CHAPTER XXIX.

VECTORS. THE COMPLEX VARIABLE. CONFORMAL REPRESENTATION.

1202. **The Operative Symbol ι.**

Let ι be defined as an operative symbol which, when applied to any straight line of given length, and lying in a given plane, has the effect of turning that line in the given plane about one of its extremities through a right angle in the positive direction of rotation, *i.e.* according to the customary convention, counter-clockwise.

Then, if OP be any length measured along the positive direction of the x-axis, ιOP will be an equal line OP_1 measured along the positive direction of the y-axis.

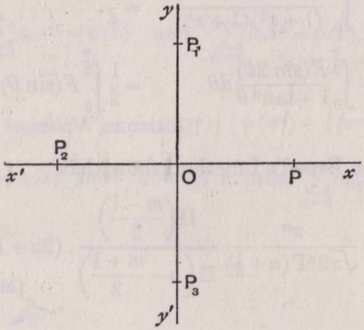

Fig. 340.

Now $\iota(\iota OP)$, or, as we may write it in analogy with algebraic custom, $\iota^2 OP$, may be interpreted as the result of *doing to ιOP what ι has done to OP*; *i.e.* OP_1 has been itself turned counter-clockwise to a position OP_2 lying along the negative direction of the x-axis, the absolute lengths of OP_2 and OP_1 being each equal to OP.

Again, $\iota\{\iota(\iota OP)\}$, *i.e.* ιOP_2 or $\iota^3 OP$ has turned OP_2 to the position OP_3 lying along the negative direction of the y-axis, the absolute lengths of OP_3 and OP being equal.

Finally, $\iota[\iota\{\iota(\iota OP)\}]$, *i.e.* ιOP_3 or $\iota^4 OP$, has turned OP_3 to the original position OP.

1203. **Interpretation of $\sqrt{-1}$.**

Let us next consider for a moment the symbol $\sqrt{-1}$, or, as it is usually called, "the square root of -1," an expression with which the student has grown familiar in algebra, in the solution of quadratic equations, factorisation, etc.

Now all arithmetical *quantities* are either positive, zero or negative. There are no others. Their squares are all either positive or zero. There is no arithmetical *quantity* whose square is negative. But the definition of $\sqrt{-1}$ is that

$$\sqrt{-1}\sqrt{-1}=-1,$$

or conforming to the usual notation and language $(\sqrt{-1})^2=-1$, and "the square of $\sqrt{-1}$" is -1. The logical inference is that $\sqrt{-1}$ is *not quantitative.*

But it is customary nevertheless to discuss and use such expressions in algebra as they arise there, and as they *obey the same fundamental laws of algebra* as are obeyed by ordinary arithmetical and algebraical *quantities*, viz. (1) the associative or distributive law, (2) the commutative law, (3) the index law, so long as they are combined with quantities which have magnitude only and no directive property.

Now, according to the usual Cartesian convention of sign to denote the relative direction of lines, if OP be regarded as a line drawn in the direction of the positive direction of the x-axis and OP_2 an equal line in the opposite direction, $OP_2=-OP$.

Thus $$\iota^2 OP=-OP=(\sqrt{-1})^2 OP.$$

We may therefore properly interpret $\sqrt{-1}$ as being identical with the operator ι, and therefore regard $\sqrt{-1}$, which is not *quantitative* at all, as being *operative* and having the property that it turns any line to which it may be applied through a right angle counter-clockwise about one of its extremities. It is not therefore commutative as regards such expressions as have direction as well as magnitude, *i.e.* such expressions as are known as "vectors," in distinction from those which have magnitude only, to which the term "scalar" is applied.

1204. **Definition of the Term "Vector."**

The terms "scalar" and "vector" are due to Sir William Rowan Hamilton.

The definition of a "vector" given by Kelland and Tait (*Quaternions*, p. 6) is, "A vector is the representative of transference through a given distance in a given direction."

In the consideration of such operative symbols and vectors we retain, as is usual, the ordinary terms addition, subtraction, multiplication, division, though the interpretation of the results will differ in some respects from the results of the corresponding common processes as applied to scalar quantities.

If a rigid lamina be displaced without rotation from one position to another position in its own plane, points $A, B, C, \dots$ of the lamina are transferred to new positions $A', B', C', \dots$, such that AA', BB', CC', etc., are all equal and parallel. A knowledge of the length and direction of any one of them would be enough to fix the second position of the lamina relatively to its original position. They are all vector quantities and equivalent. That is, they are represented by the same vector. A vector is completely defined when its magnitude and its direction are known. No account is taken of its position. In this respect a vector differs from a force which needs further description, viz. a specification of the point of application.

Hence a force is fully defined by (1) its point of application,
(2) its representative vector.

In the case of the axis of a couple the only elements necessary for its description are (1) its magnitude, (2) its direction. Hence the axis of a couple is a pure vector and needs no further description, the vector being specified.

A vector is therefore represented graphically by drawing *any* straight line in the specific direction of the vector and of the specific length indicated in the description of the vector.

And *all parallel lines of the same length, from whatever points they may be drawn, will equally represent the same vector.*

Thus, the force acting at a definite point, a velocity, an acceleration, the axis of a couple are familiar examples of vector quantities, whilst speeds, moments, energy, horse-power, are scalar quantities.

1205. **Laws of Combination of the Operator ι.**

The operator ι obeys the "associative" or distributive law of algebra. For if we apply it to the sum of two lines OA, AB (Fig. 341) which lie in the same direction, say along the x-axis, it is immaterial whether we first add the lines together and then rotate the sum through a counter-clockwise right angle,

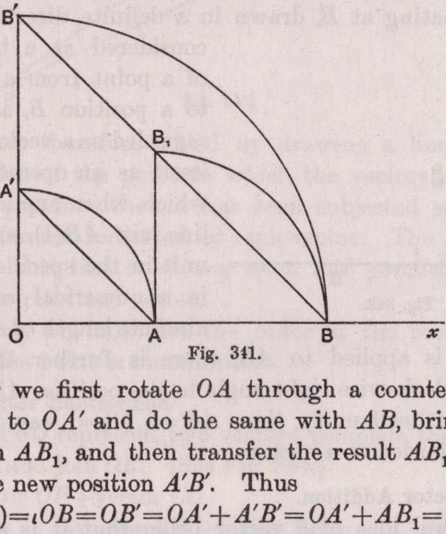

Fig. 341.

or whether we first rotate OA through a counter-clockwise right angle to OA' and do the same with AB, bringing it to the position AB_1, and then transfer the result AB_1 parallel to itself to the new position $A'B'$. Thus

$$\iota(OA+AB)=\iota OB=OB'=OA'+A'B'=OA'+AB_1=\iota OA+\iota AB.$$

1206. The same is obviously true if the operator ι be applied to the difference of two lines or to the algebraic sum of any number of lines in the same direction.

1207. Again, if a line be doubled or trebled or halved, etc., and then turned through a right angle counter-clockwise, the effect is the same as if we turn through a right angle first and then double, treble or halve, etc., *i.e.* $\iota(pOA)=pi(OA)$, p being numerical, so that ι *obeys the commutative rule* as regards numerical, that is scalar, quantities. But it is not commutative with regard to the subject of its operation, *i.e.* we cannot write ιAB as $AB\iota$ any more than we can write $\log x$ as $x \log$.

Finally, ι satisfies *the index law of algebra.* For to turn a line n times in succession through a right angle in a counter-clockwise direction brings it into the same position as it would have had if turned in the same direction through n right angles at a single operation,

i.e. $\iota^n OA = \iota . \iota . \iota \ldots$ to n operations $. OA$.

Thus ι satisfies all the fundamental laws of algebraic combination, except that it is not commutative with regard to any vector quantities upon which it is operative.

1208. The symbol AB, as denoting a line starting from A and terminating at B, drawn in a definite direction, may be considered as a transference of a point from a position A to a position B, and may be regarded as a vector, or in fact itself as an operative symbol which, when applied to a unit line, viz. $AB(1)$, extends that unit in the specified direction in a numerical ratio of the absolute length of AB to unity.

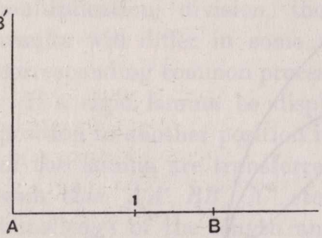

Fig. 342.

When ι is applied to AB, there is further the rotation through a clock-wise right angle to the position AB'.

If AB be itself unity, then $AB' = \iota . (1) = \iota$, say, and ι may itself be regarded as a vector.

1209. **Vector Addition.**

The general idea of a vector being that it is an operator which has the effect of transferring a point through a given distance in a given direction, we understand that "vector PQ" means that the point P is to be transferred from P to Q through a distance represented by the length of PQ in the direction specified by the direction in which the line PQ is drawn from P. This being so, it follows that

$$\text{vector } PQ + \text{vector } QP = 0,$$

for there is no change in the position of P when the whole operation has been completed.

But vector $PQ+$vector $QR=$vector PR, where the second transference (Q to R) is not made necessarily in the same direction as the first (viz. P to Q). And we must understand by the sign of equality in such a relation as this, that it stands for the words "are together equivalent to."

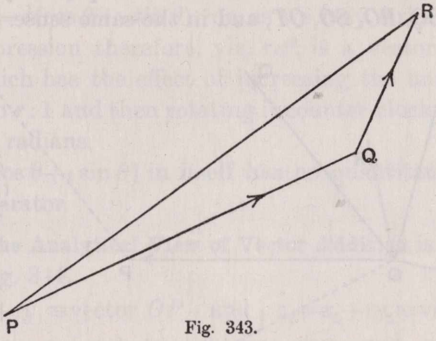

Fig. 343.

Vectors are therefore added by drawing a line from the initial position of the point to which the vectors are applied to its final position when it has been subjected successively to the transference indicated by each vector. The length and direction of this line *or of any equal and parallel line* fully represent the resultant vector.

It is clearly obvious that the order of the several transferences of the point is immaterial.

1210. **Vector Subtraction.**

If OP and OQ represent two vectors, complete the parallelogram $OPRQ$ and join OR. (See Fig. 344.)

Then vector $OP+$vector OQ

$$=\text{vector } OP+\text{vector } PR$$
$$=\text{vector } OR.$$

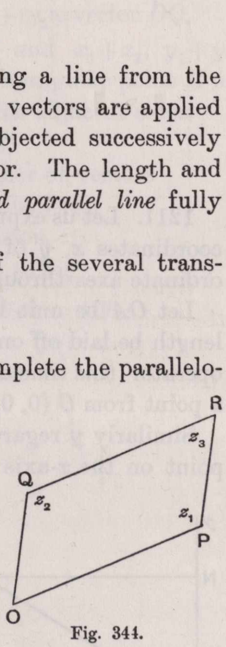

Fig. 344.

It follows that

$$\text{vector } OP=\text{vector } OR-\text{vector } OQ$$
$$=\text{vector } OR-\text{vector } PR$$
$$=\text{vector } OR+\text{vector } RP.$$

And the result of subtraction may therefore be obtained in the same way as that of addition, but drawing the subtractive vectors in the opposite direction to that in which they are drawn for addition.

Thus, if there be several vectors, OP, OQ, etc.,

$$\begin{aligned}\text{vector } OP + \text{vector } OQ - \text{vector } OR - \text{vector } OS + \text{vector } OT \\ = \text{vector } OP + \text{vector } PQ' + \text{vector } Q'R' + \text{vector } R'S' \\ + \text{vector } S'T' = \text{vector } OT',\end{aligned}$$

where PQ', $Q'R'$, $R'S'$, $S'T'$ are drawn respectively equal and parallel to OQ, RO, SO, OT, and in the same sense. (Fig. 345.)

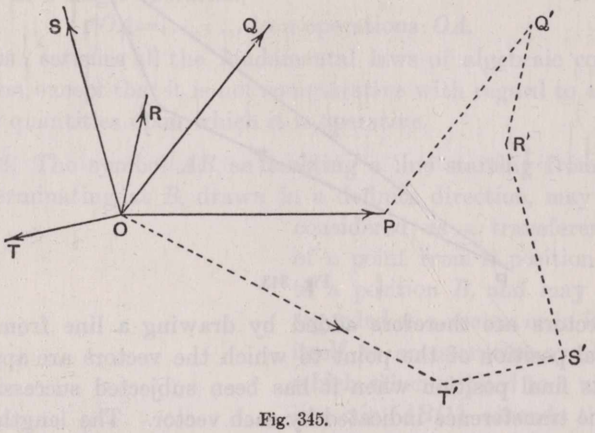

Fig. 345.

1211. Let us express the vector OP in terms of the Cartesian coordinates x, y of P referred to a pair of rectangular coordinate axes through O.

Let OA be unit length on the x-axis. Then if x units of length be laid off on the x-axis (OM), we may regard x as an operator (this time a mere numerical multiplier) which transfers a point from O (0, 0) to M (x, 0).

Similarly y regarded as an operator would transfer O to a point on the x-axis y units of length ($=ON'$) distant from O, and ιy would be the vector which would transfer a point from O an equal distance along the y-axis to N, where $ON=ON'$.

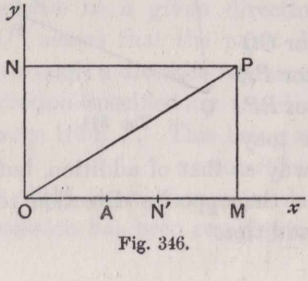

Fig. 346.

Thus, if z represent the complete operation $x+\iota y$ (Fig. 346),

$$\begin{aligned}z \equiv x + \iota y &= \text{vector } OM + \text{vector } ON \\ &= \text{vector } OM + \text{vector } MP \\ &= \text{vector } OP,\end{aligned}$$

where P is the corner opposite to O of the rectangle, with OM, ON as adjacent sides, the coordinates of P being the numerical values of x and y.

1212. If the linear magnitude of OP be r units of length and θ the angular displacement of OP from Ox, we have

$x+\iota y=r(\cos\theta+\iota\sin\theta)$, or, as we may write it, $re^{\iota\theta}$.

This expression therefore, viz. $re^{\iota\theta}$, is a vectorial operative symbol which has the effect of increasing the unit length OA in the ratio $r:1$ and then rotating it counter-clockwise through an angle θ radians.

Thus $r(\cos\theta+\iota\sin\theta)$ in itself has no quantitative meaning. It is an operator.

1213. **The Analytical View of Vector Addition** is as follows: If, in Fig. 344,

$$z_1=x_1+\iota y_1\equiv\text{vector } OP \quad\text{and}\quad z_2=x_2+\iota y_2\equiv\text{vector } OQ,$$

then $z_1+z_2=x_1+x_2+\iota(y_1+y_2)\equiv z_3$, say, and x_1+x_2, y_1+y_2 are the Cartesian coordinates of the fourth angular point R of the parallelogram drawn with OP, OQ with adjacent sides.

Thus $$z_3\equiv z_1+z_2\equiv\text{vector } OR,$$

and the rule can be extended to any number of vectors

$$z_1,\ z_2,\ z_3,\ \ldots,\ z_n,\quad\text{where } z_r=x_r+\iota y_r.$$

If Z be the resultant vector of the addition,

$$Z=z_1+z_2+z_3+\ldots+z_n=\Sigma x+\iota\Sigma y,$$

where $\Sigma x=x_1+x_2+\ldots+x_n,\quad \Sigma y=y_1+y_2+\ldots+y_n.$

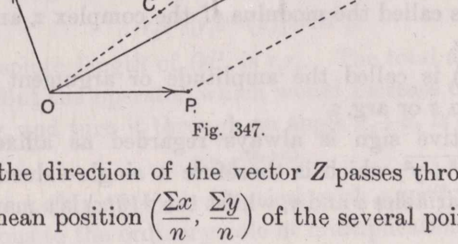

Fig. 347.

Clearly the direction of the vector Z passes through C, the centre of mean position $\left(\dfrac{\Sigma x}{n}, \dfrac{\Sigma y}{n}\right)$ of the several points P_1, P_2,

P_3, ..., whose coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., and its length is n times the distance of the centre of mean position from O, where n is the number of vectors added.

Exactly in the same way

$$z_1 - z_2 = x_1 - x_2 + \iota(y_1 - y_2),$$
$$z_1 - z_2 - z_3 + z_4 = (x_1 - x_2 - x_3 + x_4) + \iota(y_1 - y_2 - y_3 + y_4), \text{ etc.}$$

1214. In writing $z \equiv x + \iota y$, where x and y are the coordinates of a point P, we regard z as a vector which transfers a point from the origin O to P along the line OP.

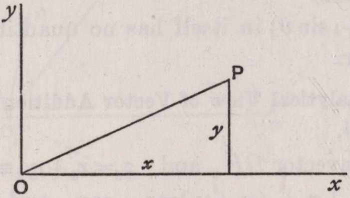

Fig. 348.

We may equally regard z as representing a label of the point P on the x-y plane, and it is then referred to as a complex variable. And in this sense every point in the plane may be represented by a complex variable, and conversely to every complex variable there is a corresponding point on the x-y plane.

