CHAPTER XXVIII.

DEFINITE INTEGRALS (III.).

1121. The Three Integrals,

$$I_1=\int_0^\pi \cos p\theta \cos q\theta\, d\theta=0\ (p\neq q);\ \text{or } \frac{\pi}{2}\ (p=q),$$

$$I_2=\int_0^\pi \sin p\theta \sin q\theta\, d\theta=0\ (p\neq q);\ \text{or } \frac{\pi}{2}\ (p=q),$$

$$I_3=\int_0^\pi \sin p\theta \cos q\theta\, d\theta=0\ (p+q \text{ even});\ \text{or } \frac{2p}{p^2-q^2}\ (p+q \text{ odd}),$$

where p and q are integers, are of very special importance in the Theory of Definite Integrals.

(i) $I_1=\int_0^\pi \cos p\theta \cos q\theta\, d\theta=\frac{1}{2}\int_0^\pi [\cos(p+q)\theta+\cos(p-q)\theta]\, d\theta$

$$=\frac{1}{2}\left[\frac{\sin(p+q)\theta}{p+q}+\frac{\sin(p-q)\theta}{p-q}\right]_0^\pi$$

$$=0, \text{ if } p \text{ and } q \text{ be unequal.}$$

But if $p=q$, $\quad Lt_{p-q=0}\left[\frac{\sin(p-q)\theta}{p-q}\right]_0^\pi=\Big[\theta\Big]_0^\pi=\pi;$

$$\therefore I_1=0 \text{ if } p\neq q \quad\text{and}\quad =\frac{\pi}{2} \text{ if } p=q.$$

In the latter case, viz. $p=q$, we may obtain the result directly without taking a limit; for

$$I_1=\int_0^\pi \cos^2 p\theta\, d\theta=\int_0^\pi \frac{1+\cos 2p\theta}{2}\, d\theta=\frac{1}{2}\left[\theta+\frac{\sin 2p\theta}{2p}\right]_0^\pi=\frac{\pi}{2}.$$

(ii) In the same way

$$I_2=\int_0^\pi \sin p\theta \sin q\theta\, d\theta=0 \text{ if } p\neq q \quad\text{or}\quad =\frac{\pi}{2} \text{ if } p=q.$$

(iii) Finally

$$I_3=\int_0^\pi \sin p\theta \cos q\theta \, d\theta=\frac{1}{2}\int_0^\pi [\sin (p+q)\theta+\sin (p-q)\theta]\, d\theta$$

$$=\frac{1}{2}\left[-\frac{\cos (p+q)\theta}{p+q}-\frac{\cos (p-q)\theta}{p-q}\right]_0^\pi$$

$$=\frac{1}{2}\left\{\frac{1-(-1)^{p+q}}{p+q}+\frac{1-(-1)^{p-q}}{p-q}\right\}$$

$$=\frac{1-(-1)^{p+q}}{2}\left\{\frac{1}{p+q}+\frac{1}{p-q}\right\}, \text{ for } (-1)^{p-q}=(-1)^{p+q},$$

$$=\{1-(-1)^{p+q}\}\frac{p}{p^2-q^2}$$

$$=0 \quad \text{or} \quad \frac{2p}{p^2-q^2},$$

according as $p+q$ is even or odd, and p, q unequal.

And if $p=q$,

$$I_3=\frac{1}{2}\int_0^\pi \sin 2p\theta \, d\theta=\frac{1}{2}\left[-\frac{\cos 2p\theta}{2p}\right]_0^\pi=\frac{1-\cos 2p\pi}{4p}=0, \; p \text{ being an integer.}$$

1122. Important Applications.

If then $F(\theta)$ be a function of θ capable of convergent expansion in a series of sines or cosines of integral multiples of θ, say,

$$F(\theta)\equiv A_0+A_1\cos\theta+A_2\cos 2\theta+\ldots+A_n\cos n\theta+\ldots,$$

we have $\displaystyle\int_0^\pi F(\theta)\cos n\theta\, d\theta=A_n\cdot\frac{\pi}{2}$ and $\displaystyle\int_0^\pi F(\theta)\, d\theta=A_0\pi.$

For upon multiplying by $\cos n\theta$ and integrating between limits 0 and π all the terms vanish except $A_n\displaystyle\int_0^\pi \cos^2 n\theta\, d\theta$, which becomes $A_n\cdot\dfrac{\pi}{2}$.

When therefore such an expansion for $F(\theta)$ is possible, this result gives a means of obtaining the several coefficients, viz.

$$A_0=\frac{1}{\pi}\int_0^\pi F(\theta)\, d\theta, \quad A_n=\frac{2}{\pi}\int_0^\pi F(\theta)\cos n\theta\, d\theta.$$

Similarly, if $F(\theta)$ be expressible in the form

$$F(\theta)=B_1\sin\theta+B_2\sin 2\theta+\ldots+B_n\sin n\theta+\ldots$$

we have $\displaystyle B_n=\frac{2}{\pi}\int_0^\pi F(\theta)\sin n\theta\, d\theta.$

In the same way, if $F(\theta) \equiv A_0 + \sum_1^\infty A_r \cos r\theta$,

then $\int_0^\pi F(\theta) \cos m\theta \cos n\theta \, d\theta = \frac{1}{2} \int_0^\pi F(\theta) \{\cos (m+n)\theta + \cos (m-n)\theta\} \, d\theta$

$$= \frac{1}{2} \cdot \frac{\pi}{2}(A_{m+n} + A_{m-n}), \quad m \neq n,$$

and $\int_0^\pi F(\theta) \cos^2 m\theta \, d\theta = \frac{1}{2} \frac{\pi}{2}(2A_0 + A_{2m})$.

Again $\int_0^\pi F(\theta) \sin 2m\theta \, d\theta = \frac{4m}{4m^2 - 1^2} A_1 + \frac{4m}{4m^2 - 3^2} A_3 + \frac{4m}{4m^2 - 5^2} A_5 + \dots$

and so on for other similar applications of the rules.

1123. There are then two cases for which the rules are particularly useful.

1. When $F(\theta)$ is a known expansion of one of the forms

$$A_0 + \sum_1^\infty A_r \cos r\theta, \quad \sum_1^\infty B_r \sin r\theta,$$

i.e. such that the coefficients $A_0, A_1, A_2, \dots$ or $B_1, B_2, \dots$ are known, the method may be used to obtain definite integrals of the forms

$$\int_0^\pi F(\theta) \frac{\cos}{\sin} p\theta \, d\theta, \quad \int_0^\pi F(\theta) \frac{\cos}{\sin} p\theta \frac{\cos}{\sin} q\theta \, d\theta, \quad \int_0^\pi F(\theta) \frac{\cos^2}{\sin^2} p\theta \, d\theta,$$

etc.

2. Conversely, if $F(\theta)$ has not been already expanded in such form, *i.e.* in a convergent series of sines or cosines of integral multiples of θ, and if such expansion be possible, and if it be possible to obtain the value of $\int_0^\pi F(\theta) \cos n\theta \, d\theta$, or of $\int_0^\pi F(\theta) \sin n\theta \, d\theta$, the values of the several coefficients may then be deduced as $A_0 = \frac{1}{\pi} \int_0^\pi F(\theta) \, d\theta$,

$$A_n = \frac{2}{\pi} \int_0^\pi F(\theta) \cos n\theta \, d\theta, \quad B_n = \frac{2}{\pi} \int_0^\pi F(\theta) \sin n\theta \, d\theta, \quad (n > 0),$$

and the expansion thus obtained holds for all values of θ *between* $\theta = 0$ and $\theta = \pi$.

1124. Again, if there be two convergent expansions of the same kind, viz.

$$F(\theta) = A_0 + A_1 \cos \theta + A_2 \cos 2\theta + A_3 \cos 3\theta + \dots,$$
$$f(\theta) = C_0 + C_1 \cos \theta + C_2 \cos 2\theta + C_3 \cos 3\theta + \dots,$$

then plainly, upon multiplication and integration between limits 0 and π,

$$A_0C_0+A_1C_1+A_2C_2+A_3C_3+\ldots=\frac{2}{\pi}\int_0^\pi f(\theta)F(\theta)\,d\theta-A_0C_0,$$

and as a case, if $f(\theta)$ and $F(\theta)$ be the same series,

$$A_0^2+A_1^2+A_2^2+A_3^2+\ldots=\frac{2}{\pi}\int_0^\pi [F(\theta)]^2\,d\theta-A_0^2.$$

1125. Further, if

$$\phi(x)\equiv A_0+A_1x+A_2x^2+A_3x^3+\ldots,$$
$$\psi(x)=C_0+C_1x+C_2x^2+C_3x^3+\ldots;$$

then writing $u=xe^{\iota\theta}$, $v=xe^{-\iota\theta}$,

$$\phi(u)+\phi(v)=2(A_0+A_1x\cos\theta+A_2x^2\cos 2\theta+A_3x^3\cos 3\theta+\ldots),$$
$$\psi(u)+\psi(v)=2(C_0+C_1x\cos\theta+C_2x^2\cos 2\theta+C_3x^3\cos 3\theta+\ldots);$$

$$\therefore\ A_0C_0\,.\,\pi+A_1C_1x^2\frac{\pi}{2}+A_2C_2x^4\frac{\pi}{2}+A_3C_3x^6\frac{\pi}{2}+\ldots$$
$$=\int_0^\pi \frac{\phi(u)+\phi(v)}{2}\cdot\frac{\psi(u)+\psi(v)}{2}\,d\theta,$$

i.e.

$$A_0C_0+A_1C_1x^2+A_2C_2x^4+A_3C_3x^6+\ldots$$
$$=\frac{1}{2\pi}\int_0^\pi [\phi(u)+\phi(v)][\psi(u)+\psi(v)]\,d\theta-A_0C_0;$$

and as a particular case, if ϕ and ψ be identical,

$$A_0^2+A_1^2x^2+A_2^2x^4+A_3^2x^6+\ldots=\frac{1}{2\pi}\int_0^\pi [\phi(u)+\phi(v)]^2\,d\theta-A_0^2,$$

i.e. when the several terms of a series can be summed, we can express the sum of the squares of these terms in the form of a definite integral, and the sum of the squares of the coefficients will be expressible by means of the same integral, putting $x=1$, provided the series is convergent for that value of x, *i.e.*

$$A_0^2+A_1^2+A_2^2+A_3^2+\ldots=\frac{1}{2\pi}\int_0^\pi [\phi(e^{\iota\theta})+\phi(e^{-\iota\theta})]^2\,d\theta-A_0^2.$$

1126. Ex. Thus for the series $(1+x)^n$, n being a positive integer,

$$A_0^2+A_1^2+A_2^2+\ldots=\frac{1}{2\pi}\int_0^\pi [(1+e^{\iota\theta})^n+(1+e^{-\iota\theta})^n]^2\,d\theta-1$$

$$=\frac{1}{2\pi}\int_0^\pi\left(e^{\frac{\iota n\theta}{2}}+e^{-\frac{\iota n\theta}{2}}\right)^2\left(e^{\frac{\iota\theta}{2}}+e^{-\frac{\iota\theta}{2}}\right)^{2n}d\theta-1=\frac{2^{2n+1}}{\pi}\int_0^\pi\left(\cos\frac{n\theta}{2}\cos^n\frac{\theta}{2}\right)^2d\theta-1.$$

Similarly for the series $e^x=1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ldots$, we have

$$1^2+\left(\frac{1}{\underline{|1}}\right)^2+\left(\frac{1}{\underline{|2}}\right)^2+\left(\frac{1}{\underline{|3}}\right)^2+\ldots\ ad.\ inf.=\frac{1}{2\pi}\int_0^\pi\left(e^{e^{\iota\theta}}+e^{e^{-\iota\theta}}\right)^2d\theta-1$$

$$=\frac{2}{\pi}\int_0^\pi e^{2\cos\theta}\cos^2(\sin\theta)\,d\theta-1.$$

1127. Again we may express as a definite integral the sum of the first r terms of any series,

$$\phi(x)=A_0+A_1x+A_2x^2+A_3x^3+\dots \textit{ ad inf.}$$

For writing as before, $u=xe^{\iota\theta}$, $v=xe^{-\iota\theta}$,

$$\frac{\phi(u)+\phi(v)}{2}=A_0+A_1x\cos\theta+A_2x^2\cos 2\theta+A_3x^3\cos 3\theta+\dots$$

to an infinite number of terms.

Also $$\frac{\sin\frac{r\theta}{2}}{\sin\frac{\theta}{2}}\cos\frac{(r-1)\theta}{2}=1+\cos\theta+\cos 2\theta+\dots+\cos(r-1)\theta.$$

Multiply and integrate from 0 to π;

$$\therefore\ A_0+A_1x+A_2x^2+A_3x^3+\dots+A_{r-1}x^{r-1}$$
$$=\frac{2}{\pi}\int_0^\pi\frac{\phi(u)+\phi(v)}{2}\,\frac{\sin\frac{r\theta}{2}}{\sin\frac{\theta}{2}}\cos\frac{(r-1)\theta}{2}\,d\theta-A_0.$$

1128. If we take as our auxiliary series,

$$\frac{\sin\frac{r\theta}{2}}{\sin\frac{\theta}{2}}\cos\frac{2k+r-1}{2}\theta=\cos k\theta+\cos(k+1)\theta+\cos(k+2)\theta+\dots \text{ to } r \text{ terms},$$

we have

$$A_kx^k+A_{k+1}x^{k+1}+\dots+A_{k+r-1}x^{k+r-1}$$
$$=\frac{2}{\pi}\int_0^\pi\frac{\phi(u)+\phi(v)}{2}\,\frac{\sin\frac{r\theta}{2}}{\sin\frac{\theta}{2}}\cos\frac{2k+r-1}{2}\theta\,d\theta,$$

i.e. the sum of r terms of $\phi(x)$ starting from any particular term, $k>0$.

Obviously other modifications may be made. And provided $\phi(x)$ remains a convergent series when $x=1$, we may put 1 for x before the integration is performed if it be required to sum the several coefficients in any of the above cases.

