacent points of it EG are connected with two adjacent points DH of the original contour forming a narrow canal. We then regard the boundary of the contour Γ' as the curve $ABCDEFGHA$, and integrate round the amended contour.

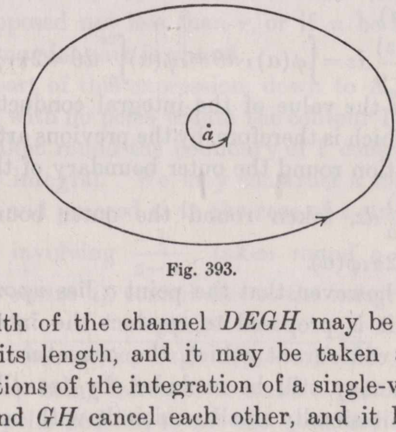

Fig. 393.

The breadth of the channel $DEGH$ may be taken as zero throughout its length, and it may be taken as straight, so that the portions of the integration of a single-valued function along DE and GH cancel each other, and it leaves us with the theorem that $\int f(z)\,dz$, round the outer boundary in the sense of the arrow at A, $+\int f(z)\,dz$ round EFG in the sense of the arrow at F, vanishes, it being supposed that $f(z)$ possesses no singularities other than that at $z=a$, which lie within the region Γ. That is, the value of $\int f(z)\,dz$, taken round the outer boundary in the positive sense, *i.e.* leaving the region always to the left-hand, is equal to $\int f(z)\,dz$, taken round the inner boundary in the same sense relatively to the region bounded by and lying within the inner contour, as indicated in Fig. 393.

1286. **The Integral** $\int \frac{\phi(z)}{z-a}dz$.

Suppose then that $f(z)\equiv\frac{\phi(z)}{z-a}$, where $\phi(z)$ has no factor $z-a$, so that there is a pole of $f(z)$ at $z=a$, at which $f(z)$ becomes infinite, and that the point a is not within an infinitesimal distance of the nearest point of the boundary.

To consider the value of $\int f(z)\,dz$, taken round a small circular contour with centre $z=a$ and small radius ρ, which will not cut the boundary, put $z=a+\rho e^{\iota\theta}$.

Then $\frac{dz}{z-a}=\iota\,d\theta$, and if ρ be infinitesimally small we may put $\phi(z)=\phi(a)$.

Hence $\int \frac{\phi(z)}{z-a}dz=\int\phi(a)\,\iota\,d\theta=\iota\phi(a)\int_0^{2\pi}d\theta=2\pi\iota\phi(a)$.

This then is the value of the integral conducted round the small circle, which is therefore, by the previous article, the value of the integration round the outer boundary of the contour.

Thus $\int \frac{\phi(z)}{z-a}dz$, taken round the outer boundary of the contour Γ, $=2\pi\iota\phi(a)$.

Supposing, however, that the point a lies *upon* the contour along which it is proposed to conduct the integration, at a point of the contour at which the curvature is finite and continuous, it may still be excluded by travelling round it along an infinitesimally small semicircle with centre at a and lying within the bounded region, cutting the contour at P and Q. Then after putting, as before, $z=a+\rho e^{\iota\theta}$, the limits for θ will now be from $-\epsilon$ to $-(\epsilon+\pi)$, where $-\epsilon$ is the value of θ at commencing the small semicircular path at P, and $-(\epsilon+\pi)$ is the value when the contour is recommenced at Q. We then have

$$\int_Q^P \frac{\phi(z)}{z-a} dz \begin{pmatrix}\text{taken round the whole contour} \\ \text{except the infinitesimal arc } PQ\end{pmatrix} + \int_{-\epsilon}^{-(\epsilon+\pi)} \phi(a)\iota\, d\theta = 0,$$

that is, Prin. Val. of $\int \frac{\phi(z)}{z-a} dz = \pi\iota\phi(a)$.

1287. **The Integral** $\int \frac{\phi(z)\, dz}{(z-a_1)(z-a_2)\dots(z-a_r)}$.

Similarly, if there be several poles of $f(z)$ lying within the contour Γ and none of them within an infinitesimal distance of the boundary.

Suppose $z=a_1$, $z=a_2$, ... $z=a_r$, to be these poles.

Let $f(z) \equiv \frac{\phi(z)}{(z-a_1)(z-a_2)\dots(z-a_r)}$, where $\phi(z)$ is of degree n, say, in z, and possesses no factors $z-a_1$, $z-a_2$, ... or $z-a_r$.

By the rules of partial fractions, we have a result of the form

$$f(z) = K_{n-r}z^{n-r} + K_{n-r-1}z^{n-r-1} + \dots + K_1 z + K_0$$
$$+ \sum_{s=1}^{s=r} \frac{\phi(a_s)}{(a_s-a_1)(a_s-a_2)\dots(a_s-a_r)} \frac{1}{z-a_s},$$

where the factor a_s-a_s is omitted from the denominator and n is supposed not less than r, or if n be less than r the integral polynomial part is absent.

The first part of this expression, down to K_0, constitutes a function of z with no poles within the contour Γ, and therefore its integral taken round the boundary of Γ contributes nothing to the whole integral. We may construct a loop for each of the infinities and proceed as in the case of a single infinity.

The term involving $\frac{1}{z-a_s}$, taken round a small circular contour with centre a_s, contributes to the integral

$$\frac{\phi(a_s)}{(a_s-a_1)(a_s-a_2)\dots(a_s-a_r)} \cdot 2\pi\iota,$$

this small circle being taken of so small a radius as to exclude all the other poles and not to cut the boundary.

Hence the whole integral taken round the contour, viz. $\int f(z)\,dz$, being equal to the sum of the integrals round the small circles which surround the several infinities,

$$=2\pi\iota\sum_1^r \frac{\phi(a_s)}{(a_s-a_1)(a_s-a_2)\dots(a_s-a_r)};$$

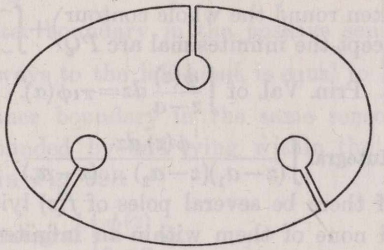

Fig. 394.

the factor a_s-a_s being omitted, $=2\pi\iota\sum_1^r \lambda_s$, say, where the value of λ_1 may be reproduced as $Lt_{\delta=0}\,\delta f(a_1+\delta)$, *i.e.*

$$Lt_{\delta=0}\frac{\phi(a_1+\delta)}{(a_1+\delta-a_2)(a_1+\delta-a_3)\dots(a_1+\delta-a_r)},$$

and similarly for λ_2, λ_3, etc.; or by the ordinary rules of partial fractions.

The effect of *pole-clusters* within a contour will be discussed in Art. 1317.

1288. **Effect of a Branch Point.**

If the function w be multiple-valued, say two-valued, but each branch being continuous and finite and possessing a differential coefficient at all points of a certain region Γ of the z-plane, Cauchy's theorem as to the integral of $\int w\,dz$ from a point A to a point B of this region along a path which does not pass beyond the boundary of Γ is still true, provided that the paths from A to B belong to the same branch of w; and as long as the paths ACB, ADB of Fig. 378 are both finite paths of the variation of w_1 lying entirely in the region Γ, or both finite paths of the variation of w_2, the theorem stated is still true, viz. that

$$\int w_1\,dz \text{ along } ACB=\int w_1\,dz \text{ along } ADB$$

and $$\int w_2\,dz \text{ along } ACB=\int w_2\,dz \text{ along } ADB.$$

When, however, the z-path encircles a branch point in one of these paths from A to B, the functions w_1 and w_2 interchange values, and the integrals of $\int w\,dz$ along two such paths may differ.

1289. For instance, in the case of the two-valued function w defined by the equation $w^2=1+z$, we have two branches

$$w_1=+\sqrt{1+z},\quad w_2=-\sqrt{1+z},$$

and there is a branch point at $z=-1$, and, as will be seen later, one also at ∞.

To examine this case, put $z=-1+re^{\iota\theta}$, and let z travel round a small circle of radius r with centre at $z=-1$, and let us start with the branch

$$w_1=+\sqrt{1+z}=+\sqrt{re^{\iota\theta}}.$$

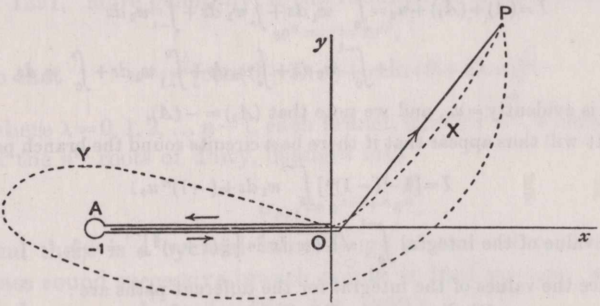

Fig. 395.

Then, in encircling the point -1, θ increases to $\theta+2\pi$ and $e^{\iota\theta}$ becomes $e^{\iota(\theta+2\pi)}$.

Hence w has changed from $\sqrt{re^{\iota\theta}}$ to $\sqrt{re^{\iota(\theta+2\pi)}}$, *i.e.* to $e^{\iota\pi}\sqrt{re^{\iota\theta}}$, and has become $-\sqrt{re^{\iota\theta}}$, *i.e.* w_2.

Now, any path from O to P will be reconcilable with (1) a number of loops round -1, (2) a straight-line path, and the integral will be

$$I=(A^n)+u_n.$$

Now, (1) in case of a path such as OXP, which is reconcilable with the straight line OP (Fig. 395), we have

$$I=\int_0^z w_1\,dz=u_0.$$

(2) In case of a single encirclement of the branch point

$$(A)=\int_0^{-1} w_1\,dz+\int_c w_1\,dz+\int_{-1}^0 w_2\,dz,$$

where $\int_c$ represents the value of the integration round the infinitesimal circle; and this $=\int_0^{2\pi}\sqrt{re^{\iota\theta}}\,(\iota re^{\iota\theta})\,d\theta$, and vanishes when r is indefinitely small.

The third integral $\int_{-1}^{0} w_2\,dz = -\int_{0}^{-1} w_2\,dz = \int_{0}^{-1} w_1\,dz$, for $w_2 = -w_1$;

$$\therefore\ (A) = 2\int_{0}^{-1} w_1\,dz.$$

We thus arrive back at O with the value $w = w_2$, and with this value must continue along the line OP.

Thus, $$u_1 = \int_{0}^{z} w_2\,dz = -u_0,$$

where u_1 is the contribution of the path OP after one encirclement of A. The whole integral is therefore

$$I = 2\int_{0}^{-1} w_1\,dz - u_0.$$

(3) If there be two circuits of the loop before reaching P, we have

$$I = (A) + (A_1) + u_2 = \int_{0}^{-1} w_1\,dz + \int_c w_1\,dz + \int_{-1}^{0} w_2\,dz$$
$$+ \int_{0}^{-1} w_2\,dz + \int_c w_2\,dz + \int_{-1}^{0} w_1\,dz + \int_{0}^{z} w_1\,dz,$$

which is evidently $= u_0$, and we note that $(A_1) = -(A)$.

(4) It will thus appear that if there be n circuits round the branch point,

$$I = [1 - (-1)^n]\int_{0}^{-1} w_1\,dz + (-1)^n u_0.$$

The value of the integral $\int_{0}^{-1} \sqrt{1+x}\,dx$ is $\left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^{-1} = -\frac{2}{3}$.

Hence the values of the integral for the different paths are:

(1) direct path, u_0;
(2) one loop + direct path, $-\frac{4}{3} - u_0$;
(3) two loops + direct path, u_0;
(4) three loops + direct path, $-\frac{4}{3} - u_0$;

and so on, alternating in value.

Hence, if $u = \int_{0}^{z} \sqrt{1+z}\,dz$, and z is thence regarded as a function of u, say $z \equiv \phi(u)$, we have $z \equiv \phi(u_0) = \phi(-\frac{4}{3} - u_0)$, indicating that two values of the argument lead to one and the same value of z.

1290. In the case of any branch point at a point $z = a$ of a function $w = f(z-a)$, which is such that $Lt_{z=a}|f(z-a)\,dz|$ is zero, as in the case considered in Art. 1289, the contribution due to the circular portion of the loop is zero, being

$$\int_{0}^{2\pi} f(re^{\iota\theta})\,\iota re^{\iota\theta}\,d\theta,$$

and vanishing with r, since $Lt_{r=0}|rf(re^{\iota\theta})|$ vanishes; and the only contribution from the loop is that due to the two banks of the canal portion of the loop.

If the function w be two-valued, it has been seen that in passing round the branch point w_1 and w_2 interchange values, and the contribution of the loop is

$$I=\int_0^a w_1\,dz+\int_c w_1\,dz+\int_a^0 w_2\,dz\,;$$

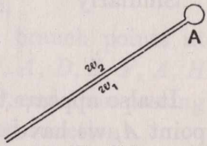

Fig. 396.

and in the case considered, viz.

$$Lt_{z=a}|w_1\,dz|=0,$$

$$\int_c w_1\,dz=0,$$

whilst $\displaystyle\int_a^0 w_2\,dz=\int_0^a w_1\,dz$ and $\displaystyle I=2\int_0^a w_1\,dz=(A).$

1291. More generally, if the function be n-valued, such as

$$w^n=z=re^{\iota\theta},$$

so that $$w=r^{\frac{1}{n}}[\cos(\theta+2\lambda\pi)+\iota\sin(\theta+2\lambda\pi)]^{\frac{1}{n}},$$

where $\lambda=0,1,2,\ldots n-1$, each branch $w_s=\alpha^s r^{\frac{1}{n}} e^{\frac{\iota\theta}{n}}$, where $\alpha=$ one of the n^{th} roots of unity, changes into

$$w_{s+1}=\alpha^{s+1} r^{\frac{1}{n}} e^{\frac{\iota\theta}{n}},$$

and there is a cyclical interchange of the value of w as we pass round successive branch points, so that $w_2=\alpha w_1$, $w_3=\alpha w_2$, and so on, and $\alpha^n=1$. (See Art. 1259.)

So in this case, $$I=\int_0^a w_1\,dz+\int_a^0 w_2\,dz$$

becomes $$I=(1-\alpha)\int_0^a w_1\,dz.$$

1292. To return to the case of a two-valued function, if after a description of the A-loop, starting from the origin with value $w=w_1$, we pass along a second loop round another branch point B, we start off along the second loop with the value w_2 and return with the value w_1, and for the two loops

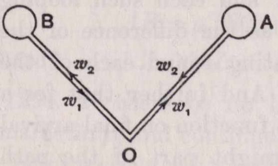

Fig. 397.

$$\begin{aligned} I&=\int_0^a w_1\,dz+\int_c w_1\,dz+\int_a^0 w_2\,dz\\ &\quad+\int_0^b w_2\,dz+\int_{c'} w_2\,dz+\int_b^0 w_1\,dz\\ &=2\int_0^a w_1\,dz-2\int_0^b w_1\,dz\\ &=(A)-(B),\ \text{say,}\end{aligned}$$

and this we shall call (AB) for shortness, so that

$$(AB)=(A)-(B).$$

Similarly $$(ABC)=(A)-(B)+(C),$$

$$(ABCD)=(A)-(B)+(C)-(D),$$

and so on.

It also appears that in a double looping of the same branch point A, we have

$$(AA)=(A)-(A)=0.$$

In a triple looping of A,

$$(AAA)=(A)-(A)+(A)=(A).$$

These peculiarities are indicated in the notation

$$(A^{2n})=0, \quad (A^{2n+1})=(A).$$

So we have

$$(AB)=(A)-(B), \quad (BA)=(B)-(A), \quad (AB)+(BA)=0,$$

$$(ABC)=(A)-(B)+(C)=(AB)+(C)=(AB)+(C)-(A)+(A)$$
$$=(AB)+(CA)+(A),$$

$$(A^2BC)=(AABC)=(A)-(A)+(B)-(C)=(BC)=(AC)+(BA),$$

$$(A^3BC)=(A)-(A)+(A)-(B)+(C)=(AB)+(C) \text{ or } (A)-(BC)$$
$$\text{or } (A)+(CB).$$

For a double looping of any pair,

$$(ABAB)=(A)-(B)+(A)-(B)=2(A)-2(B).$$

For n-encirclings of A and B we may write

$$(AB)^n=n(A-B).$$

Again, $$(B)=(B)-(A)+(A)=(BA)+(A),$$

$$(BCD)=(B)-(C)+(D)=(B)-(C)+(D)-(A)+(A)$$
$$=(BC)+(DA)+(A).$$

1293. It appears then that to integrate round any combination of these branch points, the whole can be expressed linearly in terms of integration round any one loop, say the A-loop, together with an integration round a combination of double loops round pairs of others; and each such looping of two branch points is expressible as the difference of the integrals which accrue from integrating round each of the separate branch points of the pair. And further, that for a two-valued function the value of the function on final arrival at O, and before starting on the straight part of the path from O to P, depends upon how many times the path has surrounded a branch point, and the final integration along the straight path adds $+u_0$ if an even number of circlings has been effected, and $-u_0$ if the number be odd.