When the point P moves in the plane, tracing a continuous path upon the plane, the relation between x and y is continuous, and the variation in the complex variable z is continuous.

1215. **Modulus, Amplitude.**

The letters r, θ represent the ordinary polar coordinates of the point $P(x, y)$, and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$.

$\sqrt{x^2 + y^2}$ is called the modulus of the complex z, and written $|z|$ or mod. z.

$\tan^{-1}(y/x)$ is called the amplitude or argument of z, and written amp. z or arg. z.

The positive sign is always regarded as affixed to the modulus $\sqrt{x^2 + y^2}$, which is therefore a single-valued function of the real variables x and y, whilst $\tan^{-1}(y/x)$ is a many-valued function.

The expression $\cos\theta+\iota\sin\theta$ does not change its value when any even multiple of π, say $2\lambda\pi$, is added to θ, λ being an integer, so we may regard the amplitude as $2\lambda\pi+\theta$ or $2\lambda\pi+\tan^{-1}(y/x)$, where in this latter form we are to be understood to mean by $\tan^{-1}(y/x)$ the smallest positive value of the angle whose tangent is y/x, usually called the "principal value."

1216. **Argand Diagram.**

When any relation is assigned between y and x, the Cartesian graph of this relation is called the Argand diagram of the variation of z, and is the path of the extremity of the vector OP, whose changes are defined by the given relation.

1217. **Vector Multiplication. Demoivre's Theorem.**

We use the term multiplication for want of a better term and by analogy with algebraic multiplication. But what we are about to discuss is the effect of the operation of one vector operator upon another vector operator.

Let the operators be $r_1e^{\iota\theta_1}$ and $r_2e^{\iota\theta_2}$, the original subject of the first operation being a line of unit length lying along the x-axis.

The first operation $r_1e^{\iota\theta_1}.1$ increases OA (a unit line on the x-axis) in the ratio $r_1:1$, and turns the resulting line through an angle θ_1 into a direction indicated in the figure by OP_1.

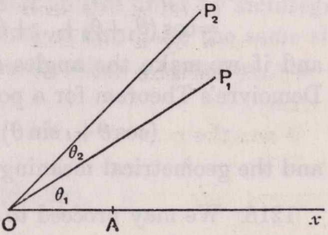

Fig. 349.

The second operation $r_2e^{\iota\theta_2}$ acting upon OP_2 does to OP_1 what $r_1e^{\iota\theta_1}$ does to unity; viz. it increases OP_1 in the ratio of $r_2:1$ and rotates the increased OP_1, which has thus become $r_2.OP_1$, through a further angle θ_2, to a position OP_2.

Thus $$r_2e^{\iota\theta_2}[r_1e^{\iota\theta_1}(1)]=OP_2.$$

The absolute length of OP_2 is r_1r_2. The total angle $x\hat{O}P_2$ is $\theta_1+\theta_2$. But the operator which would increase OA $(=1)$ to a length r_1r_2 and turn it through an angle $\theta_1+\theta_2$ is

$$r_1r_2e^{\iota(\theta_1+\theta_2)}.$$

So that $r_2e^{\iota\theta_2}[r_1e^{\iota\theta_1}(1)]$ is identical with $r_1r_2e^{\iota(\theta_1+\theta_2)}(1)$, which is analogous to the ordinary rule of multiplication in algebra.

Further, it is obvious that the order of the two operations upon unity is immaterial, so that the operations are commutative with regard to each other. It will be observed that in the multiplication of two vectors the modulus of the product is the product of their moduli, and that the amplitude of their product is the sum of the amplitudes of the original vectors.

Again we may write the result as

$$r_2(\cos\theta_2+\iota\sin\theta_2)r_1(\cos\theta_1+\iota\sin\theta_1)(1)$$
$$\equiv r_2r_1[\cos(\theta_1+\theta_2)+\iota\sin(\theta_1+\theta_2)](1),$$

which accords with what we get by the ordinary process of multiplication of $r_1(\cos\theta_1+\iota\sin\theta_1)$ by $r_2(\cos\theta_2+\iota\sin\theta_2)$.

If r_1 and r_2 be both taken unity, we obtain

$$(\cos\theta_2+\iota\sin\theta_2)(\cos\theta_1+\iota\sin\theta_1)\equiv\cos(\theta_2+\theta_1)+\iota\sin(\theta_2+\theta_1),$$

which means that to rotate a line of unit length through an angle θ_1 and then to rotate the result through a further angle θ_2 is identical with rotating the original line through a single angle $\theta_2+\theta_1$, and this can obviously be generalised for any number of angles. Thus

$$(\cos\theta_1+\iota\sin\theta_1)(\cos\theta_2+\iota\sin\theta_2)(\cos\theta_3+\iota\sin\theta_3)\dots(\cos\theta_n+\iota\sin\theta_n)$$
$$=\cos(\theta_1+\theta_2+\dots+\theta_n)+\iota\sin(\theta_1+\theta_2+\dots+\theta_n);$$

and if we make the angles $\theta_1, \theta_2, \theta_3, \dots, \theta_n$ each $=\theta$, we get Demoivre's Theorem for a positive integral index, viz.

$$(\cos\theta+\iota\sin\theta)^n=\cos n\theta+\iota\sin n\theta,$$

and the geometrical meaning of that theorem is thus shown.

1218. We may proceed to consider Demoivre's Theorem for fractional and negative indices from the same point of view.

When n is not a positive integer but $=p/q$, say, where p and q are both positive integers, $\left(\cos\frac{p}{q}\theta+\iota\sin\frac{p}{q}\theta\right)^q$ is an operator which rotates a line of length unity through q successive angles, each $=\frac{p}{q}\theta$, counter-clockwise, and therefore through an angle $p\theta$ counter-clockwise, which is therefore the same as if we rotated a line of unit length through p successive angles, each equal θ; and therefore the operators

$$\left(\cos\frac{p}{q}\theta+\iota\sin\frac{p}{q}\theta\right)^q \quad\text{and}\quad (\cos\theta+\iota\sin\theta)^p$$

are identical in their turning effect. We may therefore, consistently with the algebraic notation for indices, write

$$\cos\frac{p}{q}\theta+\iota\sin\frac{p}{q}\theta=(\cos\theta+\iota\sin\theta)^{\frac{p}{q}},$$

it being supposed that $(\cos\theta+\iota\sin\theta)^{\frac{p}{q}}$ represents an operator which, when repeated q times, gives the operator

$$(\cos\theta+\iota\sin\theta)^{p}.$$

Again, since cosines and sines are not altered if an integral multiple of 2π be added to their angle, and since to rotate a line through 2π is merely to bring it back into its original position, it will be seen that $\cos(\theta+2\lambda\pi)+\iota\sin(\theta+2\lambda\pi)$ is an operator which has the same effect as $\cos\theta+\iota\sin\theta$.

Hence the operator $\cos\frac{p}{q}(\theta+2\lambda\pi)+\iota\sin\frac{p}{q}(\theta+2\lambda\pi)$, having the same effect as $[\cos(\theta+2\lambda\pi)+\iota\sin(\theta+2\lambda\pi)]^{\frac{p}{q}}$, is the same as $(\cos\theta+\iota\sin\theta)^{\frac{p}{q}}$.

Also, the various angles $\frac{p}{q}(\theta+2\lambda\pi)$ for different values of λ, viz. $0, 1, 2, \ldots, q-1$, are such that no two differ by an integral multiple of 2π, and therefore that no two have the same sine and the same cosine. There are therefore q operators, viz.

$$\cos\frac{p}{q}\theta+\iota\sin\frac{p}{q}\theta,$$

$$\cos\frac{p}{q}(\theta+2\pi)+\iota\sin\frac{p}{q}(\theta+2\pi),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\cos\frac{p}{q}\{\theta+2(q-1)\pi\}+\iota\sin\frac{p}{q}\{\theta+2(q-1)\pi\},$$

any of which, after q of its own operations, will have the same effect as $(\cos\theta+\iota\sin\theta)^{p}$, and there are no more. For if $\lambda=q$,

$$\cos\frac{p}{q}(\theta+2q\pi)+\iota\sin\frac{p}{q}(\theta+2q\pi)=\cos\frac{p}{q}\theta+\iota\sin\frac{p}{q}\theta,$$

which is the first of the above operators over again, and so on. $\lambda=q+1$, $\lambda=q+2$, etc., give the second, third, etc., operators over again, so that other values of λ merely repeat one or other of the operators already obtained.

It is customary in the proof of Demoivre's Theorem to state this result in the form that $(\cos\theta+\iota\sin\theta)^{\frac{p}{q}}$ has q *values* and no more, these *values* being the above-mentioned expressions.

To complete the ordinary results of Demoivre's Theorem we still have to show that the operator $(\cos\theta+\iota\sin\theta)^n$ is the same as $\cos\iota\theta+\iota\sin n\theta$, where n is negative. Let $n=-m$.

Then $(\cos\theta+\iota\sin\theta)^{-m}$ is an operative symbol of inverse nature. Call its effect, when applied to unity, X.

Then $1=(\cos\theta+\iota\sin\theta)^m X$, which, by what has preceded, is the same as $(\cos m\theta+\iota\sin m\theta)X$, where m is positive and either integral or fractional.

Now, to turn a line through a counter-clockwise angle $m\theta$, and then to turn the result clockwise through the same angle, restores it to its original position, so that

$$[\cos(-m\theta)+\iota\sin(-m\theta)][\cos m\theta+\iota\sin m\theta]X=X.$$

Hence

$$[\cos(-m\theta)+\iota\sin(-m\theta)](1)=X=(\cos\theta+\iota\sin\theta)^{-m}(1),$$

i e. $\quad (\cos\theta+\iota\sin\theta)^n(1)=[\cos(-m)\theta+\iota\sin(-m)\theta](1)$

$$=(\cos n\theta+\iota\sin n\theta)(1).$$

Hence it follows that the operators

$$(\cos\theta+\iota\sin\theta)^n \quad\text{and}\quad \cos n\theta+\iota\sin n\theta$$

are identical when n is a negative integer or a negative fraction, as well as when it is a positive integer or a positive fraction, and therefore their identity has been established for any commensurable value of n.

1219. **Vector Division.**

Let $z_1\equiv r_1(\cos\theta_1+\iota\sin\theta_1)$, $z_2\equiv r_2(\cos\theta_2+\iota\sin\theta_2)$.

Then we have to consider the effect of the operator z_1/z_2.

Let $z_1=z_2z_3$, and let $z_3\equiv r_3(\cos\theta_3+\iota\sin\theta_3)$.

Then $\qquad z_1\equiv r_2r_3\{\cos(\theta_2+\theta_3)+\iota\sin(\theta_2+\theta_3)\}$,

and $\qquad z_1\equiv r_1(\cos\theta_1+\iota\sin\theta_1)$;

whence $r_1=r_2r_3$, $\theta_1=\theta_2+\theta_3$, and $r_3=r_1/r_2$, $\theta_3=\theta_1-\theta_2$.

Hence $\qquad z_3\equiv\frac{r_1}{r_2}\{\cos(\theta_1-\theta_2)+\iota\sin(\theta_1-\theta_2)\}$,

i.e. the "quotient" is a single vector whose modulus is the quotient of the moduli of the original vectors, and the amplitude of the quotient is the difference of their amplitudes.

1220. **Geometrical Meaning.**

Geometrically we may represent the result thus:

Suppose OP_2, OP_1 to be the original vectors z_2 and z_1. Con-

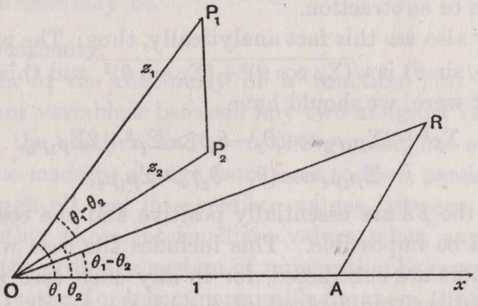

Fig. 350.

struct a triangle OAR similar to OP_2P_1, with $OA=1$ lying along the x-axis.

Then $\dfrac{OR}{OA}=\dfrac{OP_1}{OP_2}$ in magnitude and $A\hat{O}R=P_2\hat{O}P_1=\theta_1-\theta_2$.

Hence the vector OR has for modulus r_1/r_2 and for amplitude $\theta_1-\theta_2$, *i.e.* the vector OR represents the "quotient" of the vectors OP_1, OP_2.

Hence, summing up, it appears that addition, subtraction, multiplication, or division of vectors always leads to a single vector as the result of the operation.

1221. **Laws of Combination of Vectors.**

From what has been established for the addition, subtraction, multiplication and division of vector quantities, we have then the following rules as to the moduli and amplitudes of the results of these operations.

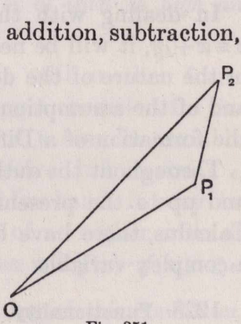

Fig. 351.

(1) The modulus of the sum, or difference, of two vectors is not greater than the sum of the moduli of the original vectors. For if OP_1, P_1P_2 represent two vectors to be added, their vector sum is represented by OP_2 and the absolute lengths of these lines are the several moduli of the vectors they represent.

Hence we have mod. $OP_2 \not> \text{mod. } OP_1 + \text{mod. } P_1P_2$.

And similarly in the case of subtraction, or of the case when more than two vectors are combined into one by the process of addition or subtraction.

We may also see this fact analytically, thus: The modulus of $\Sigma\rho(\cos\theta + \iota\sin\theta)$ is $\sqrt{(\Sigma\rho\cos\theta)^2 + (\Sigma\rho\sin\theta)^2}$, and this is $\not> \Sigma\rho$.

For if it were, we should have

$$\Sigma\rho^2 + 2\Sigma\rho_1\rho_2\cos(\theta_1 - \theta_2) > \Sigma\rho^2 + 2\Sigma\rho_1\rho_2,$$

i.e.
$$\Sigma\rho_1\rho_2\cos(\theta_1 - \theta_2) > \Sigma\rho_1\rho_2;$$

and as all the ρ's are essentially positive and the cosines < 1, this would be impossible. This includes the case when some of the vectors are subtracted, for in any such case $\pi - \theta$ may be supposed written instead of θ and the result treated as additive.

(2) The modulus of a product of complexes

$$\rho_1 e^{\iota\theta_1}\rho_2 e^{\iota\theta_2}\rho_3 e^{\iota\theta_3}\ldots\rho_n e^{\iota\theta_n}$$

is obviously $\rho_1\rho_2\rho_3\ldots\rho_n$, *i.e.* the product of the moduli, and the amplitude is $\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n$, *i.e.* the sum of the amplitudes.

(3) The modulus of a quotient, viz. $\dfrac{\rho_1 e^{\iota\theta_1}}{\rho_2 e^{\iota\theta_2}}$, *i.e.* $\dfrac{\rho_1}{\rho_2}e^{\iota(\theta_1-\theta_2)}$, is $\dfrac{\rho_1}{\rho_2}$, *i.e.* the quotient of the moduli; and the amplitude is $\theta_1 - \theta_2$, *i.e.* the difference of the amplitudes.

1222. **Revision of Definitions.**

In dealing with the functionality of a complex variable $z \equiv x + \iota y$, it will be necessary to revise our ideas of continuity, of the nature of the dependence of one function upon another and of the assumption as to the existence of a limit as used in the formation of a Differential Coefficient.

Throughout the author's treatise on the Differential Calculus and up to the present point in this account of the Integral Calculus, there have been but few references to a function of a complex variable.

1223. **Functionality.** The idea of functionality has been that when one real quantity y depends upon another real quantity x, or upon a system of real quantities x_1, x_2, x_3 in such a manner as to assume a definite value when a definite value is given to x, or when a definite system of values is given to the system of variables $x_1, x_2, x_3, \ldots$, the quantity y is then said to be a function of x, or of the system x_1, x_2, x_3, etc., as the case may be.

1224. **Continuity.**

Our idea of the continuity of a function $f(x)$ of a real independent variable x between any two assigned values of x, viz. $x=a$, the smaller, and $x=b$, the greater, has so far been that if x be made to change from $x=a$ to $x=b$, passing at least once through all real intermediate values between $x=a$ and $x=b$, whether these intermediate values when expressed by means of the ordinary system of numeration be represented by integers, fractions or incommensurable numbers, the function in question does not, as x passes through any intermediate value, suddenly change its value. And in such case its Cartesian graph has been regarded as capable of description by the motion of a material particle travelling along it from the point $\{a, f(a)\}$ to the point $\{b, f(b)\}$ without moving off the curve.

But such continuity does not also imply continuity as regards the slope of the tangent to the graph, or of continuity in the rate of bend of the curve at intermediate points.