1129. **Examples of Integrals derived from the Foregoing Principles.**

Since $$2^{2n}\cos^{2n}x=2\sum_{p=0}^{p=n-1}{}^{2n}C_p\cos(2n-2p)x+{}^{2n}C_n,$$

$$2^{2n+1}\cos^{2n+1}x=2\sum_{p=0}^{p=n}{}^{2n+1}C_p\cos(2n+1-2p)x,$$

$$(-1)^n2^{2n}\sin^{2n}x=2\sum_{p=0}^{p=n-1}(-1)^p\,{}^{2n}C_p\cos(2n-2p)x+(-1)^n\,{}^{2n}C_n,$$

and $$(-1)^n2^{2n+1}\sin^{2n+1}x=2\sum_{p=0}^{p=n}(-1)^p\,{}^{2n+1}C_p\sin(2n+1-2p)x,$$

we have, by aid of the previous article,

$$\int_0^\pi \cos^{2n} x \cos 2nx\, dx = \frac{\pi}{2^{2n}}, \quad \int_0^\pi \cos^{2n} x \cos (2n-2p) x\, dx = {}^{2n}C_p \frac{\pi}{2^{2n}},$$

$$\int_0^\pi \cos^{2n} x \cos rx\, dx = 0, \quad (r \neq 0),$$

where r is odd, or even and not lying within the range from $2n$ to $-2n$ inclusive. (A)

$$\int_0^\pi \cos^{2n+1} x \cos (2n+1) x\, dx = \frac{\pi}{2^{2n+1}},$$

$$\int_0^\pi \cos^{2n+1} x \cos (2n+1-2p) x\, dx = {}^{2n+1}C_p \cdot \frac{\pi}{2^{2n+1}},$$

$$\int_0^\pi \cos^{2n+1} x \cos rx\, dx = 0, \quad (r \neq 0),$$

where r is even, or odd and not lying within the range from $2n+1$ to $-(2n+1)$ inclusive. (B)

$$\int_0^\pi \sin^{2n} x \cos 2nx\, dx = (-1)^n \frac{\pi}{2^{2n}},$$

$$\int_0^\pi \sin^{2n} x \cos (2n-2p) x\, dx = (-1)^{n+p}\, {}^{2n}C_p \frac{\pi}{2^{2n}},$$

$$\int_0^\pi \sin^{2n} x \cos rx\, dx = 0, \quad (r \neq 0),$$

where r is odd, or even and not lying within the range from $2n$ to $-2n$ inclusive. (C)

$$\int_0^\pi \sin^{2n+1} x \sin (2n+1) x\, dx = (-1)^n \frac{\pi}{2^{2n+1}},$$

$$\int_0^\pi \sin^{2n+1} x \sin (2n+1-2p) x\, dx = (-1)^{n+p}\, {}^{2n+1}C_p \frac{\pi}{2^{2n+1}},$$

$$\int_0^\pi \sin^{2n+1} x \sin rx\, dx = 0,$$

where r is even, or odd and not lying within the range from $2n+1$ to $-(2n+1)$ inclusive. (D)

All six statements in (A) and (B) may be summed up in the result

$$\int_0^\pi \cos^\lambda x \cos \mu x\, dx = {}^\lambda C_{\frac{\lambda-\mu}{2}} \frac{\pi}{2^\lambda}, \quad (\mu \neq 0),$$

where ${}^\lambda C_{\frac{\lambda-\mu}{2}}$ is the number of combinations of λ things $\frac{\lambda-\mu}{2}$ at a time and is unity when $\mu = \lambda$, or zero if $\frac{\lambda-\mu}{2}$ be not a positive integer.

The three statements in (C) may be similarly summed up as

$$\int_0^\pi \sin^\lambda x \cos \mu x\, dx = {}^\lambda C_{\frac{\lambda-\mu}{2}} \frac{\pi}{2^\lambda} (-1)^{\frac{2\lambda-\mu}{2}} \quad (\lambda \text{ even}, \ \mu \neq 0),$$

and the three statements in (D) may be summed up as

$$\int_0^\pi \sin^\lambda x \sin \mu x\, dx = {}^\lambda C_{\frac{\lambda-\mu}{2}} \frac{\pi}{2^\lambda} (-1)^{\frac{2\lambda-\mu-1}{2}} \quad (\lambda \text{ odd}).$$

1130. Similarly, (1) $2^{2n}\int_0^\pi \cos^{2n}x\sin 2sx\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n-1}{}^{2n}C_p\cos 2(n-p)x\sin 2sx+{}^{2n}C_n\sin 2sx\right]dx$$

$=0$, by Art. 1121 (iii).

(2) $2^{2n}\int_0^\pi \cos^{2n}x\sin(2s+1)x\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n-1}{}^{2n}C_p\cos 2(n-p)x\sin(2s+1)x+{}^{2n}C_n\sin(2s+1)x\right]dx$$

$$=2\sum_{p=0}^{p=n-1}{}^{2n}C_p\cdot\frac{2(2s+1)}{(2s+1)^2-(2n-2p)^2}+{}^{2n}C_n\cdot\frac{2}{2s+1}.$$

(3) $2^{2n+1}\int_0^\pi \cos^{2n+1}x\sin 2sx\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n}{}^{2n+1}C_p\cos(2n+1-2p)x\sin 2sx\right]dx$$

$$=2\sum_{p=0}^{p=n}{}^{2n+1}C_p\frac{2\,.\,2s}{(2s)^2-(2n+1-2p)^2}.$$

(4) $2^{2n+1}\int_0^\pi \cos^{2n+1}x\sin(2s+1)x\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n}{}^{2n+1}C_p\cos(2n+1-2p)x\sin(2s+1)x\right]dx$$

$=0.$

(5) $(-1)^n 2^{2n}\int_0^\pi \sin^{2n}x\sin 2sx\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n-1}(-1)^p\,{}^{2n}C_p\cos(2n-2p)x\sin 2sx+(-1)^n\,{}^{2n}C_n\sin 2sx\right]dx$$

$=0.$

(6) $(-1)^n 2^{2n}\int_0^\pi \sin^{2n}x\sin(2s+1)x\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n-1}(-1)^p\,{}^{2n}C_p\cos(2n-2p)x\sin(2s+1)x+(-1)^n\,{}^{2n}C_n\sin(2s+1)x\right]dx$$

$$=2\sum_{p=0}^{p=n-1}(-1)^p\,{}^{2n}C_p\frac{2(2s+1)}{(2s+1)^2-(2n-2p)^2}+(-1)^n\,{}^{2n}C_n\frac{2}{2s+1}.$$

(7) $(-1)^n 2^{2n+1}\int_0^\pi \sin^{2n+1}x\cos 2sx\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n}(-1)^p\,{}^{2n+1}C_p\sin(2n+1-2p)x\cos 2sx)\right]dx$$

$$=2\sum_{p=0}^{p=n}(-1)^p\,{}^{2n+1}C_p\frac{2(2n+1-2p)}{(2n+1-2p)^2-(2s)^2}.$$

(8) $(-1)^n 2^{2n+1}\int_0^\pi \sin^{2n+1}x\cos(2s+1)x\,dx$

$$=\int_0^\pi\left[2\sum_{p=0}^{p=n}(-1)^p\,{}^{2n+1}C_p\sin(2n+1-2p)x\cos(2s+1)x\right]dx$$

$=0.$

Thus we have considered in Arts. 1129 and 1130 all cases of

$$\int_0^\pi \cos^\lambda x \cos \mu x\, dx, \qquad \int_0^\pi \cos^\lambda x \sin \mu x\, dx,$$
$$\int_0^\pi \sin^\lambda x \cos \mu x\, dx, \qquad \int_0^\pi \sin^\lambda x \sin \mu x\, dx,$$

for which λ and μ are integers, λ being positive.

1131. The eight expressions

$$\cos^{2n} x \cos 2sx, \quad \cos^{2n+1} x \cos(2s+1)x, \quad \cos^{2n} x \sin(2s+1)x, \quad \cos^{2n+1} x \sin 2sx,$$
$$\sin^{2n} x \cos 2sx, \quad \sin^{2n+1} x \cos 2sx, \quad \sin^{2n} x \sin(2s+1)x, \quad \sin^{2n+1} x \sin(2s+1)x,$$

have the same values when we put $\pi - x$ in place of x.

But the eight expressions

$$\cos^{2n} x \cos(2s+1)x, \quad \cos^{2n+1} x \cos 2sx, \quad \cos^{2n} x \sin 2sx, \quad \cos^{2n+1} x \sin(2s+1)x$$
$$\sin^{2n} x \cos(2s+1)x, \quad \sin^{2n+1} x \cos(2s+1)x, \quad \sin^{2n} x \sin 2sx, \quad \sin^{2n+1} x \sin 2sx,$$

change sign if we put $\pi - x$ in place of x.

From these considerations the integrals from 0 to $\frac{\pi}{2}$ of the eight in the first group are each half the result from 0 to π.

And the integrals of the eight in the second group from 0 to π all vanish. This is in conformity with the results found.

The integrals from 0 to $\frac{\pi}{2}$ of the eight in the second group must therefore be found by another method, viz. the reduction formulae of Arts. 249-257.

1132. We have also, by putting for $\sin^{2n} x$ its equivalent in a series of cosines of even multiples of x, say $A_0 + \sum_1^n A_{2r} \cos 2rx$,

$$\int_0^\pi x \sin^{2n} x\, dx = \int_0^\pi x(A_0 + A_2 \cos 2x + A_4 \cos 4x + \ldots + A_{2n} \cos 2nx)\, dx;$$

and therefore integrating by parts,

$$\int_0^\pi x \sin^{2n} x\, dx = \left[x\left\{A_0 x + A_2 \frac{\sin 2x}{2} + A_4 \frac{\sin 4x}{4} + \ldots + A_{2n} \frac{\sin 2nx}{2n}\right\}\right]_0^\pi$$
$$- \left[A_0 \frac{x^2}{2} - A_2 \frac{\cos 2x}{2^2} - \ldots\right]_0^\pi$$
$$= A_0\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{\pi^2}{2} A_0 = \frac{\pi^2}{2} \frac{1}{2^{2n}}\, {}^{2n}C_n = \frac{\pi^2 (2n)!}{2^{2n+1}(n!)^2},$$

with other similar results.

This may be obtained otherwise, thus:

$$\int_0^\pi x \sin^{2n} x\, dx = -\int_\pi^0 (\pi - x) \sin^{2n} x\, dx = \int_0^\pi (\pi - x) \sin^{2n} x\, dx;$$

$$\therefore \int_0^\pi x \sin^{2n} x\, dx = \frac{\pi}{2} \int_0^\pi \sin^{2n} x\, dx$$
$$= \pi \frac{2n-1}{2n} \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} = \pi^2 \frac{(2n)!}{2^{2n+1}(n!)^2}$$

1133. The former process may be extended to find $\int_0^\pi x^p \sin^{2n} x\,dx$, where p and n are positive integers.

Thus

$$\int_0^\pi x^p \sin^{2n} x\,dx = \int_0^\pi x^p \left(A_0 + \sum_1^n A_{2r}\cos 2rx\right)dx = A_0\frac{\pi^{p+1}}{p+1} + \int_0^\pi x^p \sum_1^n A_{2r}\cos 2rx\,dx$$

$$= A_0\frac{\pi^{p+1}}{p+1} + \left[x^p \sum_1^n \frac{A_{2r}}{2r}\sin 2rx - px^{p-1}\left(-\sum_1^n \frac{A_{2r}}{(2r)^2}\cos 2rx\right)\right.$$

$$+p(p-1)x^{p-2}\left(-\sum_1^n \frac{A_{2r}}{(2r)^3}\sin 2rx\right) - p(p-1)(p-2)x^{p-3}\sum_1^n \frac{A_{2r}}{(2r)^4}\cos 2rx + \dots$$

$$\left. + (-1)^p p!\sum_1^n \frac{A_{2r}}{(2r)^{p+1}}\cos\left(2rx - \overline{p+1}\frac{\pi}{2}\right)\right]_0^\pi$$

$$= A_0\frac{\pi^{p+1}}{p+1} + p\pi^{p-1}\sum_1^n \frac{A_{2r}}{(2r)^2} - p(p-1)(p-2)\pi^{p-3}\sum_1^n \frac{A_{2r}}{(2r)^4} + \dots,$$

and p being integral and positive the series will terminate.

Also

$$A_0 = \frac{1}{2^{2n}}\,{}^{2n}C_n,\quad A_2 = -\frac{1}{2^{2n-1}}\,{}^{2n}C_{n-1},\quad A_4 = \frac{1}{2^{2n-1}}\,{}^{2n}C_{n-2},\ \text{etc.},\quad A_{2r} = \frac{(-1)^r}{2^{2n-1}}\,{}^{2n}C_{n-r}.$$

Hence

$$\int_0^\pi x^p \sin^{2n} x\,dx = \frac{1}{2^{2n-1}}\left\{\frac{1}{2}\frac{\pi^{p+1}}{p+1}\,{}^{2n}C_n + p\pi^{p-1}\sum_1^n \frac{(-1)^r}{(2r)^2}\,{}^{2n}C_{n-r}\right.$$

$$\left. - p(p-1)(p-2)\pi^{p-3}\sum_1^n \frac{(-1)^r}{(2r)^4}\,{}^{2n}C_{n-r} + \dots\right\}.$$

We may obtain similar results for

$$\int_0^\pi x^p \sin^{2n+1} x\,dx,\qquad \int_0^\pi x^p \cos^{2n} x\,dx,\qquad \int_0^\pi x^p \cos^{2n+1} x\,dx,$$

or in fact for any integral of form $\int_0^\pi x^p F(x)\,dx$, where $F(x)$ can be expressed as a series of sines or cosines of integral multiples of x. For instance,

$$\int_0^\pi x^p \cos nx \frac{\sin(n+1)x}{\sin x}\,dx = \int_0^\pi x^p(1 + \cos 2x + \cos 4x + \dots + \cos 2nx)\,dx$$

$$= \frac{\pi^{p+1}}{p+1} + \left[x^p\sum_1^n \frac{\sin 2rx}{2r} - px^{p-1}\sum_1^n \frac{(-1)\cos 2rx}{(2r)^2} + p(p-1)x^{p-2}\sum_1^n \frac{(-1)\sin 2rx}{(2r)^3} - \dots\right.$$

$$\left. + (-1)^p p!\sum_1^n \frac{\cos\left(2rx - \overline{p+1}\frac{\pi}{2}\right)}{(2r)^{p+1}}\right]_0^\pi$$

$$= \frac{\pi^{p+1}}{p+1} + p\frac{\pi^{p-1}}{2^2}\sum_1^n \frac{1}{r^2} - p(p-1)(p-2)\frac{\pi^{p-3}}{2^4}\sum_1^n \frac{1}{r^4} + \dots.$$

1134. **Results derivable from Well-known Series.**

Many well-known series are established in books on Trigonometry whose terms involve sines or cosines of integral multiples of θ. And such series furnish many definite integrals by the application of the rules of Art. 1121.