Thus, if O be the origin, and there be branch points at A, B, C, D, E, F, G, H, a path in which B, C, A, D, E, F, A, H are successively looped before returning to O, and then passing to P, will give the integral of a two-branched function

$$(B)-(C)+(A)-(D)+(E)-(F)+(A)-(H)+(-1)^8 u_0,$$

and integration for a path for the loops round B, C, A, D, E will give

$$(B)-(C)+(A)-(D)+(E)-(A)+(A)+(-1)^7 u_0,$$

and these may be respectively written

$$(BC)+(AD)+(EF)+(AH)+u_0,$$
$$(BC)+(AD)+(EA)+(A)-u_0.$$

Now, if there be n critical points $A, B, C, D, \ldots$, there are $\frac{n(n-1)}{2}$ sets of differences (we omit the brackets for short),

$$\begin{array}{llll} A-B, & A-C, & A-D, & A-E, \ldots, \\ & B-C, & B-D, & B-E, \ldots, \\ & & C-D, & C-E, \ldots, \\ & & & D-E, \ldots, \end{array}$$

and only $n-1$ of them are independent, say

$$A-B, \quad B-C, \quad C-D, \quad D-E, \ldots;$$

for any other, such as $B-E$, may be expressed as

$$(B-C)+(C-D)+(D-E).$$

Hence the value of $\int w\,dz$ taken along any path from O to P must take one or other of the following forms:

$$\lambda\,(AB)+\mu\,(BC)+\nu\,(CD)+\ldots+\kappa(EF)+u_0,$$

or $$\lambda'(AB)+\mu'(BC)+\nu'(CD)+\ldots+\kappa'(EF)+(A)-u_0,$$

where $\lambda, \mu, \nu, \ldots, \lambda', \mu', \nu', \ldots$, are integers, positive or negative.

1294. If there be no branch point at infinity, and if w remains finite and continuous for all other points of the z-plane, an infinite circle, with centre at the origin, will contain all the branch points, and can be deformed into a system of loops, each passing round a branch point once, as in Fig. 398; or in case they lie in a straight line, as in Fig. 399; and the region

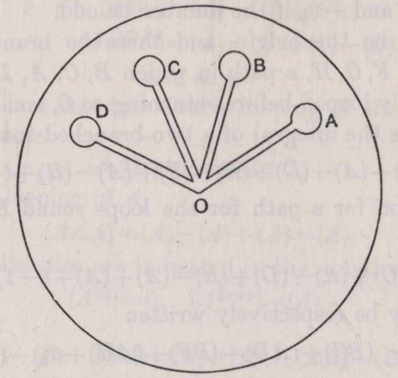

Fig. 398.

between this circle and the loop system being synectic, we have $\int w\,dz$, taken round the infinite circle, $=(A)-(B)+(C)-(D)+\ldots$, and $\int w\,dz$ round the infinite circle will be a definite quantity which, in such cases as

$$w^2=\frac{1}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)}$$

or $$w^2=\frac{1}{(z-a_1)(z-a_2)(z-a_3)(z-a_4)(z-a_5)(z-a_6)},$$

will vanish. For, taking the first of these, and putting

$$z=Re^{\iota\theta}\ (R=\infty),\quad \frac{dz}{z}=\iota\, d\theta;$$

$$\therefore \int w\,dz=\int\frac{1}{z^2}dz=\int_0^{2\pi}\frac{\iota\, d\theta}{Re^{\iota\theta}}=0, \text{ when } R=\infty;$$

and similarly in the second expression.

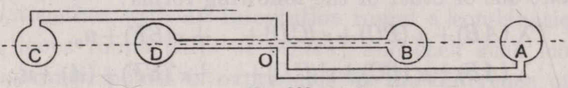

Fig. 399.

Thus in such cases there is a relation amongst these differences, viz. $(A)-(B)+(C)-(D)+\ldots=0.$

In the case of four branch points, the independent differences will reduce from three, $\{(A)-(B), (B)-(C), (C)-(D)\}$, to two, say $(A)-(B)$, $(B)-(C)$.

And the forms possible for the value of the integration along paths from O to P will be comprised in

$$I=\lambda\,(AB)+\mu\,(BC)+u_0,$$
$$I=\lambda'(AB)+\mu'(BC)+(A)-u_0.$$

1295. **Representation for Large Values of z; Branch Points at Infinity.**

To represent the nature of the function for values of z at an infinite distance from the origin, take a third variable z', such that $zz'=1$, and represent the travels of z' on a plane of its own. Then, for points z on the z-plane which are at great distance from the origin O, the points z' on the z'-plane are near the new origin O' on the z'-plane.

Taking the function

$$w_1=\frac{1}{\sqrt{(z-a_1)(z-a_2)(z-a_3)\dots(z-a_n)}},$$

which is a branch of a two-valued function, let us find the branch points.

Let O be the origin on the z-plane $A_1, A_2, \dots A_n$, the several points $z=a_1$, $z=a_2$, $z=a_3$, $\dots$, and let P be the point z.

Let $$z=a_1+r_1e^{\iota\theta_1}=a_2+r_2e^{\iota\theta_2}=a_3+r_3e^{\iota\theta_3}=\dots.$$

Then $$w_1=\frac{1}{\sqrt{r_1r_2r_3\dots e^{\iota(\theta_1+\theta_2+\theta_3+\dots)}}}.$$

Let P describe a small circle round any one of the points, say a_1. Then, after the completion of this circle, $r_1, r_2, r_3, \dots$ and $\theta_2, \theta_3, \theta_4, \dots$ have resumed their original values, but θ_1 has become $\theta_1+2\pi$.

Hence the function w_1 has become $\frac{w_1}{e^{\iota\pi}}$, *i.e.* $-w_1$ or w_2, and therefore there is a change of branch at A_1. Similarly at $A_2, A_3, \dots$. Now consider the case when $z=\infty$.

Using the other representation we have, writing $a_1=\frac{1}{a_1'}$, $a_2=\frac{1}{a_2'}$, etc.,

$$w_1=\frac{1}{\sqrt{\left(\frac{1}{z'}-\frac{1}{a_1'}\right)\left(\frac{1}{z'}-\frac{1}{a_2'}\right)\dots\left(\frac{1}{z'}-\frac{1}{a_n'}\right)}}=\frac{\sqrt{a_1'a_2'a_3'\dots a_n'}\,z'^{\frac{n}{2}}}{\sqrt{(a_1'-z')(a_2'-z')\dots(a_n'-z')}},$$

and we have to consider the behaviour of this function for values of z' near the origin O' on the z'-plane.

Putting $z'=re^{\iota\theta'}$, we have ultimately, when r is very small, $w=r^{\frac{n}{2}}e^{\iota\frac{n\theta'}{2}}$, and when z' is made to describe a small circle of radius r about the z'-origin O', θ' has changed by 2π, and the function becomes multiplied by $e^{\iota n\pi}$, *i.e.* by

$$(\cos n\pi+\iota\sin n\pi) \quad\text{or}\quad \cos n\pi.$$

Hence, if n be even, w_1 remains unchanged, but if n be odd w_1 changes into $-w_1$, *i.e.* there is a change from branch w_1 to branch w_2.

1296. Thus, in the cases

$$w=\frac{1}{\sqrt{(z-a_1)(z-a_2)}} \quad\text{and}\quad w=\frac{1}{\sqrt{(z-a_1)(z-a_2)(z-a_3)(z-a_4)}},$$

there are respectively two and four branch points, viz. $z=a_1$ and $z=a_2$ in the first, and $z=a_1$, $z=a_2$, $z=a_3$, $z=a_4$ in the second, but none at ∞.

But in the cases

$$w_1=\frac{1}{\sqrt{(z-a_1)(z-a_2)(z-a_3)}} \quad\text{and}\quad w_1=\frac{1}{\sqrt{(z-a_1)(z-a_2)(z-a_3)(z-a_4)(z-a_5)}}$$

there are branch points at a_1, a_2, a_3 in the first, and at a_1, a_2, a_3, a_4, a_5 in the second, and in both these cases there is also a branch point at ∞.

In the latter cases the loop system, when represented on the z'-plane, will be as discussed previously, the origin being also a branch point. But if represented by loops on the z-plane, we have (taking the case of three factors) a_1, a_2, a_3, ∞ as branch points at A, B, C, D respectively, the latter at infinity, and, as in Art. 1294, there are apparently three independent pairs of differences, which we may take as (AD), (BD), (CD). But writing $w=\{(z-a_1)(z-a_2)(z-a_3)\}^{-\frac{1}{2}}$, we have

$$(AD)=2\int_{a_1}^{\infty} w\,dz, \quad (BD)=2\int_{a_2}^{\infty} w\,dz, \quad (CD)=2\int_{a_3}^{\infty} w\,dz,$$

and we shall show that $(BD)=(AD)+(CD)$, which reduces the three apparently independent pairs to two really independent ones. For $\int w\,dz$ taken round any finite contour in the finite part of the z-plane, which does not include A, B or C and cannot include D, vanishes; and such a contour is deformable into an infinite contour, such as indicated in Fig. 400, with loops excluding the branch points. Therefore $\int w\,dz$ round this deformed contour also vanishes. For convenience this deformation may be taken as a circle of infinite radius centred at the origin, with four loops excluding the branch points, the canals of A, B, C being of infinite length and that of D finite. The contribution to the integral $\int w\,dz$ which accrues from these loops amounts to $(A)-(B)+(C)-(D)$, *i.e.* to $(AD)-(BD)+(CD)$. The remainder of the contour, which consists of infinite circular arcs, along each of which the same branch of w is adhered to, and which each extend from the canal of one loop to the canal of the next, contributes nothing to the integral. For taking any of these arcs, say from $\theta=\alpha$ to $\theta=\beta$, where $z=Re^{\iota\theta}$ and $\alpha<\beta<2\pi$, we have $\int w\,dz=\iota\int_\alpha^\beta zw\,d\theta$, and therefore

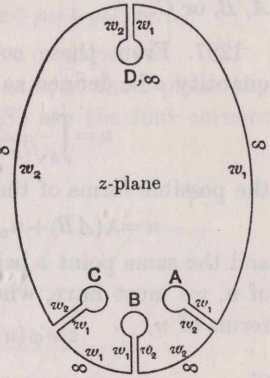

Fig. 400.

$$\text{mod.}\int w\,dz=\text{mod.}\int_\alpha^\beta zw\,d\theta \not> \int_\alpha^\beta \text{mod.}\,(zw)\,d\theta.$$

But mod. (zw) tends continually to a limit zero as mod. z is indefinitely increased, and if K be its greatest value for points on the arc from $\theta=\alpha$ to $\theta=\beta$, $\int_\alpha^\beta \text{mod.}\,(zw)\,d\theta$ is positive and $<K(\beta-\alpha)$, and therefore also tends to a zero limit. Hence the whole integral for the deformed contour is that due to the four loops only, viz. $(AD)-(BD)+(CD)$, which therefore vanishes. It follows that the only possible values of the integral $u=\int_z^\infty \frac{dz}{\sqrt{(z-a_1)(z-a_2)(z-a_3)}}$ are of one or other of the forms

$$p(AD)+q(BD)+r(CD)+u_0,$$

or

$$p'(AD)+q'(BD)+r'(CD)+(A)-u_0,$$

where p, q, etc., are integers, and that by virtue of the relation $(BD)=(AD)+(CD)$ these further reduce to

$$\lambda(AD)+\mu(CD)+u_0 \quad \text{or} \quad \lambda'(AD)+\mu'(CD)+(A)-u_0,$$

where λ, μ, λ', μ' are integers, and u_0 is the value of $\int_z^\infty w\,dz$ by any straight-line path from z to ∞, which does not pass through A, B, or C.

1297. From these considerations it will follow that, if a quantity z be defined as $\phi(u)$, and given by

$$u=\int_0^z \frac{dz}{\sqrt{(z-a_1)(z-a_2)}}=\int_0^z w\,dz, \text{ say,}$$

the possible forms of the result being limited to

$$u=\lambda(AB)+u_0, \quad \text{or} \quad u=\lambda(AB)+(A)-u_0,$$

and the same point z being attained for either of these values of u, we must have, when we regard z as being expressed in terms of u,

$$z\equiv\phi(u)=\phi[\lambda(AB)+u_0],$$

or

$$=\phi[\lambda(AB)+(A)-u_0].$$

ϕ must therefore be a periodic function such that an addition of (AB), *i.e.* $(A)-(B)$, to the argument any number of times makes no difference, and also that, if (A) be added to any number of sets of integrals round double loops (AB), the same will be true if the sign of u_0 be changed.

In the cases

$$u=\int_z^\infty \frac{dz}{\sqrt{(z-a_1)(z-a_2)(z-a_3)}} \quad \text{and} \quad u=\int_0^z \frac{dz}{\sqrt{(z-a_1)(z-a_2)(z-a_3)(z-a_4)}},$$

since

$$u=\lambda(AB)+\mu(BC)+u_0,$$

or

$$\lambda'(AB)+\mu(BC)+(A)-u_0$$

in both cases, for A, B, C are *any* three of the four branch points, we have

$$\phi(u)=\phi[\lambda(AB)+\mu(BC)+u_0],$$

or

$$=\phi[\lambda'(AB)+\mu'(BC)+(A)-u_0],$$

and a double periodicity of $z\equiv\phi(u)$ is established.

1298. **Period Parallelograms.**

A geometrical illustration of this double periodicity may be given.

Let $\phi(z)$ be a doubly periodic function of a single complex variable z with independent periods ω, ω', viz.

$$\omega=a+\iota\beta, \quad \omega'=a'+\iota\beta',$$

so that $\phi(z)=\phi(z+\omega)=\phi(z+2\omega)=\dots$
$$=\phi(z+\omega')=\phi(z+2\omega')=\dots$$
$$=\phi(z+\omega+\omega')=\dots=\phi(z+p\omega+q\omega')=\dots,$$
where p and q are any integers, positive or negative.

Referred to any set of rectangular axes in the z-plane, the points $(0, 0)$, (α, β), $(\alpha+\alpha', \beta+\beta')$, (α', β') are the four corners of a parallelogram (Fig. 401).

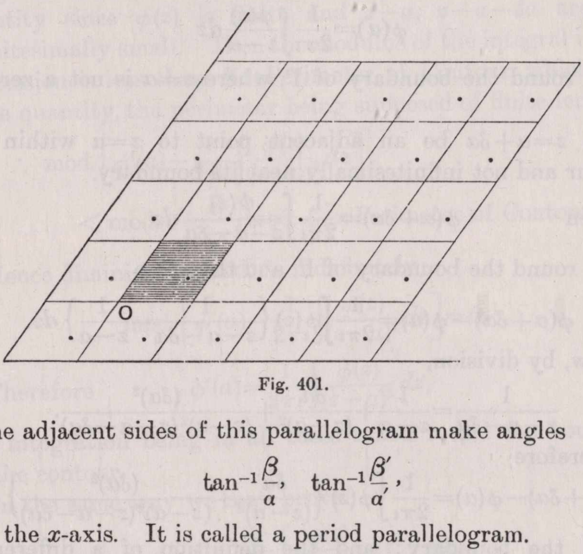

Fig. 401.

The adjacent sides of this parallelogram make angles
$$\tan^{-1}\frac{\beta}{\alpha},\quad \tan^{-1}\frac{\beta'}{\alpha'},$$
with the x-axis. It is called a period parallelogram.

The four points, $p\alpha+\iota q\beta$, $(p+1)\alpha+\iota(q+1)\beta$,
$$\{(p+1)\alpha+\alpha'\}+\iota\{(q+1)\beta+\beta'\},\quad (p\alpha+\alpha')+\iota(q\beta+\beta'),$$
will equally form the angular points of a parallelogram of the same size and shape as before. The whole z-plane may be regarded as mapped out into a network of such equal parallelograms by giving to p and q all integral values. As z travels over the region bounded by any one of these parallelograms, $\phi(z)$ ranges through all the values it is capable of assuming. If z travels into other parallelograms on the z-plane the values of $\phi(z)$ are merely repetitions of the values it attained at corresponding points within the first parallelogram. Thus points similarly situated with regard to any elementary parallelogram of the network give the same value of $\phi(z)$.

1299. **If $\phi(z)$ be Synectic throughout Γ, so also are its Differential Coefficients.**

We shall next show that when $\phi(z)$ is synectic within and upon the boundary of a given region bounded by a closed finite contour Γ, all its differential coefficients are synectic within that region.

We have seen that if a be a point within the region and not within an infinitesimal distance of the boundary,

$$\phi(a)=\frac{1}{2\pi\iota}\int\frac{\phi(z)}{z-a}\,dz$$

taken round the boundary of Γ, where $z=a$ is not a zero of $\phi(z)$.

Let $z=a+\delta a$ be an adjacent point to $z=a$ within the contour and not infinitesimally near its boundary.

Then
$$\phi(a+\delta a)=\frac{1}{2\pi\iota}\int\frac{\phi(z)}{z-a-\delta a}\,dz$$

taken round the boundary of Γ, and therefore

$$\phi(a+\delta a)-\phi(a)=\frac{1}{2\pi\iota}\int\phi(z)\left\{\frac{1}{z-a-\delta a}-\frac{1}{z-a}\right\}dz.$$

Now, by division,

$$\frac{1}{z-a-\delta a}=\frac{1}{z-a}+\frac{\delta a}{(z-a)^2}+\frac{(\delta a)^2}{(z-a)^2(z-a-\delta a)}.$$

Therefore

$$\phi(a+\delta a)-\phi(a)=\frac{1}{2\pi\iota}\int\phi(z)\left\{\frac{\delta a}{(z-a)^2}+\frac{(\delta a)^2}{(z-a)^2(z-a-\delta a)}\right\}dz$$

round the boundary; and the definition of a differential coefficient is that it is the limit, if there be one, of

$$\frac{\phi(a+\delta a)-\phi(a)}{\delta a}\quad\text{(Art. 1239)},$$

when $|\delta a|$ is made indefinitely small. Hence we may put

$$\phi(a+\delta a)-\phi(a)=\{\phi'(a)+\epsilon\}\,\delta a,$$

where ϵ is something whose modulus ultimately vanishes with $|\delta a|$.