1225. From a purely analytical point of view we may regard a function $f(x)$ as being continuous at a point $x=x_0$, if *when any infinitesimal change is made in x the consequent change in $f(x)$ is itself also an infinitesimal, and of at least as high an order.*

1226. We may put this condition into still another form, which will be more helpful in enunciating a condition for the continuity of a single-valued function of a *complex* variable later, viz. that for any assignable positive infinitesimal ϵ, however small, which may be chosen beforehand, it may be possible to choose another infinitesimal δ of no higher order of smallness than ϵ, so that if $x-x_0<\delta$, then will $f(x) \sim f(x_0) < \epsilon$.

1227. To examine the geometrical meaning of this condition, imagine two lines AB, CD drawn parallel to the x-axis at an arbitrary infinitesimal distance ϵ apart, and let these lines cut the graph of the function $y=f(x)$ at points P, Q respectively.

Let the coordinates of P and Q be $x_0, f(x_0)$ and $x_0+\delta, f(x_0+\delta)$ respectively. Let P_1 be a point on the graph between P and Q, the coordinates of P_1 being $x, f(x)$. Let P_1N, QM be drawn at right angles to AB. Then $PN=x-x_0$, $PM=\delta$, $MQ=\epsilon$, $NP_1=f(x)-f(x_0)$. Then if, however small MQ be taken, NP_1 is $<MQ$ for all positions of N from P to M, where PM is of no higher a degree of smallness than QM, there cannot be a break in the curve at the point P.

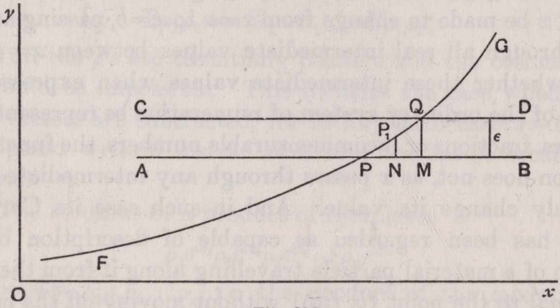

Fig. 352.

If this be so for all points x_0 between $x=x_1$ and $x=x_2$, $f(x)$ will be continuous for all values of x between these limits.

The figure is drawn for the case $f(x)>f(x_0)$.

1228. **Definition of Functionality of a Complex Variable.** The nature and representation of an independent complex variable having been explained, we may proceed as in the case of a real variable to explain what is meant by the term Function as used in the case of complex variables. When one complex variable w is connected with another complex variable z in such a manner that for each value that may be assigned to z, w will itself take up a definite value, or a system of definite values, which can be derived from the value of z by some combination of the fundamental arithmetical rules, then w will be said to be a function of z, and will be denoted by an equation of the form $w=f(z)$ or $f(w, z)=0$. Here z stands for $x+\iota y$, and x, y are themselves supposed to be real and may be regarded as the Cartesian coordinates of some arbitrary point referred to a given pair of rectangular axes in the z-plane.

If one value of x and one value of y give rise always to one value of w and no more, then w is said to be a *single-valued* or *uniform* function of z, *i.e.* of $x+\iota y$. Such functions as $w\equiv Az^n+Bz^{n-1}+\ldots+C$, where n is a positive integer, $\sin z$, $\cos z$, $\tan z$, e^z, $e^z \sin z$, etc., are single-valued functions.

But if several values of w result from one value of x and one value of y, then w is said to be a *many-valued* or *multiple-valued* function of z.

Thus $w\equiv az^{\frac{p}{q}}$ is a q-valued function, for there are q separate q^{th} roots of z^p (p and q are supposed positive integers prime to each other). So also $w\equiv \sin^{-1}z$, $\tan^{-1}z$, $e^z\tan^{-1}z, \ldots$ are multiple-valued functions of z, as also $w\equiv \log z$, for w may be written $\log(ze^{2\iota\lambda\pi})=2\iota\lambda\pi+\log z$, where λ is any integer.

1229. Continuity of a Single-Valued or Uniform Function of z.

Suppose that the point z ranges over a definite region Γ on the z-plane, and that z_0 is a definite point in this region. Let w be any single-valued function of z, which takes the value w_0 when z assumes the value z_0. Then if, for any positive infinitesimal ϵ of however high an order which may be arbitrarily chosen, another small positive infinitesimal ζ be assignable, such that if $|z-z_0|<\zeta$, we also have $|w-w_0|<\epsilon$; then w is a continuous function of z at $z=z_0$, and if this be true for all points z_0 which lie in the definite region Γ on the z-plane, w is said to be continuous for all such points, *i.e.* throughout the region.

1230. Geometrical Illustration.

Illustrating this geometrically, let P and P_0 be the two points z and z_0 in the z-plane, and let Q and Q_0 be the two corresponding points in the w-plane. Let Γ and Γ' be the corresponding regions on the two planes for which we are to discuss the continuity of the function. Draw a small circle with radius ζ and centre P_0, and another small circle with radius ϵ and centre Q_0. Then, if ζ can be so chosen that when P lies within the ζ-circle, Q lies within the ϵ-circle for all points P within the ζ-circle, when ϵ is arbitrarily chosen smaller than anything that can be conceived beforehand, however small; then w is said to be a continuous function of z at the point z_0, and for all points z_0 which lie within the region Γ for which the same is true.

If, then, for every small change in the modulus of either of two variables, there be a small change of at least the same

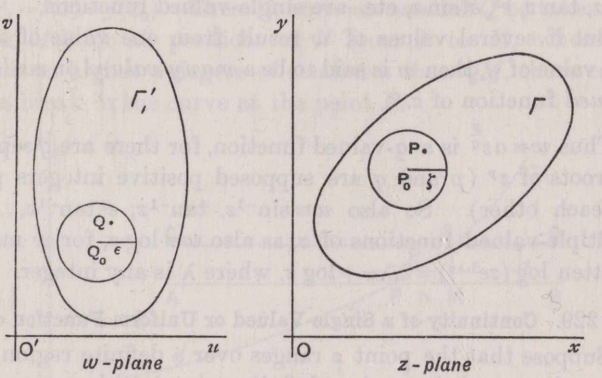

Fig. 353.

order of smallness in the modulus of the other, the second of these variables is a continuous function of the first.

1231. **Positive Integral Powers of a Complex are continuous.**

It follows from the definition of continuity above that all positive integral powers of z are continuous. Consider for instance $w=z^3$.

Then if w_0 and z_0 be corresponding points and $z-z_0=\rho$,

$$w-w_0=z^3-z_0{}^3=3\rho z_0{}^2+3\rho^2 z_0+\rho^3.$$

Hence

$$\text{mod.}\,(w-w_0) \not> 3(\text{mod.}\,\rho)(\text{mod.}\,z_0{}^2) + 3(\text{mod.}\,\rho^2)(\text{mod.}\,z_0)+(\text{mod.}\,\rho^3).$$

Now if we take (mod. ρ) small enough, say ζ, we can make the whole of the right-hand side less than any quantity assignable beforehand, however small.

Hence ζ can be chosen so that when

$$(\text{mod.}\,\rho) < \zeta, \quad \text{mod.}\,(w-w_0) < \epsilon,$$

any assignable quantity, however small, and therefore w is a continuous function of z for all values of z in the z-plane.

Similarly we may show that any other positive integral power of z is continuous for all values of z.

1232. Continuity of a Finite Series.

If w, w', w'', ... be a set of one-valued functions of a complex variable z, finite in number, and each continuous for values of z lying within a given contour on the z-plane, then their sum Σw will be continuous for values of z lying in that region.

For if w_0, w_0', w_0'', ... be the values of w, w', w'', ... respectively, corresponding to $z=z_0$, it is by hypothesis possible to determine the positive quantities ζ, ζ', ζ'', ..., so that for a given assigned small positive quantity ϵ,

$$\text{when mod.}\,(z-z_0)<\zeta, \text{ we have mod.}\,(w-w_0)<\epsilon,$$
$$\text{when mod.}\,(z-z_0)<\zeta', \text{ we have mod.}\,(w'-w_0')<\epsilon, \text{ etc.};$$

and if $\bar{\zeta}$, say, be the smallest of the quantities ζ, ζ', ζ'', ..., then it is possible to find $\bar{\zeta}$, so that when

$$\text{mod.}\,(z-z_0)<\bar{\zeta}, \quad \text{we have } \Sigma\,\text{mod.}\,(w-w_0)<n\epsilon,$$

where n is the number of functions; and therefore, since the modulus of a sum is not greater than the sum of the moduli, mod. $(\Sigma w-\Sigma w_0)<n\epsilon$ for all values of $n\epsilon$, however small. Hence Σw is a continuous function of z.

1233. As a case of this result any integral polynomial function of z,

$$a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots+a_n,$$

is a continuous function of z, n being a positive integer.

1234. Discontinuity.

To examine the continuity of the function $w=\dfrac{1}{z-a}$ *in the region near* $z=a$ *and elsewhere.*

This function becomes ∞ when $z=a$, and therefore it is impossible to assign an infinitesimal ζ such that when

$$\text{mod.}\,(z-a)<\zeta, \quad \text{mod.}\left(\frac{1}{z-a}-\frac{1}{0}\right)$$

is less than any assignable quantity ϵ, and the function is discontinuous at $z=a$.

But at any other point z_0 in the z-plane the function is continuous.

For if $z=z_0+h$, where $z_0\neq a$,

$$\text{mod.}\left(\frac{1}{z_0+h-a}-\frac{1}{z_0-a}\right)=\text{mod.}\left[\frac{-h}{(z_0-a)(z_0+h-a)}\right],$$

which can be made as small as we like by sufficiently diminishing mod. h, *i.e.* by sufficiently diminishing mod. $(z-z_0)$.

1235. Conformal Representation.

Let us consider the equation $w=f(z)$.

We have $z\equiv x+\iota y$, and if $f(z)$ be separated into its real and unreal parts, say $f_1(x, y)+\iota f_2(x, y)$, we may write w in the form $u+\iota v$, where

$$u=f_1(x, y) \quad \text{and} \quad v=f_2(x, y).$$

If we superimpose a relation $y=F(x)$ between x and y, we shall have, by elimination of x between the equations,

$$u=f_1\{x, F(x)\}, \quad v=f_2\{x, F(x)\},$$

a resultant relation of the form $v=\phi(u)$.

And to represent this to the eye we shall require two sets of rectangular axes, not necessarily in the same plane. Call these planes the z-plane and the w-plane.

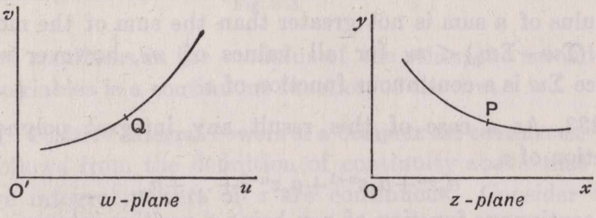

Fig. 354.

Then when a point $P(x, y)$ traverses the graph of $y=F(x)$, in the z-plane the corresponding point $Q(u, v)$ will traverse the graph of $v=\phi(u)$ in the w-plane.

When no such relation as $y=F(x)$ is superimposed connecting the values of x and y, there will be no relation between the coordinates u and v of the corresponding point in the w-plane.

If there be more than one value of w for a single value of z, then each value of w is said to constitute a "branch" of w. For instance, in the equation $w^n=z$ the function w is many-valued, and is said to have n branches. (See Art. 1256.)

Such a representation by means of the z-plane and the w-plane of the associated z and w-loci is generally spoken of as a "conform" or "conformal" representation of these loci;

and it will be remembered that, u and v being conjugate functions of x and y, the curves $u=\text{const.}$ and $v=\text{const.}$ cut each other orthogonally. (*Diff. Calc.*, Art. 195.)

1236. Two Important Cases.

There are two very well-known cases of conformal representation in Elementary Conic Sections.

1. If $w=a\cos z=X+\iota Y$ say (see Art. 590),

$$X+\iota Y=a\cos(x+\iota y)=a(\cos x\cosh y-\iota\sin x\sinh y),$$

$$X=a\cos x\cosh y,\quad Y=-a\sin x\sinh y\,;$$

$$\therefore\quad \frac{X^2}{a^2\cosh^2 y}+\frac{Y^2}{a^2\sinh^2 y}=1\ \ldots\ldots(\alpha)\quad\text{and}\quad \frac{X^2}{a^2\cos^2 x}-\frac{Y^2}{a^2\sin^2 x}=1.\ \ldots\ldots(\beta)$$

And for z-loci of the form $y=\text{constant}$ we have confocal ellipses in the w-plane, whilst for loci of the form $x=\text{constant}$ in the z-plane we have confocal hyperbolae in the w-plane; and the ordinary property of orthogonalism of these two families of conics manifestly follows.

2. The other case is $\qquad w=a\tan z,$

i.e. $\qquad x+\iota y=\tan^{-1}\dfrac{X+\iota Y}{a}$ and $x-\iota y=\tan^{-1}\dfrac{X-\iota Y}{a}$;

whence $\qquad 2x=\tan^{-1}\dfrac{2aX}{a^2-X^2-Y^2}$; $\quad 2y=\tanh^{-1}\dfrac{2aY}{a^2+X^2+Y^2}$,

i.e. $\qquad a^2-X^2-Y^2=2aX\cot 2x$ and $a^2+X^2+Y^2=2aY\coth 2y$,

i.e. $$(X+a\cot 2x)^2+Y^2=a^2\operatorname{cosec}^2 2x$$

and $$X^2+(Y-a\coth 2y)^2=a^2\operatorname{cosech}^2 2y,$$

so that for the z-loci $x=\text{const.}$ and $y=\text{const.}$ the w-loci are a pair of families of coaxial circles, the two families of course being orthogonal to each other.

Other examples will be discussed in due course.

1237. Case of Non-Existence of a Limit.

In the definition of a differential coefficient of a function of a real variable as $Lt_{h=0}\dfrac{f(x+h)-f(x)}{h}$, it was presupposed that such a limit existed, and this supposition was sufficient for the time.

It is possible, however, for a function to exist for which the expression in question, viz. $\dfrac{f(x+h)-f(x)}{h}$, does not approach any determinate limit, finite or infinite, when h is indefinitely diminished, although such a function may be continuous.

For instance, let us consider the case of a function of x in which the infinitesimally close ordinates of the graph terminate at points $P_1, P_2, P_3, P_4, \ldots$, such as shown in the figure, the consecutive angles $P_1\hat{P}_2P_3$, $P_2\hat{P}_3P_4$, $P_3\hat{P}_4P_5$, etc., being alternately $<$ and $>\pi$, and the nature of the function being such that each of the elements of the graph between these successive ordinates can again be themselves divided up into an infinite number of portions having the same peculiarity,

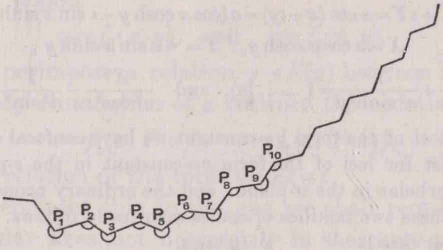

Fig. 355.

the distances between the new subdividing ordinates being infinitesimals of a higher order than the infinitesimal distances between the first set, and so on with further subdivisions. It will be clear that the direction of the line which we please to call the tangent at any point P will depend upon the order of the infinitesimal closeness of the ordinates, and may or may not have a limiting position.

1238. **Weierstrass' Example.**

An example is given by Weierstrass, viz. the case of

$$y=\sum_0^\infty b^n \cos a^n \pi x,$$

where a is an odd positive integer, b positive and <1, and $ab>1+\dfrac{3\pi}{2}$, which, though continuous at every point, *has no differential coefficient determinable at any point.* See Harkness and Morley, *Theory of Functions*, p. 59, or Forsyth, *Theory of Functions*, pp. 133-136, where the student will find the case discussed at length.

1239. **Differentiation of a Function of a Complex Variable.**

It has been seen that in order to define a complex variable $z(\equiv x+\iota y)$, the values of x and y must both be separately assigned. They are independent of each other. Any law connecting them may be arbitrarily assigned. But so long as such law is unassigned z depends upon a doubly infinite system of values. But when x and y have once been assigned, then z becomes known. That is, to a definite value of z corresponds a definite point whose Cartesian coordinates are x, y on the x-y plane, and this point it is usual to designate as the point z.

Conversely to any value specified for z, a definite specification of x and y is implied. When z changes its value to z', and in consequence x and y change to x' and y', say, the value of z' does not depend in any way upon the manner in which the point x, y has travelled to the point x', y', no relation having been assigned to hold between x and y. Hence the vector $z'-z$ is independent of any particular law which may be arbitrarily assigned, connecting x and y. If w be any single-valued function of z, defined as in Art. 1228, and expressed as $w=f(z)$, then when z becomes z', w becomes w', where $w'=f(z')$. Thus $w'-w=f(z')-f(z)$, and is independent of any particular path by which z' is made to approach z on the x-y plane.

Suppose the points z' and z to be infinitesimally near points on the z-plane, and let z' be written $z+\delta z$, and w' be written $w+\delta w$. Then $\delta w=f(z+\delta z)-f(z)$.