For convenience we quote a number of the more important :

1. $\dfrac{1-a^2}{1-2a\cos\theta+a^2} = 1+2a\cos\theta+2a^2\cos 2\theta+2a^3\cos 3\theta+\dots,\qquad a^2<1,$

or $\quad = -1-\dfrac{2}{a}\cos\theta-\dfrac{2}{a^2}\cos 2\theta-\dfrac{2}{a^3}\cos 3\theta-\dots,\qquad a^2>1,$

2. $\dfrac{\sin\theta}{1-2a\cos\theta+a^2} = \sin\theta+a\sin 2\theta+a^2\sin 3\theta+\dots,\qquad a^2<1,$

or $\quad = \dfrac{1}{a^2}\sin\theta+\dfrac{1}{a^3}\sin 2\theta+\dfrac{1}{a^4}\sin 3\theta+\dots,\qquad a^2>1,$

3. $\dfrac{1-a\cos\theta}{1-2a\cos\theta+a^2} = 1+a\cos\theta+a^2\cos 2\theta+a^3\cos 3\theta+\dots,\qquad a^2<1,$

or $\quad = -\dfrac{1}{a}\cos\theta-\dfrac{1}{a^2}\cos 2\theta-\dfrac{1}{a^3}\cos 3\theta-\dots,\qquad a^3>1,$

4. $\dfrac{\cos\theta}{1-2a\cos\theta+a^2} = \dfrac{a}{1-a^2}+\dfrac{1+a^2}{1-a^2}(\cos\theta+a\cos 2\theta+a^2\cos 3\theta+\dots),\qquad a^2<1,$

or $\quad = \dfrac{1}{a(a^2-1)}+\dfrac{1}{a^2}\dfrac{a^2+1}{a^2-1}\left(\cos\theta+\dfrac{1}{a}\cos 2\theta+\dfrac{1}{a^2}\cos 3\theta+\dots\right),\qquad a^2>1.$

5. $\log(1-2a\cos\theta+a^2) = -2(a\cos\theta+\tfrac{1}{2}a^2\cos 2\theta+\tfrac{1}{3}a^3\cos 3\theta+\dots),\qquad a^2<1,$

or $\quad = \log a^2-2\left(\dfrac{1}{a}\cos\theta+\dfrac{1}{2a^2}\cos 2\theta+\dfrac{1}{3a^3}\cos 3\theta+\dots\right),\qquad a^2>1.$

6. $\tan^{-1}\dfrac{a\sin\theta}{1-a\cos\theta} = a\sin\theta+\tfrac{1}{2}a^2\sin 2\theta+\tfrac{1}{3}a^3\sin 3\theta+\dots,\qquad a^2<1,$

or $\quad = \pi-\theta-\left(\dfrac{1}{a}\sin\theta+\dfrac{1}{2a^2}\sin 2\theta+\dfrac{1}{3a^3}\sin 3\theta+\dots\right),\qquad a^2>1$

and in each of these cases a may be changed to $-a$.

We also have

7. $\log\left(2\cos\dfrac{\theta}{2}\right) = \cos\theta-\dfrac{1}{2}\cos 2\theta+\dfrac{1}{3}\cos 3\theta-\dots,\quad (-\pi<\theta<\pi).$

8. $\log\left(2\sin\dfrac{\theta}{2}\right) = -\cos\theta-\dfrac{1}{2}\cos 2\theta-\dfrac{1}{3}\cos 3\theta-\dots,\quad (0<\theta<2\pi).$

9. $\log(2\sin\theta) = -\cos 2\theta-\tfrac{1}{2}\cos 4\theta-\tfrac{1}{3}\cos 6\theta-\dots,\quad (0<\theta<\pi).$

10. $\dfrac{\theta}{2} = \sin\theta-\dfrac{1}{2}\sin 2\theta+\dfrac{1}{3}\sin 3\theta-\dots,\quad (-\pi<\theta<\pi).$

11. $\dfrac{\pi-\theta}{2} = \sin\theta+\dfrac{1}{2}\sin 2\theta+\dfrac{1}{3}\sin 3\theta+\dots,\quad (0<\theta<\pi).$

12. $\dfrac{\pi}{4} = \sin\theta+\dfrac{1}{3}\sin 3\theta+\dfrac{1}{5}\sin 5\theta+\dots,\quad (0<\theta<\pi).$

It will be noted that if $n<1$,

$$\log(1-n\cos\theta) \text{ is a case of } \log\left(1-\frac{2a}{1+a^2}\cos\theta\right),$$

the value of a being given by $\quad 1+a^2=\dfrac{2a}{n},$

or putting $a=\tan\dfrac{\alpha}{2},\qquad n=\sin\alpha.$

1135. Derivation of Other Series.

Other series may be obtained by differentiation with regard to θ.

Let $u \equiv 1-2a\cos\theta+a^2$.

Taking the series

$$\frac{1-a^2}{u}=1+2a\cos\theta+2a^2\cos 2\theta+2a^3\cos 3\theta+\dots \quad \dots\dots\dots(1),\ a^2<1,$$

and
$$\frac{a\sin\theta}{u}=a\sin\theta+a^2\sin 2\theta+a^3\sin 3\theta+\dots \quad \dots\dots\dots(2),\ a^2<1.$$

Differentiate (1) with regard to θ,

$$\frac{2a(1-a^2)\sin\theta}{u^2}=2a\sin\theta+4a^2\sin 2\theta+6a^3\sin 3\theta+\dots, \quad a^2<1,$$

i.e. $$(1-a^2)\frac{\sin\theta}{u^2}=\sin\theta+2a\sin 2\theta+3a^2\sin 3\theta+\dots+na^{n-1}\sin n\theta+\dots \ (3),\ a^2<1,$$

and differentiating (2) with regard to θ,

$$\frac{(1+a^2)\cos\theta-2a}{u^2}=\cos\theta+2a\cos 2\theta+3a^2\cos 3\theta+\dots+na^{n-1}\cos n\theta+\dots \ (4),\ a^2<1.$$

Equation (1) may be written,

$$\frac{(1-a^2)(1-2a\cos\theta+a^2)}{u^2}=1+2a\cos\theta+2a^2\cos 2\theta+\dots+2a^n\cos n\theta+\dots \quad \dots\dots(5),\ a^2<1.$$

Multiply (4) and (5) by $2a(1-a^2)$ and $1+a^2$ respectively, and add, then

$$\frac{(1-a^2)^3}{u^2}=1+a^2+4a\cos\theta+2a^2(3-a^2)\cos 2\theta+2a^3(4-2a^2)\cos 3\theta+\dots$$
$$+2a^n\{n(1-a^2)+(1+a^2)\}\cos n\theta+\dots \quad \dots\dots\dots(6),\ a^2<1,$$

and so on with further differentiations.

And similarly when a^2 is >1, we have

$$\frac{a^2-1}{u}=1+\frac{2}{a}\cos\theta+\frac{2}{a^2}\cos 2\theta+\frac{2}{a^3}\cos 3\theta+\dots, \quad \dots\dots\dots(1')$$

$$\frac{a\sin\theta}{u}=\frac{1}{a}\sin\theta+\frac{1}{a^2}\sin 2\theta+\frac{1}{a^3}\sin 3\theta+\dots. \quad \dots\dots\dots(2')$$

Differentiate (1′) with regard to θ,

$$\frac{2a(a^2-1)\sin\theta}{u^2}=\frac{2}{a}\sin\theta+\frac{4}{a^2}\sin 2\theta+\frac{6}{a^3}\sin 3\theta+\dots,$$

or
$$\frac{(a^2-1)\sin\theta}{u^2}=\frac{1}{a^2}\sin\theta+\frac{2}{a^3}\sin 2\theta+\frac{3}{a^4}\sin 3\theta+\dots, \quad \dots\dots\dots(3')$$

and differentiating (2′) with regard to θ,

$$\frac{(1+a^2)\cos\theta-2a}{u^2}=\frac{1}{a^2}\cos\theta+\frac{2}{a^3}\cos 2\theta+\frac{3}{a^4}\cos 3\theta+\dots, \quad \dots\dots\dots(4')$$

and equation (1′) may be written,

$$\frac{(a^2-1)(1-2a\cos\theta+a^2)}{u^2}=1+\frac{2}{a}\cos\theta+\frac{2}{a^2}\cos 2\theta+\frac{2}{a^3}\cos 3\theta+\dots. \quad \dots\dots(5')$$

Multiply (4′) and (5′) by $2a(a^2-1)$ and a^2+1 respectively, and add, then

$$\frac{(a^2-1)^3}{u^2}=a^2+1+4a\cos\theta+\frac{2(3a^2-1)}{a^2}\cos 2\theta+\ldots+\frac{2\{n(a^2-1)+(a^2+1)\}}{a^n}\cos n\theta+\ldots \quad \ldots(6')$$

etc.

1136. Successive Derivation of Further Series.

Again we have

$$\frac{d^2}{d\theta^2}\frac{1}{(A+B\cos\theta)^m}=\frac{d}{d\theta}\frac{mB\sin\theta}{(A+B\cos\theta)^{m+1}}=\frac{mB\cos\theta(A+B\cos\theta)+m(m+1)B^2(1-\cos^2\theta)}{(A+B\cos\theta)^{m+2}}$$

$$=\frac{\lambda+\mu(A+B\cos\theta)+\nu(A+B\cos\theta)^2}{(A+B\cos\theta)^{m+2}}, \text{ say,}$$

where $\left.\begin{aligned}\lambda+\mu A+\nu A^2&=m(m+1)B^2,\\ \mu B+2\nu AB&=mAB,\\ \nu B^2&=-m^2B^2,\end{aligned}\right\}$ giving $\left.\begin{aligned}\lambda&=-m(m+1)(A^2-B^2),\\ \mu&=m(2m+1)A,\\ \nu&=-m^2,\end{aligned}\right\}$

i.e. $\dfrac{m(m+1)(A^2-B^2)}{u^{m+2}}-\dfrac{m(2m+1)A}{u^{m+1}}+\dfrac{m^2}{u^m}=-\dfrac{d^2}{d\theta^2}\dfrac{1}{u^m}$, where $u=A+B\cos\theta$.

Hence when series for $\dfrac{1}{u^m}$ and $\dfrac{1}{u^{m+1}}$ in terms of cosines of integral multiples of θ have been found, a series of the same kind can be deduced for $\dfrac{1}{u^{m+2}}$.

Thus, putting $A=1+a^2$ and $B=-2a$, we have

$$\frac{m(m+1)(1-a^2)^2}{u^{m+2}}=\frac{m(2m+1)(1+a^2)}{u^{m+1}}-\frac{m^2}{u^m}-\frac{d^2}{d\theta^2}\frac{1}{u^m}. \quad \ldots\ldots\ldots\ldots(1)$$

Putting $m=1$ and taking the case $a^2<1$,

$$\frac{1.2(1-a^2)^2}{u^3}=\frac{1.3(1+a^2)}{u^2}-\frac{1}{u}-\frac{d^2}{d\theta^2}\left(\text{expansion of }\frac{1}{u}\right)$$

$$=\frac{3(1+a^2)}{(1-a^2)^3}\left[(1+a^2)+\sum_1^\infty 2a^n\{(n+1)-(n-1)a^2\}\cos n\theta\right]$$

$$-\frac{1}{1-a^2}\left[1+\sum_1^\infty 2a^n\cos n\theta\right]$$

$$+\frac{1}{1-a^2}\left[\sum_1^\infty 2n^2a^n\cos n\theta\right]$$

$$=\frac{2(1+4a^2+a^4)}{(1-a^2)^3}+\sum_1^\infty 2a^n\left[\frac{3(1+a^2)}{(1-a^2)^3}\{(n+1)-(n-1)a^2\}+\frac{n^2-1}{1-a^2}\right]\cos n\theta,$$

i.e. $$\frac{(1-a^2)^5}{u^3}=(1+4a^2+a^4)+\sum_1^\infty A_n\cos n\theta,$$

where $A_n=a^n[(1-a^2)^2n^2+3(1-a^4)n+2(1+4a^2+a^4)]$.

And further applications of the formula (1), viz. putting $m=2, 3$, etc., will furnish successively the series for $\dfrac{1}{u^4}$, $\dfrac{1}{u^5}$, etc.; and similarly in the case when $a^2>1$.

1137. Moreover the differentiation of any one of these series furnishes another, *e.g.* $\dfrac{1}{u^{m-1}}$ furnishes the series for $\dfrac{\sin\theta}{u^m}$ in terms of series of sines of integral multiples of θ, as was seen in equation (3) of Art. 1135.