We may therefore write

$$\phi'(a)+\epsilon=\frac{1}{2\pi\iota}\int\phi(z)\left\{\frac{1}{(z-a)^2}+\frac{\delta a}{(z-a)^2(z-a-\delta a)}\right\}dz$$

or
$$\phi'(a)-\frac{1}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^2}\,dz=-\epsilon+\frac{\delta a}{2\pi\iota}\int\frac{\phi(z)\,dz}{(z-a)^2(z-a-\delta a)},\ \ldots(1)$$

and therefore the moduli of the two sides of this equation are equal. And since the modulus of the sum of two complex quantities is less than the sum of their moduli, and the modulus of the product is the product of the moduli, we have

$$\text{mod.}[\text{right-hand side}] < \text{mod.}\,\epsilon + \frac{\text{mod.}\,\delta a}{2\pi}\,\text{mod.}\int\frac{\phi(z)}{(z-a)^2(z-a-da)}\,dz.$$

Let K be the greatest of the moduli of the values of the integrand as we travel round the boundary, which is a finite quantity since $\phi(z)$ is finite and $z-a$, $z-a-\delta a$ are not infinitesimally small. Then the modulus of the integral in this expression is less than $K\times$ Perimeter of Contour, which is a finite quantity, the perimeter being supposed of finite length;

$$\therefore\ \text{mod.}\left[\phi'(a)-\frac{1}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^2}\,dz\right]$$
$$< \text{mod.}\,\epsilon+\frac{K}{2\pi}\,.\,\text{mod.}\,\delta a\times\text{Perimeter of Contour.}$$

Hence diminishing mod. δa indefinitely,

$$\text{mod.}\left[\phi'(a)-\frac{1}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^2}\,dz\right]=0.$$

Therefore $$\phi'(a)=\frac{1}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^2}\,dz,$$

the integration being in all cases taken round the boundary of the contour.

In the same way we may prove

$$\phi''(a)=\frac{2!}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^3}\,dz,\quad\text{etc.}$$

For if $z=a+\delta a$ be a point within the contour and not within an infinitesimal distance of the boundary, we have

$$\phi'(a+\delta a)=\frac{1}{2\pi\iota}\int\frac{\phi(z)}{(z-a-\delta a)^2}\,dz,$$

and $$\frac{\phi'(a+\delta a)-\phi'(a)}{\delta a}=\frac{1}{2\pi\iota}\int\phi(z)\left[\frac{1}{(z-a-\delta a)^2}-\frac{1}{(z-a)^2}\right]\frac{dz}{\delta a}$$
$$=\frac{1}{2\pi\iota}\int\phi(z)\left[\frac{2}{(z-a)^3}\right]dz+\theta,$$

where mod. θ vanishes with mod. δa,

$$=\frac{2!}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^3}\,dz+\theta.$$

It appears therefore

(1) that $\dfrac{\phi'(a+\delta a)-\phi'(a)}{\delta a}$ approaches to and ultimately differs by less than any conceivable quantity from $\dfrac{2!}{2\pi\iota}\displaystyle\int\frac{\phi(z)}{(z-a)^3}dz$, when mod. δa is made to diminish indefinitely without reference to the way in which the indefinite approach of the point $a+\delta a$ to the point a is conducted. Hence $\phi'(a)$ is a function of a which *possesses a differential coefficient*;

(2) since $\phi(a)$ and $\phi(a+\delta a)$ are by supposition single-valued, the expression $\dfrac{\phi(a+\delta a)-\phi(a)}{\delta a}$ is also single-valued, and also its limit; so $\phi'(a)$ is *single-valued*;

(3) $\phi'(a)$ is *finite*; for its equivalent $\dfrac{1}{2\pi\iota}\displaystyle\int\frac{\phi(z)}{(z-a)^2}dz$ is such that the integrand is finite for all points upon the contour, since the point a is not at an infinitesimal distance from the boundary, and the boundary itself is of finite length by supposition;

(4) for any positive infinitesimal change in $|\delta a|$ there is a change

$$|\{\phi'(a+\delta a)-\phi'(a)\}| \ngtr |\delta a| \left|\left\{\frac{2!}{2\pi a}\int\frac{\phi(z)}{(z-a)^2}+\theta\right\}\right|$$

of the same order as $|\delta a|$ in $|\phi'(a)|$. Hence $\phi'(a)$ is *continuous*.

Hence $\phi'(a)$ has a *differential coefficient* at the point a, is *single-valued*, is *finite* and is *continuous*. It is therefore synectic at any point a within the specified region for which $\phi(a)$ is synectic.

Also
$$\phi''(a)=\frac{2!}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^3}dz,$$

the integration proceeding, as before, round the boundary. And the argument may now be repeated with this result to establish the successive equations,

$$\phi'''(a)=\frac{3!}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^4}dz \ldots \phi^{(n)}(a)=\frac{n!}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^{n+1}}dz, \ldots,$$

all of which functions are synectic in the region for which $\phi(a)$ is synectic.

1300. **Taylor's and Maclaurin's Theorem.**

We may now proceed to establish Taylor's Theorem for the expansion of $f(a+h)$. Let $f(z)$ be any function of z which is synectic within and upon a given circle C with centre at $z=a$ and radius ρ, and suppose $z=a$ not to be a zero of $f(z)$. Let $a+h$ be another point within this contour and not within an infinitesimal distance of the boundary.

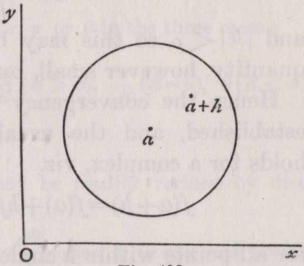

Fig. 402.

Then

$$f(a+h)=\frac{1}{2\pi\iota}\int\frac{f(z)}{z-a-h}\,dz,$$

the integration being conducted round the boundary.

Now, by division,

$$\frac{1}{z-a-h}=\frac{1}{z-a}+\frac{h}{(z-a)^2}+\frac{h^2}{(z-a)^3}+\dots$$
$$+\frac{h^n}{(z-a)^{n+1}}+\frac{h^{n+1}}{(z-a)^{n+1}}\frac{1}{z-a-h};$$

$$\therefore\ f(a+h)=\frac{1}{2\pi\iota}\int f(z)\left[\frac{1}{z-a}+\frac{h}{(z-a)^2}+\frac{h^2}{(z-a)^3}+\dots\right.$$
$$\left.+\frac{h^n}{(z-a)^{n+1}}+\frac{h^{n+1}}{(z-a)^{n+1}}\frac{1}{z-a-h}\right]dz$$
$$=\frac{1}{2\pi\iota}\left[\int\frac{f(z)}{z-a}\,dz+h\int\frac{f(z)}{(z-a)^2}\,dz+h^2\int\frac{f(z)}{(z-a)^3}\,dz+\dots\right.$$
$$\left.+h^n\int\frac{f(z)}{(z-a)^{n+1}}\,dz\right]+\frac{h^{n+1}}{2\pi\iota}\int\frac{f(z)dz}{(z-a)^{n+1}(z-a-h)}$$
$$=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots+\frac{h^n}{n!}f^{(n)}(a)+R_n,$$

where $R_n=\dfrac{h^{n+1}}{2\pi\iota}\displaystyle\int\frac{f(z)}{(z-a)^{n+1}(z-a-h)}\,dz$ taken round the circle; and putting $z=a+\rho e^{\iota\theta}$, we have

$$R_n=\frac{h^{n+1}}{2\pi}\,\frac{1}{\rho^n}\int\frac{f(z)}{z-a-h}\,e^{-n\iota\theta}\,d\theta.$$

Let the greatest value of $\left|\dfrac{f(z)}{z-a-h}\,e^{-n\iota\theta}\right|$ be K, which is finite since $|f(z)|$ is finite at all points within the circle, and the point $z=a+h$ is not within an infinitesimal distance of the boundary.

Then $$|R_n| \ngtr \frac{1}{2\pi}\left|\frac{h^{n+1}}{\rho^n}\right|\int_0^{2\pi} K\,d\theta,$$

i.e. $$|R_n| \ngtr \left|\frac{h}{\rho}\right|^n . |h| . K,$$

and $|h| < \rho$, so this may be made less than any assignable quantity, however small, by increasing n indefinitely.

Hence the convergency within the circle of radius ρ is established, and the usual form of Taylor's theorem still holds for a complex, viz.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots \text{ to } \infty$$

for all points within a circle of centre a and radius $> |(a+h)|$, provided $f(z)$ is synectic for all points within this region.

If the origin be at the point $z=a$, *i.e.* $a=0$, we have the same result as for Maclaurin's theorem for a real variable, viz.

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \dots,$$

with the same limitations as before.

1301. **Definite Integrals obtained by Contour Integration.**

Cauchy's Theorem of Art. 1275 is of great use in establishing in a rigorous manner many results in definite integrals and in furnishing new results. In such investigations the form of w as a function of z is at our choice, and the particular contour of integration is also at our choice.

Consider the integration of $\int \frac{dz}{z-a}$ *round any closed contour, a being supposed real.*

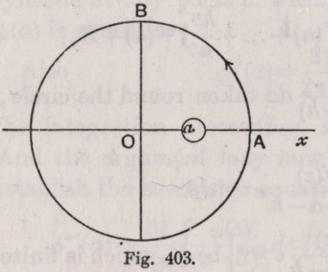

Fig. 403.

It follows from Arts. 1275 and 1286, that the result of this integration is

(1) $2\pi\iota$, (2) $\pi\iota$ or (3) 0,

according as

(1) the contour encloses the point $z=a$;

(2) the contour passes through $z=a$ with continuous curvature at the point;

(3) the contour is such that $z=a$ lies outside it.

Take as contour a circle of radius R (drawn as $> a$ in the figure) and centred at the origin.

Put $z=Re^{\iota\theta}$; then $dz=\iota Re^{\iota\theta}\,d\theta$;

$$\therefore \int_0^{2\pi}\frac{Re^{\iota\theta}.\,\iota\,d\theta}{Re^{\iota\theta}-a}=2\pi\iota,\ \pi\iota \text{ or } 0, \text{ as } R>a,\ =a \text{ or } <a;$$

whence $$\int_0^{2\pi}\frac{Re^{\iota\theta}(Re^{-\iota\theta}-a)}{R^2-2aR\cos\theta+a^2}\,d\theta=2\pi,\ \pi \text{ or } 0 \text{ in the three cases};$$

whence $$\int_0^{2\pi}\frac{R-a\cos\theta}{R^2-2aR\cos\theta+a^2}\,d\theta=\frac{2\pi}{R}\,(R>a),\ \ \frac{\pi}{R}\,(R=a),\ \ 0\,(R<a),$$

and $$\int_0^{2\pi}\frac{\sin\theta}{R^2-2aR\cos\theta+a^2}\,d\theta=0$$

in any of the cases, results which may be readily verified by direct integration.

1302. *Consider the integration of* $w\equiv\dfrac{e^{\iota kz}}{z}$, *where k is real and positive, round a contour bounded by* (1) *an infinite semicircle BCD, centre at the origin of the x-y axes, radius R* $(=\infty)$, (2) *a small semicircle EFA, centre at the origin and radius r, concave in the same direction as the former, and* (3) *the two intercepted portions of the x-axis, viz. DE and AB.*

w has a pole at the origin. The small semicircle excludes this pole. Examine the behaviour of the function when z is infinite.

Let $z=Re^{\iota\theta}$. Then $w=\dfrac{e^{\iota kRe\iota\theta}}{Re^{\iota\theta}}=\dfrac{e^{-kR\sin\theta}\{\cos(kR\cos\theta)+\iota\sin(kR\cos\theta)\}}{Re^{\iota\theta}}$,

and therefore vanishes in the limit when R is increased indefinitely, so long as $\sin\theta$ is not negative; that is from $\theta=0$ to $\theta=\pi$ inclusive. There is no pole in the region described, and w is synectic throughout the region. The total integral $\int w\,dz$ taken round this perimeter therefore vanishes. To estimate this we consider the integrations:

(1) from r to R $(=\infty)$ along the x-axis;

(2) from $\theta=0$ to $\theta=\pi$ round the great semicircle BCD;

(3) from $-R$ to $-r$ along the x-axis;

(4) from $\theta=\pi$ to $\theta=0$ round the small semicircle EFA.

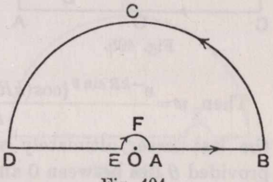

Fig. 404.

(1) Along AB, $y=0$ and $dz=dx$, and the corresponding contribution to the whole integral is $\displaystyle\int_r^\infty \frac{e^{\iota kx}}{x}\,dx$.

(2) Along BCD, $R=$constant, $z=Re^{\iota\theta}$, $\dfrac{dz}{z}=\iota\,d\theta$, and the contribution to the whole is

$$\int\frac{e^{\iota kz}}{z}\,dz=\int_0^\pi e^{\iota kRe^{\iota\theta}}\,\iota\,d\theta=\int_0^\pi \iota e^{-kR\sin\theta}\{\cos(kR\cos\theta)+\iota\sin(kR\cos\theta)\}\,d\theta,$$

which ultimately vanishes when R increases indefinitely. Therefore there is no contribution from this part of the integration.

(3) Along DE, $\int \frac{e^{\iota kz}}{z} dz = \int_{-\infty}^{-r} \frac{e^{\iota kx}}{x} dx$, and as x is negative we write $-x$ for x,

$$= -\int_r^{\infty} \frac{e^{-\iota kx}}{x} dx,$$

which is the contribution for this portion DE of the integration.

(4) Round the small semicircle the contribution is $\int_{\pi}^{0} e^{\iota k r e^{\iota\theta}} \iota\, d\theta$, and r being infinitesimally small this becomes $-\int_0^{\pi} \iota\, d\theta = -\pi\iota$.

Hence, summing up,

$$\int_r^{\infty} \frac{e^{\iota kx}}{x} dx + 0 - \int_r^{\infty} \frac{e^{-\iota kx}}{x} dx - \pi\iota = 0,$$

i.e. in the limit when r is indefinitely diminished,

$$\int_0^{\infty} \frac{e^{\iota kx} - e^{-\iota kx}}{x} dx = \iota\pi \quad \text{or} \quad \int_0^{\infty} \frac{\sin kx}{x} dx = \frac{\pi}{2},$$

k being supposed positive, which is in accord with the result of Art. 993.

1303. *Consider* $\int \frac{e^{\iota kz}}{z-a} dz$, *where* k *is a real positive quantity and* a *is a complex, viz.* $\alpha + \iota\beta$, *in which* β *is positive.*

We take as contour the x-axis, an infinite semicircle whose centre is at the origin and radius R $(=\infty)$, and an infinitesimal circle of radius r, and centre at the real point (α, β), which, since β is positive, lies within the great semicircle.

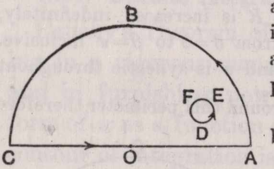

Fig. 405.

There is a pole at $z=a$, which is excluded by the small circle. Examine the behaviour of $w = \frac{e^{\iota kz}}{z-a}$, when z is infinite. Put $z = Re^{\iota\theta}$.

Then $w = \frac{e^{-kR\sin\theta}\{\cos(kR\cos\theta) + \iota\sin(kR\sin\theta)\}}{Re^{\iota\theta} - a}$, and therefore, as in the last case, ultimately vanishes when R is indefinitely increased, provided θ lies between 0 and π inclusive.

There is no pole in the region *between* the two circles, and w is synectic throughout it; and $\int w\, dz = 0$ when taken round the boundaries in opposite directions.

(1) Along the x-axis $z=x$, and we have as the part contributed by integrating from C to A, *i.e.* $-\infty$ to ∞,

$$\int_{-\infty}^{\infty} \frac{e^{\iota kx}}{x - \alpha - \iota\beta} dx = \int_{-\infty}^{\infty} \frac{(x - \alpha + \iota\beta)(\cos kx + \iota \sin kx)}{(x-\alpha)^2 + \beta^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{\{(x-\alpha)\cos kx - \beta \sin kx\}}{(x-\alpha)^2 + \beta^2} dx + \iota \int_{-\infty}^{\infty} \frac{(x-\alpha)\sin kx + \beta\cos kx}{(x-\alpha)^2 + \beta^2} dx.$$

(2) Round the infinite semicircle, we have a contribution

$$\int_0^\pi \frac{e^{\iota k Re^{\iota\theta}}}{Re^{\iota\theta}-a} Re^{\iota\theta}\iota\, d\theta = \int_0^\pi \frac{e^{-kR\sin\theta}\{\cos(kR\cos\theta)+\iota\sin(kR\cos\theta)\}}{Re^{\iota\theta}-a} Re^{\iota\theta}\iota\, d\theta,$$

which, by virtue of the ultimately zero factor $e^{-kR\sin\theta}$, adds nothing, R being absolutely infinite and $\sin\theta$ positive.

(3) Round the infinitesimal circle DEF, put $z=a+re^{\iota\theta}$.

The integration round the perimeter must give $2\pi\iota e^{\iota k(a+\iota\beta)}$, according to the general result of Art. 1286, *i.e.* $=2\pi(\iota\cos ka-\sin ka)e^{-k\beta}$; whence, as $\int f(z)\,dz$ round the outer boundary $ABCOA$ is equal to that round DEF in the same sense, we have by equating real and imaginary parts,

$$\left.\begin{aligned}\int_{-\infty}^{\infty}\frac{(x-a)\cos kx-\beta\sin kx}{(x-a)^2+\beta^2}dx &= -2\pi e^{-k\beta}\sin ka,\\ \int_{-\infty}^{\infty}\frac{(x-a)\sin kx+\beta\cos kx}{(x-a)^2+\beta^2}dx &= 2\pi e^{-k\beta}\cos ka,\end{aligned}\right\}$$

which may be written

$$\left.\begin{aligned}\int_{-\infty}^{\infty}\frac{\cos\left(kx+\tan^{-1}\dfrac{\beta}{x-a}\right)}{\sqrt{(x-a)^2+\beta^2}}dx &= -2\pi e^{-k\beta}\sin ka,\\ \int_{-\infty}^{\infty}\frac{\sin\left(kx+\tan^{-1}\dfrac{\beta}{x-a}\right)}{\sqrt{(x-a)^2+\beta^2}}dx &= 2\pi e^{-k\beta}\cos ka.\end{aligned}\right\}$$

1304. *In the case where $\beta=0$, the centre of the small circle lies on the x-axis* and a semicircular arc DEF, of radius r and centre at a, 0, replaces the complete small circle before considered.