We shall define $Lt\dfrac{f(z+\delta z)-f(z)}{\delta z}$, when δz is made indefinitely small, as the differential coefficient of $f(z)$ or w with regard to z, *provided such limit exists independent of the way in which the point $z+\delta z$ is made to approach the point z indefinitely closely*, that is, independent of any particular path which may be assigned to pass through the points x, y and $x+\delta x$, $y+\delta y$.

We shall denote this limit by $\dfrac{dw}{dz}$ or $f'(z)$.

It follows that $\dfrac{dw}{dz}$ is independent of $\dfrac{dy}{dx}$ by definition.

1240. Before assuming the functional relation $w=f(z)$, but assuming that u and v are functions of x and y, and that $w\equiv u+\iota v$ and $z\equiv x+\iota y$, we might enquire what relation, if any, must subsist between u and v in order that $Lt\,\frac{\delta w}{\delta z}$ should be independent of $Lt\,\frac{\delta y}{\delta x}$.

Proceeding from this point of view, we have

$$Lt\,\frac{\delta w}{\delta z}=\frac{dw}{dz}=\frac{d(u+\iota v)}{d(x+\iota y)}=\frac{u_x\,dx+u_y\,dy+\iota(v_x\,dx+v_y\,dy)}{dx+\iota\,dy}$$

$$=\frac{(u_x+\iota v_x)\,dx+\iota(-\iota u_y+v_y)\,dy}{dx+\iota\,dy};$$

and in order that this should be independent of $\frac{dy}{dx}$, we must have

$$u_x+\iota v_x=-\iota u_y+v_y,$$

i.e. $\quad u_x=v_y \quad$ and $\quad u_y=-v_x$;

whence $\quad u_{xx}+u_{yy}=0 \quad$ and $\quad v_{xx}+v_{yy}=0.$

So that u and v must be conjugate functions of x and y satisfying the Laplacian equation $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$, whose general solution is $\phi=F_1(x+\iota y)+F_2(x-\iota y)$, where F_1 and F_2 are arbitrary functional forms. It appears therefore that in putting

$$w=f(z), \quad \textit{i.e. } u+\iota v=f(x+\iota y),$$

the property of independence of $\frac{dw}{dz}$ and $\frac{dy}{dx}$ is implied; and further, that $\frac{dw}{dz}=u_x+\iota v_x$ or $-\iota u_y+v_y$, *i.e.* $\frac{u_y+\iota v_y}{\iota}$.

Also it is understood in defining $\frac{dw}{dz}$ as $Lt_{\delta z=0}\frac{f(z+\delta z)-f(z)}{\delta z}$, provided *such limit be existent*, that the function $f(z)$ is continuous at all points within a small circle on the x-y plane, of which z is the centre, and whose radius is not less than the modulus of δz. Also it is presumed that either $f(z)$ is a single-valued function of z, or if not so, that in passing from the point $z+\delta z$ to the point z, we adhere to the same branch of w.

For example, in the case $w^2=z$, so that $w=\sqrt{z}$ or $-\sqrt{z}$, it is to be understood that we keep to the same sign in both cases, viz. $w=\sqrt{z}$ and $w+\delta w=\sqrt{z+\delta z}$, or $w=-\sqrt{z}$ and $w+\delta w=-\sqrt{z+\delta z}$, and that the gradation of values from $\sqrt{z}$ to $\sqrt{z+\delta z}$ is a continuous gradation.

1241. The Standard Forms.

It will be found that the ordinary "standard forms" of differentiation still hold good when the independent variable z is a complex. That is, we still have

$$\frac{dz^n}{dz}=nz^{n-1}, \quad \frac{d}{dz}\sin z=\cos z, \quad \frac{d}{dz}\log z=\frac{1}{z}, \text{ etc.}$$

Also the rules for the differentiation of a product or a quotient still hold good, viz. the same for complex variables as for real ones.

And in due course it will be shown that Taylor's expansion of $f(z+h)$ also holds.

1242. Geometrical Meaning of Differentiation.

Let OP, OQ represent the vectors z and $z+\delta z$ on the z-plane, and $O'P'$, $O'Q'$ the corresponding vectors w and $w+\delta w$, as determined from the equation $w=f(z)$ on the w-plane.

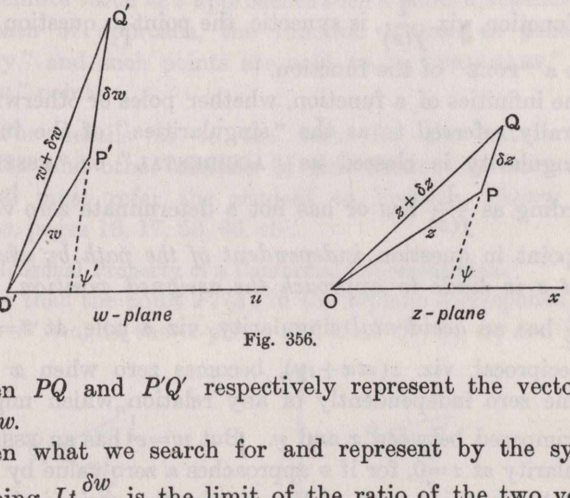

Fig. 356.

Then PQ and $P'Q'$ respectively represent the vectors δz and δw.

Then what we search for and represent by the symbol $\frac{dw}{dz}$, being $Lt\,\frac{\delta w}{\delta z}$, is the limit of the ratio of the two vectors $P'Q'$, PQ, when PQ is indefinitely diminished. This is therefore itself a vector quantity; and if the tangents to the z-path and the w-path make respectively angles ψ and ψ' with the axes Ox and $O'u$, the modulus of this vector is $Lt\,\frac{|\delta w|}{|\delta z|}$, and the amplitude is $\psi'-\psi$ (Art. 1220).

1243. Zeros, Infinities, Singularities of a Function.

When $w=f(z)$, and a value of z, say $z=a$, gives w a zero value, $z=a$ is said to be a "root" of $w=0$, or a "ZERO" of the function w.

When $z=a$ gives an infinite value to w, $z=a$ is called an INFINITY of the function.

The equations $f(z)=0$, $\frac{1}{f(z)}=0$ therefore respectively give the ZEROS and the INFINITIES of the function $f(z)$.

A single-valued or uniform function $f(z)$ which possesses a differential coefficient, and which is finite and continuous for all values of z for points within and upon the boundary of a definite region Γ of the plane of x-y is said to be "SYNECTIC" for that region.

1244. If an infinity of the function be such that at all points in the immediate neighbourhood of the infinity the reciprocal of the function, viz. $\frac{1}{f(z)}$, is synectic, the point in question is said to be a "POLE" of the function.

The infinities of a function, whether poles or otherwise, are generally referred to as the "singularities" of the function. A singularity is classed as "ACCIDENTAL" or "ESSENTIAL" according as $\frac{1}{f(z)}$ has or has not a determinate zero value at the point in question, *independent of the path by which the point z is made to approach the assigned position.* Thus, $w\equiv\frac{1}{z}$ has an *accidental* singularity, viz. a pole, at $z=0$; for its reciprocal, viz. $z(\equiv x+\iota y)$, becomes zero when x and y become zero independently of any relation which might be superimposed between x and y. But $w=e^{\frac{1}{z}}$ has an ESSENTIAL singularity at $z=0$, for if z approaches a zero value by a path along the positive part of the x-axis, the reciprocal of the function, viz. $\frac{1}{e^{\frac{1}{z}}}$, approaches the value $\frac{1}{e^{+\frac{1}{0}}}=\frac{1}{e^{+\infty}}$, that is $\frac{1}{\infty}$ or zero; but if the approach be along the negative portion of the x-axis, $\frac{1}{e^{\frac{1}{z}}}$ approaches the value $\frac{1}{e^{-\frac{1}{0}}}$, *i.e.* $\frac{1}{e^{-\infty}}$ or e^{∞}, *i.e.* ∞.

1245. The term Synectic is due to CAUCHY. The terms HOLOMORPHIC or INTEGRAL are also used to denote the possession by a function of the same properties. The former term is due to BRIOT and BOUQUET, the latter to HALPHEN. These terms are applied to describe such functions in distinction from functions which the same authors respectively term "MEROMORPHIC" or "FRACTIONAL," and which are characterised by the possession of singularities at a point or at points within the contour, viz. poles or ESSENTIAL singularities.

Thus $\sin z$, $\cos z$, $\exp z$, are synectic or holomorphic functions of z for all points of the z-plane; whilst $\dfrac{\sin z}{z-a}$, $\cot z$, etc., are meromorphic at certain regions of the plane by virtue of the existence of the pole at $z=a$ in the first case, or of the poles at the zeros of $\sin z$ in the second case.

At points of the region Γ of the z-plane, for which w takes a single definite value as z approaches such a point independent of the path of approach, the function is said to behave "regularly," and such points are said to be "ORDINARY" or "REGULAR" points.

1246. For details as to the tests for the nature of singularities and other matters of this nature, we have no space, and must refer the student to Forsyth, *Theory of Functions*, pages 16, 17, 53, 66, etc.

1247. Isogonal Property of a Conformal Representation.

Suppose that the point P, (z), in the z-plane corresponds to the point P', (w), in the w-plane, and that Q_1, Q_2, (z_1 and z_2),

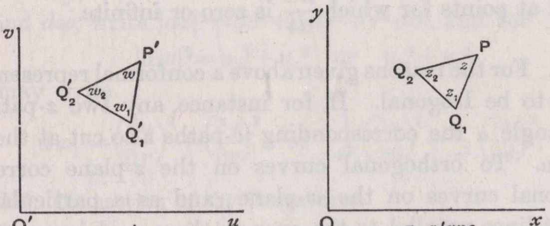

Fig. 357.

are adjacent points to z in the z-plane, whilst Q_1', Q_2', (w_1 and w_2), are the corresponding points in the w-plane;

then, since the value of $\dfrac{dw}{dz}$ is to be independent of the direction of the differential element dz, we must have

$$Lt\,\frac{w-w_1}{z-z_1}=Lt\,\frac{w-w_2}{z-z_2},$$

when the vectors $z-z_1$, $z-z_2$ are infinitesimally small.

Hence $$Lt\,\frac{w-w_1}{w-w_2}=Lt\,\frac{z-z_1}{z-z_2}.$$

Let the moduli and amplitudes of $z-z_1$, $z-z_2$, $w-w_1$, $w-w_2$ be respectively (ρ_1, θ_1), (ρ_2, θ_2), (ρ_1', θ_1'), (ρ_2', θ_2').

Then in the limit

$$\frac{\rho_1'}{\rho_2'}e^{\iota(\theta_1'-\theta_2')}=\frac{\rho_1}{\rho_2}e^{\iota(\theta_1-\theta_2)}, \quad \text{whence } \frac{\rho_1'}{\rho_2'}=\frac{\rho_1}{\rho_2},\ \theta_1'-\theta_2'=\theta_1-\theta_2,$$

i.e. $$P'Q_1' : P'Q_2' = PQ_1 : PQ_2 \quad \text{and} \quad Q_1'\hat{P}'Q_2' = Q_1\hat{P}Q_2.$$

Hence, in any such representation, infinitesimal triangles, and therefore any other *elements*, preserve their similarity, and angles are unaltered in such a transformation. But the moduli of z and w vary with the position of P, and therefore the ratio of such infinitesimal elements is not preserved as a constant in general throughout any *finite* regions in the two planes.

1248. It is also to be noted that it has been assumed that the ratios $(w-w_1)/(z-z_1)$, $(w-w_2)/(z-z_2)$ do not become zero or infinite within an infinitesimal distance of the points P, P' considered. That is to say, that the theorem is not to be applied at points for which $\dfrac{dw}{dz}$ is zero or infinite.

1249. For the reasons given above a conformal representation is said to be Isogonal. If, for instance, any two z-paths cut at an angle α the corresponding w-paths also cut at the same angle α. To orthogonal curves on the z-plane correspond orthogonal curves on the w-plane; and as a particular case straight lines parallel to the axes on the one plane correspond to curves which cut at right angles on the other plane. To two curves which touch one another in the one plane correspond curves which touch on the other plane, but as straight lines do not in general correspond to straight lines in the conformal representation, linear tangents do not become linear tangents, but curvilinear tangents.

1250. **Ratio of Elements of Area.**

Again, the ratio of the infinitesimal areas $P'Q_1'Q_2'$, PQ_1Q_2 is that of the squares of the moduli of dw and dz, *i.e.* if

$$z=x+\iota y \quad \text{and} \quad w=u+\iota v=f(x+\iota y),$$

$$\frac{\text{the } w\text{-element of area}}{\text{the } z\text{-element of area}}=\frac{|dw|^2}{|dz|^2}=\frac{|du+\iota\, dv|^2}{|dx+\iota\, dy|^2}$$

$$=\frac{|u_x dx+u_y dy+\iota(v_x dx+v_y dy)|^2}{|dx+\iota\, dy|^2}=\frac{(u_x dx+u_y dy)^2+(v_x dx+v_y dy)^2}{dx^2+dy^2},$$

and since $u_x=v_y$ and $u_y=-v_x$, this ratio becomes

$$u_x^2+v_x^2 \text{ or } u_y^2+v_y^2 \text{ or } u_x^2+u_y^2 \text{ or } v_x^2+v_y^2 \text{ or } u_x v_y-u_y v_x,$$

i.e. $J\begin{pmatrix}u, v\\ x, y\end{pmatrix}$, where J is the Jacobian of u, v with regard to x, y. Or again, it may be written as

$$(u_x+\iota v_x)(u_x-\iota v_x), \quad \text{i.e. } f'(x+\iota y)f'(x-\iota y).$$

Thus the ratio of the corresponding elements at u, v and at x, y is that of $J\begin{pmatrix}u, v\\ x, y\end{pmatrix}:1$.

It follows of course at once that the inverse ratio is

$$J'\begin{pmatrix}x, y\\ u, v\end{pmatrix}:1,$$

and therefore that $JJ'=1$, as is otherwise well known. (*Diff. Calc.*, Art. 540.)

We may, if desirable to use a polar form for the moduli of dz and dw, write $|dz|^2=ds^2$ or $dr^2+r^2\,d\theta^2$, and for

$$|dw|^2=u_x^2+v_x^2 \quad \text{or} \quad u_y^2+v_y^2,$$

we may write

$$|dw|^2=\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial v}{\partial r}\right)^2 \quad \text{or} \quad \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2+\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2, \text{ etc.}$$

1251. **Connection of the Curvatures.**

The curvatures of the companion w and z curves may be connected as follows.

Let ρ and ρ' be the radii of curvature at corresponding points P, P'.

Then $|dz|$ and $|dw|$ are the lengths of the corresponding infinitesimal arcs.

Let ψ and ψ' be the corresponding angles which the two tangents make respectively with the x and u axes, θ and θ' the polar angular coordinates of the points and ϕ, ϕ' the angles between the tangents at P and P' and their respective polar radii r, r'.

Then $z = re^{\iota\theta}$, $w = r'e^{\iota\theta'}$, $\psi = \theta + \phi$, $\psi' = \theta' + \phi'$,

whilst $\theta = \text{amp.}\, z$ and $\theta' = \text{amp.}\, w$ are the respective amplitudes.

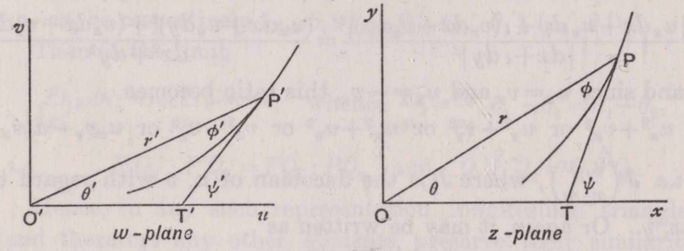

Fig. 358.

Then, since $w = f(z)$, we have $r'e^{\iota\theta'} = f(re^{\iota\theta})$,

and $\qquad dr'e^{\iota\theta'} + \iota r'e^{\iota\theta'}\,d\theta' = f'(re^{\iota\theta})(dre^{\iota\theta} + \iota re^{\iota\theta}\,d\theta)$.

Put $f'(re^{\iota\theta}) \equiv Re^{\iota\Theta}$, say, R and Θ being the modulus and amplitude of $f'(re^{\iota\theta})$, *i.e.* $\Theta \equiv \text{amp.}\, f'(z)$.