Thus, since

$$\frac{(1-a^2)^3}{u^2}=1+a^2+\sum_1^\infty 2a^n[n(1-a^2)+(1+a^2)]\cos n\theta, \quad a^2<1,$$

or
$$\frac{(a^2-1)^3}{u^2}=a^2+1+\sum_1^\infty \frac{2}{a^n}[n(a^2-1)+(a^2+1)]\cos n\theta, \quad a^2>1,$$

we have, by differentiating,

$$\frac{\sin\theta}{u^3}=\sum_1^\infty \frac{na^{n-1}}{2(1-a^2)^3}[n(1-a^2)+(1+a^2)]\sin n\theta, \quad a^2<1,$$

or
$$=\sum_1^\infty \frac{n}{2(a^2-1)^3}\frac{1}{a^{n+1}}[n(a^2-1)+(a^2+1)]\sin n\theta, \quad a^2>1,$$

and so on.

Again a series for $\frac{\cos\theta}{u^m}$ may be found in terms of the series for $\frac{1}{u^m}$ and $\frac{1}{u^{m-1}}$.

For
$$\frac{\cos\theta}{u^m}=\frac{1}{2a}\frac{1+a^2-u}{u^m}=\frac{1+a^2}{2a}\cdot\frac{1}{u^m}-\frac{1}{2a}\frac{1}{u^{m-1}}.$$

1138. Other powers of $\sin\theta$ or $\cos\theta$ in the numerator may be readily arranged for.

Thus, since $\frac{\sin\theta}{u^2}=\frac{1}{1-a^2}\sum_1^\infty na^{n-1}\sin n\theta$, $(a^2<1)$, we have

$$\frac{\sin^2\theta}{u^2}=\frac{1}{2(1-a^2)}\sum_1^\infty na^{n-1}2\sin\theta\sin n\theta$$
$$=\frac{1}{2(1-a^2)}\sum_1^\infty na^{n-1}\{\cos(n-1)\theta-\cos(n+1)\theta\}$$
$$=\frac{1}{2}\frac{1}{1-a^2}[1+2a\cos\theta+(3a^2-1)\cos 2\theta+(4a^3-2a)\cos 3\theta$$
$$+(5a^4-3a^2)\cos 4\theta+\ldots], \quad a^2<1.$$

And if $a^2>1$, a similar result may be obtained. These results are mainly interesting from the definite integrals which may be obtained from them by the aid of the results of Art. 1121; and to this matter we now turn.

1139. Definite Integrals immediately derivable.

By the application of the rules of Art. 1121 to the series of Art. 1134, we have at once the following definite integrals. Put $1-2a\cos\theta+a^2\equiv u$, and consider in each case n to be a positive integer.

$$\left.\begin{aligned}&\left.\begin{aligned}&(1)\quad \int_0^\pi\frac{d\theta}{u}=\frac{\pi}{1-a^2}\\&(2)\quad \int_0^\pi\frac{\cos n\theta}{u}d\theta=\frac{\pi}{1-a^2}a^n\end{aligned}\right\}a^2<1\\&\left.\begin{aligned}&(1')\quad \int_0^\pi\frac{d\theta}{u}=\frac{\pi}{a^2-1}\\&(2')\quad \int_0^\pi\frac{\cos n\theta}{u}d\theta=\frac{\pi}{a^2-1}\frac{1}{a^n}\end{aligned}\right\}a^2>1\end{aligned}\right\}\text{from Series 1.}$$

$$\left.\begin{array}{l}
\left.\begin{aligned}
&(3)\ \int_0^\pi \frac{\sin^2\theta}{u}\,d\theta=\frac{\pi}{2}\\
&(4)\ \int_0^\pi \frac{\sin\theta\sin n\theta}{u}\,d\theta=\frac{\pi}{2}\,a^{n-1}
\end{aligned}\right\}a^2<1\\
\left.\begin{aligned}
&(3')\ \int_0^\pi \frac{\sin^2\theta}{u}=\frac{\pi}{2a^2}\\
&(4')\ \int_0^\pi \frac{\sin\theta\sin n\theta}{u}=\frac{\pi}{2}\,\frac{1}{a^{n+1}}
\end{aligned}\right\}a^2>1
\end{array}\right\}\text{from Series 2.}$$

$$\left.\begin{array}{l}
\left.\begin{aligned}
&(5)\ \int_0^\pi \frac{1-a\cos\theta}{u}\,d\theta=\pi\\
&(6)\ \int_0^\pi \frac{(1-a\cos\theta)\cos n\theta}{u}\,d\theta=\frac{\pi}{2}\,a^n\ \ (n>0)
\end{aligned}\right\}a^2<1\\
\left.\begin{aligned}
&(5')\ \int_0^\pi \frac{1-a\cos\theta}{u}\,d\theta=0\\
&(6')\ \int_0^\pi \frac{(1-a\cos\theta)\cos n\theta}{u}\,d\theta=-\frac{\pi}{2}\cdot\frac{1}{a^n}\ \ (n>0)
\end{aligned}\right\}a^2>1
\end{array}\right\}\text{from Series 3.}$$

$$\left.\begin{array}{l}
\left.\begin{aligned}
&(7)\ \int_0^\pi \frac{\cos\theta}{u}\,d\theta=\frac{\pi a}{1-a^2}\\
&(8)\ \int_0^\pi \frac{\cos\theta\cos n\theta}{u}\,d\theta=\frac{\pi}{2}\,\frac{1+a^2}{1-a^2}\,a^{n-1}\ \ (n>0)
\end{aligned}\right\}a^2<1\\
\left.\begin{aligned}
&(7')\ \int_0^\pi \frac{\cos\theta}{u}\,d\theta=\frac{\pi}{a^2-1}\cdot\frac{1}{a}\\
&(8')\ \int_0^\pi \frac{\cos\theta\cos n\theta}{u}\,d\theta=\frac{\pi}{2}\,\frac{a^2+1}{a^2-1}\,\frac{1}{a^{n+1}}\ \ (n>0)
\end{aligned}\right\}a^2<1
\end{array}\right\}\text{from Series 4.}$$

$$\left.\begin{array}{l}
\left.\begin{aligned}
&(9)\ \int_0^\pi \log u\,d\theta &&=0^*\\
&(10)\ \int_0^\pi \cos n\theta\log u\,d\theta &&=-\frac{\pi}{n}\,a^n\dagger
\end{aligned}\right\}a^2<1\\
\left.\begin{aligned}
&(9')\ \int_0^\pi \log u\,d\theta &&=\pi\log a^2{}^*\\
&(10')\ \int_0^\pi \cos n\theta\log u\,d\theta &&=-\frac{\pi}{n}\,\frac{1}{a^n}\dagger
\end{aligned}\right\}a^2>1
\end{array}\right\}\text{from Series 5.}$$

$$(11)\ \int_0^\pi \log u\,d\theta=0^*,\ \text{when } a=1,\quad \text{from Series 9.}$$

$$\left.\begin{aligned}
&(12)\ \int_0^\pi \sin n\theta\tan^{-1}\frac{a\sin\theta}{1-a\cos\theta}\,d\theta=\frac{\pi}{2n}\,a^n,\quad a^2<1\\
&(13)\ \int_0^\pi \sin n\theta\tan^{-1}\frac{\sin\theta}{a-\cos\theta}\,d\theta=\frac{\pi}{2n}\,\frac{1}{a^n},\quad a^2>1
\end{aligned}\right\}\text{from Series 6.}$$

* Poisson, *Journal de l'École Polytechnique*, xvii.

† Legendre, *Exercices*, vol. ii., p. 123.

(14) $\int_0^\pi \cos n\theta \log\left(2\cos\frac{\theta}{2}\right)d\theta = (-1)^{n-1}\frac{\pi}{2n}$, from Series 7.

(15) $\int_0^\pi \cos n\theta \log\left(2\sin\frac{\theta}{2}\right)d\theta = -\frac{\pi}{2n}$, from Series 8.

Illustrative Examples.

1140. Denoting $1-2a\cos\theta+a^2$ by u:

1. Deduce from $\int_0^\pi \log u\, d\theta = 0$ or $\pi\log a^2$, as a^2 is $<$ or $>$ 1, by integration by parts,

$$\int_0^\pi \frac{\theta\sin\theta}{u}d\theta = \frac{\pi}{2a}\log(1+a)^2 \quad (a^2<1)$$

or

$$= \frac{\pi}{2a}\log\left(1+\frac{1}{a}\right)^2 \quad (a^2<1).$$

2. Deduce from Series 3 and 3′, Art. 1135,

$$\int_0^\pi \frac{\sin\theta}{u^2}d\theta = \frac{2}{(1-a^2)^2},\quad (a^2<1),\ \text{or}\ \frac{2}{(a^2-1)^2},\quad (a^2>1).$$

3. Show by direct integration that

$$\int_0^\pi \frac{\sin\theta}{u^n}d\theta = \frac{1}{2a(n-1)}\left\{\frac{1}{(a-1)^{2(n-1)}} - \frac{1}{(a+1)^{2(n+1)}}\right\} \quad (n\neq 1),$$

$$\int_0^\pi \frac{\sin\theta}{u}d\theta = \frac{1}{a}\log\frac{1+a}{1-a} \quad (a^2<1)$$

or

$$= \frac{1}{a}\log\frac{a+1}{a-1} \quad (a^2>1).$$

4. Prove that $\int_0^\pi \frac{\sin\theta\sin n\theta}{u^2}d\theta = \frac{n\pi}{2}\frac{a^{n-1}}{1-a^2} \quad (a^2<1)$.

or

$$= \frac{n\pi}{2}\frac{a^{-n-1}}{a^2-1} \quad (a^2>1).$$

5. Prove that $\int_0^\pi \frac{d\theta}{u^3} = \pi\frac{1+4a^2+a^4}{(1-a^2)^5} \quad (a^2<1)$.

6. Prove that

$$\int_0^\pi \frac{\cos n\theta}{u^3}d\theta = \frac{\pi}{2}\frac{a^n}{(1-a^2)^5}\{(1-a^2)^2n^2+3(1-a^4)n+2(1+4a^2+a^4)\} \quad (a^2<1).$$

7. From the formulae of Art. 1137, deduce

$$\int_0^\pi \frac{\sin^2\theta}{u^3}d\theta = \frac{\pi}{2}\frac{1}{(1-a^2)^3} \quad (a^2<1)$$

or

$$= \frac{\pi}{2}\frac{1}{(a^2-1)^3} \quad (a^2>1).$$

$$\int_0^\pi \frac{\sin\theta\sin n\theta}{u^3}d\theta = \frac{\pi}{4}\frac{na^{n-1}}{(1-a^2)^3}[n(1-a^2)+(1+a^2)] \quad (a^2<1)$$

or

$$= \frac{\pi}{4}\frac{na^{-n-1}}{(a^2-1)^3}[n(a^2-1)+(a^2+1)] \quad (a^2>1).$$

1141. **Series for Evaluation when the Integral is *not* expressible in Finite Terms.**

Again we may obtain the values of many definite integrals of this class in the form of series which, though they may not be capable of summation, will nevertheless serve for their numerical calculation.

For instance, $\int_0^\pi \sin 2\theta \log(1-2a\cos\theta+a^2)\,d\theta \quad (a^2<1)$

$$=-2\int_0^\pi \sin 2\theta\left(a\cos\theta+\frac{1}{2}a^2\cos 2\theta+\frac{1}{3}a^3\cos 3\theta+\dots\right)d\theta$$

$$=-2\left[\frac{4a}{2^2-1^2}+\frac{1}{3}\cdot\frac{4a^3}{2^2-3^2}+\frac{1}{5}\cdot\frac{4a^5}{2^2-5^2}+\dots\right]$$

$$=8\left[\frac{a}{(-1)1.3}+\frac{a^3}{1.3.5}+\frac{a^5}{3.5.7}+\frac{a^7}{5.7.9}+\frac{a^9}{7.9.11}+\dots\right]$$

1142. Again, since $\sin(p+1)\theta-\sin(p-1)\theta=2\sin\theta\cos p\theta$ we have

$$\int_0^\pi \frac{\sin(p+1)\theta}{\sin\theta}\,d\theta-\int_0^\pi \frac{\sin(p-1)\theta}{\sin\theta}\,d\theta=2\int_0^\pi \cos p\theta\,d\theta=0,$$

when p is integral.

That is, putting $u_p=\int_0^\pi \frac{\sin p\theta}{\sin\theta}\,d\theta$, we have

$$u_{p+1}=u_{p-1}=u_{p-3}=\text{etc.},$$

and $u_1=\int_0^\pi \frac{\sin\theta}{\sin\theta}\,d\theta=\pi,\quad u_2=\int_0^\pi \frac{\sin 2\theta}{\sin\theta}\,d\theta=\int_0^\pi 2\cos\theta\,d\theta=0;$

$$\therefore\ u_{2n}=0,\quad u_{2n+1}=\pi.$$

Again, p and q being integral,

$$\int_0^\pi \frac{\sin p\theta}{\sin\theta}\cos q\theta\,d\theta=\frac{1}{2}\int_0^\pi \frac{\sin(p+q)\theta+\sin(p-q)\theta}{\sin\theta}\,d\theta$$

$=0$ if $p+q$ be even, or if $p+q$ be odd and $p<q$,

$=\pi$ if $p+q$ be odd and $p>q$.

Hence if $F(\theta)$ be a function capable of convergent expansion as a series of cosines of multiples of θ, say

$$F(\theta)=A_0+A_1\cos\theta+A_2\cos 2\theta+\dots+A_r\cos r\theta+\dots,$$

$$\int_0^\pi \frac{\sin 2p\theta}{\sin\theta}F(\theta)\,d\theta=(A_1+A_3+\dots+A_{2p-1})\pi$$

and $$\int_0^\pi \frac{\sin(2p+1)\theta}{\sin\theta}F(\theta)\,d\theta=(A_0+A_2+A_4+\dots+A_{2p})\pi.$$

ILLUSTRATIVE EXAMPLES.