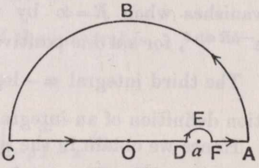

Fig. 406.

To consider the effect of this, we integrate:

(1) from C to D, (2) round DEF,

(3) from F to A, (4) round ABC.

For (1) and (3), we have

$$\left(\int_{-\infty}^{a-r}+\int_{a+r}^{\infty}\right)\frac{e^{\iota kx}}{x-a}dx,$$

i.e. when r is infinitesimally small, viz. the Principal Value of

$$\int_{-\infty}^{\infty}\frac{e^{\iota kx}}{x-a}dx.$$

For (2), putting $z=a+re^{\iota\theta}$, $\dfrac{dz}{z-a}=\iota\, d\theta$, and the contribution is

$$\int_\pi^0 e^{\iota k(a+re^{\iota\theta})}\iota\, d\theta = -\pi\iota e^{\iota ka},$$

r being infinitesimal.

For (4) we have, as before, a contribution nil.

Hence ultimately, r being indefinitely small,

$$\int_{-\infty}^{\infty} \frac{\cos kx + \iota \sin kx}{x-a}\, dx - \pi(\iota \cos ka - \sin ka) = 0,$$

i.e. $$\left.\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos kx}{x-a}\, dx &= -\pi \sin ka,\\ \int_{-\infty}^{\infty} \frac{\sin kx}{x-a}\, dx &= \pi \cos ka,\end{aligned}\right\}$$ Principal Values being taken in each case.

1305. *Consider the integration* $\int \frac{e^{\iota az} - e^{\iota bz}}{z} dz$, *a and b being real and positive, taken round a contour consisting of*

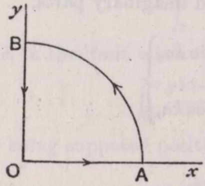

Fig. 407.

(1) *the positive portion of the x-axis;*

(2) *an infinite quadrantal arc, centre at the origin and radius* $R\ (=\infty)$;

(3) *the positive portion of the y-axis;*

As in the last two cases, the function vanishes in the limit when $|z| = \infty$, and it will be clear that there is no pole in the region round which it is proposed to integrate.

We have then

$$\int_0^R \frac{e^{\iota ax} - e^{\iota bx}}{x}\, dx + \int_0^{\frac{\pi}{2}} \frac{e^{\iota aRe^{\iota\theta}} - e^{\iota bRe^{\iota\theta}}}{1}\, \iota\, d\theta + \int_R^0 \frac{e^{-ay} - e^{-by}}{y}\, dy = 0.$$

The first integral $= \int_0^R \frac{(\cos ax - \cos bx) + \iota(\sin ax - \sin bx)}{x}\, dx.$

The second integral $= \int_0^{\frac{\pi}{2}} [e^{-aR\sin\theta} e^{\iota aR\cos\theta} - e^{-bR\sin\theta} e^{\iota bR\cos\theta}]\, \iota\, d\theta$, which vanishes when $R = \infty$ by virtue of the exponential factors $e^{-aR\sin\theta}$ $e^{-bR\sin\theta}$, for $\sin\theta$ is positive.

The third integral $= -\log\frac{b}{a}$ by Frullani's Theorem, or by the summation definition of an integration as in Ex. 1, Art. 16.

Hence we obtain in the limit, when $R = \infty$,

$$\int_0^{\infty} \frac{\cos ax - \cos bx}{x}\, dx = \log\frac{b}{a}, \quad \int_0^{\infty} \frac{\sin ax - \sin bx}{x}\, dx = 0,$$

results previously established.

1306. *Consider the integral* $\int \frac{z^{a-1}}{1+z} dz$, *where a is real and* <1 *and* >0, *where by* z^{a-1} *we understand that particular one of its values whose amplitude is* $(a-1)$ *times that of z.*

There are two poles, $z = 0$ and $z = -1$. There are also branch points at the origin and at ∞.

Take as contour an infinitely large semicircle, radius $R\ (=\infty)$ and centre at O, the origin; an infinitesimally small semicircle of radius ρ and centre

O; an infinitesimally small semicircle with centre at $z=-1$ and radius ρ, the concavities of the circles all being in the same direction; and the remaining portions of the boundary being the intercepted portions of the x-axis; the whole making the figure $ABCDEFGHIJA$ (Fig. 408), within which, with the meaning indicated for z^{a-1}, the function is synectic.

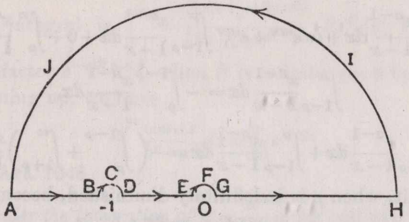

Fig. 408.

The poles are then excluded from the contour, and the integration is to be conducted along the six parts AB, BCD, DE, EFG, GH, $HIJA$ indicated in the figure.

(1) Along AB the integral is $\int_{-R}^{-1-\rho} \frac{x^{a-1}}{1+x} dx$, or changing x to $-x$,

$$-\int_R^{1+\rho} \frac{(-1)^{a-1} x^{a-1}}{1-x} dx \quad \text{or} \quad -e^{\iota a\pi} \int_{1+\rho}^{\infty} \frac{x^{a-1}}{1-x} dx.$$

(2) Along the semicircle BCD, put $z=-1+\rho e^{\iota\theta}$; $\therefore \ \frac{dz}{z+1}=\iota\, d\theta$.

The contribution is then $\int_\pi^0 (-1+\rho e^{\iota\theta})^{a-1} \iota\, d\theta$, or since ρ is infinitesimally small,

$$(-1)^{a-1} \int_\pi^0 \iota\, d\theta = (-1)^a \iota\pi = \iota\pi e^{\iota a\pi}.$$

(3) Along the straight line DE the portion of the integral is

$$\int_{-1+\rho}^{-\rho} \frac{x^{a-1}}{1+x} dx, \quad \text{or changing } x \text{ to } -x,$$

$$-\int_{1-\rho}^{\rho} \frac{(-1)^{a-1} x^{a-1}}{1-x} dx \quad \text{or} \quad e^{\iota a\pi} \int_{1-\rho}^{\rho} \frac{x^{a-1}}{1-x} dx.$$

(4) Along the semicircle EFG we have, putting $z=\rho e^{\iota\theta}$,

$$\int_\pi^0 \frac{(\rho e^{\iota\theta})^{a-1} \iota \rho e^{\iota\theta}\, d\theta}{1+\rho e^{\iota\theta}},$$

which vanishes, ρ being an infinitesimal and $1>a>0$.

(5) The contribution from GH is $\int_\rho^\infty \frac{x^{a-1}}{1+x} dx$.

(6) For the semicircle $HIJA$ we have, putting $z=Re^{\iota\theta}$,

$$\int_0^\pi \frac{(Re^{\iota\theta})^{a-1} \iota Re^{\iota\theta}}{Re^{\iota\theta}+1} d\theta,$$

which vanishes, since R is infinite and $1>a>0$.

Let I_1 and I_2 be the Principal Values of $\int_0^\infty \frac{x^{a-1}}{1+x}dx$ and $\int_0^\infty \frac{x^{a-1}}{1-x}dx$, *i.e.*

$$Lt_{\rho=0}\int_\rho^\infty \frac{x^{a-1}}{1+x}dx \quad\text{and}\quad Lt_{\rho=0}\left[\int_0^{1-\rho}+\int_{1+\rho}^\infty\right]\frac{x^{a-1}}{1-x}dx \text{ respectively;}$$

we then have, summing up the six portions,

$$-e^{\iota a\pi}\int_{1+\rho}^\infty \frac{x^{a-1}}{1-x}dx+\iota\pi e^{\iota a\pi}+e^{\iota a\pi}\int_{1-\rho}^\rho \frac{x^{a-1}}{1-x}dx+0+\int_\rho^\infty \frac{x^{a-1}}{1+x}dx+0=0$$

and
$$\int_{1-\rho}^\rho \frac{x^{a-1}}{1-x}dx=-\int_\rho^{1-\rho}\frac{x^{a-1}}{1-x}dx,$$

so that
$$-\int_{1+\rho}^\infty \frac{x^{a-1}}{1-x}dx+\int_{1-\rho}^\rho \frac{x^{a-1}}{1-x}dx=-\left(\int_\rho^{1-\rho}+\int_{1+\rho}^\infty\right)\frac{x^{a-1}}{1-x}dx,$$

and in the limit, when ρ is indefinitely diminished, becomes $=-I_2$;

$$\therefore\quad -e^{\iota a\pi}I_2+\iota\pi e^{\iota a\pi}+I_1=0,$$

i.e.
$$-(\cos a\pi+\iota\sin a\pi)I_2+\pi(\iota\cos a\pi-\sin a\pi)+I_1=0;$$

whence
$$\left.\begin{aligned} I_1-\cos a\pi I_2&=\pi\sin a\pi,\\ -I_2\sin a\pi+\pi\cos a\pi&=0;\end{aligned}\right\}$$

therefore $I_1=\pi\operatorname{cosec}a\pi$ and $I_2=\pi\cot a\pi$.

These are the results of Articles 871 and 1103.

1307. *Consider* $\int \frac{e^{\iota az}}{b^2+z^2}dz$ *for real and positive values of a and b.*

There are poles at $z=\pm\iota b$; and when $|z|=\infty$ the integrand vanishes.

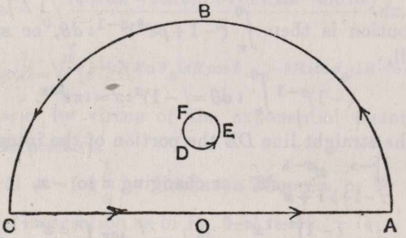

Fig. 409.

Integrate round an infinite semicircle with centre at the origin O and radius $R\,(=\infty)$, and round a circle of infinitesimal radius ρ with centre at the pole ιb.

Then the integral taken round the outer boundary = the integral taken in the same sense round the inner boundary, and the latter is

$$2\pi\iota\frac{e^{\iota a(\iota b)}}{\iota b+\iota b}=\frac{\pi}{b}e^{-ab}.\quad\text{(Art. 1286.)}$$

Over the outer boundary we have

$$\int_{-\infty}^0 \frac{e^{\iota ax}}{b^2+x^2}dx+\int_0^\infty \frac{e^{\iota ax}}{b^2+x^2}dx+\int_0^\pi \frac{e^{\iota aRe^{\iota\theta}}}{b^2+R^2e^{2\iota\theta}}\,.\,\iota Re^{\iota\theta}\,d\theta.$$

Writing $-x$ for x in the first integral, it becomes

$$-\int_{\infty}^{0}\frac{e^{-\iota ax}}{b^2+x^2}dx,\quad i.e.\int_0^{\infty}\frac{e^{-\iota ax}}{b^2+x^2}dx,$$

and the first two integrals combine to give $\int_0^{\infty}\frac{2\cos ax}{b^2+x^2}dx.$

The third integral is $\int_0^{\pi}\frac{e^{-aR\sin\theta}e^{\iota aR\cos\theta}}{b^2+R^2e^{2\iota\theta}}\iota Re^{\iota\theta}d\theta$, and vanishes by virtue of the factor $e^{-aR\sin\theta}$, when R is infinite, $\sin\theta$ being positive.

Thus, summing up, we have

$$\int_0^{\infty}\frac{\cos ax}{b^2+x^2}dx=\frac{\pi}{2b}e^{-ab},$$

the result of Art. 1048.

1308. *Consider the integration of* $w=\frac{ze^{\iota az}}{b^2+z^2}$ *for real and positive values of a and b.*

The poles are at $z=\pm\iota b$; and when $|z|=\infty$ the integrand vanishes. Take the same contour as in the last example.

The integral round the small circle, whose centre is ιb,

$$=2\pi\iota\frac{\iota be^{\iota a(\iota b)}}{\iota b+\iota b}=\pi\iota e^{-ab}.$$

Over the outer boundary we have

$$\int_{-\infty}^{0}\frac{xe^{\iota ax}}{b^2+x^2}dx+\int_0^{\infty}\frac{xe^{\iota ax}}{b^2+x^2}dx+\int_0^{\pi}\frac{Re^{\iota\theta}e^{\iota aRe^{\iota\theta}}}{b^2+R^2e^{2\iota\theta}}\iota Re^{\iota\theta}d\theta.$$

Writing $-x$ for x in the first integral, it becomes

$$\int_{\infty}^{0}\frac{xe^{-\iota ax}}{b^2+x^2}dx=-\int_0^{\infty}\frac{xe^{-\iota ax}}{b^2+x^2}dx,$$

which combines with the second integral to give $\int_0^{\infty}\frac{2\iota x\sin ax}{b^2+x^2}dx.$

The third integral, as in the last case, contains the factor $e^{-aR\sin\theta}$ in the integrand, and therefore vanishes when R is ∞, $\sin\theta$ being positive.

Hence, as the integral round the outer boundary is equal to that round the inner in the same sense,

$$\int_0^{\infty}\frac{x\sin ax}{b^2+x^2}dx=\frac{\pi}{2}e^{-ab}.$$

1309. *Consider the integration of* $w=\frac{e^{\iota az}}{z(b^2+z^2)}$ *for real and positive values of a and b.*

There are poles at $z=0$ and $z=\pm\iota b$; and when $|z|=\infty$ the integrand vanishes.

Take the same contour as in the last two cases, with the addition of a small semicircle of radius ρ, with centre at the origin, to exclude the pole at $z=0$.

Integrate, as before, round the boundary $CDEFABC$, and equate to the integral round the small circle encircling $z=\iota b$ in the same sense.

Thus

$$\int_{-\infty}^{-\rho} \frac{e^{\iota ax}\,dx}{x(b^2+x^2)} + \int_{\pi}^{0} \frac{e^{\iota a\rho e^{\iota\theta}}\,\iota\rho e^{\iota\theta}\,d\theta}{\rho e^{\iota\theta}(b^2+\rho^2 e^{2\iota\theta})} + \int_{\rho}^{\infty} \frac{e^{\iota ax}\,dx}{x(b^2+x^2)} + \int_{0}^{\pi} \frac{e^{\iota aRe^{\iota\theta}}\,\iota Re^{\iota\theta}\,d\theta}{Re^{\iota\theta}(b^2+R^2e^{2\iota\theta})}$$

$$= 2\pi\iota\frac{e^{\iota a(\iota b)}}{2\iota^2 b^2} = -\frac{\pi\iota}{b^2}\,e^{-ab}.$$

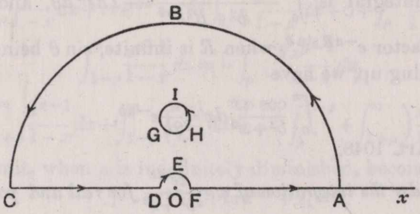

Fig. 410.

Then writing $-x$ for x in the first integral, it combines with the third to give $\int_0^\infty \frac{2\iota \sin ax}{x(b^2+x^2)}$.

Since ρ is infinitesimal the second integral $= \int_\pi^0 \frac{\iota}{b^2}\,d\theta = -\frac{\pi\iota}{b^2}$.

The fourth integral vanishes for the same reason as in the last two cases.

Hence $$\int_0^\infty \frac{\sin ax}{x(b^2+x^2)}\,dx = \frac{\pi}{2b^2}(1-e^{-ab}).$$

1310. *Consider* $\int \frac{e^{\iota az}}{b^{2n}+z^{2n}}\,dz$, *a and b being real and positive.*

The poles are given by

$$z^{2n}+b^{2n} \equiv \prod_{s=0}^{s=n-1}\left(z^2 - 2bz\cos\frac{2s+1}{2n}\pi + b^2\right) = 0,$$

i.e. $$z = b\left(\cos\frac{2s+1}{2n}\pi \pm \iota\sin\frac{2s+1}{2n}\pi\right) = be^{\pm\frac{2s+1}{2n}\iota\pi},$$

and lie upon a circle of radius b at equal angular intervals $\frac{\pi}{n}$, the x-axis being an axis of symmetry with regard to the poles and not passing through any of them. Also if $|z| = \infty$ the integrand ultimately vanishes.

We take the same contour as before, viz. an infinite semicircle of radius $R\,(=\infty)$ and centre at the z-origin O, the x-axis and infinitesimal circles of radius ρ drawn round each pole as centre.