Then, since $dr' = ds' \cos\phi'$, $r'd\theta' = ds' \sin\phi'$, etc., we have

$$\sqrt{dr'^2 + r'^2\,d\theta'^2}\,e^{\iota\theta'}e^{\iota\phi'} = \sqrt{dr^2 + r^2\,d\theta^2}\,e^{\iota\theta}e^{\iota\phi}Re^{\iota\Theta};$$

that is $\qquad |dw|\,e^{\iota\psi'} = |dz|\,Re^{\iota(\psi+\Theta)}$,

whence

$$|dw| = R|dz| \text{ or } |f'(z)\,dz| \quad \text{and} \quad \psi' - \psi = \Theta = \text{amp.}\, f'(z),$$

whence $\qquad d\psi' - d\psi = d\,\text{amp.}\, f'(z)$;

and since $\rho = \dfrac{|dz|}{d\psi}$ and $\rho' = \dfrac{|dw|}{d\psi'}$, we obtain

$$\frac{|dw|}{\rho'} - \frac{|dz|}{\rho} = d\,\text{amp.}\, f'(z)$$

or

$$\frac{|f'(z)\,dz|}{\rho'} - \frac{|dz|}{\rho} = d\,\text{amp.}\, f'(z). \quad \ldots\ldots\ldots\ldots\ldots(\text{A})$$

In many cases of conformal representation, the z-curve is taken as one of simple nature, usually a well-known curve, and the w curve is often one which is of more or less complicated nature, and the labour of applying the ordinary formulae to obtain ρ' in such cases, may generally be *avoided by the use of this connection between the curvatures.*

1252. **Illustrations.**

Ex. 1. Taking $aw=z^2$, where a is real and positive, we have $ar'e^{\iota\theta'}=r^2e^{2\iota\theta}$, whence $ar'=r^2$, $\theta'=2\theta$.

Here

$$f(z)=\frac{z^2}{a},\quad f'(z)=\frac{2z}{a},\quad \text{amp.}\, f(z)=\text{amp.}\frac{2r}{a}e^{\iota\theta}=\theta,\quad d\,\text{amp.}\, f'(z)=d\theta,$$

$$|dz|=\sqrt{dr^2+r^2\,d\theta^2},\quad |f'(z)\,dz|=\left|\frac{2z}{a}\right|.|dz|=\frac{2r}{a}.|dz|;$$

$$\therefore\quad \frac{2r}{a\rho'}-\frac{1}{\rho}=\frac{d\theta}{\sqrt{dr^2+r^2\,d\theta^2}}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{-\frac{1}{2}}.$$

To verify in the simplest case, take the z-curve as $r=a$; then

$$\rho=a,\ \frac{dr}{d\theta}=0;\quad \therefore\quad \frac{2}{\rho'}-\frac{1}{a}=\frac{1}{a},\quad i.e.\ \rho'=a,$$

which is obviously correct. For if $r=a$, $r'=\dfrac{a^2}{a}=a$, and the w-curve is also a circle of radius a but described twice as fast as the z-circle, since $\theta'=2\theta$, and therefore is traced twice over for one tracing of the z-circle.

Ex. 2. Consider $w=+\sqrt{a^2+bz}$, a and b being both real. We have

$$r'e^{\iota\theta'}=\sqrt{a^2+bre^{\iota\theta}}=\sqrt[4]{a^4+2a^2br\cos\theta+b^2r^2}\,e^{\frac{\iota}{2}\tan^{-1}\frac{br\sin\theta}{a^2+br\cos\theta}},$$

i.e. $\quad r'^4=a^4+2a^2br\cos\theta+b^2r^2\quad$ and $\quad\tan 2\theta'=br\sin\theta/(a^2+br\cos\theta)$.

Also $\qquad dw=f'(z)\,dz=b\,dz/2\sqrt{a^2+bz}=\dfrac{b}{2r'}e^{-\iota\theta'}dz,$

$$|dw|=\frac{b}{2r'}|dz|,\quad |dz|=\sqrt{dr^2+r^2\,d\theta^2},\quad \text{amp.}\, f'(z)=-\theta',$$

and $\qquad d\theta'=\{a^2b\sin\theta\,dr+br(a^2\cos\theta+br)d\theta\}/2r'^4;$

whence

$$\frac{b}{2r'\rho'}-\frac{1}{\rho}=-\left\{a^2b\sin\theta\frac{dr}{d\theta}+br(a^2\cos\theta+br)\right\}\Big/2r'^4\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\ \ldots,\quad (1)$$

which will be the general formula connecting the curvatures of the z and w curves in any transformation by means of $w=\sqrt{a^2+bz}$.

For instance, take the z-curve to be the circle $r=c$. Then the w-curve is a Cassinian oval. For $r'^2e^{2\iota\theta'}=a^2+bce^{\iota\theta}$, *i.e.*

$$r'^2\cos 2\theta'=a^2+bc\cos\theta,\quad r'^2\sin 2\theta'=bc\sin\theta,$$

and $\qquad r'^4-2a^2r'^2\cos 2\theta'+a^4=b^2c^2\quad$ [see *Diff. Calc.*, Art. 458],

that is, if S, H be the foci and P any point on the curve, $SP\,.\,HP=bc$.

Putting $r=\rho=c$, $\dfrac{dr}{d\theta}=0$, in Equation (1), and substituting for $\cos\theta$,

$$\frac{b}{2r'\rho'}-\frac{1}{c}=-\frac{b}{2r'^4}\left(bc+a^2\frac{r'^2\cos 2\theta'-a^2}{bc}\right)=-\frac{r'^4-a^4+b^2c^2}{4cr'^4},$$

i.e. $\rho'=2bcr'^3/(3r'^4+a^4-b^2c^2)$, for the Cassinian.

If $a^2=bc$, we have the case of Bernoulli's Lemniscate, and $\rho'=2a^2/3r'$.

In the case just considered, it will be seen that since

$$(w-a)(w+a)=bz,$$

we have
$$\text{mod.}(w-a)\,\text{mod.}(w+a)=b\,\text{mod.}\,z\,;$$

and therefore that if mod. z be constant, *i.e.* if the z curve be chosen as above to be a circle of radius c and centre at the origin, the corresponding w-curve has the property that the product of its bi-focal radii SP, HP is constant, the coordinates of the foci S, H being $(a, 0)$ and $(-a, 0)$, and therefore it is one of the class of the Cassinian ovals $r_1r_2=bc$. This result is therefore obvious as the immediate interpretation of the w-z equation without reference to the polar form.

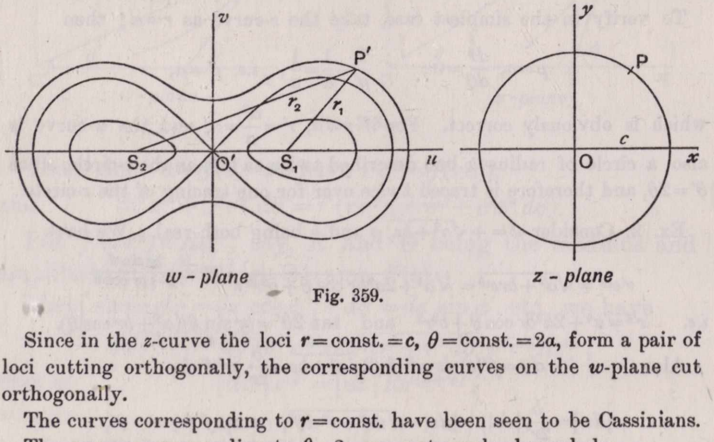

Fig. 359.

Since in the z-curve the loci $r=\text{const.}=c$, $\theta=\text{const.}=2\alpha$, form a pair of loci cutting orthogonally, the corresponding curves on the w-plane cut orthogonally.

The curves corresponding to $r=\text{const.}$ have been seen to be Cassinians.

The curves corresponding to $\theta=2\alpha$ are rectangular hyperbolae.

For since $r'^2e^{2\iota\theta'}-a^2=bre^{\iota\theta}=bre^{2\iota\alpha}$,

$$r'^2\cos 2\theta'-a^2=br\cos 2\alpha, \quad r'^2\sin 2\theta'=br\sin 2\alpha,$$

that is,
$$r'^2\sin 2(\theta'-\alpha)+a^2\sin 2\alpha=0.$$

These hyperbolae for a parameter α are therefore the orthogonal trajectories of the Cassinians $r_1r_2=\text{const.}$

Further, it may be remarked that in considering the transformation $w^2-a^2=bz$, we have really considered any transformation of the form $Aw^2+Bw+C=z$; for by putting $w=w'-\dfrac{B}{2A}$, we have

$$Aw'^2-\frac{B^2}{4A}+C=z,$$

which is of the form $w^2-a^2=bz$.

Hence the results for $Aw^2+Bw+C=z$ are the same as those considered, with a mere transformation of the position of the axes.

1253. Curvature; the Form for Cartesians.

We may put the curvature formula of Art. 1251 into another form more particularly useful for a Cartesian z-locus.

For $$w=f(z)=f(x+\iota y),\quad dw=f'(z)\,dz,$$

$$|dz|=\sqrt{dx^2+dy^2},\quad |dw|=|f'(z)|\,.\,|dz|;$$

whence $$\frac{|f'(z)|}{\rho'}-\frac{1}{\rho}=\frac{\dfrac{d}{dx}\operatorname{amp}.f'(z)}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}.\ \ldots\ldots\ldots\ldots\ldots(\mathrm{B})$$

1254. Thus, if the z-locus is a straight line for instance, say $y=mx+c$,

$$\rho=\infty,\quad \frac{dy}{dx}=m\quad\text{and}\quad \rho'=\frac{|f'(z)|\sqrt{1+m^2}}{\dfrac{d}{dx}\operatorname{amp}.f'(z)}.$$

1255. **Illustrative Examples.**

(1) For example, *in the case* $w=a\cos z$ considered in Art. 1236, for which $X=a\cos x\cosh y$, $Y=-a\sin x\sinh y$, so that $y=c$ gives the ellipse $\dfrac{X^2}{a^2\cosh^2 c}+\dfrac{Y^2}{a^2\sinh^2 c}=1$, we have

$$f(z)=a\cos z,$$

$$f'(z)=-a\sin z=-a(\sin x\cosh y+\iota\cos x\sinh y),$$

$$|f'(z)|=a\sqrt{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y}=a\sqrt{\cosh 2y-\cos 2x}/\sqrt{2},$$

$$\operatorname{amp}.f'(z)=\tan^{-1}(\cot x\tanh y),$$

$$\frac{d}{dx}\operatorname{amp}.f'(z)=\frac{\sin 2x\dfrac{dy}{dx}-\sinh 2y}{\cosh 2y-\cos 2x};$$

and in our case for $y=c$, we have $\rho=\infty$, $\dfrac{dy}{dx}=0$, $m=0$, and the radius of curvature of the derived curve is

$$\frac{a}{\sqrt{2}}\frac{(\cosh 2c-\cos 2x)^{\frac{3}{2}}}{\sinh 2c},\quad\text{where }\cos x=\frac{X}{a\cosh c},$$

which may be readily verified directly for the ellipse.

(2) (A) *In the case* $w=\dfrac{z^n}{a^{n-1}}$ (*a real*), we have

$$r'e^{\iota\theta'}=\frac{r^n e^{\iota n\theta}}{a^{n-1}},\quad r'=\frac{r^n}{a^{n-1}},\quad \theta'=n\theta.$$

Hence to any z-locus $F(r,\theta)=0$ corresponds a w-locus

$$F\left(a^{\frac{n-1}{n}}\,r'^{\frac{1}{n}},\ \frac{\theta'}{n}\right)=0.$$

In this case, since $\theta'=n\theta$, a z-line through the origin corresponds to a w-line through the origin, and in consequence in this case $\phi=\phi'$, *i.e.* the angles which the tangents make with their radii vectores are equal.

Hence to an equiangular spiral in the z-plane and with pole at the origin corresponds another and equal equiangular spiral in the w-plane with its pole at the new origin.

(B) Moreover, since $\psi-\theta=\psi'-\theta'$, we have $\psi'-\psi=\theta'-\theta$, whence

$$\frac{|dw|}{\rho'}-\frac{|dz|}{\rho}=d\,\text{amp.}\,w-d\,\text{amp.}\,z,$$

which is what the curvature formula of Art. 1251 reduces to, since

$$f'(z)=\frac{nr^{n-1}}{a^{n-1}}e^{\iota(n-1)\theta} \text{ and amp.} f'(z)=(n-1)\,\theta=\text{amp.}\,w-\text{amp.}\,z.$$

(C) In this group of results, if we take the z-locus as the straight line $r\cos\theta=a$, we have

$$\phi'=\phi=\frac{\pi}{2}-\theta=\frac{\pi}{2}-\frac{\theta'}{n},$$

which gives the well-known property of all curves of the form

$$r^{-\frac{1}{n}}=a^{-\frac{1}{n}}\cos\left(-\frac{1}{n}\theta\right),$$

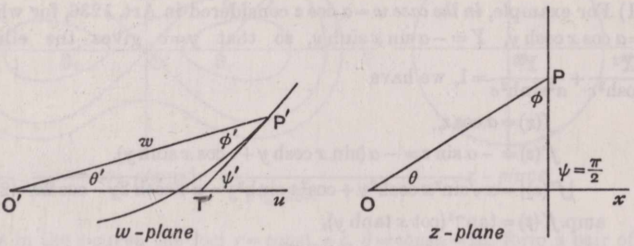

Fig. 360.

which include as particular cases the Parabola ($n=2$), the Rectangular Hyperbola ($n=\frac{1}{2}$), Bernoulli's Lemniscate ($n=-\frac{1}{2}$), the Cardioide ($n=-2$), the Straight Line ($n=1$) and the Circle ($n=-1$).

(D) To any curve $r^p=a^p\cos p\theta$ corresponds the curve

$$(r'a^{n-1})^{\frac{p}{n}}=a^p\cos\frac{p\theta'}{n}, \quad \textit{i.e. } r^q=a^q\cos q\theta, \quad \text{where } \frac{p}{q}=n.$$

Hence to $r^{\frac{n-1}{k}}=a^{\frac{n-1}{k}}\cos\frac{n-1}{k}\theta$ corresponds its own k^{th} pedal curve, for the k^{th} pedal is got by substituting for the present index and multiple of θ

$$\frac{\frac{n-1}{k}}{1+k\frac{n-1}{k}} \text{ for } \frac{n-1}{k}, \quad \textit{i.e. } \frac{n-1}{kn} \text{ for } \frac{n-1}{k},$$

which gives the ratio $n:1$ for the indices and multiple of θ as required.

(E) Quasi-Inversion.

The conformal representation of $w=\frac{k^2}{z}$, where k is real, is very important.

We have at once $r'e^{\iota\theta'}=\frac{k^2}{re^{\iota\theta}}=\frac{k^2}{r}e^{-\iota\theta}$; whence $r'r=k^2$ and $\theta'=-\theta$.

Hence, if the same axes be taken for the z and w curves, we have a combination of inversion and reflexion in the x-axis. This process is known as Quasi-Inversion. The name is due to Cayley.

Now, reflexion with regard to a straight line makes no difference in the nature of a curve. Hence the usual rules of inversion apply, viz. a straight line which does not pass through the origin inverts into a circle through the origin. If the straight line pass through the origin it inverts into a straight line through the origin. To a circle through the origin corresponds a straight line not through the origin. To a circle which does not pass through the origin corresponds another circle which does not pass through the origin. To a parabola with focus at the origin corresponds a Cardioide with pole at the origin. To a conic with focus at the origin corresponds a Limaçon with pole at the origin, and so on.

Hence when the z-curve is given, the w-curve is at once known and can be constructed by the reflexion of the curve traced by a Peaucellier cell linkage arrangement as explained in *Diff. Calc.*, Art. 232.

(F) The Homographic Relation.

Consider next the conformal representation of $w=\frac{az+b}{cz+d}$.

This is the general linear transformation. It is known as a "Homographic" relation between w and z.

Obviously $$cwz+dw-az-b=0,$$

or $$\left(w-\frac{a}{c}\right)\left(z+\frac{d}{c}\right)=\frac{b}{c}-\frac{ad}{c^2}=\frac{bc-ad}{c^2}.$$

Now this transformation is unaltered by changes in a, b, c, d, which preserve the ratios. In fact, there are only three constants, namely the ratios $a:b:c:d$. There is therefore no loss of generality in taking $bc-ad=1$.

This being done, let $w=\frac{a}{c}+w'$, $z=-\frac{d}{c}+z'$, which merely shifts the origins of w and z, retaining axes parallel to their original directions; for if $\frac{a}{c}=\alpha+\iota\beta$, say, and $-\frac{d}{c}=\gamma+\iota\delta$, the new origins will be the points (α, β) and (γ, δ) respectively; we then have $w'z'=\frac{1}{c^2}$, *i.e.* another quasi-inversion connection between the z and w loci.

(G) Obviously, if when $w=\frac{az+b}{cz+d}$, z is itself connected with a third variable t by another homographic relation $z=\frac{pt+q}{rt+s}$, then upon substituting for z, w is of the form $\frac{At+B}{Ct+D}$, whether the variables and constants involved be real or complex.

That is, if w be homographic with regard to z and z be homographic with regard to t, then w is homographic with regard to t, and so on for any number of variables.

The relation may obviously be thrown into the form

$$\frac{\lambda}{wz}+\frac{\mu}{w}+\frac{\nu}{z}+1=0,$$

where λ, μ, ν are constants. This relation is of much use in the theory of geometrical optics, in various forms, the quantity λ being there usually zero.