1143. 1. Thus, since

$$\cos^{2n}\theta=\frac{1}{2^{2n-1}}\left[\frac{1}{2}\,{}^{2n}C_n+{}^{2n}C_{n+1}\cos 2\theta+{}^{2n}C_{n+2}\cos 4\theta+\ldots+{}^{2n}C_{2n}\cos 2n\theta\right],$$

we have, if $p>n$,

$$\int_0^\pi \frac{\sin(2p+1)\theta}{\sin\theta}\cos^{2n}\theta\,d\theta=\frac{\pi}{2^{2n-1}}\left[\frac{1}{2}\,{}^{2n}C_n+{}^{2n}C_{n+1}+\ldots+{}^{2n}C_{2n}\right]$$

$$=\frac{\pi}{2^{2n}}\left[{}^{2n}C_0+{}^{2n}C_1+\ldots+{}^{2n}C_{2n}\right]=\frac{\pi}{2^{2n}}(1+1)^{2n}=\pi,$$

whilst, if $p<n$,

$$\int_0^\pi \frac{\sin(2p+1)\theta}{\sin\theta}\cos^{2n}\theta\,d\theta=\frac{\pi}{2^{2n-1}}\left[\frac{1}{2}\,{}^{2n}C_n+{}^{2n}C_{n+1}+\ldots+{}^{2n}C_{n+p}\right]=\frac{\pi}{2^{2n}}\sum_{r=n-p}^{r=n+p}{}^{2n}C_r.$$

2. Apply Art. 1142 to show that, if $u\equiv 1-2a\cos\theta+a^2$,

$$\int_0^\pi \frac{\sin 2n\theta}{\sin\theta}\,\frac{\cos\theta}{u}\,d\theta=\pi\,\frac{1+a^2}{1-a^2}\cdot\frac{1-a^{2n}}{1-a^2}\quad(a^2<1).$$

3. Prove that

$$\int_0^\pi \frac{\sin 2n\theta}{\sin\theta}\log u\,d\theta=-2\pi\left\{\frac{a}{1}+\frac{a^3}{3}+\frac{a^5}{5}+\ldots+\frac{a^{2n-1}}{2n-1}\right\}\quad(a^2<1).$$

1144. A Reduction Formula.

Let $u\equiv 1-2a\cos\theta+a^2$.

We have seen that

$$I_1\equiv\int_0^\pi\frac{\cos p\theta}{u}=\frac{\pi a^p}{1-a^2}\ (a^2<1)\ \text{and}\ \frac{\pi a^{-p}}{a^2-1}\ (a^2>1),$$

p being a positive integer.

Let $$I_n=\int_0^\pi\frac{\cos p\theta}{u^n}\,d\theta.$$

Then $$\frac{dI_n}{da}=2n\int_0^\pi\frac{\cos p\theta}{u^{n+1}}(\cos\theta-a)\,d\theta$$

$$=n\int_0^\pi\frac{\cos p\theta}{u^{n+1}}\,\frac{1-a^2-u}{a}\,d\theta=n\frac{1-a^2}{a}I_{n+1}-\frac{n}{a}I_n;$$

$$\therefore\ I_{n+1}=\frac{1}{1-a^2}\left(I_n+\frac{a}{n}\frac{dI_n}{da}\right),\quad i.e.\ I_{n+1}=\frac{1}{1-a^2}\,\frac{d}{da^n}(a^nI_n),\ \ldots.(1)$$

an equation by means of which the successive values of I_2, I_3, I_4, etc., may be deduced.

1145. We have

$$I_2=\frac{1}{1-a^2}\,\frac{d}{da}(aI_1)=\frac{\pi}{1-a^2}\,\frac{d}{da}\,\frac{a^{p+1}}{1-a^2}$$

$$=\frac{\pi a^p}{(1-a^2)^3}K_2,\ \text{where}\ K_2=(p+1)-(p-1)a^2,$$

$I_3 = \frac{1}{1-a^2}\frac{d}{da^2}(a^2 I_2)$, which after a little reduction takes the form

$\frac{1}{2!}\frac{\pi a^p}{(1-a^2)^5}K_3$, where $K_3 = (p+1)(p+2) - 2(p+2)(p-2)a^2 + (p-2)(p-1)a^4$,

$I_4 = \frac{1}{1-a^2}\frac{d}{da^3}(a^3 I_3)$, which after reduction becomes $\frac{1}{3!}\frac{\pi a^p}{(1-a^2)^7}K_4$,

where $K_4 = (p+1)(p+2)(p+3) - 3(p+2)(p+3)(p-3)a^2$
$+ 3(p+3)(p-3)(p-2)a^4 - (p-3)(p-2)(p-1)a^6$,

and so on, the law of formation of the successive values of K_n being obvious, and it may be verified inductively by substitution in Equation (1) that the general form of the result is

$$I_n = \frac{\pi a^p}{(1-a^2)^{2n-1}}\,{}^{n+p-1}C_p\Big[1 + {}^{n-1}C_1\frac{n-1-p}{1+p}a^2 + {}^{n-1}C_2\frac{(n-1-p)(n-2-p)}{(1+p)(2+p)}a^4 + {}^{n-1}C_3\frac{(n-1-p)(n-2-p)(n-3-p)}{(1+p)(2+p)(3+p)}a^6 + \dots\Big],$$

a form due to Legendre (*Exercices*, p. 374).

If we replace ${}^{n+p-1}C_p$ by its equivalent $\frac{(p+1)(p+2)\dots(p+n-1)}{1.2.3\dots(n-1)}$ the same formula, with the sign changed and $-p$ written for p, will suffice for the calculation of the corresponding integrals in the case when $a^2 > 1$.

1146. As particular cases we have, if $a^2 < 1$,

$$\int_0^\pi \frac{\cos p\theta}{u^2}d\theta = \frac{\pi a^p}{(1-a^2)^3}(p+1)\left[1 + \frac{1-p}{1+p}a^2\right] = \frac{\pi a^p}{(1-a^2)^3}[(p+1) - (p-1)a^2],$$

$$\int_0^\pi \frac{\cos p\theta}{u^3}d\theta = \frac{\pi a^p}{(1-a^2)^5}\frac{(p+2)(p+1)}{1.2}\left[1 + 2\frac{2-p}{1+p}a^2 + \frac{(2-p)(1-p)}{(1+p)(2+p)}a^4\right]$$
$$= \frac{1}{2!}\frac{\pi a^p}{(1-a^2)^5}[(p+1)(p+2) - 2(p+2)(p-2)a^2 + (p-2)(p-1)a^4],$$

etc. ;

and if $a^2 > 1$,

$$\int_0^\pi \frac{\cos p\theta}{u^2}d\theta = \frac{\pi a^{-p}}{(a^2-1)^3}[(1-p) + (1+p)a^2],$$

$$\int_0^\pi \frac{\cos p\theta}{u^3}d\theta = \frac{1}{2!}\frac{\pi a^{-p}}{(a^2-1)^5}[(1-p)(2-p) + 2(2-p)(2+p)a^2 + (2+p)(1+p)a^4],$$

etc.

1147. Some Special Cases.

The special cases when $p=0$ and $p=n-1$ are interesting.

If $p=0$,

$$\int_0^\pi \frac{d\theta}{u^n} = \frac{\pi}{(1-a^2)^{2n-1}}[1 + {}^{n-1}C_1{}^2a^2 + {}^{n-1}C_2{}^2a^4 + {}^{n-1}C_3{}^2a^6 + \dots],$$

the several coefficients being the squares of those of the binomial expansion of $(1+z)^{n-1}$.

Thus
$$\int_0^\pi \frac{d\theta}{u} = \frac{\pi}{1-a^2},$$
$$\int_0^\pi \frac{d\theta}{u^2} = \frac{\pi}{(1-a^2)^3}(1+a^2),$$
$$\int_0^\pi \frac{d\theta}{u^3} = \frac{\pi}{(1-a^2)^5}(1+2^2a^2+a^4),$$
$$\int_0^\pi \frac{d\theta}{u^4} = \frac{\pi}{(1-a^2)^7}(1+3^2a^2+3^2a^4+a^6),$$
$$\int_0^\pi \frac{d\theta}{u^5} = \frac{\pi}{(1-a^2)^9}(1+4^2a^2+6^2a^4+4^2a^6+a^8),$$
etc.

If $p=n-1$, we have
$$\int_0^\pi \frac{\cos(n-1)\theta}{u^n}\,d\theta = \frac{\pi a^{n-1}}{(1-a^2)^{2n-1}}\,{}^{2n-2}C_{n-1}.$$

Cases where $a^2>1$. Take for instance $I_2=\int_0^\pi \frac{d\theta}{u^2}$.

Here $\quad p=0\quad$ and $\quad I_2 = -\frac{\pi}{(1-a^2)^3}(1+a^2) = \frac{\pi}{(a^2-1)^3}(1+a^2).$

Again, $\quad I_3=\int_0^\pi \frac{d\theta}{u^3} = \frac{\pi}{(a^2-1)^5}(1+2^2a^2+a^4)$, etc.;

and it will appear generally that in the case of $p=0$, the only change necessary in the previous results will be to replace $1-a^2$ by a^2-1.

1148. **Extension of the Reduction Formula.**

It may be remarked that any integral of the form
$$I_n=\int_0^\pi \frac{F(\theta)}{u^n}\,d\theta$$
is subject to the same reduction formula as that used in the last article, viz.
$$I_{n+1}=\frac{1}{1-a^2}\,\frac{d}{da^n}(a^nI_n).$$

For $\displaystyle \frac{dI_n}{da} = 2n\int_0^\pi \frac{F(\theta)}{u^{n+1}}(\cos\theta-a)\,d\theta = n\int_0^\pi \frac{F(\theta)}{u^{n+1}}\,\frac{1-a^2-u}{a}\,d\theta$
$$= n\,\frac{1-a^2}{a}I_{n+1}-\frac{n}{a}I_n,$$
giving, as before, $\quad I_{n+1}=\frac{1}{1-a^2}\,\frac{d}{da^n}(a^nI_n).$

Hence in all such cases, if I_1 can be obtained in finite terms, so also can all the rest of the group I_2, I_3, I_4, etc.

1149. We shall show for instance that this is the case with the class of integrals

$$I_n=\int_0^\pi \frac{\sin p\theta}{u^n}\,d\theta,\ p \text{ being a positive integer.}$$

To do this it is only necessary to show that I_1 is expressible in finite terms, and we shall find that

$$\frac{1-a^2}{2}\int_0^\pi \frac{\sin p\theta}{u}\,d\theta=\frac{a^{p-1}-a^{-(p-1)}}{1}+\frac{a^{p-3}-a^{-(p-3)}}{3}+\frac{a^{p-5}-a^{-(p-5)}}{5}+\dots$$

$$\text{to } \frac{p}{2} \text{ or } \frac{p+1}{2} \text{ terms } -(a^p-a^{-p})\tanh^{-1}a. \quad \dots\dots(1),\ (a^2<1). \dots(1).$$

Take the case p odd $=2\lambda+1$, say,

$$\frac{1-a^2}{2}\int_0^\pi \frac{\sin(2\lambda+1)\theta}{u}\,d\theta=\int_0^\pi \sin(2\lambda+1)\theta\left[\tfrac{1}{2}+a\cos\theta+a^2\cos 2\theta+\dots\right]d\theta$$

$$=\frac{1}{2\lambda+1}+2(2\lambda+1)\left[\frac{a^2}{(2\lambda+1)^2-2^2}+\frac{a^4}{(2\lambda+1)^2-4^2}+\dots+\frac{a^{2\lambda}}{(2\lambda+1)^2-(2\lambda)^2}\right]$$

$$-2(2\lambda+1)\left[\frac{a^{2\lambda+2}}{(2\lambda+2)^2-(2\lambda+1)^2}+\frac{a^{2\lambda+4}}{(2\lambda+4)^2-(2\lambda+1)^2}+\dots \text{ ad inf.}\right]$$

$$=\frac{1}{2\lambda+1}-\left[a^2\left(\frac{1}{1-2\lambda}-\frac{1}{3+2\lambda}\right)+a^4\left(\frac{1}{3-2\lambda}-\frac{1}{5+2\lambda}\right)+\dots+a^{2\lambda}\left(\frac{1}{-1}-\frac{1}{4\lambda+1}\right)\right]$$

$$-\left[a^{2\lambda+2}\left(\frac{1}{1}-\frac{1}{4\lambda+3}\right)+a^{2\lambda+4}\left(\frac{1}{3}-\frac{1}{4\lambda+5}\right)+\dots \text{ ad inf.}\right]$$

$$=\frac{a^{2\lambda}}{1}+\frac{a^{2\lambda-2}}{3}+\dots+\frac{a^2}{2\lambda-1}+\frac{1}{2\lambda+1}+\frac{a^2}{2\lambda+3}+\dots+\frac{a^{2\lambda}}{4\lambda+1}$$

$$-\left[a^{2\lambda+1}\tanh^{-1}a-\frac{1}{a^{2\lambda+1}}\left(\tanh^{-1}a-\frac{a^1}{1}-\frac{a^3}{3}-\dots-\frac{a^{2\lambda-1}}{2\lambda-1}-\frac{a^{2\lambda+1}}{2\lambda+1}-\dots-\frac{a^{4\lambda+1}}{4\lambda+1}\right)\right],$$

i.e.
$$\frac{1-a^2}{2}\int_0^\pi \frac{\sin(2\lambda+1)\theta}{u}\,d\theta=\frac{a^{2\lambda}-a^{-2\lambda}}{1}+\frac{a^{2\lambda-2}-a^{-(2\lambda-2)}}{3}+\dots$$

$$+\frac{a^2-a^{-2}}{2\lambda-1}-\{a^{2\lambda+1}-a^{-(2\lambda+1)}\}\tanh^{-1}a.$$

And in exactly the same way, if p be even $=2\lambda$,

$$\frac{1-a^2}{2}\int_0^\pi \frac{\sin 2\lambda\theta}{u}\,d\theta=\frac{a^{2\lambda-1}-a^{-(2\lambda-1)}}{1}+\frac{a^{2\lambda-3}-a^{-(2\lambda-3)}}{3}+\dots+\frac{a-a^{-1}}{2\lambda-1}$$

$$-\{a^{2\lambda}-a^{-2\lambda}\}\tanh^{-1}a \quad (a^2<1),$$

which establishes the result stated.