Now $$\frac{1}{z^{2n}+b^{2n}} = \sum_{s=0}^{s=n-1} \frac{1}{2n\left(be^{\iota\frac{2s+1}{2n}\pi}\right)^{2n-1}} \frac{1}{\left(z - be^{\iota\frac{2s+1}{2n}\pi}\right)}$$

$$+ \sum_{s=0}^{s=n-1} \frac{1}{2n\left(be^{-\iota\frac{2s+1}{2n}\pi}\right)^{2n-1}} \frac{1}{\left(z - be^{-\iota\frac{2s+1}{2n}\pi}\right)},$$

the poles of the second group lying outside the contour of integration, and therefore contributing nothing. The pole $z=be^{\iota\frac{2s+1}{2n}\pi}$ contributes

$$2\iota\pi\frac{e^{\iota abe^{\iota\frac{2s+1}{2n}\pi}}}{2n\left(be^{\iota\frac{2s+1}{2n}\pi}\right)^{2n-1}}.$$

Hence the poles within the contour contribute in the aggregate

$$\sum_{s=0}^{s=n-1}\frac{\iota\pi}{n}\frac{e^{\iota abe^{\iota\frac{2s+1}{2n}\pi}}}{\left(be^{\iota\frac{2s+1}{2n}\pi}\right)^{2n-1}},$$

i.e. $\quad -\sum_0^{n-1}\frac{\iota\pi}{nb^{2n-1}}e^{\iota\frac{2s+1}{2n}\pi}e^{\iota abe^{\iota\frac{2s+1}{2n}\pi}}$

$$=-\sum_0^{n-1}\frac{\iota\pi}{nb^{2n-1}}e^{-ab\sin\frac{2s+1}{2n}\pi}\left[\cos\left(\frac{2s+1}{2n}\pi+ab\cos\frac{2s+1}{2n}\pi\right)\right.$$
$$\left.+\iota\sin\left(\frac{2s+1}{2n}\pi+ab\cos\frac{2s+1}{2n}\pi\right)\right]. \quad (1)$$

For the outer contour we have

$$\int_{-\infty}^{0}\frac{e^{\iota ax}}{b^{2n}+x^{2n}}dx+\int_0^\infty\frac{e^{\iota ax}}{b^{2n}+x^{2n}}dx+\int_0^\pi\frac{e^{\iota aRe^{\iota\theta}}\iota Re^{\iota\theta}d\theta}{b^{2n}+R^{2n}e^{\iota 2n\theta}}.$$

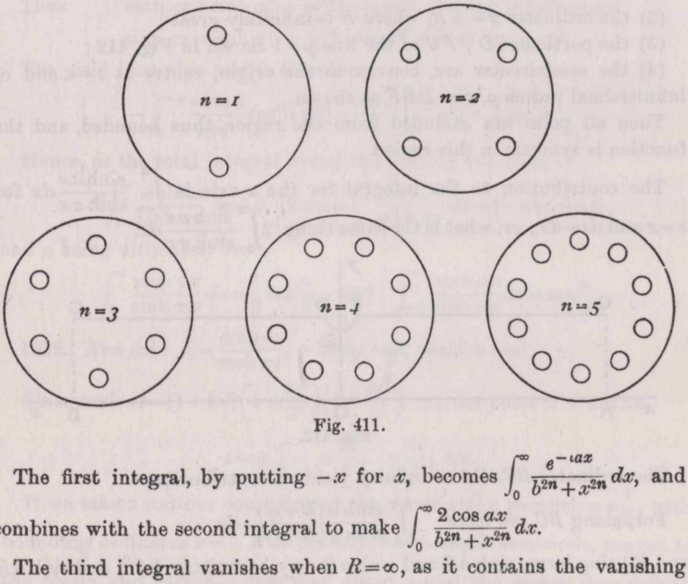

Fig. 411.

The first integral, by putting $-x$ for x, becomes $\int_0^\infty\frac{e^{-\iota ax}}{b^{2n}+x^{2n}}dx$, and combines with the second integral to make $\int_0^\infty\frac{2\cos ax}{b^{2n}+x^{2n}}dx.$

The third integral vanishes when $R=\infty$, as it contains the vanishing factor $e^{-aR\sin\theta}$; and since the integral round the outer boundary of the contour is equal to the sum of the integrals round the small circles which contain the poles which lie within the great semicircle,

$$\int_0^\infty \frac{\cos ax}{b^{2n}+x^{2n}}\,dx=\frac{\pi}{2nb^{2n-1}}\sum_0^{n-1} e^{-ab\sin\frac{2s+1}{2n}\pi}\sin\left[\frac{2s+1}{2n}\pi+ab\cos\left(\frac{2s+1}{2n}\pi\right)\right], \quad (2)$$

which is the result established in Art. 1067.

It will be noted that in the summation above in equation (1), that the imaginary portion vanishes, the poles being symmetrically situated about the y-axis.

The arrangement of the poles in the cases $n=1$, $n=2$, $n=3$, $n=4$, $n=5$, is shown in Fig. 411.

1311. *Consider* $w=\dfrac{\sinh az}{\sinh \pi z}$, *$a$ real, positive and* $<\pi$.

Since the limit of this expression when $|z|=0$ is $\dfrac{a}{\pi}$, there will be no pole at the origin; and when $|z|=\infty$ the integrand ultimately becomes zero, since $a<\pi$.

Since $\sinh \pi z=\pi z\left(1+\dfrac{z^2}{1^2}\right)\left(1+\dfrac{z^2}{2^2}\right)\ldots$, there are poles at $z=\pm\iota$, $z=\pm 2\iota$, $z=\pm 3\iota, \ldots$, which are all situated on the y-axis in the z-plane.

Take for the contour round which the integration $\int w\,dz$ is to be conducted:

(1) the complete x-axis;

(2) the ordinates $x=\pm R$, where R is infinitely great;

(3) the portions CD; FG of the line $y=1$ shown in Fig. 412;

(4) the semicircular arc, convex to the origin, centre at $z=\iota$ and of infinitesimal radius ρ, viz. DEF as shown.

Then all poles are excluded from the region thus bounded, and the function is synectic in this region.

The contribution to the integral for the x-axis is $\displaystyle\int_{-\infty}^{\infty}\frac{\sinh ax}{\sinh \pi x}\,dx$ for $z=x$ and $dz=dx$; or, what is the same thing, $\displaystyle 2\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\,dx$.

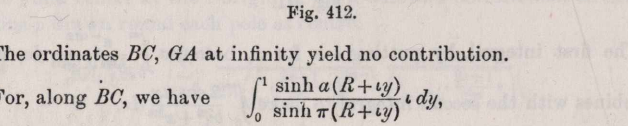

Fig. 412.

The ordinates BC, GA at infinity yield no contribution.

For, along BC, we have $\displaystyle\int_0^1 \frac{\sinh a(R+\iota y)}{\sinh \pi(R+\iota y)}\,\iota\,dy$,

and R being large, $\sinh aR$ and $\cosh aR$ may be written $\frac{1}{2}e^{aR}$, and $\sinh \pi R$ and $\cosh \pi R$ may be written $\frac{1}{2}e^{\pi R}$.

Hence the integration along BC reduces to $\int_0^1 \frac{e^{aR}e^{\iota ay}}{e^{\pi R}e^{\iota\pi y}}\iota\,dy$, *i.e.*

$$\int_0^1 e^{(a-\pi)R}e^{(a-\pi)\iota y}\iota dy,$$

which vanishes by virtue of the zero factor $e^{(a-\pi)R}$ in the integrand, since $a-\pi$ is negative and R is infinite. Similarly for the portion GA.

For the portions CD and FG we have respectively

$$\int_\infty^\rho \frac{\sinh a(\iota+x)}{\sinh\pi(\iota+x)}dx \quad\text{and}\quad \int_{-\rho}^{-\infty}\frac{\sinh a(\iota+x)}{\sinh\pi(\iota+x)}dx.$$

Considering the first of these integrals,

$$\sinh a(\iota+x)=\iota\sin a\cosh ax+\cos a\sinh ax,$$
$$\sinh\pi(\iota+x)= \qquad\qquad - \qquad \sinh\pi x;$$

$\therefore$ the integral becomes $\int_\rho^\infty \frac{\iota\sin a\cosh ax+\cos a\sinh ax}{\sinh\pi x}dx$;

and writing $-x$ for x in the second integral, it becomes

$$-\int_\rho^\infty \frac{\sinh a(\iota-x)}{\sinh\pi(\iota-x)}dx=-\int_\rho^\infty\frac{\iota\sin a\cosh ax-\cos a\sinh ax}{\sinh\pi x}dx,$$

and CD, FG together yield $2\cos a\int_\rho^\infty\frac{\sinh ax}{\sinh\pi x}dx.$

To consider the contribution of the infinitesimal semicircle DEF, put $z=\iota+\rho e^{\iota\theta}$, and integrate from $\theta=0$ to $\theta=-\pi$.

Thus $\qquad \sinh az=\sinh a(\iota+\rho e^{\iota\theta})=\iota\sin a$, ρ being infinitesimal,

$$\sinh\pi z=\sinh\pi(\iota+\rho e^{\iota\theta})=\pi\rho e^{\iota\theta}\cosh\pi\iota=-\pi\rho e^{\iota\theta}.$$

The yield from this part is therefore

$$-\int_0^{-\pi}\frac{\iota\sin a}{\pi\rho e^{\iota\theta}}(\rho e^{\iota\theta}\iota\,d\theta)=\frac{\sin a}{\pi}\int_0^{-\pi}d\theta=-\sin a.$$

Hence, as the total integral round the contour vanishes,

$$2\int_0^\infty\frac{\sinh ax}{\sinh\pi x}dx+0+2\cos a\int_\rho^\infty\frac{\sinh ax}{\sinh\pi x}dx+(-\sin a)=0;$$

and ρ being ultimately zero,

$$\int_0^\infty\frac{\sinh ax}{\sinh\pi x}dx=\frac{1}{2}\tan\frac{a}{2},\quad\text{and}\quad\int_{-\infty}^\infty\frac{\sinh ax}{\sinh\pi x}dx=\tan\frac{a}{2}.$$

1312. *Now take* $w=\frac{\cosh az}{\cosh\pi z}$, *a being real, positive and* $<\pi$.

Since $\cosh\pi z=(1+4z^2)\left(1+\frac{4z^2}{3^2}\right)\left(1+\frac{4z^2}{5^2}\right),\ldots$, the poles of w are at

$$z=\pm\frac{\iota}{2},\quad\pm\frac{3\iota}{2},\quad\pm\frac{5\iota}{2},\text{ etc.}$$

If we take a contour consisting of the x-axis and a parallel, $y=\frac{1}{2}$, with bounding ordinates $x=\pm R$ at infinity, and a small semicircle, convex to the origin and radius ρ, described about $z=\frac{\iota}{2}$; the region thus defined excludes the poles, and w is a synectic within it, so that $\int w\,dz=0$ when the integration is conducted along the contour of this region.

The points B, C, shown in the figure, are supposed at ∞, and A, G at $-\infty$, and DEF is the infinitesimal semicircle about $z=\frac{\iota}{2}$ (Fig. 413).

The x-axis contributes $\int_{-\infty}^{\infty}\frac{\cosh ax}{\cosh \pi x}dx$, that is, $2\int_0^{\infty}\frac{\cosh ax}{\cosh \pi x}dx$.

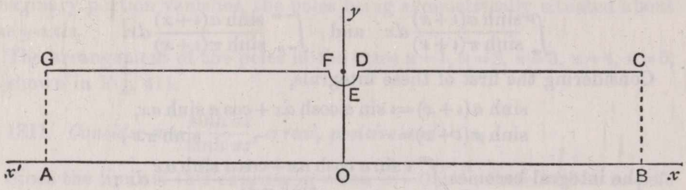

Fig. 413.

The ordinates at infinity contribute

$$\int_0^{\frac{1}{2}}\frac{\cosh a(R+\iota y)}{\cosh \pi(R+\iota y)}\iota\,dy \quad\text{and}\quad \int_{\frac{1}{2}}^{0}\frac{\cosh a(-R+\iota y)}{\cosh \pi(-R+\iota y)}\iota\,dy\,;$$

and, as in the former case,

$$\cosh aR,\quad \sinh aR,\quad \cosh \pi R,\quad \sinh \pi R$$

may be replaced by $\frac{1}{2}e^{aR}$, $\frac{1}{2}e^{aR}$, $\frac{1}{2}e^{\pi R}$, $\frac{1}{2}e^{\pi R}$, respectively, since R is infinitely large; and we may write

$$\cosh a(R+\iota y)=\tfrac{1}{2}e^{aR}e^{\iota ay},\quad \cosh \pi(R+\iota y)=\tfrac{1}{2}e^{\pi R}e^{\iota\pi y},$$
$$\cosh a(-R+\iota y)=\tfrac{1}{2}e^{aR}e^{-\iota ay}\quad\text{and}\quad \cosh \pi(-R+\iota y)=\tfrac{1}{2}e^{\pi R}e^{-\iota\pi y}\,;$$

and the two integrals become

$$\int_0^{\frac{1}{2}}e^{(a-\pi)R}e^{\iota(a-\pi)y}\iota\,dy \quad\text{and}\quad -\int_0^{\frac{1}{2}}e^{(a-\pi)R}e^{-\iota(a-\pi)y}\iota\,dy,$$

which both vanish when R is infinite by virtue of the ultimately zero factor $e^{(a-\pi)R}$ in the integrands, a being $<\pi$. Hence the yield from the two ordinates is nil.

The parts CD and FG respectively contribute

$$\int_{\infty}^{\rho}\frac{\cosh a\left(x+\frac{\iota}{2}\right)}{\cosh \pi\left(x+\frac{\iota}{2}\right)}dx \quad\text{and}\quad \int_{-\rho}^{-\infty}\frac{\cosh a\left(x+\frac{\iota}{2}\right)}{\cosh \pi\left(x+\frac{\iota}{2}\right)}dx,$$

and
$$\cosh a\left(x+\frac{\iota}{2}\right)=\cosh ax\cos\frac{a}{2}+\iota\sinh ax\sin\frac{a}{2},$$
$$\cosh \pi\left(x+\frac{\iota}{2}\right)=\qquad\qquad \iota\sinh \pi x,$$

and the first integral becomes $-\int_{\rho}^{\infty}\frac{\cosh ax\cos\frac{a}{2}+\iota\sinh ax\sin\frac{a}{2}}{\iota\sinh \pi x}dx\,;$

and similarly writing $-x$ for x in the second integral, it becomes

$$-\int_\rho^\infty \frac{\cosh a\left(\frac{\iota}{2}-x\right)}{\cosh \pi\left(\frac{\iota}{2}-x\right)}dx=\int_\rho^\infty \frac{\cosh ax\cos\frac{a}{2}-\iota\sinh ax\sin\frac{a}{2}}{\iota\sinh\pi x}dx.$$

Hence, in the aggregate, these two terms yield $-2\sin\frac{a}{2}\int_\rho^\infty \frac{\sinh ax}{\sinh \pi x}dx.$

To find what accrues from the semicircle DEF, we put $z=\frac{\iota}{2}+\rho e^{\iota\theta}$, and integrate with regard to θ from $\theta=0$ to $\theta=-\pi$.

Thus, since $\cosh a\left(\frac{\iota}{2}+\rho e^{\iota\theta}\right)=\cos\frac{a}{2}$ to the first term, ρ being infinitesimal, and $\cosh\pi\left(\frac{\iota}{2}+\rho e^{\iota\theta}\right)=\pi\rho\iota e^{\iota\theta}$,

$$\int\frac{\cosh az}{\cosh \pi z}dz \text{ round the semicircle}=\int_0^{-\pi}\frac{\cos\frac{a}{2}}{\pi\rho\iota e^{\iota\theta}}\iota\rho e^{\iota\theta}\,d\theta=-\cos\frac{a}{2},$$

and the total integral round the contour $=0$, since w is synectic throughout the region bounded ; hence

$$2\int_0^\infty\frac{\cosh ax}{\cosh\pi x}dx+0-2\sin\frac{a}{2}\int_\rho^\infty\frac{\sinh ax}{\sinh\pi x}dx-\cos\frac{a}{2}=0\,;$$

and ρ being ultimately zero,

$$2\int_0^\infty\frac{\cosh ax}{\cosh\pi x}dx=\cos\frac{a}{2}+2\sin\frac{a}{2}\cdot\frac{1}{2}\tan\frac{a}{2}=\sec\frac{a}{2};$$

and therefore $\int_0^\infty\frac{\cosh ax}{\cosh\pi x}dx=\frac{1}{2}\sec\frac{a}{2}$, and $\int_{-\infty}^\infty\frac{\cosh ax}{\cosh\pi x}dx=\sec\frac{a}{2}$.

1313. *Consider* $w=\frac{e^{\iota az}}{\cosh\pi z}$, *where* a *is a complex constant* $=\alpha+\iota\beta$, *in which* β *is not negative.*

The poles are, as before, $z=\pm\frac{\iota}{2}, \pm\frac{3\iota}{2}, \pm\frac{5\iota}{2}$, etc., and in addition, since

$$e^{\iota(\alpha+\iota\beta)(x+\iota y)}=e^{-\beta x-\alpha y}e^{\iota(\alpha x-\beta y)},$$

the function becomes infinite if $\beta x+\alpha y=-\infty$. Hence we must take a contour which excludes all such points.

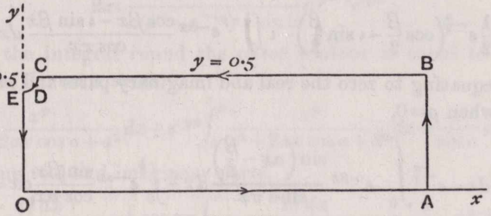

Fig. 414.

The region bounded by the positive direction of the x-axis, an ordinate $x=R$ where $R=\infty$, the straight line $y=\frac{1}{2}$, the quadrant of a circle of centre $z=\frac{\iota}{2}$ and infinitesimal radius ρ, viz. CDE and the portion EO of the y-axis, contains no pole and the function w is synectic throughout it (Fig. 414).

The x-axis contributes $\displaystyle\int_0^\infty \frac{e^{-\beta x}e^{\iota\alpha x}}{\cosh \pi x}\,dx.$

The ordinate AB at infinity contributes nothing, for the integrand contains the factor $e^{-\beta x}$, which vanishes when $x=\infty$.

The path $y=\frac{1}{2}$ from $x=R$ to $x=\rho$ contributes

$$\int_\infty^\rho \frac{e^{-\frac{\alpha}{2}}e^{-\beta x}e^{\iota\left(\alpha x-\frac{\beta}{2}\right)}}{\iota \sinh \pi x}\,dx, \quad \text{for } \cosh\pi\left(\frac{\iota}{2}+x\right)=\iota\sinh\pi x.$$

For the infinitesimal quadrantal arc with centre $\frac{\iota}{2}$, put $z=\frac{\iota}{2}+\rho e^{\iota\theta}$ and integrate from $\theta=0$ to $\theta=-\frac{\pi}{2}$.