The equation $w=\dfrac{az+b}{cz+d}$ may be written further in the form

$$\frac{w-\lambda}{w+\lambda}=\frac{(a-\lambda c)z+(b-\lambda d)}{(a+\lambda c)z+(b+\lambda d)}=k\frac{z-\mu}{z+\mu}, \text{ say};$$

so $$\frac{|w-\lambda|}{|w+\lambda|}=|k|\frac{|z-\mu|}{|z+\mu|}.$$

And if we use bi-focal coordinates in each system, viz. (R, R') and (r, r'), the two foci on the two planes being λ, $-\lambda$ in the w-plane and μ, $-\mu$ in the z-plane, then $\dfrac{R}{R'}=|k|\dfrac{r}{r'}$, so that when z describes a circle in the z-plane, viz. $r:r'=$constant, w will describe a circle in the w-plane, viz. $R:R'=$constant, a result which has been already stated.

The case $\dfrac{w-a}{w-b}=z$ is a case of the above quasi-inversion.

We have $\dfrac{|w-a|}{|w-b|}=|z|$, and if the z-locus is the fixed circle $|z|=$constant, the w-locus is a fixed circle.

(H) *Consider next the conformal representation of the equation*

$$w=Az^\alpha+Bz^\beta+Cz^\gamma+\dots,$$

where $A, B, C, \dots$ and $\alpha, \beta, \gamma, \dots$ are all real positive quantities.

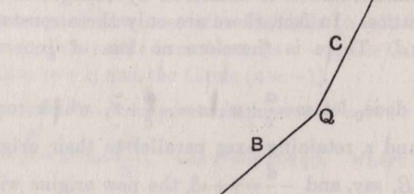

Fig. 361.

Putting, as in previous cases, $z=re^{\iota\theta}$, $w=r'e^{\iota\theta'}=X+\iota Y$,

$$X=Ar^\alpha\cos\alpha\theta+Br^\beta\cos\beta\theta+Cr^\gamma\cos\gamma\theta+\dots$$
$$Y=Ar^\alpha\sin\alpha\theta+Br^\beta\sin\beta\theta+Cr^\gamma\sin\gamma\theta+\dots$$

If we take the z-curve to be a circle of radius unity, then for the w-curve $X=\Sigma A\cos\alpha\theta$, $Y=\Sigma A\sin\alpha\theta$, and this locus can be constructed as the locus of a point carried on one of a set of hinged rods OP, PQ, QR, ... of lengths A, B, C, etc., the carried point being considered as the end of the last rod and one end of the first rod fixed at a point O, the whole system moving in a plane and the several rods rotating with angular velocities in the ratio $\alpha:\beta:\gamma:$ etc., ... ; in fact, what is usually known as an epicyclic train of linkages.

(I) *Consider the case of two terms* $w=Az^{\alpha}+Bz^{\beta}$.

Let Q be a point attached to a circle of centre P and radius b, which rolls without sliding upon the outside of the circumference of a fixed

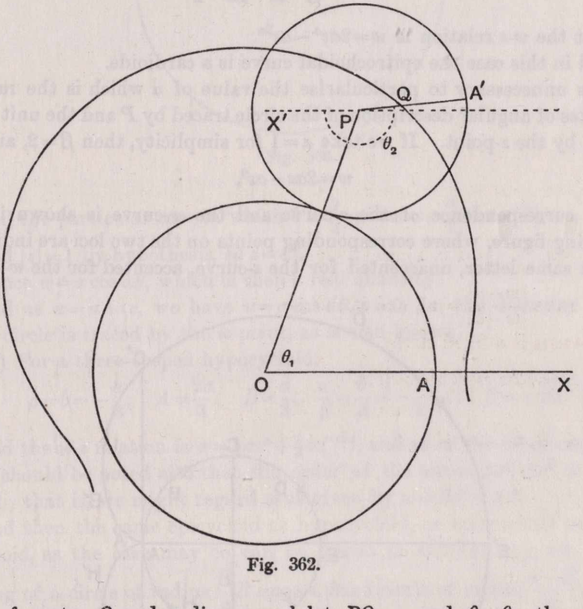

Fig. 362.

circle of centre O and radius a, and let $PQ=\rho$, and θ_1, θ_2, the angles which OP and PQ have turned through since A', the extremity of the radius which passes through Q of the moving circle, was in contact with the fixed circle at A. Let PX' be parallel to AO. Then the angle $X'PA'$ (marked in the figure as $>\pi$) is θ_2.

Then, for pure rolling,

$$a\theta_1=b(\theta_2-\theta_1) \quad \text{or} \quad (a+b)\theta_1=b\theta_2.$$

Let $\theta_1=\alpha\theta$, $\theta_2=\beta\theta$, and take $A=a+b$, $B=-\rho$.

$$\therefore\ \frac{a+b}{\beta}=\frac{b}{\alpha}=\frac{A}{\beta}, \quad \textit{i.e.}\ b=\frac{\alpha}{\beta}A \quad \text{and} \quad a=\frac{\beta-\alpha}{\beta}A.$$

Then the coordinates of Q are

$$X = A\cos\alpha\theta + B\cos\beta\theta, \quad Y = A\sin\alpha\theta + B\sin\beta\theta.$$

So $w = Az^{\alpha} + Bz^{\beta}$ gives in this case a trochoidal locus for w corresponding to the circular locus for z, the trochoid being traced by the motion of a point at distance $\rho\ (= -B)$ from the centre of a circle of radius $b\left(=\frac{\alpha}{\beta}A\right)$ rolling upon a fixed circle of radius $a\left(=\frac{\beta-\alpha}{\beta}A\right)$. If $\rho = b$, an epicycloid is traced by the w-point, supposing b to be positive.

In the case $a = b = \rho$ we have

$$A = 2a, \quad B = -a, \quad \text{and} \quad \frac{\alpha}{\beta} = \frac{b}{A} = \frac{a}{2a} = \frac{1}{2}, \quad i.e.\ \beta = 2\alpha,$$

so that the w-z relation is $w = 2az^{\alpha} - az^{2\alpha}$.

And in this case the epitrochoidal curve is a cardioide.

It is unnecessary to particularise the value of α which is the ratio of the rates of angular description of the circle traced by P and the unit circle traced by the z-point. If we take $\alpha = 1$ for simplicity, then $\beta = 2$, and we have

$$w = 2az - az^2.$$

The correspondence of the z-curve and the w-curve is shown in the adjoining figure, where corresponding points on the two loci are indicated by the same letter, unaccented for the z-curve, accented for the w-curve.

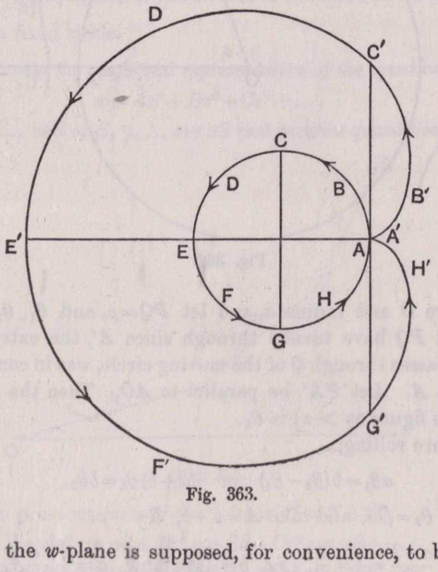

Fig. 363.

In the figure the w-plane is supposed, for convenience, to be superposed upon the z-plane.

(J) If b be negative and $\rho=b=-b'$, we have a hypocycloid traced, and

$$A=a-b',\quad B=b',\quad \frac{\beta-a}{\beta}=\frac{a}{a-b'},\quad i.e.\ \beta=\frac{b'-a}{b'}a,$$

giving
$$w=(a-b')z^a+b'z^{-\frac{a-b'}{b'}a}$$

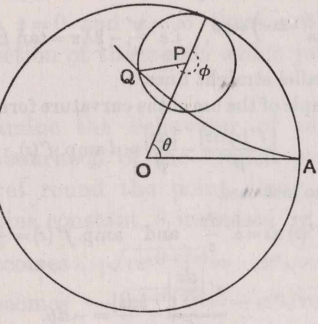

Fig. 364.

And the particular case in which $b'=\frac{a}{2}$ gives $w=\frac{a}{2}(z^a+z^{-a})$.

And $|z|=1$ by hypothesis, so $z=e^{\iota\theta}$.

Hence $w=a\cos a\theta$, which is then a real quantity.

And as $w=u+\iota v$, we have $u=a\cos a\theta$, $v=0$, *i.e.* the diameter of the fixed circle is traced by the w-point, as is well known.

(K) For a three-cusped hypocycloid,

$$\rho=b=-\frac{a}{3},\quad A=\frac{2a}{3},\quad B=\frac{a}{3},\quad \frac{a}{\beta}=\frac{b}{A}=-\frac{1}{2};\quad \therefore\ \beta=-2a.$$

And the w-z relation is $w=\frac{2}{3}az^a+\frac{1}{3}az^{-2a}$, and so on for other cases.

It should be noted also that the order of the terms Az^α, Bz^β is immaterial; that is, we might regard w as given by $w=Bz^\beta+Az^\alpha$.

And then the same epicycloid or hypocycloid, or epitrochoid or hypotrochoid, as the case may be, can be traced in another way, viz. by the rolling of a circle of radius $\frac{\beta}{\alpha}B$ upon a fixed circle of radius $\frac{\alpha-\beta}{\alpha}B$.

(L) *The case* $\frac{w}{\alpha'}=\log\frac{z}{\alpha}$, *where* α, α' *are real constants.*

This case gives $\frac{r'}{\alpha'}e^{\iota\theta'}=\log\left(\frac{r}{\alpha}e^{\iota\theta}\right)=\log\frac{r}{\alpha}+\iota(\theta+2\lambda\pi)$;

whence
$$\log\frac{r}{\alpha}=\frac{r'}{\alpha'}\cos\theta'=\frac{x'}{\alpha'},\quad \theta+2\lambda\pi=\frac{r'}{\alpha'}\sin\theta'=\frac{y'}{\alpha'}.$$

So that to a circle $r=$const. on the z-plane corresponds a straight line parallel to the y-axis on the w-plane; and to a straight line through the origin, $\theta=$const., on the z-plane corresponds a family of straight lines parallel to the x-axis on the w-plane.

Corresponding to the Archimedean Spiral $r=a\theta$ on the z-plane, we have, on the w-plane, a family of logarithmic curves, viz.

$$\frac{y'}{a'}-2\lambda\pi=\frac{a}{a}e^{\frac{x'}{a'}}.$$

Corresponding to the Equiangular Spiral $r=ae^{\theta\cot\beta}$ on the z-plane, we have, on the w-plane, the family

$$ae^{\frac{x'}{a'}}=ae^{\left(\frac{y'}{a'}-2\lambda\pi\right)\cot\beta}, \quad i.e. \ \frac{y'}{a'}-2\lambda\pi=\tan\beta\left(\frac{x'}{a'}+\log\frac{a}{a}\right),$$

viz. a family of parallel straight lines.

As a further example of the use of the curvature formula of Art. 1251, viz.

$$\frac{|f'(z)\,dz|}{\rho'}-\frac{|dz|}{\rho}=d\operatorname{amp.}f'(z),$$

let us apply it in the last case.

We have $\qquad f'(z)\,dz=a'\dfrac{dz}{z}\quad$ and $\quad \operatorname{amp.}f'(z)=-\theta$;

$$\therefore\ \frac{a'\left|\dfrac{dz}{z}\right|}{\rho'}-\frac{|dz|}{\rho}=-d\theta.$$

In the particular case where the z-curve is the equiangular spiral,

$$z=ae^{\theta(\cot\beta+\iota)},\quad \frac{dz}{z}=(\cot\beta+\iota)\,d\theta,\quad dz=re^{\iota\theta}(\cot\beta+\iota)\,d\theta$$

$$=\frac{r}{\sin\beta}e^{\iota(\theta+\beta)}\,d\theta;$$

and $\qquad \left|\dfrac{dz}{z}\right|=\dfrac{d\theta}{\sin\beta},\quad |dz|=\dfrac{r}{\sin\beta}d\theta\quad$ and $\quad \rho'=\infty.$

Thus the formula reduces to $\rho=r\operatorname{cosec}\beta$, which is the well-known result for an equiangular spiral.

1256. Branches and Branch Points.

In the case of a multiple-valued function, where each value of the independent variable z leads to more than one value of the dependent variable w, the several values of w are said to be branches of the function Thus, if the equation connecting w and z be $F(w, z)=0$, and if upon solution for w we find

$$w_1=f_1(z),\quad w_2=f_2(z),\quad w_3=f_3(z), \text{ etc.},$$

each of these forms being now single-valued, then w_1, w_2, w_3, etc., are called the "branches" of w.

When z traces any curve in the (x, y) plane, each of the functions w_1, w_2, w_3, ... traces out a corresponding curve in the (u, v) plane, and each curve is a graph of its own branch.

If for any point z two values of w become equal, such point is said to be a "branch point" of w. A line which connects two and only two branch points is called a branch line or cross line.

1257. The simplest example is the case when $w^2=z$. Here w is a two-valued function. The function has "branches" $w_1=+\sqrt{z}$, $w_2=-\sqrt{z}$.

At the points $z=0$ and $z=\infty$ there are "branch points." The positive direction of the x-axis which joins $z=0$ to $z=\infty$ is a branch line.

1258. To examine the behaviour of w_1 and w_2 in the immediate neighbourhood of the branch point at $z=0$, put $z=re^{\iota\theta}$, and travel round the point along a small circle of radius r; r remains constant, θ increases by 2π.

$$w_1=+\sqrt{re^{\iota\theta}} \text{ becomes } \sqrt{re^{\iota(\theta+2\pi)}}= e^{\iota\pi}\sqrt{re^{\iota\theta}}=-\sqrt{re^{\iota\theta}}=w_2,$$
$$w_2=-\sqrt{re^{\iota\theta}} \text{ becomes } -\sqrt{re^{\iota(\theta+2\pi)}}=-e^{\iota\pi}\sqrt{re^{\iota\theta}}= \sqrt{re^{\iota\theta}}=w_1.$$

Hence in passing once round the branch point $z=0$, and therefore crossing the branch line, each branch changes into the other.

1259. Similarly for the case $w^q=z$, where q is a positive integer.

Here w is a q-valued function of z, and we have

$$w=z^{\frac{1}{q}}\left(\cos\frac{2\lambda\pi}{q}+\iota\sin\frac{2\lambda\pi}{q}\right), \text{ where } \lambda=1, 2, 3, \ldots \text{ or } q.$$

Let the q q^{th} roots of unity be called $\alpha, \alpha^2, \alpha^3, \ldots \alpha^q$.

Then the branches of the function may be written

$$w_1=\alpha z^{\frac{1}{q}},\quad w_2=\alpha^2 z^{\frac{1}{q}},\quad w_3=\alpha^3 z^{\frac{1}{q}}, \ldots\; w_q=\alpha^q z^{\frac{1}{q}},$$

where by $z^{\frac{1}{q}}$ we mean any definite q^{th} root of z, the same to be taken throughout.

The points $z=0$ and $z=\infty$ are branch points, and the positive portion of the x-axis is a branch line.

In passing once round a small circle of radius r encircling a branch point, say that at $z=0$, w_s changes from being $\alpha^s(re^{\iota\theta})^{\frac{1}{q}}$ to being $\alpha^s[re^{\iota(\theta+2\pi)}]^{\frac{1}{q}}$, that is to

$$\alpha^s e^{\iota\frac{2\pi}{q}}(re^{\iota\theta})^{\frac{1}{q}} \quad\text{or}\quad \alpha^{s+1}(re^{\iota\theta})^{\frac{1}{q}};$$

therefore w_s changes to w_{s+1}.

Thus the system of branches changes from

$$w_1, w_2, w_3, \dots w_{q-1}, w_q \quad \text{to} \quad w_2, w_3, w_4, \dots w_q, w_1,$$

and a second encircling of this small contour will cause the further change to $w_3, w_4, w_5, \dots w_1, w_2$, and so on. So that when z has travelled q times round the branch point at $z=0$, the original order will have been restored.

Similarly also for the case $w^q = z^p$, where p and q are positive integers prime to each other,

1260. Reverting to the case $w^2 = az$, where a is positive and real, put

$$z = re^{\iota\theta}, \quad w_1 = r_1 e^{\iota\theta_1}, \quad w_2 = r_2 e^{\iota\theta_2}.$$

Then $w_1 \equiv r_1 e^{\iota\theta_1} = +\sqrt{are^{\iota\theta}}, \quad w_2 \equiv r_2 e^{\iota\theta_2} = -\sqrt{are^{\iota\theta}} = \sqrt{are^{\iota(\theta+2\pi)}}$;

$$r_1 = \sqrt{ar}, \quad \theta_1 = \frac{\theta}{2}; \quad r_2 = \sqrt{ar}, \quad \theta_2 = \pi + \frac{\theta}{2}.$$

We show separate w-planes for the separate branches. (Fig. 365.) Take as the z-curve the circle $r = a$.