If we write $a=e^{-\gamma}$ we may exhibit the result as

$$\int_0^\pi \frac{\sin p\theta}{u}\,d\theta=2\frac{\sinh p\gamma}{\sinh\gamma}\frac{\tanh^{-1}a}{a}-\frac{2}{a}\operatorname{cosech}\gamma\left[\frac{\sinh(p-1)\gamma}{1}+\frac{\sinh(p-3)\gamma}{3}+\dots\right.$$

$$\left.\text{to } \frac{p}{2} \text{ or } \frac{p+1}{2} \text{ terms}\right],$$

according as p is even or odd.

1150. Particular Cases.

The particular cases when $p=1$, 2, 3, etc., are

$$\int_0^\pi \frac{\sin\theta}{u}\,d\theta = -\frac{2}{1-a^2}\left(a-\frac{1}{a}\right)\tanh^{-1}a = \frac{2}{a}\tanh^{-1}a,$$

$$\int_0^\pi \frac{\sin 2\theta}{u}\,d\theta = \frac{2}{1-a^2}\left[\left(a-\frac{1}{a}\right)-\left(a^2-\frac{1}{a^2}\right)\tanh^{-1}a\right] = 2\frac{1+a^2}{a^2}\tanh^{-1}a-\frac{2}{a},$$

$$\int_0^\pi \frac{\sin 3\theta}{u}\,d\theta = \frac{2}{1-a^2}\left[\left(a^2-\frac{1}{a^2}\right)-\left(a^3-\frac{1}{a^3}\right)\tanh^{-1}a\right] = 2\frac{1+a^2+a^4}{a^3}\tanh^{-1}a-2\frac{1+a^2}{a^2},$$

etc.

1151. General Conclusion derived.

It appears then that $\int_0^\pi \frac{\sin p\theta}{u}\,d\theta$ is in all cases, when p is a positive integer and $a^2<1$, of the form

$$P+Q\tanh^{-1}a,$$

where P and Q are known algebraical functions of a.

And in any such case the reduction formula

$$I_{n+1}=\frac{1}{1-a^2}\frac{d}{da^n}(a^nI_n)$$

may be used to determine I_2, I_3, I_4, etc.

It will be observed that the first case of this result follows at once from the series for $\frac{\sin\theta}{u}$ (No. 2 of Art. 1134).

For $$\int_0^\pi \frac{\sin\theta}{u}\,d\theta = \int_0^\pi (\sin\theta + a\sin 2\theta + a^2\sin 3\theta+\dots)\,d\theta \quad (a^2<1)$$

$$=2\left(1+\frac{a^2}{3}+\frac{a^4}{5}+\dots\right)=\frac{2}{a}\tanh^{-1}a.$$

If a^2 be >1,

$$\int_0^\pi \frac{\sin\theta}{u}\,d\theta = \int_0^\pi \left(\frac{1}{a^2}\sin\theta+\frac{1}{a^3}\sin 2\theta+\frac{1}{a^4}\sin 3\theta+\dots\right)d\theta$$

$$=2\left(\frac{1}{a^2}+\frac{1}{3}\frac{1}{a^4}+\frac{1}{5}\frac{1}{a^6}+\dots\right)$$

$$=\frac{2}{a}\tanh^{-1}\frac{1}{a}=\frac{2}{a}\coth^{-1}a.$$

The general case when $a^2>1$ for $\int_0^\pi \frac{\sin p\theta}{u}\,d\theta$ may be investigated as in the case $a^2<1$, using the series

$$\frac{a^2-1}{1-2a\cos\theta+a^2}=1+\frac{2}{a}\cos\theta+\frac{2}{a^2}\cos 2\theta+\dots,$$

and it will be clear that all that will be necessary to modify equation (1) of Art. 1149 will be to replace $1-a^2$ by a^2-1 on the left-hand side and a by a^{-1} on the right, which leaves the formula for $\int_0^\pi \frac{\sin p\theta}{u}\,d\theta$ unchanged, except that $\tanh^{-1}a$ will be replaced by $\coth^{-1}a$.

Thus, in all cases whether $a^2 >$ or < 1, and p a positive integer, we have

$$\frac{1-a^2}{2}\int_0^\pi \frac{\sin p\theta}{u}\,d\theta = \frac{a^{p-1}-a^{-(p-1)}}{1}+\frac{a^{p-3}-a^{-(p-3)}}{3}+\dots$$

$$\text{to } \frac{p}{2} \text{ or } \frac{p+1}{2} \text{ terms } -(a^p-a^{-p})X,$$

where $X=\tanh^{-1}a$ or $\coth^{-1}a$, according as $a^2 <$ or > 1.

1152. General Formulae.

Let the expressions $\int_0^\pi \frac{\cos p\theta}{u^n}\,d\theta$ and $\int_0^\pi \frac{\sin p\theta}{u^n}\,d\theta$ be respectively called $C(p, n)$ and $S(p, n)$.

Then

$$\int_0^\pi \frac{\cos p\theta\cos q\theta}{u^n}\,d\theta = \frac{1}{2}\int_0^\pi \frac{\cos(p+q)\theta+\cos(p-q)\theta}{u^n}\,d\theta \quad = \tfrac{1}{2}[\ \ C(p+q, n)+C(p-q, n$$

$$\int_0^\pi \frac{\sin p\theta\sin q\theta}{u^n}\,d\theta = \frac{1}{2}\int_0^\pi \frac{-\cos(p+q)\theta+\cos(p-q)\theta}{u^n}\,d\theta = \tfrac{1}{2}[-C(p+q, n)+C(p-q, n$$

$$\int_0^\pi \frac{\cos p\theta\sin q\theta}{u^n}\,d\theta = \frac{1}{2}\int_0^\pi \frac{\sin(p+q)\theta-\sin(p-q)\theta}{u^n}\,d\theta \quad = \tfrac{1}{2}[\ \ S(p+q, n)-S(p-q, n)$$

$$\int_0^\pi \frac{\sin p\theta\cos q\theta}{u^n}\,d\theta = \frac{1}{2}\int_0^\pi \frac{\sin(p+q)\theta+\sin(p-q)\theta}{u^n}\,d\theta \quad = \tfrac{1}{2}[\ \ S(p+q, n)+S(p-q, n$$

Hence all such integrals can be computed, p, q and n being positive integers.

1153. Integrals of the Class $\int_0^\pi u^n \cos p\theta\,d\theta$ (Legendre, *Exercices*, p. 375), n a positive integer.

We have

$$u^n=(1-2a\cos\theta+a^2)^n=(1-ae^{\iota\theta})^n(1-ae^{-\iota\theta})^n$$
$$=(K_0+K_1e^{\iota\theta}+K_2e^{2\iota\theta}+\dots)(K_0+K_1e^{-\iota\theta}+K_2e^{-2\iota\theta}+\dots),$$

where $K_p=(-1)^p a^p \dfrac{n(n-1)\dots(n-p+1)}{1.2\dots p}$ and $K_0=1$.

The coefficients of $e^{p\iota\theta}$ and $e^{-p\iota\theta}$ in the product are each

$$K_pK_0+K_{p+1}K_1+K_{p+2}K_2+K_{p+3}K_3+\dots,$$

giving rise to the term

$$(K_pK_0+K_{p+1}K_1+K_{p+2}K_2+\dots)2\cos p\theta,$$

and in the integration this is the only term we shall require, for all the others vanish by virtue of the theorem of Art. 1121.

Hence $I\equiv\int_0^\pi u^n\cos p\theta\,d\theta=\pi(K_pK_0+K_{p+1}K_1+K_{p+2}K_2+\dots)$.

Now $\dfrac{K_{p+1}}{K_p}=-a\dfrac{n-p}{p+1},\ \dfrac{K_{p+2}}{K_p}=a^2\dfrac{(n-p)(n-p-1)}{(p+1)(p+2)}$, etc.,

and $\qquad K_1=-\dfrac{n}{1}a,\qquad K_2=\dfrac{n(n-1)}{1.2}a^2$, etc.;

$$\therefore\ I=(-1)^p\pi a^p\frac{n(n-1)\dots(n-p+1)}{1.2\dots p}\left[1+\frac{n}{1}\cdot\frac{n-p}{p+1}a^2+\frac{n(n-1)}{1.2}\cdot\frac{(n-p)(n-p-1)}{(p+1)(p+2)}a^4+\dots\right].$$

1154. **The Particular Case** $p=0$ gives

$$I=\pi(K_0^2+K_1^2+K_2^2+K_3^2+\dots),$$

i.e. $$\int_0^\pi u^n\,d\theta=\pi(1+{}^nC_1^2a^2+{}^nC_2^2a^4+{}^nC_3^2a^6+\dots).$$

We have seen (Art. 1147) that

$$\int_0^\pi\frac{d\theta}{u^{n+1}}=\frac{\pi}{(1-a^2)^{2n+1}}(1+{}^nC_1^2a^2+{}^nC_2^2a^4+{}^nC_3^2a^6+\dots);$$

whence it follows that

$$\int_0^\pi u^n\,d\theta=(1-a^2)^{2n+1}\int_0^\pi\frac{d\theta}{u^{n+1}}\quad\text{(see Art. 1155)};\ \dots\dots\dots\dots(1)$$

and more generally, since

$$\int_0^\pi\frac{\cos p\theta\,d\theta}{u^{n+1}}=\frac{\pi a^p}{(1-a^2)^{2n+1}}\,\frac{(p+1)(p+2)\dots(p+n)}{1.2.3\dots n}\times\left(1+\frac{n}{1}\cdot\frac{n-p}{p+1}a^2+\frac{n(n-1)}{1.2}\cdot\frac{(n-p)(n-p-1)}{(p+1)(p+2)}a^4+\dots\right),$$

by writing $n+1$ for n in the formula of Art. 1145, we have, by comparison with the result proved above for $\int_0^\pi u^n\cos p\theta\,d\theta$,

$$\int_0^\pi\frac{\cos p\theta}{u^{n+1}}\,d\theta=\frac{(-1)^p}{(1-a^2)^{2n+1}}\,\frac{(n+1)(n+2)\dots(n+p)}{n(n-1)\dots(n-p+1)}\int_0^\pi u^n\cos p\theta\,d\theta,$$

or

$$\int_0^\pi u^n\cos p\theta\,d\theta=(-1)^p(1-a^2)^{2n+1}\frac{n(n-1)\dots(n-p+1)}{(n+1)(n+2)\dots(n+p)}\int_0^\pi\frac{\cos p\theta}{u^{n+1}}\,d\theta.\ (2)$$

In the value of $\int_0^\pi u^n\cos p\theta\,d\theta$ established in Art. 1153, it is to be noted that p has been assumed not greater than n.

If p be $>n$ no term containing $\cos p\theta$ would occur in the expansion of u^n; $\therefore\ \int_0^\pi u^n\cos p\theta\,d\theta=0\quad(p>n)$.

If $n=p$, we have $\int_0^\pi u^n\cos n\theta\,d\theta=(-1)^n\pi a^n$.

The results of this article are due to Euler (vol. iv., *Calc. Intég.*, p. 194, etc.). The method of proof is that of Legendre (*Exercices*, p. 576).

1155. The Equation $\int_0^\pi u^n\,d\theta=(1-a^2)^{2n+1}\int_0^\pi \frac{d\theta}{u^{n+1}}$ may be established directly by the transformation

$$(1-2a\cos\theta+a^2)(1-2a^2\cos\theta'+a^2)=(1-a^2)^2,$$

which has an interesting geometrical interpretation due to the late Dr. N. M. Ferrers.*

Moreover, so far it has been assumed that n is a positive integer. It will be seen from what follows that this limitation is no longer necessary.

Take a circle of radius a and centre O and a point B within the circle at a distance b from the centre.

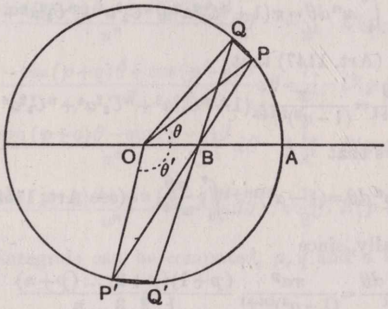

Fig. 336.

Let PBP' be any chord through B, and let the portions PB, BP' subtend angles θ, θ' at the centre; then

$$PB^2=a^2+b^2-2ab\cos\theta,$$
$$BP'^2=a^2+b^2-2ab\cos\theta',$$

and

$$(a^2+b^2-2ab\cos\theta)(a^2+b^2-2ab\cos\theta')=PB^2.BP'^2=(a^2-b^2)^2.$$

Also, if QBQ' be a contiguous position of the chord, the elementary triangles BPQ, $BQ'P'$ are similar; hence

$$\frac{d\theta}{-d\theta'}=Lt\frac{PQ}{P'Q'}=Lt\frac{BP}{BQ'}=\frac{BP}{BP'}=\left(\frac{a^2+b^2-2ab\cos\theta}{a^2+b^2-2ab\cos\theta'}\right)^{\frac{1}{2}}=\frac{a^2-b^2}{a^2+b^2-2ab\cos\theta'};$$

$$\therefore\ (a^2+b^2-2ab\cos\theta)^n\,d\theta=-\frac{(a^2-b^2)^{2n}}{(a^2+b^2-2ab\cos\theta')^n}\,\frac{(a^2-b^2)}{a^2+b^2-2ab\cos\theta'}\,d\theta'$$
$$=-\frac{(a^2-b^2)^{2n+1}}{(a^2+b^2-2ab\cos\theta')^{n+1}}\,d\theta'.$$

* See *Solutions of Senate House Problems and Riders*, 1878. Edited by Mr. J. W. L. Glaisher.