This yields $$\int_0^{-\frac{\pi}{2}} \frac{e^{\iota(\alpha+\iota\beta)\left(\frac{\iota}{2}+\rho e^{\iota\theta}\right)}}{\cosh\pi\left(\frac{\iota}{2}+\rho e^{\iota\theta}\right)}\,\iota\rho e^{\iota\theta}\,d\theta,$$

i.e. ρ being infinitesimal,

$$\int_0^{-\frac{\pi}{2}} \frac{e^{-\frac{1}{2}(\alpha+\iota\beta)}}{\pi}\,d\theta=-\tfrac{1}{2}e^{-\frac{1}{2}(\alpha+\iota\beta)}.$$

The portion EO of the y-axis contributes

$$\int_{\frac{1}{2}-\rho}^0 \frac{e^{-\alpha y}e^{-\iota\beta y}}{\cosh \iota\pi y}\,\iota\,dy=-\iota\int_0^{\frac{1}{2}-\rho}\frac{e^{-\alpha x}e^{-\iota\beta x}}{\cos\pi x}\,dx.$$

Hence, as the total integral $\int w\,dz$ vanishes,

$$\int_0^\infty \frac{e^{-\beta x}(\cos\alpha x+\iota\sin\alpha x)}{\cosh\pi x}\,dx-\int_0^\infty\frac{e^{-\frac{\alpha}{2}}e^{-\beta x}\left[\cos\left(\alpha x-\frac{\beta}{2}\right)+\iota\sin\left(\alpha x-\frac{\beta}{2}\right)\right]}{\iota\sinh\pi x}\,dx$$
$$-\frac{1}{2}e^{-\frac{\alpha}{2}}\left(\cos\frac{\beta}{2}-\iota\sin\frac{\beta}{2}\right)-\iota\int_0^{\frac{1}{2}-\rho}e^{-\alpha x}\frac{\cos\beta x-\iota\sin\beta x}{\cos\pi x}\,dx=0.$$

Hence, equating to zero the real and imaginary parts and proceeding to the limit when $\rho=0$,

$$\left.\begin{aligned}\int_0^\infty e^{-\beta x}\frac{\cos\alpha x}{\cosh\pi x}\,dx-e^{-\frac{\alpha}{2}}\int_0^\infty e^{-\beta x}\frac{\sin\left(\alpha x-\frac{\beta}{2}\right)}{\sinh\pi x}\,dx-\int_0^{\frac{1}{2}}e^{-\alpha x}\frac{\sin\beta x}{\cos\pi x}\,dx&=\frac{1}{2}e^{-\frac{\alpha}{2}}\cos\frac{\beta}{2},\\ \int_0^\infty e^{-\beta x}\frac{\sin\alpha x}{\cosh\pi x}\,dx+e^{-\frac{\alpha}{2}}\int_0^\infty e^{-\beta x}\frac{\cos\left(\alpha x-\frac{\beta}{2}\right)}{\sinh\pi x}\,dx-\int_0^{\frac{1}{2}}e^{-\alpha x}\frac{\cos\beta x}{\cos\pi x}\,dx&=-\frac{1}{2}e^{-\frac{\alpha}{2}}\sin\frac{\beta}{2}.\end{aligned}\right\}$$

If we put $\beta=0$ in the first, we have

$$\left.\begin{aligned}&\int_0^\infty \frac{\cos ax}{\cosh \pi x}\,dx - e^{-\frac{a}{2}}\int_0^\infty \frac{\sin ax}{\sinh \pi x}\,dx = \tfrac{1}{2}e^{-\frac{a}{2}},\\ &\text{and changing the sign of } a,\\ &\int_0^\infty \frac{\cos ax}{\cosh \pi x}\,dx + e^{\frac{a}{2}}\int_0^\infty \frac{\sin ax}{\sinh \pi x}\,dx = \tfrac{1}{2}e^{\frac{a}{2}},\end{aligned}\right\}$$

and solving these equations,

$$\int_0^\infty \frac{\cos ax}{\cosh \pi x}\,dx = \frac{1}{2}\operatorname{sech}\frac{a}{2},\quad \int_0^\infty \frac{\sin ax}{\sinh \pi x}\,dx = \frac{1}{2}\tanh\frac{a}{2}.$$

1314. *Consider* $w=\dfrac{z^p}{z^2-2az\cos\alpha+a^2}$, *where* $1>p>0$, a *real and* $\pi>\alpha>0$.

There are poles at $z=ae^{\pm\iota\alpha}=a\cos\alpha\pm\iota a\sin\alpha$. Take as contour an infinite semicircle, radius R $(=\infty)$ and centre at the origin O; the x-axis; and a small circle, radius ρ and centre at $z=ae^{\iota\alpha}$, *i.e.* $(a\cos\alpha, a\sin\alpha)$ (Fig. 405).

The contribution from integrating along the x-axis is

$$\int_{-\infty}^\infty \frac{x^p}{x^2-2ax\cos\alpha+a^2}\,dx = \left(\int_{-\infty}^0+\int_0^\infty\right)\frac{x^p}{x^2-2ax\cos\alpha+a^2}\,dx;$$

and putting $-x$ for x in the first integral,

$$=\int_0^\infty \frac{x^p}{x^2-2ax\cos\alpha+a^2}\,dx + \int_0^\infty \frac{(-1)^p x^p}{x^2+2ax\cos\alpha+a^2}\,dx.$$

Round the infinite semicircle we have

$$\int_0^\pi \frac{R^p e^{\iota p\theta}}{R^2e^{2\iota\theta}-2aRe^{\iota\theta}\cos\alpha+a^2}\,Re^{\iota\theta}\,.\,\iota\,d\theta,$$

which vanishes, since $p<1$.

For the infinitesimal circle put $z=ae^{\iota\alpha}+\rho e^{\iota\theta}$. The result is, by Art. 1286

$$2\pi\iota\,\frac{(ae^{\iota\alpha}+\rho e^{\iota\theta})^p}{ae^{\iota\alpha}+\rho e^{\iota\theta}-ae^{-\iota\alpha}};$$

and ρ being infinitesimal, this becomes

$$2\pi\iota\,\frac{a^p e^{\iota p\alpha}}{a(e^{\iota\alpha}-e^{-\iota\alpha})}=\frac{\pi}{\sin\alpha}\,a^{p-1}e^{\iota p\alpha};$$

and since the integral round the outer contour is equal to that round the inner in the same sense,

$$\int_0^\infty \frac{x^p}{x^2-2ax\cos\alpha+a^2}\,dx + e^{\iota p\pi}\int_0^\infty \frac{x^p}{x^2+2ax\cos\alpha+a^2}\,dx = \frac{\pi}{\sin\alpha}\,a^{p-1}e^{\iota p\alpha},$$

and equating real and imaginary parts,

$$\int_0^\infty \frac{x^p\,dx}{x^2-2ax\cos\alpha+a^2} + \cos p\pi\int_0^\infty \frac{x^p\,dx}{x^2+2ax\cos\alpha+a^2} = \frac{\pi}{\sin\alpha}\,a^{p-1}\cos p\alpha,$$

$$\sin p\pi\int_0^\infty \frac{x^p\,dx}{x^2+2ax\cos\alpha+a^2} = \frac{\pi}{\sin\alpha}\,a^{p-1}\sin p\alpha.$$

Hence

$$\left.\begin{aligned}\int_0^\infty \frac{x^p dx}{x^2+2ax\cos\alpha+a^2}&=\frac{\pi}{\sin\alpha}a^{p-1}\frac{\sin p\alpha}{\sin p\pi},\\ \int_0^\infty \frac{x^p dx}{x^2-2ax\cos\alpha+a^2}&=\frac{\pi}{\sin\alpha}a^{p-1}\frac{\sin p(\pi-\alpha)}{\sin p\pi},\end{aligned}\right\}\begin{matrix}1>p>0,\\ \pi>\alpha>0,\end{matrix}$$

the latter of which follows also from the former by writing $\pi-\alpha$ for α.

1315. *Consider* $w=\dfrac{e^{\iota az}}{\cosh z-\cos b}$, *where a and b are real.* $(0<b<\pi.)$

The poles are given by $\cosh z=\cos b$, that is

$$e^{2z}-2\cos b e^z+1=0,\quad e^z=\cos b\pm\iota\sin b,\quad z=\iota(2n\pi\pm b),$$

where n is any integer.

These poles are all situated upon the y-axis at distances from the origin $\pm b$, $\pm 2\pi\pm b$, etc.

Take as contour the entire x-axis, the ordinates $x=\pm R$ ($R=\infty$), the straight line $y=\pi$, and an infinitesimal circle, radius ρ and centre $z=\iota b$. Then the function w is synectic in the region thus bounded, the only pole ($z=\iota b$) which lies within the outer boundary being excluded by the inner.

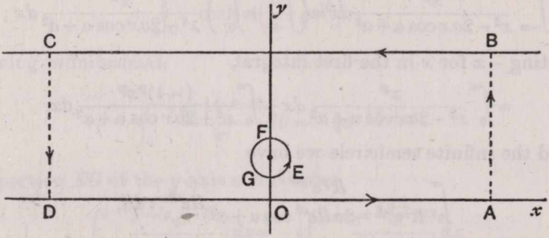

Fig. 415.

The contributions from the various parts are:

(1) From the x axis DA, $\displaystyle\int_{-\infty}^{\infty}\frac{e^{\iota ax}}{\cosh x-\cos b}dx$.

(2) From the ordinate AB,

$$\int_0^\pi \frac{e^{\iota a(R+\iota y)}}{\dfrac{e^{R+\iota y}+e^{-R-\iota y}}{2}-\cos b}\,\iota\,dy=2\iota\int_0^\pi \frac{e^{-ay}(\cos aR+\iota\sin aR)}{e^{R+\iota y}+e^{-R-\iota y}-2\cos b}dy=0,$$

where $R=\infty$; therefore AB contributes nothing. Similarly CD gives no contribution.

(3) From BC, viz. $y=\pi$, we have

$$z=x+\iota\pi,\quad dz=dx,\quad \cosh z=-\cosh x\quad\text{and}\quad e^{\iota az}=e^{-\pi a}\,.\,e^{\iota ax}.$$

Hence BC renders

$$\int_\infty^{-\infty}\frac{e^{-\pi a}e^{\iota ax}}{-\cosh x-\cos b}dx=e^{-\pi a}\int_{-\infty}^{\infty}\frac{e^{\iota ax}}{\cosh x+\cos b}dx.$$

(4) The integration round the small circle gives

$$2\pi\iota\frac{e^{\iota a(\iota b)}}{\sinh\iota b},\quad i.e.\ 2\pi\frac{e^{-ab}}{\sin b},$$

and the integration round the outer contour is equal to that round the small circle in the same sense. Hence

$$\int_{-\infty}^{\infty}\frac{e^{\iota ax}dx}{\cosh x-\cos b}+e^{-\pi a}\int_{-\infty}^{\infty}\frac{e^{\iota ax}dx}{\cosh x+\cos b}=\frac{2\pi}{\sin b}e^{-ab}.$$

Let $$I_1=\int_{-\infty}^{\infty}\frac{\cos ax}{\cosh x-\cos b}dx,\quad I_2=\int_{-\infty}^{\infty}\frac{\sin ax}{\cosh x-\cos b}dx,$$

$$I_1'=\int_{-\infty}^{\infty}\frac{\cos ax}{\cosh x+\cos b}dx,\quad I_2'=\int_{-\infty}^{\infty}\frac{\sin ax}{\cosh x+\cos b}dx.$$

Then $$I_1+\iota I_2+e^{-\pi a}(I_1'+\iota I_2')=\frac{2\pi}{\sin b}e^{-ab},$$

and therefore $I_1+e^{-\pi a}I_1'=\dfrac{2\pi}{\sin b}e^{-ab}$ and $I_2+e^{-\pi a}I_2'=0$.

Also, if we write $\pi-b$ for b, the accented and unaccented letters are interchanged. Hence

$$I_1'+e^{-\pi a}I_1=\frac{2\pi}{\sin b}e^{-a(\pi-b)}\quad\text{and}\quad I_2'+e^{-\pi a}I_2=0\ ;$$

and solving these four equations,

$$I_1\equiv\int_{-\infty}^{\infty}\frac{\cos ax}{\cosh x-\cos b}dx=\frac{2\pi}{\sin b}\,\frac{\sinh a(\pi-b)}{\sinh a\pi},\ \text{.............(1)}$$

$$I_1'\equiv\int_{-\infty}^{\infty}\frac{\cos ax}{\cosh x+\cos b}dx=\frac{2\pi}{\sin b}\,\frac{\sinh ab}{\sinh a\pi},\ \text{.............(2)}$$

and $I_2=I_2'=0$, as is indeed obvious beforehand, since, in integrating from $-\infty$ to ∞ elements of the integrands for which x only differs in sign cancel each other.

Obviously other results may be deduced from these by various selections of a and b, combined with addition or subtraction of the results.

For instance, in the formulae for I_1 and I_1', the integrands are not affected if the sign of x be changed, so that

$$\int_0^{\infty}\frac{\cos ax}{\cosh x-\cos b}dx=\frac{\pi}{\sin b}\,\frac{\sinh a(\pi-b)}{\sinh a\pi},\ \text{....................(3)}$$

$$\int_0^{\infty}\frac{\cos ax}{\cosh x+\cos b}dx=\frac{\pi}{\sin b}\,\frac{\sinh ab}{\sinh a\pi}.\ \text{.........................(4)}$$

Changing b to $\dfrac{\pi}{2}-b$ in (3) and (4),

$$\int_0^{\infty}\frac{\cos ax}{\cosh x-\sin b}dx=\frac{\pi}{\cos b}\,\frac{\sinh a\left(\frac{\pi}{2}+b\right)}{\sinh a\pi},\ \text{..................(5)}$$

$$\int_0^{\infty}\frac{\cos ax}{\cosh x+\sin b}dx=\frac{\pi}{\cos b}\,\frac{\sinh a\left(\frac{\pi}{2}-b\right)}{\sinh a\pi}.\ \text{..................(6)}$$

Putting $a=1$ in (3) and (4),

$$\int_0^\infty \frac{\cos x}{\cosh x-\cos b}\,dx=\frac{\pi}{\sin b}\,\frac{\sinh(\pi-b)}{\sinh\pi}, \quad\text{.....................(7)}$$

$$\int_0^\infty \frac{\cos x}{\cosh x+\cos b}\,dx=\frac{\pi}{\sin b}\,\frac{\sinh b}{\sinh\pi}. \quad\text{.....................(8)}$$

Adding (3) and (4),

$$\int_0^\infty \frac{\cos ax\cosh x}{\cosh 2x-\cos 2b}\,dx=\frac{\pi}{4\sin b}\,\frac{\sinh a(\pi-b)+\sinh ab}{\sinh a\pi}$$

$$=\frac{\pi}{4\sin b}\,\frac{\cosh a\left(\frac{\pi}{2}-b\right)}{\cosh\frac{a\pi}{2}}. \quad\text{.....................(9)}$$

Subtracting (4) from (3),

$$\int_0^\infty \frac{\cos ax}{\cosh 2x-\cos 2b}\,dx=\frac{\pi}{2\sin 2b}\,\frac{\sinh a\left(\frac{\pi}{2}-b\right)}{\sinh\frac{a\pi}{2}}. \quad\text{...........(10)}$$

Writing $\frac{\pi}{2}-b$ for b in (9) and (10),

$$\int_0^\infty \frac{\cos ax\cosh x}{\cosh 2x+\cos 2b}\,dx=\frac{\pi}{4\cos b}\,\frac{\cosh ab}{\cosh\frac{a\pi}{2}}, \quad\text{.....................(11)}$$

$$\int_0^\infty \frac{\cos ax}{\cosh 2x+\cos 2b}\,dx=\frac{\pi}{2\sin 2b}\,\frac{\sinh ab}{\sinh\frac{a\pi}{2}}, \quad\text{.....................(12)}$$

and so on with other cases.

1316. *Consider* $w=\frac{e^{az}}{1-e^z}$, a being real and $1>a>0$.

Here there are poles wherever $e^z=1$, *i.e.* $z=\log(e^{2\iota\lambda\pi})=2\lambda\pi\iota$ for any integral value of λ.

Take as contour a rectangle of infinite length, one side along the x-axis and extending from $x=-\infty$ to $x=\infty$; two ordinates, one at ∞, one at $-\infty$; the line $y=\pi$ and an infinitesimal semicircle excluding the origin. Then, integrating round this contour, no pole being in the region surrounded, we have, with the notation of preceding cases,

$$\int_{-\infty}^{-\rho}\frac{e^{ax}}{1-e^x}\,dx+\int_\pi^0\frac{e^{a\rho e^{\iota\theta}}}{1-e^{\rho e^{\iota\theta}}}\,\iota\rho e^{\iota\theta}\,d\theta+\int_\rho^\infty\frac{e^{ax}}{1-e^x}\,dx+\int_0^\pi\frac{e^{a(R+\iota y)}}{1-e^{(R+\iota y)}}\,\iota\,dy$$

$$+\int_\infty^{-\infty}\frac{e^{a(x+\iota\pi)}}{1-e^{(x+\iota\pi)}}\,dx+\int_\pi^0\frac{e^{a(-R+\iota y)}}{1-e^{(-R+\iota y)}}\,\iota\,dy=0.$$

In the limit, when ρ is indefinitely small and R infinitely great, the first and third integrals together give the Principal Value of $\int_{-\infty}^\infty\frac{e^{ax}}{1-e^x}\,dx$.

The second integral $=\int_\pi^0(-\iota)\,d\theta$ when ρ becomes indefinitely small, $=\iota\pi$.

The fourth vanishes, since it is ultimately

$$-Lt_{R=\infty}\int_0^{\pi} e^{(a-1)(R+\iota y)}\iota\, dy \quad \text{and} \quad a<1.$$

The fifth integral $=\int_{\infty}^{-\infty}\frac{(\cos a\pi+\iota\sin a\pi)e^{ax}}{1+e^x}dx.$

The sixth integral ultimately vanishes when R increases without limit.