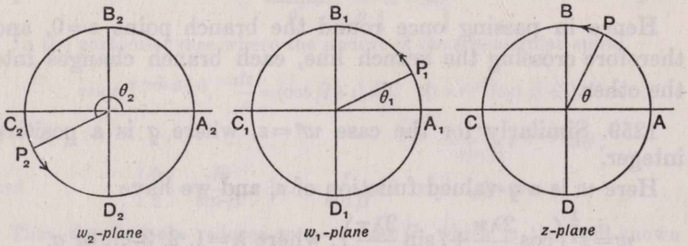

Fig. 365.

Here, as P, (z), moves round the circumference $ABCD$ of the circle $r=a$, the points P_1, (w_1), and P_2, (w_2), respectively describe two semicircles shown in the accompanying figure, viz. the upper half circle $A_1B_1C_1$ for w_1 and the lower half circle $C_2D_2A_2$ for w_2. When P traverses its path a second time, P_1 proceeds to describe the lower half circle of w_1, viz. $C_1D_1A_1$, whilst P_2 describes the upper half $A_2B_2C_2$ for w_2.

1261. **Sheets, Riemann's Surface.**

In order to avoid the inconvenience of the same value of z indicating two or more values of w, the following device is adopted.

Imagine the x-y plane upon which the point z travels to be split into as many parallel sheets as there are values of w to which any one value of z gives rise. Let these sheets still carry with them the tracings of the original axes, and let them be separated from each other by infinitesimal distances ϵ, the origins lying in a line perpendicular to the several planes and the axes remaining parallel, and let the same point z be marked upon each plane. Let the several planes be designated as No. 1, No. 2, No. 3, etc., and be associated with the several functions $w=w_1$, $w=w_2$, $w=w_3$, etc., to which the value of z gives rise, so that when z travels on plane No. 1, the graph of w_1 is traced on the w-plane, when z travels on plane No. 2 the graph of w_2 is traced on the w-plane, and so on. In this way each value of z with its particularising plane gives rise only to one value of w, so that w may now be looked upon as a single-valued function of z, and z requires for its description not only the values of x and y, but also the number or label of its particularising plane.

Now it will be inferred from the examples considered that when z in its travel upon the original x-y plane in continuous motion crosses a branch line AB in that plane there is a change in the branch of the function, w_1 to w_2 say. In order to represent the continuous motion of z in our new system of sheets from plane (1) to plane (2) it will be necessary to suppose the existence of a plane bridge extending from A to B, and terminating at these points and leading from plane (1) on which A, B lie to plane (2) on which A', B' lie where A', B' are the new positions of A, B on plane (2), so that in passing from z_1 on plane (1) to z_2 on plane (2) the point z passes down the bridge of infinitesimal length from the one plane to the other without changing its value in so passing.

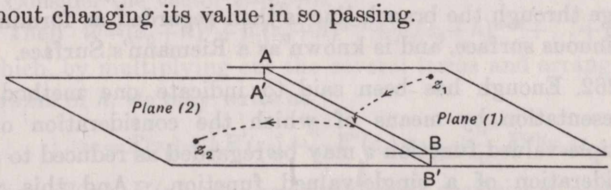

Fig. 366.

And in the case when there are only two branch points and one branch line, we shall consider the several z-sheets to be nowhere else connected. Thus, as z passes over this bridge from plane (1) to plane (2), w_1 changes to w_2. After travelling in plane (2) the point z must again cross the bridge to get back to its original position z_1, for there is no other connection between the planes (2) and (1). The excursion of z from plane (1) to plane (2) and back again may be indicated to an eye looking endwise along the branch line from B to A, as in the diagram No. 367, the bridge being represented in duplicate as PQ or $P'Q'$ for convenience.

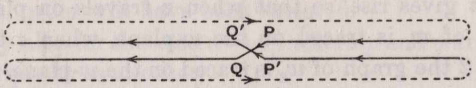

Fig. 367.

Thus, in the case of $w^2=z$, we have the diagram of the change indicated in Fig. 368.

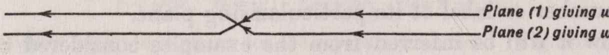

Fig. 368.

In the case of $w^q=z$ the cyclic order of changes as z passes the branch line is indicated in Fig. 369 (taking, for example, $q=5$).

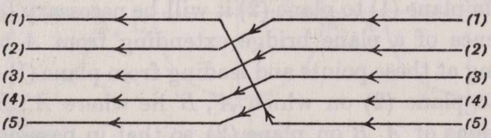

Fig. 369.

The whole system of sheets thus connected by means of a bridge through the branch line is then regarded as forming a continuous surface, and is known as a Riemann's Surface.

1262. Enough has been said to indicate one method of representation by means of which the consideration of a multiple-valued function z may be regarded as reduced to the consideration of a single-valued function. And this will suffice for our purposes in this book. The whole theory of Branch points, Branch lines and Riemann's representation would occupy far more space than is at our disposal, and we must refer the student to treatises on the Theory of Functions, *e.g.* Forsyth, *Theory of Functions*, Chapter XV., or Harkness and Morley, *Theory of Functions*, Chapter VI., where this very interesting matter will be found fully discussed.

1263. **Any Algebraic Equation of the n^{th} degree has n roots, n being a positive integer.**

Let $w \equiv F(z) = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \ldots + p_n = 0$, where z and the several coefficients may be real or complex and n is a positive integer.

Whilst z travels over the whole of the z-plane it is obvious that w will travel over at any rate some part of the w-plane.

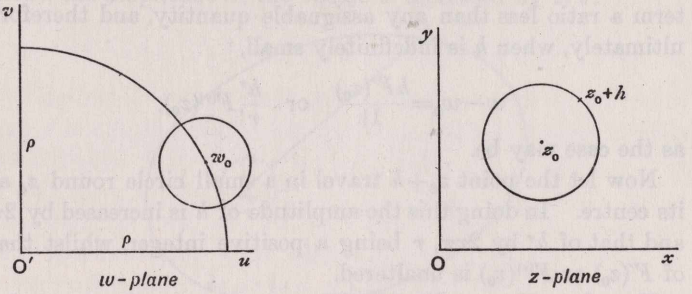

Fig. 370.

Let O and O' be the two origins. Then we shall show that w must reach O' in its travels over the w-plane. For, if there were any finite limit of the nearness of approach of w to O', let ρ be that limit. Let z_0 be the value of z for which w arrives at its limiting value, w_0 say, which must lie somewhere on the circumference of a circle of radius ρ in the w-plane and having O' for its centre.

Consider the vector $z = z_0 + h$.

Then $w = (z_0 + h)^n + p_1(z_0 + h)^{n-1} + p_2(z_0 + h)^{n-2} + \ldots + p_n$, which, by multiplying out the several terms and arranging in powers of h, we may write as

$$w = F(z_0) + hF'(z_0) + \frac{h^2}{2!}F''(z_0) + \ldots + \frac{h^n}{n!}F^{(n)}(z_0),$$

where $F(z_0)$, $F'(z_0)$, etc., are the several coefficients occurring, and are functions of z_0 alone, finite so long as z_0 is finite. Then obviously $w_0 = F(z_0)$, and therefore

$$w - w_0 = hF'(z_0) + \frac{h^2}{2!}F''(z_0) + \ldots + \frac{h^n}{n!}F^{(n)}(z_0) = hF'(z_0) + \xi, \text{ say.}$$

Then, provided $F'(z_0)$ does not vanish, we can, by making h sufficiently small, make the ratio $\xi : hF'(z_0)$ less than any assignable quantity.

And even if $F'(z_0)$ does vanish, as well as

$$F''(z_0),\ F'''(z_0) \dots F^{(r-1)}(z_0),\ \text{say},$$

so that $\frac{h^r}{r!}F^{(r)}(z_0)$ is the first term which does not vanish, we can in the same way, by taking h sufficiently small, make the remainder of the series beyond the term $\frac{h^r}{r!}F^{(r)}(z_0)$ bear to this term a ratio less than any assignable quantity, and therefore ultimately, when h is indefinitely small,

$$w-w_0=\frac{hF'(z_0)}{1!} \quad \text{or} \quad \frac{h^r}{r!}F^{(r)}(z_0),$$

as the case may be.

Now let the point z_0+h travel in a small circle round z_0 as its centre. In doing this the amplitude of h is increased by 2π and that of h^r by $2r\pi$, r being a positive integer, whilst that of $F'(z_0)$ or $F^{(r)}(z_0)$ is unaltered.

Therefore the amplitude of $w-w_0$ increases by 2π or by $2r\pi$, and the point w describes some curve about w_0 which returns into itself after one or r complete circuits, as z describes a small circle about z_0. Hence it must penetrate at least once into the circle of radius ρ in its travel about w_0. And this contradicts the hypothesis that there is an inferior limit to the closeness of approach of w to O'

There must therefore be at least one value of z, say $z=z_1$, for which w coincides with the origin O' and makes $F(z)$ vanish.

Hence $z-z_1$ must be a factor of $F(z)$.

Dividing out $z-z_1$ from $F(z)$ we get an expression of degree $n-1$ in powers of z to which the same process can be applied.

And, proceeding in this way, it is clear that $F(z)$ *must have n zeros.*

And, if $z_1, z_2, z_3, \dots z_n$ be the values of z for which $F(z)$ vanishes, we get $w=A(z-z_1)(z-z_2)(z-z_3)\dots(z-z_n)$, where A is independent of z, but may be a complex constant.

Thus $$\text{mod. } w=\text{mod. } A \, . \prod_{r=1}^{r=n} \text{mod. } (z-z_r),$$

and $$\text{amp. } w=\text{amp. } A+\sum_{r=1}^{r=n} \text{amp. } (z-z_r).$$

1264. Number of roots within a given Contour.

We are now in a position to assign the number of roots of $w=0$ which lie within a given contour in the w-plane.

When z travels in a closed curve once round z_0 the amplitude of the vector $z-z_0$ is increased by 2π, and if the closed curve encircles z_0 r times before returning to the starting point, the amplitude of the vector is increased by $2r\pi$.

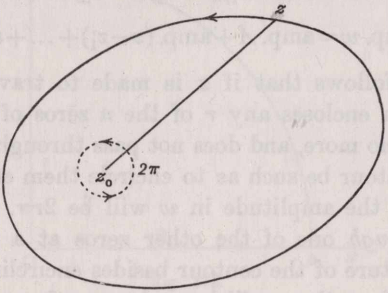

Fig. 371.

When z travels round a closed contour which does not enclose z_0 the amplitude of $z-z_0$ increases by a certain amount, and then decreases again till it assumes its original value when the whole circuit of the contour has been traversed, so that there is no change in the amplitude.

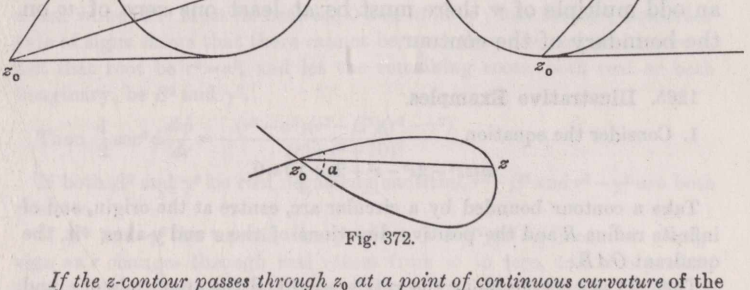

Fig. 372.

If the z-contour passes through z_0 at a point of continuous curvature of the contour instead of surrounding it, there is a change of π in the amplitude of $z-z_0$. If z_0 be *situated at a node* of the z-curve, then, when z describes a loop starting from the node by one of the branches which passes through z_0 and returning to the node by another branch, the change in the amplitude of $z-z_0$ is a, where a is the angle between the directions of the two tangents at the node between which the loop lies.

Remembering that if

$$w=A(z-z_1)(z-z_2)(z-z_3)\dots(z-z_n)$$

we have $\quad \text{amp.}\,w=\text{amp.}\,A+\text{amp.}\,(z-z_1)+\dots+\text{amp.}\,(z-z_n),$

it obviously follows that if z is made to travel round any contour which encloses any r of the n zeros of w, viz. $z_1, z_2, z_3, \dots z_n$, and no more, and does not pass through any of them, and if the contour be such as to encircle them each once only, the change of the amplitude in w will be $2r\pi$. If, however, it passes *through* one of the other zeros at a point of continuous curvature of the contour besides encircling the r zeros considered before, there will be a change of amplitude to the extent of $(2r+1)\pi$. Conversely, if as z passes along the perimeter of any region S it be observed that the change of amplitude is $2r\pi$, we infer either that there are r zeros of w within that region or $r-2p$ zeros within and $2p$ upon the boundary, and that, if the change of amplitude be $(2r+1)\pi$, there will be r zeros within and one upon the boundary or $r-2p$ zeros within and $2p+1$ upon the boundary, so that in the one case there are r roots within or upon the boundary, and in the other there are $r+1$ roots within or upon the boundary, and the number upon the boundary is even in the first case, odd in the second, and if the change of amplitude be an odd multiple of π there must be at least one zero of w on the boundary of the contour.

1265. **Illustrative Examples.**

1. Consider the equation

$$w\equiv z^4-2z^3-z^2+2z+10=0.$$

Take a contour bounded by a circular arc, centre at the origin, and of infinite radius R and the positive directions of the x and y-axes, viz. the quadrant OAB.

Then (1) as z travels along the x-axis, $y=0$ and the amplitude of z, and therefore also of w is zero, in moving from O to A.

(2) As z travels along the quadrantal arc AB of the infinite circle,

$$w=R^4\left(e^{4\iota\theta}-2\frac{e^{3\iota\theta}}{R}-\frac{e^{2\iota\theta}}{R^2}+2\frac{e^{\iota\theta}}{R^3}+\frac{10}{R^4}\right)=R^4e^{4\iota\theta} \text{ ultimately,}$$

and as θ changes from 0 to $\frac{\pi}{2}$ the increase of amplitude is $4\cdot\frac{\pi}{2}=2\pi$.

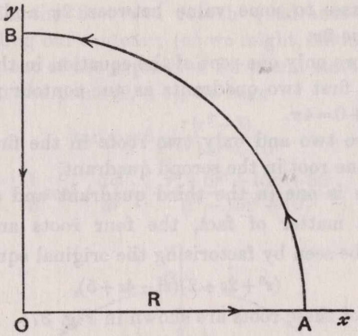

Fig. 373.

(3) As z travels from $z=\infty$ at B down the y-axis to O, $x=0$, and $z=\iota y=\iota r$, say, and $w=r^4+2\iota r^3+r^2+2\iota r+10=\rho(\cos\phi+\iota\sin\phi)$, say, where

$$\tan\phi=2\frac{r^3+r}{r^4+r^2+10}, \quad \text{.............................}(a)$$

so that $\tan\phi$ remains positive as r decreases from ∞ to zero, vanishing at both limits. To find where it attains its maximum value, we have by differentiation

$$\frac{1}{2}\sec^2\phi\frac{d\phi}{dr}=-\frac{r^6+2r^4-29r^2-10}{(r^4+r^2+10)^2}, \quad \text{....................}(b)$$

and the equation to find the stationary values of $\tan\phi$ is

$$r^6+2r^4-29r^2-10=0, \quad \text{..............................}(c)$$

which being a cubic for r^2 must have one value of r^2 real. Moreover, as $r^2=\infty$ makes the left-hand member positive, and $r^2=0$ makes it negative, a real value of r^2 must lie between 0 and infinity; and further, Descartes' rule of signs shows that there cannot be more than one real positive root. Let that root be $r^2=\alpha^2$, and let the remaining roots, both real or both imaginary, be β^2 and γ^2.

Then $\dfrac{1}{2}\sec^2\phi\dfrac{d\phi}{dr}=-\dfrac{(r^2-\alpha^2)(r^2-\beta^2)(r^2-\gamma^2)}{(r^4+r^2+10)^2}.$

If both β^2 and γ^2 be real negative quantities, $r^2-\beta^2$ and $r^2-\gamma^2$ are both positive.

If β^2 and γ^2 be unreal, the product $(r^2-\beta^2)(r^2-\gamma^2)$ cannot change sign as r changes through real values from ∞ to zero, and this product is ultimately r^4 when r is infinite. Hence in either case $(r^2-\beta^2)(r^2-\gamma^2)$ is positive.

Also r is *decreasing*. Hence

from $r=R$ to $r=a$, we have $\frac{d\phi}{dr}=(-)^{ve}$, therefore $\tan\phi$ is increasing,

and from $r=a$ to $r=0$, $\frac{d\phi}{dr}=(+)^{ve}$, therefore $\tan\phi$ is decreasing.

But at $r=R$ the amplitude ϕ is 2π.

Hence ϕ increases to some value between 2π and $2\pi+\frac{\pi}{2}$, and then returns to its value 2π.

There is therefore only one root of the equation in the first quadrant.

If we take the first two quadrants as our contour we get a change of amplitude $0+4\pi+0=4\pi$.

Hence there are two and only two roots in the first two quadrants. That is, there is one root in the second quadrant.