If the chord be allowed to rotate so that θ increases from $\theta=0$ to $\theta=\pi$, then θ' decreases from $\theta'=\pi$ to $\theta'=0$. Hence, integrating and replacing θ' by θ,

$$\int_0^\pi (a^2-2ab\cos\theta+b^2)^n\,d\theta=(a^2-b^2)^{2n+1}\int_0^\pi \frac{d\theta}{(a^2-2ab\cos\theta+b^2)^{n+1}}.$$

Taking the radius a to be unity and replacing b by a, we have the equation established otherwise by Euler and Legendre above.

Writing $c\cos\frac{\alpha}{2}$, $c\sin\frac{\alpha}{2}$ for a and b respectively, the equation may be thrown into the compact form

$$\int_0^\pi (1-\sin\alpha\cos\theta)^n\,d\theta=(\cos\alpha)^{2n+1}\int_0^\pi \frac{d\theta}{(1-\sin\alpha\cos\theta)^{n+1}}.$$

1156. Another Interpretation of the Integral.

The integral may also be interpreted in connection with the angles known in elliptic motion as the True and Eccentric Anomalies.

Let S and C be the focus and centre of an ellipse, A' the end of the major axis most remote from S, and A the nearer

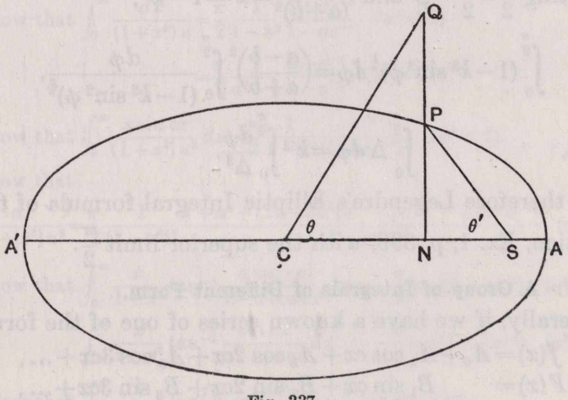

Fig. 337.

end, P a point on the curve, NP its ordinate, and Q the corresponding point on the auxiliary circle. Then $A'SP$ is the supplement of the "true anomaly," and SCQ is the "eccentric anomaly." Let these angles be θ' and θ respectively.

Then, from the polar equation of the ellipse,

$$\frac{CA\,(1-e^2)}{SP}=1-e\cos\theta',$$

and also $SP=CA-e\,.\,CN=CA\,(1-e\cos\theta)$.

Hence $(1-e\cos\theta)(1-e\cos\theta')=1-e^2$;

and if we write $\sin\alpha$ for e, *i.e.*

$$e=\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\frac{2ab}{a^2+b^2}\quad\left(\text{where }\tan\frac{\alpha}{2}=\frac{b}{a}\right),$$

we have

$$(a^2+b^2-2ab\cos\theta)(a^2+b^2-2ab\cos\theta')=(a^2-b^2)^2 \text{ as before.}$$

The case when $n=\frac{1}{2}$, viz.

$$\int_0^\pi\sqrt{a^2-2ab\cos\theta+b^2}\,d\theta=(a^2-b^2)^2\int_0^\pi\frac{d\theta}{(a^2-2ab\cos\theta+b^2)^{\frac{3}{2}}}d\theta,$$

may be written

$$\int_0^\pi\sqrt{(a+b)^2-4ab\cos^2\frac{\theta}{2}}\,d\theta=(a^2-b^2)^2\int_0^\pi\frac{d\theta}{\left((a+b)^2-4ab\cos^2\frac{\theta}{2}\right)^{\frac{3}{2}}},$$

or putting $\frac{\theta}{2}=\frac{\pi}{2}-\phi$ and $\frac{4ab}{(a+b)^2}=k^2=1-k'^2$,

$$\int_0^{\frac{\pi}{2}}(1-k^2\sin^2\phi)^{\frac{1}{2}}\,d\phi=\left(\frac{a-b}{a+b}\right)^2\int_0^{\frac{\pi}{2}}\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}},$$

that is,
$$\int_0^{\frac{\pi}{2}}\Delta\,d\phi=k'^2\int_0^{\frac{\pi}{2}}\frac{d\phi}{\Delta^3},$$

and is therefore Legendre's Elliptic Integral formula of transformation, Ex. 1, p. 399, with the superior limit $\frac{\pi}{2}$.

1157. A Group of Integrals of Different Form.

Generally, if we have a known series of one of the forms

$$f(x)=A_0+A_1\cos cx+A_2\cos 2cx+A_3\cos 3cx+\dots,$$
$$F(x)=\qquad B_1\sin cx+B_2\sin 2cx+B_3\sin 3cx+\dots,$$

we have, by the integrals of Arts. 1048 ... 1051, viz.

$$\int_0^\infty\frac{\sin cx}{x(1+x^2)}\,dx=\frac{\pi}{2}(1-e^{-c});\quad \int_0^\infty\frac{\cos cx}{1+x^2}\,dx=\frac{\pi}{2}e^{-c};$$

$$\int_0^\infty\frac{x\sin cx}{1+x^2}\,dx=\frac{\pi}{2}e^{-c},$$

where c is positive,

$$\int_0^\infty \frac{f(x)}{1+x^2}\,dx \quad = \frac{\pi}{2}(A_0 + A_1 e^{-c} + A_2 e^{-2c} + A_3 e^{-3c} + \ldots),$$

$$\int_0^\infty \frac{F(x)}{x(1+x^2)}\,dx = \frac{\pi}{2}[B_1(1-e^{-c}) + B_2(1-e^{-2c}) + B_3(1-e^{-3c}) + \ldots],$$

$$\int_0^\infty \frac{xF(x)}{1+x^2}\,dx \quad = \frac{\pi}{2}(B_1 e^{-c} + B_2 e^{-2c} + B_3 e^{-3c} + \ldots).$$

Accordingly, taken in conjunction with the particular class of series given in Art. 1134, we obtain another numerous group of definite integrals.

Illustrative Examples. (c positive throughout.)

1158. 1. Since $\frac{\sin cx}{u} = \sin cx + a \sin 2cx + a^2 \sin 3cx + \ldots$ $(a^2 < 1)$, where $u \equiv 1 - 2a\cos cx + a^2$, we have

$$\int_0^\infty \frac{x}{1+x^2}\frac{\sin cx}{u}\,dx = \int_0^\infty \frac{x}{1+x^2}(\sin cx + a\sin 2cx + a^2 \sin 3cx + \ldots)\,dx$$

$$= \frac{\pi}{2}(e^{-c} + ae^{-2c} + a^2 e^{-3c} + \ldots)$$

$$= \frac{\pi}{2}\frac{e^{-c}}{1-ae^{-c}} = \frac{\pi}{2}\frac{1}{e^c - a}.$$

[Legendre, *Exercices*, vol. ii., p. 123.]

2. Show that $\int_0^\infty \frac{dx}{(1+x^2)u} = \frac{\pi}{2}\frac{1}{1-a^2}\frac{1+ae^{-c}}{1-ae^{-c}}$ $(a^2 < 1)$

or $= \frac{\pi}{2}\frac{1}{a^2-1}\frac{a+e^{-c}}{a-e^{-c}}$ $(a^2 > 1)$.

3. Show that $\int_0^\infty \frac{x \sin cx}{(1+x^2)u^2}\,dx = \frac{\pi}{2}\frac{1}{1-a^2}\frac{e^{-c}}{(1-ae^{-c})^2}$ $(a^2 < 1)$.

4. Show that

$$\int_0^\infty \frac{dx}{(1+x^2)u^2} = \frac{\pi}{2}\frac{1}{(1-a^2)^3}\frac{1+a^2+(2a-3a^2)e^{-c} - 3a^2 e^{-2c} + 8a^3 e^{-3c}}{(1-ae^{-c})^2} \quad (a^2<1).$$

5. Show that $\int_0^\infty \frac{x}{1+x^2}\tan^{-1}\frac{a\sin cx}{1-a\cos cx}\,dx = -\frac{\pi}{2}\log(1-ae^{-c})$ $(a^2 < 1)$,

$$\int_0^\infty \frac{x}{1+x^2}\tan^{-1}\frac{\sin cx}{a - \cos cx}\,dx = -\frac{\pi}{2}\log\left(1 - \frac{1}{a}e^{-c}\right) \quad (a^2 > 1).$$

6. Show that $\int_0^\infty \frac{1}{1+x^2}\log\left(2\cos\frac{cx}{2}\right)dx = \frac{\pi}{2}\log(1+e^{-c})$.

7. Show that $\int_0^\infty \frac{1}{1+x^2}\log\left(2\sin\frac{cx}{2}\right)dx = \frac{\pi}{2}\log(1-e^{-c})$.

8. Show that $\int_0^\infty \frac{\log u}{1+x^2}\,dx = \pi\log(1-ae^{-c})$ $(a^2 \leqslant 1)$

or $= \pi\log(a - e^{-c})$ $(a^2 > 1)$.

9. From the last example deduce

$$\int_0^\infty \log\tan\frac{cx}{2}\frac{dx}{1+x^2}=\frac{\pi}{2}\log\frac{1-e^{-c}}{1+e^{-c}}.$$

[GEORGES BIDONE, *Mém. de Turin*, vol. XX.]

EXAMPLES.

1. Show that $$\int_0^\pi \frac{dx}{u^2}=\pi\frac{1+a^2}{(1-a^2)^3},$$
where $u\equiv 1-2a\cos x+a^2$ and $a^2<1$.

2. Show that $\displaystyle\int_0^\pi \frac{\cos nx}{u^2}dx=\pi\frac{a^n}{(1-a^2)^3}\{(n+1)-(n-1)a^2\}\quad(a^2<1).$

3. Show that
$$\int_0^\pi \frac{\sin x\sin nx}{u^3}dx=\frac{\pi}{4}\frac{na^{n-1}}{(1-a^2)^3}\{(n+1)-(n-1)a^2\}\quad(a^2<1).$$

4. Show that $\displaystyle\int_0^\pi \frac{\cos nx\,dx}{(b^2+x^2)u}=\frac{\pi}{2b}\frac{1}{1-a^2}\frac{(1-a^2)e^{-nb}-2a^{n+1}\sinh b}{1-2a\cosh b+a^2}.$

5. Show that $\displaystyle\int_0^\infty \frac{\log\tan x}{1+x^2}dx=\frac{\pi}{2}\log\tanh e.$

6. Show that $\displaystyle\int_0^\pi \frac{\cos n\theta}{25-24\cos\theta}d\theta=\frac{\pi}{7}\left(\frac{3}{4}\right)^n.$

7. Show that $\displaystyle\int_0^\pi \log(25-24\cos\theta)\,d\theta=4\pi\log 2.$

8. Show that (a) $\displaystyle\int_0^\pi \frac{\sin\theta}{25-24\cos\theta}d\theta=\frac{1}{12}\log 7;$

(b) $\displaystyle\int_0^\pi \frac{\sin\theta}{(25-24\cos\theta)^n}d\theta=\frac{1}{24}\cdot\frac{1}{n-1}\left(1-\frac{1}{49^{n-1}}\right).$

9. Show that $\displaystyle\int_0^\pi \frac{\theta\sin\theta}{5-4\cos\theta}d\theta=\frac{\pi}{2}\log\frac{3}{2}.$

10. Show that $\displaystyle\int_0^\pi \frac{\sin\theta}{(5-4\cos\theta)^2}d\theta=\frac{2}{9}.$

11. Show that $\displaystyle\int_0^\pi \frac{\sin\theta\sin n\theta}{(5-4\cos\theta)^3}d\theta=\frac{n(3n+5)}{2^{n+3}}\cdot\frac{\pi}{27}.$

12. Show that $\displaystyle\int_0^\pi \frac{\sin^2\theta}{(5-3\cos\theta)^3}d\theta=\frac{\pi}{2^7}.$

13. Show that $\displaystyle\int_0^\pi \sin p\theta\log u\,d\theta$

$$=-\sum_1^\infty\frac{1}{n}a^n[1-(-1)^{p+n}]\frac{2p}{p^2-n^2}\quad(a^2<1)$$

or $$=\frac{1-\cos p\pi}{p}\log a^2-\sum_1^\infty\frac{1}{na^n}[1-(-1)^{p+n}]\frac{2p}{p^2-n^2}\quad(a^2>1),$$

where the term for which $n=p$ is omitted in the summation, p being a positive integer.

14. Show that

$$\int_0^\pi \frac{\sin p\theta}{u^3}\,d\theta=\frac{1}{(1-a^2)^5}\left[(1+4a^2+a^4)\frac{1-\cos p\pi}{p}+\sum_1^\infty A_n\{1-(-1)^{p+n}\}\frac{p}{p^2-n^2}\right]$$

$(a^2<1)$,

the term where $n=p$ being omitted in the summation (Art. 1136).

1159. On the Transition from a Real Value of k to a Complex Value of k in the Formula for $\int_0^\infty e^{-kx}x^{n-1}\,dx$. M. Serret's Investigation.

In establishing the result

$$\int_0^\infty e^{-kx}x^{n-1}\,dx=\frac{\Gamma(n)}{k^n},\quad (n>0),\quad \text{(Art. 864)},$$

it was assumed throughout the proof that k was real. We cannot therefore assume the theorem as still true for complex values of k without further investigation. We consider the integral

$$I\equiv\int_0^\infty e^{-(a-\iota b)x}x^{n-1}\,dx,\quad \text{where } \iota\equiv\sqrt{-1}.$$

Then I will be finite if a be positive.