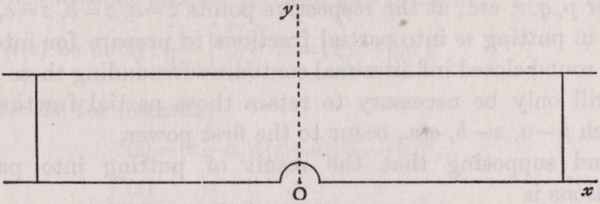

Fig. 416.

Thus, Prin. Val. of $\int_{-\infty}^{\infty}\frac{e^{ax}}{1-e^x}dx+(\cos a\pi+\iota\sin a\pi)\int_{\infty}^{-\infty}\frac{e^{ax}}{1+e^x}dx+\iota\pi=0.$

Hence $\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}dx=\pi\operatorname{cosec} a\pi,$

and the Principal Value of

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1-e^x}dx=\pi\cot a\pi.$$

This result is, however, only a transformation of that of Art. 1306.

1317. Effect of Pole-Clusters within a Contour.

If several poles, say n, be clustered together at one point of the z-plane, the point is said to be a pole of multiplicity n, or to possess polarity of the n^{th} order at the point $z=a$.

It is useful to note that in applying the theorem

$$\phi^{(n-1)}(a)=\frac{(n-1)!}{2\pi\iota}\int\frac{\phi(z)}{(z-a)^n}dz$$

to the case in which

$$w\equiv f(z)\equiv\frac{\phi(z)}{(z-a)^n}=\frac{1}{(z-a)^n},$$

where n is a positive integer, we have $\phi(z)=1$, and all its differential coefficients with regard to z are zero.

Hence $\int\frac{dz}{(z-a)^n}$ round the multiple pole $z=a$ is zero for all positive integral values of n except $n=1$, and when $n=1$ we have

$$\int\frac{dz}{z-a}=2\pi\iota.$$

It follows that if w be of the form

$$\frac{\phi(z)}{(z-a)^p(z-b)^q(z-c)^r\dots},$$

where $\phi(z)$ does not contain any of the factors $z-a$, $z-b$, $z-c, \dots$, but is rational and algebraic, there is polarity of order p, q, r, etc., at the respective points $z=a$, $z=b$, $z=c$, etc., and in putting w into partial fractions to prepare for integration round closed infinitesimal contours surrounding these poles it will only be necessary to retain those partial fractions in which $z-a$, $z-b$, etc., occur to the first power.

And supposing that the result of putting into partial fractions is

$$w=K_nz^n+K_{n-1}z^{n-1}+\dots+K_1z+K_0+\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}+\dots$$
$$+\sum_{r=2}^{r=p}\frac{A'}{(z-a)^r}+\sum_{r=2}^{r=q}\frac{B'}{(z-b)^r}+\dots,$$

then, in integrating round any closed contour which encloses all these critical points and no others,

$$\int w\,dz=2\pi\iota(A+B+C+\dots).$$

1318. Moreover, when the numerator of w, supposed rational and algebraic, is of degree in z at least two lower than the degree of the denominator, $A+B+C+\dots=0$ (Art. 149), and therefore in such cases $\int w\,dz=0$, however many critical points may be enclosed within the contour, and whatever the degree of their polarity, provided the contour of integration contains all the poles.

It is worth notice that if

$a_1, a_2, a_3, \dots$ be the zeros, of multiplicity p, q, r, etc.,

and $a_1', a_2', a_3', \dots$ be the poles, of multiplicity p_1', q_1', r_1', etc., of a function $f(z)$, so that

$$f(z)=\frac{(z-a_1)^p(z-a_2)^q(z-a_3)^r\dots}{(z-a_1')^{p'}(z-a_2')^{q'}(z-a_3')^{r'}\dots},$$

we have
$$\frac{f'(z)}{f(z)}=\Sigma\frac{p}{z-a_1}-\Sigma\frac{p'}{z-a_1'};$$

whence, if $\phi(z)$ be any other function of z which has none of the factors $z-a_1'$, $z-a_2'$, etc., then

$$\frac{1}{2\pi\iota}\int \phi(z)\frac{f'(z)}{f(z)}\,dz=[\Sigma p\phi(a_1)-\Sigma p'\phi(a_1')],$$

the integral being taken round a contour which contains all the poles without passing through any of them;

or if $\phi(z)$ be unity, $\dfrac{1}{2\pi\iota}\displaystyle\int\frac{f'(z)}{f(z)}\,dz=(\Sigma p-\Sigma p')$.

1319. If, for instance,

$$f(z)=(z-a_1)^p(z-a_2)^q(z-a_3)^r\ldots,$$

$$\frac{f'(z)}{f(z)}=\frac{p}{z-a_1}+\frac{q}{z-a_2}+\frac{r}{z-a_3}+\ldots;$$

and if we integrate round any contour which contains some or all of the roots,

$$\frac{1}{2\pi\iota}\int\frac{f'(z)}{f(z)}\,dz=\frac{1}{2\pi\iota}\left[p\int\frac{dz}{z-a_1}+q\int\frac{dz}{z-a_2}+\ldots\right],$$

for all the roots within the contour

$$=p+q+\ldots$$

$=$ the number of roots within the contour,

counting each root as many times over as it occurs in $f(z)$.

1320. Again, if in integrating round the perimeter of a closed curve which possesses no singularities and lies entirely in a region of the z-plane in which w is a synectic function, then if w be constant along the boundary of this curve it is constant for all points lying in the region thus bounded; for if $z=\zeta$ be any point of this bounded region, then if $f(\zeta)$ be the value of w at the point ζ, then

$$f(\zeta)=\frac{1}{2\pi\iota}\int\frac{f(z)}{z-\zeta}\,dz,$$

where z is a point on the boundary; and if $f(z)=\text{const.}=A$, say, at all points of the boundary,

$$f(\zeta)=\frac{1}{2\pi\iota}\int\frac{A}{z-\zeta}\,dz=\frac{1}{2\pi\iota}\cdot A\,.\,2\pi\iota=A,$$

for ζ is a pole of the function $\dfrac{f(z)}{z-\zeta}$.

Hence, for all points ζ which lie within the boundary, the function $w \equiv f(\zeta)$ has the same value as when ζ lies on the boundary.

1321. Further, if we are given the value of w at all points of the contour of a region within which w is to be assumed synectic, the equation

$$f(\zeta)=\frac{1}{2\pi\iota}\int\frac{f(z)}{z-\zeta}\,d\zeta$$

may be used to find the value of $f(\zeta)$ at all points within the contour For if $f(z)$ takes the form $\chi(z)$ at the boundary, the value of $f(\zeta)$ for a point within the boundary is

$$\frac{1}{2\pi\iota}\int\frac{\chi(z)}{z-\zeta}\,dz.$$

1322. Ex. Supposing that at all points of the circular contour $r=1$ a certain function known to be synectic within the circle takes the value $\cos 3\theta - a^2\cos\theta+\iota(\sin 3\theta - a^2\sin\theta)$, what is the function?

Putting this into the form $e^{3\iota\theta}-a^2e^{\iota\theta}$, and writing $z=e^{\iota\theta}$, $dz=\iota e^{\iota\theta}$,

$$f(\zeta)=\frac{1}{2\pi\iota}\int_0^{2\pi}\frac{e^{3\iota\theta}-a^2e^{\iota\theta}}{e^{\iota\theta}-\zeta}\,\iota e^{\iota\theta}\,d\theta$$

$$=\frac{1}{2\pi\iota}\int_0^{2\pi}\left[e^{2\iota\theta}+\zeta e^{\iota\theta}+\zeta^2-a^2+\frac{\zeta(\zeta^2-a^2)}{e^{\iota\theta}-\zeta}\right]\iota e^{\iota\theta}\,d\theta$$

$$=\frac{1}{2\pi\iota}\left[\frac{e^{3\iota\theta}}{3}+\zeta\frac{e^{2\iota\theta}}{2}+(\zeta^2-a^2)e^{\iota\theta}+\zeta(\zeta^2-a^2)\log(e^{\iota\theta}-a^2)\right]_0^{2\pi}$$

$$=\frac{1}{2\pi\iota}\,\zeta(\zeta^2-a^2)\log 1\,;$$

and $\log 1$ being $\log e^{2\lambda\pi\iota}$, where λ is an integer, we have $f(z)=\lambda\zeta(\zeta^2-a^2)$, where the proper integral value of λ is to be chosen; and putting $\zeta=e^{\iota\theta}$, we have the contour value $\lambda(e^{3\iota\theta}-a^2e^{\iota\theta})$. Hence $\lambda=1$ and $f(z)=z(z^2-a^2)$ for any point z within the contour $r=1$.

1323. (1) *Consider* $w=\dfrac{e^{\iota az}}{z^n}$, *n being greater than* 0 *and less than* 1, *and a real and positive.*

Here there is a pole at $z=0$. We may avoid this pole by taking a contour consisting of the portion of the x-axis from $x=\rho$ to $x=R$, a quadrant with centre at the origin and radius R; the portion of the y-axis from $y=R$ to $y=\rho$, and a quadrant with centre at the origin and radius ρ. And we shall choose R to be ∞ and ρ to be infinitesimal. Then w is synectic in the region thus bounded, and we have

$$\int_\rho^R\frac{e^{\iota ax}}{x^n}\,dx+\int_0^{\frac{\pi}{2}}\frac{e^{\iota aRe^{\iota\theta}}}{(Re^{\iota\theta})^{n-1}}\,\iota\,d\theta+\int_R^\rho\frac{e^{-ay}}{(\iota y)^n}\,\iota\,dy+\int_{\frac{\pi}{2}}^0\frac{e^{\iota a\rho e^{\iota\theta}}}{(\rho e^{\iota\theta})^{n-1}}\,\iota\,d\theta=0.$$

The second integral contains the factor $\frac{e^{-aR\sin\theta}}{R^{n-1}}$, in which $\sin\theta$ is positive, and vanishes when R is infinite.

The fourth integral vanishes when ρ is infinitesimal since $n<1$.

Hence, proceeding to the limit $R=\infty$ and $\rho=0$,

$$\int_0^\infty \frac{e^{\iota ax}}{x^n}\,dx=\iota^{1-n}\int_0^\infty \frac{e^{-ay}}{y^n}\,dy=\iota^{1-n}\int_0^\infty y^{-n}e^{-ay}\,dy,$$

$$\int_0^\infty \frac{\cos ax+\iota\sin ax}{x^n}\,dx=\left[\cos(1-n)\frac{\pi}{2}+\iota\sin(1-n)\frac{\pi}{2}\right]\int_0^\infty y^{-n}e^{-ay}\,dy;$$

$$\left.\begin{aligned}\therefore\quad \int_0^\infty \frac{\cos ax}{x^n}\,dx&=\cos(1-n)\frac{\pi}{2}\,\frac{\Gamma(1-n)}{a^{1-n}}=\frac{\sin\frac{n\pi}{2}}{\Gamma(n)}\cdot\frac{1}{a^{1-n}}\cdot\frac{\pi}{\sin n\pi}=\frac{\pi}{2\Gamma(n)a^{1-n}}\,\frac{1}{\cos\frac{n\pi}{2}},\\ \int_0^\infty \frac{\sin ax}{x^n}\,dx&=\sin(1-n)\frac{\pi}{2}\,\frac{\Gamma(1-n)}{a^{1-n}}=\frac{\cos\frac{n\pi}{2}}{\Gamma(n)}\cdot\frac{1}{a^{1-n}}\cdot\frac{\pi}{\sin n\pi}=\frac{\pi}{2\Gamma(n)a^{1-n}}\,\frac{1}{\sin\frac{n\pi}{2}},\end{aligned}\right\}$$

giving the well-known integrals of Fresnel (Art. 1166).

1324. (2) *Consider* $w=\frac{1}{(z^2+b^2)^{n+1}}$.

Here there are poles of the $n+1^{\text{th}}$ order at $z=\iota b$ and at $z=-\iota b$.

Taking the contour to be the infinite semicircle, the x-axis, and the small circle about $z=\iota b$ and radius ρ, as before, we have

$$w\equiv f(z)=\frac{\phi(z)}{(z-\iota b)^{n+1}},$$

where $\phi(z)=\frac{1}{(z+\iota b)^{n+1}}$ and $\phi^{(n)}(z)=\frac{(-1)^n(n+1)(n+2)\dots(2n)}{(z+\iota b)^{2n+1}}$,

i.e. $$\phi^{(n)}(\iota b)=(-1)^n\frac{(2n)!}{n!}\,\frac{1}{(2\iota b)^{2n+1}}=\frac{1}{\iota}\,\frac{(2n)!}{(n)!}\,\frac{1}{(2b)^{2n+1}}.$$

Hence $\int\frac{dz}{(z^2+b^2)^{n+1}}=\frac{2\pi}{(2b)^{2n+1}}\,\frac{(2n)!}{(n!)^2}$ round the multiple pole ιb.

The integration along the x-axis is $\int_{-\infty}^\infty\frac{dx}{(x^2+b^2)^{n+1}}$ or $2\int_0^\infty\frac{dx}{(x^2+b^2)^{n+1}}$.

Round the infinite semicircle we have $\int_0^\pi\frac{\iota Re^{\iota\theta}\,d\theta}{(R^2e^{2\iota\theta}+b^2)^{n+1}}$, which obviously vanishes if R be made infinite.

Hence $$\int_0^\infty\frac{dx}{(x^2+b^2)^{n+1}}=\frac{\pi}{(2b)^{2n+1}}\,\frac{(2n)!}{(n!)^2}.$$

The result is readily verified by putting $x=b\tan\theta$, when the integral becomes

$$\frac{1}{b^{2n+1}}\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\,d\theta.$$

1325. Instead of using the formula $\int\frac{\phi(z)}{(z-a)^{n+1}}\,dz=\frac{\phi^{(n)}(a)}{n!}2\pi\iota$, as above, we might follow the method of Art. 1317, and put $\frac{1}{(z-\iota b)^{n+1}(z+\iota b)^{n+1}}$ into Partial fractions so far as is required to find the Partial fraction of the form $\frac{A}{z-\iota b}$. We then proceed thus (Art. 144): put $z=\iota b+y$. We then have

$$\frac{1}{y^{n+1}}\frac{1}{(2\iota b+y)^{n+1}}=\frac{1}{y^{n+1}}\frac{1}{(2\iota b)^{n+1}}\Big[1-(n+1)\frac{y}{2\iota b}+\ldots$$
$$+\frac{(n+1)(n+2)\ldots(2n)}{1.2\ldots n}(-1)^n\left(\frac{y}{2\iota b}\right)^n+\ldots\Big];$$

whence
$$A=\frac{1}{(2\iota b)^{n+1}}(-1)^n\frac{(n+1)(n+2)\ldots(2n)}{1.2\ldots n}\cdot\frac{1}{(2\iota b)^n}$$
$$=\frac{1}{\iota}\frac{1}{(2b)^{2n+1}}\frac{(2n)!}{(n!)^2},$$

and the value required is $A.2\pi\iota$, *i.e.* round the multiple pole at $z=\iota b$ the integral is $\frac{2\pi}{(2b)^{2n+1}}\frac{(2n)!}{(n!)^2}$, as before.

1326. *Consider* $w\equiv f(z)\equiv\frac{e^{\iota az}}{(b^2+z^2)^{n+1}}$, *a real and positive.*

There is polarity of the $(n+1)^{\text{th}}$ order at the points $z=\pm\iota b$.

Take the contour as before, viz. an infinite semicircle centred at the origin, the x-axis and an infinitesimal circle round ιb.

We have, putting $f(z)\equiv\frac{\phi(z)}{(z-\iota b)^{n+1}}$, $\phi(z)=\frac{e^{\iota az}}{(z+\iota b)^{n+1}}$,

and
$$\phi^{(n)}(z)=(\iota a)^n\frac{e^{\iota az}}{(z+\iota b)^{n+1}}-\frac{n}{1}(\iota a)^{n-1}e^{\iota az}\frac{(n+1)}{(z+\iota b)^{n+2}}+\frac{n(n-1)}{1.2}(\iota a)^{n-2}e^{\iota az}\frac{(n+1)(n+2)}{(z+\iota b)^{n+3}}$$
$$-\frac{n(n-1)(n-2)}{1.2.3}(\iota a)^{n-3}\frac{e^{\iota az}(n+1)(n+2)(n+3)}{(z+\iota b)^{n+4}}-\ldots+e^{\iota az}(-1)^n\frac{(n+1)(n+2)\ldots(2n)}{(z+\iota b)^{2n+1}}.$$

And since $\int\frac{\phi(z)}{(z-\alpha)^{n+1}}dz$, round a multiple pole of the n^{th} order, $=\frac{2\pi\iota}{n!}\phi^{(n)}(\alpha)$, we have, putting ιb for α,

$$\int f(z)dz=\int\frac{\phi(z)}{(z-\iota b)^{n+1}}dz=\frac{2\pi\iota}{n!}\Big[(\iota a)^n\frac{e^{-ab}}{(2\iota b)^{n+1}}-\frac{n}{1}(\iota a)^{n-1}\frac{e^{-ab}(n+1)}{(2\iota b)^{n+2}}$$
$$+\frac{n(n-1)}{1.2}(\iota a)^{n-2}e^{-ab}\frac{(n+1)(n+2)}{(2\iota b)^{n+3}}-\ldots+e^{-ab}(-1)^n\frac{(2n)!}{n!(2\iota b)^{2n+1}}\Big]$$
$$=\frac{2\pi e^{-ab}}{n!}\Big[\frac{a^n}{(2b)^{n+1}}+\frac{(n+1)n}{1}\cdot\frac{a^{n-1}}{(2b)^{n+2}}+\frac{(n+2)(n+1)n(n-1)}{2!}\frac{a^{n-2}}{(2b)^{n+3}}+\ldots$$
$$+\frac{(2n)!}{n!}\frac{1}{(2b)^{2n+1}}\Big].$$

Round the outer contour we have

$$\int_{-\infty}^{0}\frac{e^{\iota ax}}{(b^2+x^2)^{n+1}}dx+\int_0^{\infty}\frac{e^{\iota ax}}{(b^2+x^2)^{n+1}}dx+\int_0^{\pi}\frac{e^{\iota aR(\cos\theta+\iota\sin\theta)}}{(b^2+R^2e^{2\iota\theta})}\iota Re^{\iota\theta}d\theta.$$

Putting $-x$ for x in the first and combining the result with the second, we get $2\int_0^{\infty}\frac{\cos ax}{(b^2+x^2)^{n+1}}dx$. The third integral vanishes as the integrand contains the factor $e^{-aR\sin\theta}$, which vanishes when $R=\infty$, $\sin\theta$ never becoming negative. Hence we obtain

$$\int_0^\infty \frac{\cos ax}{(b^2+x^2)^{n+1}}\,dx=\frac{\pi}{n!}\,\frac{e^{-ab}}{(2b)^{2n+1}}\Big[(2ab)^n+\frac{(n+1)n}{1!}(2ab)^{n-1}+\frac{(n+2)(n+1)n(n-1)}{2!}(2ab)^{n-2}+\ldots+\frac{(2n)!}{n!}\Big],$$

which agrees with the result of Art. 1057, writing n for $n+1$ in the present result.