Similarly there is one in the third quadrant and one in the fourth quadrant. As a matter of fact, the four roots are $-1\pm\sqrt{-1}$ and $2\pm\sqrt{-1}$, as may be seen by factorising the original equation as

$$(z^2+2z+2)(z^2-4z+5),$$

and the localities of these roots are shown in Fig. 374.

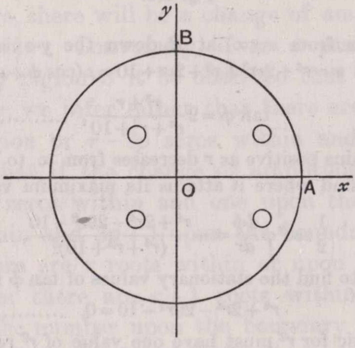

Fig. 374.

2. Consider next the equation

$$w\equiv z^6-6z^5+16z^4-24z^3+25z^2-18z+10=0.$$

Take the same contour as in the last case.

(1) Along the x-axis from O to A $z=x$, and there is no change in the amplitude, which remains zero.

(2) Along the infinite circle w is ultimately $R^6e^{6\iota\theta}$, and there is a change of amplitude $6\times\frac{\pi}{2}=3\pi$ in passing from A to B.

(3) Down the y-axis from B to O, $z=\iota r$, say.

Hence $\quad w=-r^6-6\iota r^5+16r^4+24\iota r^3-25r^2-18\iota r+10$

$$=\rho(\cos\phi+\iota\sin\phi), \text{ say.}$$

Then $\tan\phi = \dfrac{6r^5 - 24r^3 + 18r}{r^6 - 16r^4 + 25r^2 - 10} = \dfrac{6(r^2-1)(r^3-3r)}{(r^2-1)(r^4-15r^2+10)}$.

This indicates a peculiarity at $r = \pm 1$, *i.e.* $z = \pm\iota$; and it will appear from $w \equiv z^6 - 6z^5 + \ldots + 10$ that z^2+1 is a factor and two of the roots are $z = \pm\iota$.

To exclude these roots we draw two small semicircles of radius r' with centres $(0, \pm 1)$ in the first and fourth quadrants as shown in the figure, thus amending our contour; (or we might, having discovered these roots, divide z^2+1 out of the expression for w and start again).

Hence, except at the point $(0, \pm 1)$, we have

$$\tan\phi = 6\frac{r(r^2-3)}{r^4 - 15r^2 + 10}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

whence
$$\frac{1}{6}\sec^2\phi\frac{d\phi}{dr} = -\frac{r^6 + 6r^4 + 15r^2 + 30}{(r^4 - 15r^2 + 10)^2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots(b)$$

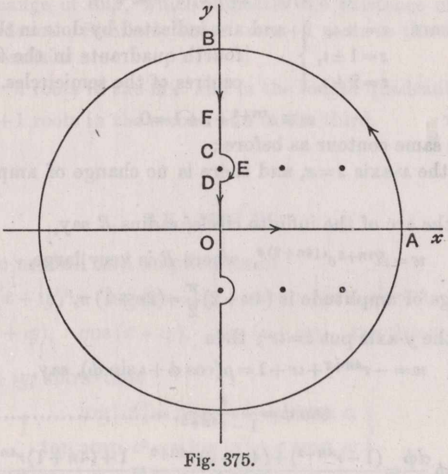

Fig. 375.

so that $\dfrac{d\phi}{dr}$ is negative for all positive values of r, and therefore as r decreases along the y-axis ϕ increases, with the exception of in the immediate neighbourhood of the point where $r=1$; and $\tan\phi$ vanishes both at $r = R = \infty$ and at $r=0$ as well as at $r = \sqrt{3}$.

To consider what happens in the neighbourhood of $r=1$, about which the small semicircle is drawn, put $z = \iota + r'e^{\iota\theta'}$. Then to first powers of r',

$$w \equiv (-1 + 6\iota r'e^{\iota\theta'}) - 6(\iota + 5r'e^{\iota\theta'}) + 16(1 - 4\iota r'e^{\iota\theta'}) - 24(-\iota - 3r'e^{\iota\theta'})$$
$$+ 25(-1 + 2\iota r'e^{\iota\theta'}) - 18(\iota + r'e^{\iota\theta'}) + 10 = 8(3-\iota)r'e^{\iota\theta'},$$

and the variable portion of the amplitude diminishes from $\theta' = \dfrac{\pi}{2}$ to

$\theta' = -\frac{\pi}{2}$ as z traverses the semicircle CED from C to D; otherwise along the y-axis the value of the amplitude is always increasing from $\phi = 3\pi$ at ∞, where $\tan\phi = 0$ to $\phi = 4\pi$ at $r = \sqrt{3}$, where $\tan\phi = 0$ again, and except for the semicircle CED to $\phi = 5\pi$ at $r = 0$, where $\tan\phi$ has again become zero, besides the loss of π in passing round the small semicircle.

Hence the change of amplitude round the whole contour is

0 from O to A, 3π from A to B, π from B to F, where $OF = \sqrt{3}$,

π from F to O except round the semicircle CED, $-\pi$ round CED;

i.e. in all, the change of amplitude is 4π, which indicates the existence of two roots in the first quadrant, besides the root $z = \iota$ on the boundary.

In the same way, it can be shown that there is another root $z = -\iota$, and two others in the fourth quadrant, but none in the second and third.

As a matter of fact, the expression when factorised becomes

$$(z^2+1)(z^2-2z+2)(z^2-4z+5),$$

and the roots are $\left.\begin{array}{l} z = \pm\iota, \\ z = 1 \pm \iota, \\ z = 2 \pm \iota, \end{array}\right\}$ and are indicated by dots in the second and fourth quadrants in the figure and the centres of the semicircles.

3. Consider $$w \equiv z^{4n+2} + z + 1 = 0.$$

Taking the same contour as before:

(1) Along the x-axis $z = x$, and there is no change of amplitude in z or in w.

(2) Along the arc of the infinite circle, radius R say,

$$w = R^{4n+2} e^{\iota(4n+2)\theta}, \text{ where } R \text{ is very large,}$$

and the change of amplitude is $(4n+2)\frac{\pi}{2} = (2n+1)\pi$.

(3) Along the y-axis put $z = \iota r$; then

$$w = -r^{4n+2} + \iota r + 1 = \rho(\cos\phi + \iota\sin\phi), \text{ say,}$$

and $$\tan\phi = \frac{r}{1 - r^{4n+2}}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

$$\sec^2\phi\,\frac{d\phi}{dr} = \frac{(1 - r^{4n+2}) + (4n+2)r^{4n+2}}{(1 - r^{4n+2})^2} = \frac{1 + (4n+1)r^{4n+2}}{(1 - r^{4n+2})^2},$$

which is positive for all positive values of r. Hence, as r is decreasing as z travels from B to O down the y-axis, ϕ is also decreasing, and the decrease is from $(2n+1)\pi$ through $(2n+1)\pi - \frac{\pi}{2}$ at $r = 1$, where $\tan\phi = \infty$, to $(2n+1)\pi - \pi$ at O. That is, the total change of amplitude in passing round this contour is $2n\pi$, which indicates the existence of n roots in the first quadrant.

(4) If we take the first two quadrants as contour with an infinite semicircular boundary, the change of amplitude is

$$0 + (4n+2)\pi + 0 = (4n+2)\pi.$$

Hence there are $2n+1$ roots in the first and second quadrants, *i.e.* $(n+1)$ roots in the second quadrant.

(5) Consider next the behaviour in the fourth quadrant.

For the variation of z down the y-axis, OB', put $z=-\iota r$,

$$w=-r^{4n+2}-\iota r+1=\rho'(\cos\phi'+\iota\sin\phi'),\text{ say,}$$

$$\tan\phi'=\frac{r}{r^{4n+2}-1},$$

$$\sec^2\phi'\frac{d\phi'}{dr}=-\frac{(4n+1)r^{4n+2}+1}{(r^{4n+2}-1)^2},$$

which is essentially negative; and r is increasing, therefore ϕ' is decreasing, and $\phi'=0$ at O, and again at B', where $r=\infty$, and there is a loss of π in the amplitude.

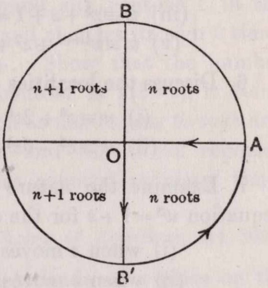

Fig. 376.

In traversing $B'A$ there is, as before, an increase of $(2n+1)\pi$ in the amplitude, whilst in traversing AO there is no change. This gives a change of $2n\pi$, which indicates the existence of n roots in the fourth quadrant. Similarly there are $n+1$ roots in the third.

Hence the localities are:

n roots in the first and in the fourth quadrants;

$n+1$ roots in the second and in the third.

EXAMPLES.

1. Find the moduli and amplitudes of

$$(x+\iota y)^n,\quad \log(x+\iota y),\quad a^{x+\iota y},\quad (x+\iota y)^{a+\iota b},$$

$$\sin(x+\iota y),\quad \cos(x+\iota y),\quad \sec(x+\iota y),\quad \tan^{-1}(x+\iota y).$$

2. If $z\equiv x+\iota y$, show that

$$\left.\begin{aligned}\log|c^z|&=x\log|c|-y\text{ amp. }c,\\ \tan\text{ amp. }c^z&=y\log|c|+x\text{ amp. }c.\end{aligned}\right\}$$

3. How are $\sin z$, $\log z$, $\tan^{-1}z$ defined when $z=x+\iota y$? Show that if $z=x+\iota y$,

$$\frac{dz^n}{dz}=nz^{n-1},\quad \frac{d\sin z}{dz}=\cos z,\quad \frac{d\log z}{dz}=\frac{1}{z},\quad \frac{d}{dz}\tan^{-1}z=\frac{1}{1+z^2}.$$

4. Discuss the locality of the roots of the equations:

(i) $w\equiv z^4-2z^3+4z+12=0$;

(ii) $w\equiv z^4+2z^3-4z+12=0$;

(iii) $w\equiv z^4+6z^3+16z^2+20z+12=0$;

(iv) $w\equiv z^4-6z^3+16z^2-20z+12=0$;

stating in each case how many roots lie in each quadrant.

5. Find how many roots lie in each quadrant in the following cases:

(i) $w \equiv z^4 + z + 1 = 0$; (ii) $w \equiv z^{4n} + z + 1 = 0$;

(iii) $w \equiv z^5 + z + 1 = 0$; (iv) $w \equiv z^{4n+1} + z + 1 = 0$;

(v) $w \equiv z^{4n+1} + z^2 + 1 = 0$; (vi) $w \equiv z^{4n+2} + z^2 + 1 = 0$;

6. Discuss the localities of the roots of the equations:

(i) $w \equiv z^6 + 2z^5 + 7z^4 + 10z^3 + 14z^2 + 8z + 8 = 0$;

(ii) $w \equiv z^5 - 6z^4 + 5z^3 - 30z^2 + 4z - 24 = 0$.

7. Examine the nature of the conformal representation of the equation $w^2 = 1 + z$ for the cases:

(i) when z moves on the circle mod. $z = c$;

(ii) when z moves on the straight line $y = 1 + x$;

(iii) when z moves on the straight line $y = c$.

8. Find the radius of curvature of the hyperbola

$$x^2 \sec^2 c - y^2 \operatorname{cosec}^2 c = a^2$$

by a consideration of the conformal representation of the equation $w = a \cos z$, taking for the z-path the straight line $x = c$.

9. Supposing $a^2 w = z^3$, and a to be real, show that if z traces the curve $(x^2 + y^2)^3 = a^3(x^3 - 3xy^2)$, then w traces a circle at three times the angular rate. Deduce a formula for the radius of curvature of the above z-locus, and verify your result directly.

10. Taking the equation $w + 1 = (z + 1)^2$, show that the w-path corresponding to mod. $z = 1$ is a cardioide.

11. Examine the w-locus in the case $w = \cosh \log z$, when the z-locus is mod. $z = 1$.

12. Taking the relation $w^3 - 3w = z$, show, by putting $w = t + \dfrac{1}{t}$, that if t describes the circle mod. $t = k$:

(1) the z point describes an ellipse;

(2) the three w-points corresponding to any value of t^3 describe a confocal ellipse and form the angular points of a maximum inscribed triangle. [HARKNESS AND MORLEY, *Theory of Functions*, p. 39.]

13. Discuss the conformal representations arising from the equation

$$w = \log z,$$

and show that the curvature at any point of the w-locus is proportional to the value of $\dfrac{1}{r}\dfrac{ds}{d\phi}$ at the corresponding point of the z-locus, ϕ being the angle between the tangent and the radius r, and ds an element of arc of the z-locus.

14. Suppose w to be any rational function of $z(\equiv x+\iota y)$, and that w is put into the form $p+\iota q$ where p and q are real. Suppose that as z travels in the positive direction round any contour Γ in the x-y plane, p/q passes through the value 0 and changes its sign k times from + to − and l times from − to +. Show that the number of roots of $w=0$ which lie within the contour is $\frac{1}{2}(k-l)$, it being further supposed that the contour is such as not to *pass through* any point for which both p and q vanish, and that when repeated imaginary roots of $w=0$ occur they are counted as many times over as they occur.

[CAUCHY. (See TODHUNTER, *Theory of Equations*, Art. 308.)]

15. If ϕ be the longitude and λ the latitude of a place on the surface of a sphere and $\theta\equiv\text{gd}^{-1}\lambda$:

(i) Show that the coordinates of a point X_s, Y_s of the stereographic projection of ϕ, λ are

$$\left.\begin{aligned}X_s&=ae^{-\theta}\cos\phi,\\ Y_s&=ae^{-\theta}\sin\phi,\end{aligned}\right\}\ i.e.\ X_s+\iota Y_s=ae^{\iota(\phi+\iota\theta)}.$$

(ii) If X_m, Y_m be the coordinates of the same point in a Mercator projection defined as

$$X_m=a\phi,\quad Y_m=a\theta,$$

express X_s and Y_s in terms of X_m and Y_m.

(iii) Considering the equation $w/a=e^{\iota z/a}$ (a real), show that w is the stereographic projection of a point on the sphere, whose Mercator projection is z.

(iv) Show that the magnification in the stereographic projection $\propto(1+\sin\lambda)^{-1}$, and in the Mercator projection $\propto\sec\lambda$

(v) Examine the stereographic and Mercator projections of:

(a) the meridians; (b) the parallels of latitude; (c) a rhumb line.

16. If $\xi+\iota\eta=(x+\iota y)^{\frac{1}{n}}$, prove that the systems of curves $r^n\cos n\theta=a^n$, $r^n\sin n\theta=b^n$, in the plane ξ-η correspond to straight lines parallel to the axes in the plane x-y, and find the value of the integral $\int r^{2n-2}\,dA$ for the rectangular space included between any four of them, dA denoting an element of area. [ST. JOHN'S, 1890.]

17. In the relation $w=c\sin z$, show that the w-curve which corresponds to a rectangle $x=\pm\pi/2$, $y=\pm k$ on the z-plane is an ellipse with two narrow canals extending from the extremities of the major axis to the nearer foci, and that the interiors of the respective regions correspond. [FORSYTH, *Th. of F.*, p. 504.]

18. Writing $Z = X + \iota Y$, where X and Y are real, and taking $Z = \sin z$, determine a simply-connected region of the plane of z which is transformed conformally into the half plane $Y > 0$.

[MATH. TRIP., 1913.]

19. For the equation $\sqrt{X + \iota Y} = \tan(\frac{1}{4}\pi\sqrt{x + \iota y})$, show that we have as corresponding areas the area within the circle $X^2 + Y^2 = 1$, and that within the parabola $y^2 = 4(1 - x)$. Examine also the nature of the correspondence as regards

(i) the points on the circumference of the circle; (ii) those on the diameter $Y = 0$.

[MATH. TRIP., 1887.]

20. If $z = \sin^2\frac{1}{2}Z = \sin^2\frac{1}{2}(X + \iota Y)$, show that the lines $X = \text{const.}$, $Y = \text{const.}$ correspond to a system of confocal conics, and that the ratio of the areas of the triangles z_1, z_2, z_3 and Z_1, Z_2, Z_3 is proportional to the product of the distances z_1 (or z_2 or z_3) from the common foci of the system, the points Z_1, Z_2, Z_3 being the vertices of an infinitesimal triangle in the Z-plane and z_1, z_2, z_3 the vertices of the corresponding triangle on the z-plane.

[OX. II. P., 1913.]

21. Show that $\zeta = (z + a)^2/(z - a)^2$ gives one conformal representation of the semi-circular area $x^2 + y^2 \leqq a^2$, $y \geqq 0$ on the plane of $z = x + \iota y$, upon the upper half $\eta \geqq 0$ of the plane $\zeta = \xi + \iota y$. Explain how to modify the formula so that $x = h$, $y = 0$ become $\xi = 0$, $\eta = 0$, and $x = x_0$, $y = y_0$ become $\xi = 0$, $\eta = 1$ ($h^2 \leqq a^2$, $x_0{}^2 + y_0{}^2 < a^2$).

[MATH. TRIP. II., 1919.]