1327. *Consider the case* $w=z^{n-1}e^{-kz}$, *where* k *is a complex constant* $\equiv a-\iota b$, *in which* a *is positive,* b *positive and not both zero, and* $1>n>0$.

Since $n<1$, there is a pole at the origin. Writing $z=re^{\iota\theta}$, $k=\rho e^{-\iota\beta}$, where β is $\not>\pi/2$, we have $w=r^{n-1}e^{\iota(n-1)\theta}e^{-\rho r\cos(\theta-\beta)}e^{-\iota\rho r\sin(\theta-\beta)}$, which cannot become infinite, except at $z=0$, unless $\cos(\theta-\beta)$ be negative, *i.e.* $\theta>\beta+\frac{\pi}{2}$, or $<\beta-\frac{\pi}{2}$, in which case an infinite value of r would make w infinite.

We shall avoid these poles if we take a contour consisting of a sectorial area bounded by $\theta=0$, $\theta=a(<\pi/2)$ and by arcs $r=R_1$, $r=R_2$, where R_1 is infinitely large and R_2 infinitesimally small. The region thus bounded is such that w is synectic within it, and we have

$$\int_{R_2}^{R_1} x^{n-1}e^{-(a-\iota b)x}dx+\int_0^a (R_1e^{\iota\theta})^n e^{-(a-\iota b)R_1e^{\iota\theta}}\iota\,d\theta$$
$$+\int_{R_1}^{R_2}(re^{\iota a})^{n-1}e^{-\rho e^{-\iota\beta}re^{\iota a}}e^{\iota a}dr+\int_a^0 (R_2e^{\iota\theta})^n e^{-(a-\iota b)R_2e^{\iota\theta}}\iota\,d\theta=0.$$

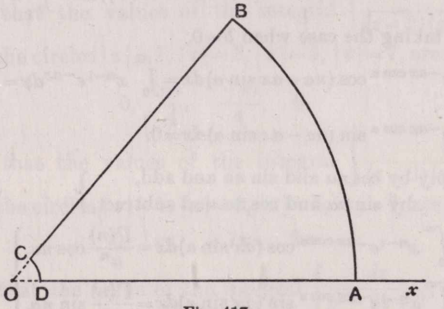

Fig. 417.

The second and fourth integrals contribute nothing, for in the second the integrand contains the factor $R_1{}^n e^{-\rho R_1\cos(\theta-\beta)}$, which vanishes when R_1 is infinite, since we are supposing $a<\pi/2$, and therefore, θ being $<a$, $\theta-\beta<\pi/2$; and in the fourth, the integrand contains the factor $R_2{}^n e^{-\rho R_2\cos(\theta-\beta)}$, which vanishes when R_2 is infinitesimally small.

Hence, proceeding to the limit when $R_1\to\infty$, $R_2\to 0$, we have

$$\int_0^\infty x^{n-1}e^{-ax}e^{\iota bx}dx=e^{n\iota a}\int_0^\infty r^{n-1}e^{-\rho re^{\iota(a-\beta)}}dr. \quad\ldots\ldots\ldots\ldots\ldots(1)$$

If now we choose the angle of the sector, viz. α, to be β, *i.e.* $\tan^{-1}\frac{b}{a}$, we have

$$\int_0^\infty x^{n-1}e^{-ax}e^{\iota bx}dx = e^{n\iota\beta}\int_0^\infty r^{n-1}e^{-\rho r}dr, \quad \text{where } \rho=\sqrt{a^2+b^2},$$

$$= e^{n\iota\beta}\frac{\Gamma(n)}{\rho^n}, \quad \rho \text{ being real},$$

i.e.
$$\int_0^\infty x^{n-1}e^{-(a-\iota b)x}dx = \frac{\Gamma(n)}{(a-\iota b)^n},$$

which shows that the theorem $\int_0^\infty x^{n-1}e^{-kx}dx = \frac{\Gamma(n)}{k^n}$ is true for a complex constant $k=a-\iota b$ as well as for a real one, a being positive (see Art. 1159).

Also
$$\left.\begin{aligned}\int_0^\infty x^{n-1}e^{-ax}\cos bx\,dx &= \frac{\Gamma(n)}{(a^2+b^2)^{\frac{n}{2}}}\cos\left(n\tan^{-1}\frac{b}{a}\right),\\ \int_0^\infty x^{n-1}e^{-ax}\sin bx\,dx &= \frac{\Gamma(n)}{(a^2+b^2)^{\frac{n}{2}}}\sin\left(n\tan^{-1}\frac{b}{a}\right).\end{aligned}\right\}\ldots\ldots\ldots\ldots(2)$$

1328. Equation (1) of the previous article gives

$$\int_0^\infty x^{n-1}e^{-ax}e^{\iota bx}dx = \int_0^\infty x^{n-1}e^{-(a\cos\alpha+b\sin\alpha)x}e^{\iota\{n\alpha-x(a\sin\alpha-b\cos\alpha)\}}dx\,;$$

whence

$$\left.\begin{aligned}\int_0^\infty x^{n-1}e^{-(a\cos\alpha+b\sin\alpha)x}\cos\{n\alpha-x(a\sin\alpha-b\cos\alpha)\}dx &= \int_0^\infty x^{n-1}e^{-ax}\cos bx\,dx\\ \int_0^\infty x^{n-1}e^{-(a\cos\alpha+b\sin\alpha)x}\sin\{n\alpha-x(a\sin\alpha-b\cos\alpha)\}dx &= \int_0^\infty x^{n-1}e^{-ax}\sin bx\,dx,\end{aligned}\right\}$$

and therefore taking the case when $b=0$,

$$\left.\begin{aligned}\int_0^\infty x^{n-1}e^{-ax\cos\alpha}\cos(n\alpha-ax\sin\alpha)dx &= \int_0^\infty x^{n-1}e^{-ax}dx = \frac{\Gamma(n)}{a^n},\\ \int_0^\infty x^{n-1}e^{-ax\cos\alpha}\sin(n\alpha-ax\sin\alpha)dx &= 0.\end{aligned}\right\}$$

If we multiply by $\cos n\alpha$ and $\sin n\alpha$ and add,
and by $\sin n\alpha$ and $\cos n\alpha$ and subtract,

we obtain
$$\left.\begin{aligned}\int_0^\infty x^{n-1}e^{-ax\cos\alpha}\cos(ax\sin\alpha)dx &= \frac{\Gamma(n)}{a^n}\cos n\alpha,\\ \int_0^\infty x^{n-1}e^{-ax\cos\alpha}\sin(ax\sin\alpha)dx &= \frac{\Gamma(n)}{a^n}\sin n\alpha.\end{aligned}\right\}$$

[Cf. Briot and Bouquet.]

If γ be any other angle, we have upon multiplication by $\cos\gamma$, $\sin\gamma$ and subtracting, and by $\sin\gamma$, $\cos\gamma$ and adding,

$$\left.\begin{aligned}\int_0^\infty x^{n-1}e^{-ax\cos\alpha}\cos(ax\sin\alpha+\gamma)dx &= \frac{\Gamma(n)}{a^n}\cos(n\alpha+\gamma),\\ \int_0^\infty x^{n-1}e^{-ax\cos\alpha}\sin(ax\sin\alpha+\gamma)dx &= \frac{\Gamma(n)}{a^n}\sin(n\alpha+\gamma).\end{aligned}\right\}$$

$(\alpha<\pi/2,\ 1>n>0,\ a\ +^{\text{ve}}.)$

PROBLEMS.

1. If $w^2 = z - 1$, examine the value of $\int_0^{z_1} w\,dz$,

(i) *via* the branch $w = \sqrt{z-1}$ by any path which does not encircle the branch-point at $z = 1$;

(ii) *via* a path starting with the same branch and encircling the branch-point once.

2. Find the values of

$$\int \frac{\sin z}{z-a}\,dz, \quad \int \frac{\sin z}{(z-a)^2}\,dz, \quad \int \frac{\sin z}{(z-a)^3}\,dz,$$

taken round a small circle whose centre is at $z = a$.

3. Find the values of

$$\int \frac{z}{z-a}\,dz, \int \frac{z^2}{(z-a)^2}\,dz, \int \frac{z^2}{(z-a)^3}\,dz \quad \text{and} \quad \int \frac{z^2}{(z-a)^4}\,dz,$$

taken round a small circle whose centre is at $z = a$.

4. Show that the values of the integral $\int \frac{dz}{(z-2)(z-4)}$, taken round the circles $|z| = 1$, $|z| = 3$, $|z| = 5$, are respectively

$$0, \quad -\pi\iota \quad \text{and} \quad 0.$$

5. Show that the values of the integral $\int \frac{dz}{(z-2)(z-4)(z-6)}$, taken round the circles $|z| = 1$, $|z| = 3$, $|z| = 5$, $|z| = 7$, are respectively

$$0, \quad \frac{\pi\iota}{4}, \quad \frac{-\pi\iota}{4}, \quad 0.$$

6. Show that the values of the integral $\int \frac{z^2\,dz}{(z-2)(z-4)(z-6)}$, taken round the circles $|z| = 1$, $|z| = 3$, $|z| = 5$, $|z| = 7$, are respectively

$$0, \quad \pi\iota, \quad -7\pi\iota, \quad 2\pi\iota.$$

7. Show that the value of the integral $\int \frac{dz}{z^2 - 2z + 2}$, taken round a contour consisting of the x-axis, the y-axis and the arc of the circle $|z| = 2$, which lies in the first quadrant, is π.

8. Show that the value of the integral $\int \frac{z^2\,dz}{(z-1)^4(z^3+1)}$, taken round a contour consisting of a semicircle of radius greater than unity, with centre at the origin and its diameter the y-axis and lying towards the positive side of the x-axis, is $-\frac{\pi\iota}{24}$, and the

same integral, taken round the entire circumference of the circle $x^2+y^2+2x=0$, is $\frac{\pi\iota}{24}$. Show also that the same integral, taken round the rectangle bounded by $x=0$, $x=0\cdot75$, $y=\pm 1$, is $-\frac{2\pi\iota}{3}$.

9. Show that the integral $\int \frac{dz}{(z^3+1)^2}$, taken round a contour which consists of the y-axis and that part of any semicircle $|z|>1$, which lies on the positive side of the y-axis, is $-\frac{4}{9}\pi\iota$.

[FORSYTH, *Th. Funct.*, p. 42.]

10. If p and q be positive integers, show by integrating $\int \frac{z^{2p}}{1+z^{2q}}dz$ round the perimeter of a semicircle of radius a (supposed >1), having its diameter coincident with the axis of x and its centre at the origin, that

$$\int_{-a}^{a} \frac{x^{2p}}{1+x^{2q}}dx + \iota\int_0^{\pi} \frac{a^{2p+1}e^{(2p+1)\iota\theta}}{1+a^{2q}e^{2q\iota\theta}}d\theta = \frac{\pi}{q\sin\frac{2p+1}{2q}\pi},$$

and deduce that if $1>a>0$,

$$\int_0^{\infty} \frac{x^{a-1}}{1-x}dx = \frac{\pi}{\sin a\pi}.$$

[MATH. TRIP., 1887.]

11. When is a function said to have a pole? Distinguish between a pole and an *essential* singularity; show that a function which is everywhere regular is a constant.

From consideration of the integral $\int \frac{e^{\iota z}\,dz}{(z-a)^2+b^2}$, where a and b are real positive quantities, taken round a suitable boundary, show that

$$\int_0^{\infty} \frac{\cos x}{(x-a)^2+b^2}dx + 2a\int_0^{\infty} \frac{e^{-y}y\,dy}{(a^2+b^2-y^2)^2+4a^2y^2} = \frac{\pi\cos a}{be^b},$$

$$\int_0^{\infty} \frac{\sin x}{(x-a)^2+b^2}dx - \int_0^{\infty} \frac{e^{-y}(a^2+b^2-y^2)\,dy}{(a^2+b^2-y^2)^2+4a^2y^2} = \frac{\pi\sin a}{be^b},$$

[I. C. S., 1908.]

12. Determine a function which shall be regular within the circle $|z|=1$, and shall have at the circumference of this circle the value

$$\frac{(a^2-1)\cos\theta+\iota(a^2+1)\sin\theta}{a^4-2a^2\cos 2\theta+1},$$

where $a^2>1$, θ denoting the vectorial angle. [I. C. S., 1909.]

13. Establish by contour integration the result

$$\int_0^{\infty} \frac{x^2\,dx}{(x^2-a^2)^2+b^2x^2} = \frac{\pi}{2b},$$

b being positive. [I. C. S., 1910.]

14. By considering the contour integral

$$\int \frac{e^{az}}{1-e^z}\,dz, \quad (0 < a < 1),$$

round a rectangle of infinite length ($x = -\infty$ to $+\infty$), and finite breadth ($y = 0$ to π) with a small semicircle excluding the origin, prove that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \pi \operatorname{cosec} \pi a.$$

[I. C. S., 1903.]

15. If a, b be two quantities each of the form $\alpha + \beta\iota$, explain the meaning of the integration $\int_a^b \phi(z)\,dz$, and point out in what cases the value of the integral is dependent on the path chosen between the limits. [ST. JOHN'S COLL., 1881.]

16. Prove that, a being positive,

$$\int_0^\infty e^{-2ax}\cos x^2\,dx = \int_0^\infty \sin(a'^2 - a^2)\,da';$$

$$\int_0^\infty e^{-2ax}\sin x^2\,dx = \int_0^\infty \cos(a'^2 - a^2)\,da'.$$

[SMITH'S PRIZE, 1876.]

17. Evaluate the integral $\int \frac{\sin z}{z^3 - a^3}\,dz$, taken round the unit circle in the counter-clockwise sense, where a is any real number other than ± 1. [MATH. TRIP., PT. II., 1920.]

18. Evaluate the integral $\int \frac{\log(z-a)}{z-a}\,dz$, taken round the unit circle in the counter-clockwise sense, where a is any real number other than ± 1, and the logarithm has its principal value.

[MATH. TRIP., PT. II., 1920.]

19. Explain what is meant by a period of an integral of a function, and investigate the periods of the integrals

$$\int \frac{dz}{1+z^2}, \quad \int (1-z^2)^{-\frac{1}{2}}\,dz, \quad \int (1-z^2)^{\frac{1}{2}}\,dz.$$

[MATH. TRIP., PT. II., 1913.]

20. Show, by contour integration round an infinite semicircle and its diameter, that

$$\int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{x^2+x+1} = \frac{\pi}{\sqrt{3}}, \qquad \int_0^\infty \frac{x^{\frac{1}{2}}\,dx}{x^2-x+1} = \pi,$$

$$\int_0^\infty \frac{x^{\frac{1}{3}}\,dx}{x^2+x+1} = \frac{4\pi}{3}\sin\frac{\pi}{9}, \quad \int_0^\infty \frac{x^{\frac{1}{3}}\,dx}{x^2-x+1} = \frac{4\pi}{3}\sin\frac{2\pi}{9}.$$

21. Discuss, by contour integration round an infinite semicircle and its diameter, $\int \frac{z^p\,dz}{z^2+2z\cos\alpha+1}$, where p lies between ± 1 and $0<\alpha<\pi$.

22. Prove that $\int_0^{\frac{\pi}{2}} \log\cos\theta\,d\theta = \frac{\pi}{2}\log\frac{1}{2}$, by consideration of the integral $\int \log\frac{1}{2}\left(z+\frac{1}{z}\right)\frac{dz}{z}$ taken round a suitable contour.

23. By consideration of the integration $\int e^{-a^2z^2}\,dz$ round the perimeter of an infinite rectangle of breadth b/a^2, establish Laplace's integral of Art. 1041, a being real.

24. By consideration of $\int e^{-a^4z^4}\,dz$ round an infinite rectangle of breadth b, a being real and positive, prove that

$$\int_0^\infty e^{-a^4x^2(x^2-6b^2)}\cos\{4a^4bx(x^2-b^2)\}\,dx = \frac{e^{a^4b^4}}{4a}\Gamma\left(\tfrac{1}{4}\right).$$

25. By integration of $\int \frac{e^{\iota kz}}{z^4+4a^4}\,dz$ round an infinite quadrant, where a and k are real and positive, show that

$$\int_0^\infty \frac{\cos kx}{x^4+4a^4}\,dx = \frac{\pi}{8a^3}e^{-ka}(\sin ka+\cos ka);$$

$$\int_0^\infty \frac{\sin kx - e^{-kx}}{x^4+4a^4}\,dx = \frac{\pi}{8a^3}e^{-ka}(\sin ka-\cos ka).$$