Since $e^{-(a-\iota b)x}=e^{-ax}(\cos bx+\iota\sin bx)$ the integral consists of two separate integrals, viz.

$$\int_0^\infty e^{-ax}\cos bx\,x^{n-1}\,dx+\iota\int e^{-ax}\sin bx\,x^{n-1}\,dx.$$

Let R, Φ be respectively the modulus and argument of I. Thus

$$Re^{\iota\Phi}=\int_0^\infty e^{-(a-\iota b)x}x^{n-1}\,dx.$$

Let $b=a\tan\phi$, ϕ lying between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, so that

$$Re^{\iota\Phi}=\int_0^\infty e^{-ax}e^{\iota ax\tan\phi}x^{n-1}\,dx.$$

Then differentiating with regard to ϕ,

$$Re^{\iota\Phi}\left[\frac{\partial\log R}{\partial\phi}+\iota\frac{\partial\Phi}{\partial\phi}\right]=\iota a\sec^2\phi\int_0^\infty e^{-ax}e^{\iota ax\tan\phi}x^n\,dx.$$

Integrating by parts,

$$\int_0^\infty e^{-(a-\iota b)x}x^n\,dx=\left[\frac{e^{-(a-\iota b)x}x^n}{-(a-\iota b)}\right]_0^\infty+\frac{n}{a-\iota b}\int_0^\infty e^{-(a-\iota b)x}x^{n-1}\,dx,$$

and the portion between square brackets vanishes at both limits, a being positive.

Hence $\quad Re^{\iota\Phi}\left(\dfrac{\partial \log R}{\partial\phi}+\iota\dfrac{\partial\Phi}{\partial\phi}\right)=\dfrac{n}{a-\iota b}\iota a\sec^2\phi Re^{\iota\Phi}$

$$=n(\iota-\tan\phi)Re^{\iota\Phi};$$

$$\therefore\ \frac{\partial\log R}{\partial\phi}=-n\tan\phi \qquad\text{and}\qquad \frac{\partial\Phi}{\partial\phi}=n;$$

$$\therefore\ \log R=n\log\cos\phi+\log A \quad\text{and}\quad \Phi=n\phi+B,$$

where A and B are independent of ϕ.

i.e. $\qquad R=A\cos^n\phi,\quad \Phi=n\phi+B.$

But when ϕ vanishes $b=0$, and the integral is

$$\int_0^\infty e^{-ax}x^{n-1}\,dx=\frac{\Gamma(n)}{a^n},\ \text{ and } \Phi \text{ vanishes.}$$

Hence $B=0$ and $A=\dfrac{\Gamma(n)}{a^n}$; hence $R=\dfrac{\Gamma(n)}{a^n}\cos^n\phi$, $\Phi=n\phi$.

Hence

$$I=\frac{\Gamma(n)\cos^n\phi}{a^n}(\cos n\phi+\iota\sin n\phi)=\frac{\Gamma(n)}{a^n(1-\iota\tan\phi)^n}=\frac{\Gamma(n)}{(a-\iota b)^n}.$$

So the theorem $\quad \displaystyle\int_0^\infty e^{-kx}x^{n-1}\,dx=\frac{\Gamma(n)}{k^n}$

*still holds when k is complex, provided the real part a of the complex is positive.**

If n be a fractional quantity, $\dfrac{p}{q}$, $(a-\iota b)^n$ will be susceptible of q values and no more, if its argument be unrestricted in value. We must then obtain the argument of $(a-\iota b)^n$ by multiplying by n the argument of $a-\iota b$ taken between the limits $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

1160. We then have the two integrals

$$\left.\begin{aligned}\int_0^\infty e^{-ax}\cos bx\,x^{n-1}\,dx&=\frac{\Gamma(n)}{a^n}\cos^n\phi\cos n\phi=\frac{\Gamma(n)}{b^n}\sin^n\phi\cos n\phi,\\ \int_0^\infty e^{-ax}\sin bx\,x^{n-1}\,dx&=\frac{\Gamma(n)}{a^n}\cos^n\phi\sin n\phi=\frac{\Gamma(n)}{b^n}\sin^n\phi\sin n\phi,\end{aligned}\right\}\ldots\text{(A)}$$

i.e. $$\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx=\frac{\Gamma(n)}{(a^2+b^2)^{\frac{n}{2}}}\cos\left(n\tan^{-1}\frac{b}{a}\right),$$

$$\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx=\frac{\Gamma(n)}{(a^2+b^2)^{\frac{n}{2}}}\sin\left(n\tan^{-1}\frac{b}{a}\right).$$

* See Serret, *Calcul Intégral*, p. 193.

These results (A) are then so far established on the understanding that a is a positive quantity.

1161. When a vanishes the integral $\int_0^\infty e^{\iota bx} x^{n-1}\,dx$ may still be finite if n be a positive proper fraction.

Consider either integral, say $\int_0^\infty e^{-ax}\sin bx\, x^{n-1}\,dx$ $(b, +^{\text{ve}})$.

This is equal to

$$\left[\int_0^{\frac{\pi}{b}}+\int_{\frac{\pi}{b}}^{\frac{2\pi}{b}}+\int_{\frac{2\pi}{b}}^{\frac{3\pi}{b}}+\dots+\int_{\frac{r\pi}{b}}^{\frac{(r+1)\pi}{b}}+\dots\right]e^{-ax}\sin bx\, x^{n-1}\,dx.$$

Let $(-1)^r u_r=\int_{\frac{r\pi}{b}}^{\frac{(r+1)\pi}{b}} e^{-ax}\sin bx\, x^{n-1}\,dx$, and write $\dfrac{z+r\pi}{b}$ for x,

$$(-1)^r u_r=\int_0^\pi e^{-a\frac{z+r\pi}{b}}\sin(z+r\pi)\cdot\left(\frac{z+r\pi}{b}\right)^{n-1}\frac{dz}{b},$$

i.e.
$$u_r=\frac{1}{b^n}\int_0^\pi e^{-\frac{a}{b}(z+r\pi)}\sin z\,.\,(z+r\pi)^{n-1}\,dz,$$

and the whole integral $\int_0^\infty e^{-ax}\sin bx\, x^{n-1} dx$ is made up of such terms as this with alternate signs, viz. $\sum_0^\infty (-1)^r u_r$, *i.e.*

$$=u_0-u_1+u_2-u_3+\dots,$$

which is convergent if $a>0$, for the terms diminish as r increases and are of alternate sign. But in the case when $a=0$, u_r becomes $u_r'\equiv\dfrac{1}{b^n}\int_0^\pi \sin z(z+r\pi)^{n-1}\,dz$, and when r becomes indefinitely large this does not ultimately vanish unless $n<1$. When this is so, the series

$$u_0'-u_1'+u_2'-u_3'+\dots$$

is convergent, and its sum will be the same as the sum

$$u_0-u_1+u_2-u_3+\dots$$

for the value $a=0$, $n<1$.

For if
$$S=u_0-u_1+u_2-u_3+\dots \textit{ ad inf.},$$
$$S'=u_0'-u_1'+u_2'-u_3'+\dots,$$

and S_m, S'_m be the sums of the first m terms and R_m, R'_m the remainders respectively,

$$S=S_m+R_m, \quad S'=S'_m+R'_m,$$

i.e.
$$S-S'=S_m-S'_m+R_m-R'_m.$$

But $S_m-S'_m=0$ when $a=0$, and R_m, R'_m separately diminish indefinitely as m increases indefinitely. Hence $S-S'=0$ when $a=0$ and $0<n<1$.

Hence formulae (A) become, when $a=0$, and therefore $\phi=\frac{\pi}{2}$,

$$\left.\begin{aligned}\int_0^\infty x^{n-1}\cos bx\,dx&=\frac{\Gamma(n)}{b^n}\cos\frac{n\pi}{2},\\ \int_0^\infty x^{n-1}\sin bx\,dx&=\frac{\Gamma(n)}{b^n}\sin\frac{n\pi}{2},\end{aligned}\right\}\text{(B), where } n \text{ is a positive proper fraction } (b \text{ positive}).$$

1162. Putting $x=z^\lambda$ and $n\lambda=p$, we have

$$\left.\begin{aligned}\int_0^\infty z^{p-1}\cos bz^\lambda\,dz&=\frac{\Gamma\left(\frac{p}{\lambda}\right)}{\lambda b^{\frac{p}{\lambda}}}\cos\frac{p\pi}{2\lambda},\\ \int_0^\infty z^{p-1}\sin bz^\lambda\,dz&=\frac{\Gamma\left(\frac{p}{\lambda}\right)}{\lambda b^{\frac{p}{\lambda}}}\sin\frac{p\pi}{2\lambda},\end{aligned}\right\}\text{(B}'\text{), where } p<\lambda \text{ and both are positive } (b \text{ positive}).$$

1163. Since $\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}$, the integrals (B) may be written

$$\left.\begin{aligned}\int_0^\infty x^{n-1}\cos bx\,dx&=\frac{\pi}{\sin\frac{n\pi}{2}}\,\frac{1}{2b^n\Gamma(1-n)},\\ \int_0^\infty x^{n-1}\sin bx\,dx&=\frac{\pi}{\cos\frac{n\pi}{2}}\,\frac{1}{2b^n\Gamma(1-n)}.\end{aligned}\right\}\text{............(C)}$$

1164. M. Serret points out that the latter integral remains finite when n is indefinitely diminished to zero, and that the formula then reduces to

$$\int_0^\infty \frac{\sin bx}{x}\,dx=\frac{\pi}{2} \quad (b \text{ positive}).$$

1165. If we write $1-n=m$, m being a positive proper fraction, the formulae (C) take the form

$$\left.\begin{aligned}\int_0^\infty \frac{\cos bx}{x^m}dx&=\frac{\pi}{\cos\frac{m\pi}{2}}\frac{b^{m-1}}{2\Gamma(m)},\\ \int_0^\infty \frac{\sin bx}{x^m}dx&=\frac{\pi}{\sin\frac{m\pi}{2}}\frac{b^{m-1}}{2\Gamma(m)},\end{aligned}\right\}\begin{aligned}&0<m<1\\&(b \text{ positive}).\end{aligned}\quad \text{(D)}$$

1166. The case $m=\frac{1}{2}$ gives

$$\left.\begin{aligned}\int_0^\infty \frac{\cos bx}{\sqrt{x}}dx&=\frac{\pi}{\cos\frac{\pi}{4}}\frac{b^{-\frac{1}{2}}}{2\Gamma(\frac{1}{2})}=\frac{\sqrt{\pi}}{\sqrt{2b}},\\ \int_0^\infty \frac{\sin bx}{\sqrt{x}}dx&=\frac{\pi}{\sin\frac{\pi}{4}}\frac{b^{-\frac{1}{2}}}{2\Gamma(\frac{1}{2})}=\frac{\sqrt{\pi}}{\sqrt{2b}},\end{aligned}\right\}(b \text{ positive}).\quad \text{...(E)}$$

Putting $x=z^2$ in these integrals,

$$\int_0^\infty \cos bz^2dz=\int_0^\infty \sin bz^2dz=\frac{1}{2}\sqrt{\frac{\pi}{2b}}\quad (b \text{ positive});$$

and if we put $b=\frac{\pi}{2}$, we have

$$\int_0^\infty \cos\frac{\pi z^2}{2}dz=\int_0^\infty \sin\frac{\pi z^2}{2}dz=\frac{1}{2}.\quad \text{...........(F)}$$

These two integrals are known as Fresnel's Integrals, and will be considered more fully in Art. 1169.

The groups of integrals of these articles are due to Euler (*Calc. Intégral*, vol. iv., p. 337, etc.). They are also discussed by Laplace, vol. viii., *Journal de l'École Polytechnique*, p. 244, etc., by Legendre, *Exercices*, p. 367, etc., by Serret, *Calc. Intég.*, p. 193, etc.

1167. **Further Results.**

Returning to formulae (A), viz.

$$\left.\begin{aligned}\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx&=\frac{\Gamma(n)}{a^n}\cos^n\phi\cos n\phi,\\ \int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx&=\frac{\Gamma(n)}{a^n}\cos^n\phi\sin n\phi,\end{aligned}\right\}\text{ where } b=a\tan\phi,$$

and putting $n=1$, we have the well-known results

$$\left.\begin{aligned}\int_0^\infty e^{-ax}\cos bx\,dx&=\frac{a}{a^2+b^2},\\ \int_0^\infty e^{-ax}\sin bx\,dx&=\frac{b}{a^2+b^2}.\end{aligned}\right\}$$

Again remembering that $b=a\tan\phi$, we have $b^m=a^m\tan^m\phi$, and keeping a constant,

$$b^{m-1}db=a^m\tan^{m-1}\phi\sec^2\phi\,d\phi.$$

Hence multiplying the integrals by the sides of this identity, and integrating with regard to b from $b=0$ to $b=\infty$, and therefore with regard to ϕ from $\phi=0$ to $\phi=\frac{\pi}{2}$, and taking $1>m>0$,

$$\Gamma(n)\,a^{m-n}\int_0^{\frac{\pi}{2}}\sin^{m-1}\phi\cos^{n-m-1}\phi\cos n\phi\,d\phi=\int_0^\infty\int_0^\infty e^{-ax}x^{n-1}b^{m-1}\cos bx\,dx\,db,$$

and

$$\Gamma(n)\,a^{m-n}\int_0^{\frac{\pi}{2}}\sin^{m-1}\phi\cos^{n-m-1}\phi\sin n\phi\,d\phi=\int_0^\infty\int_0^\infty e^{-ax}x^{n-1}b^{m-1}\sin bx\,dx\,db.$$

The right-hand sides of these integrals are respectively (taking $n>m$),

$$\int_0^\infty e^{-ax}x^{n-1}\frac{\Gamma(m)}{x^m}\cos\frac{m\pi}{2}\,dx \quad\text{and}\quad \int_0^\infty e^{-ax}x^{n-1}\frac{\Gamma(m)}{x^m}\sin\frac{m\pi}{2}\,dx$$

by formulae (B),

i.e. $$\frac{\Gamma(n-m)}{a^{n-m}}\cdot\Gamma(m)\cos\frac{m\pi}{2}\quad\text{and}\quad\frac{\Gamma(n-m)}{a^{n-m}}\Gamma(m)\sin\frac{m\pi}{2};$$

whence we obtain

$$\left.\begin{aligned}&\int_0^{\frac{\pi}{2}}\sin^{m-1}\phi\cos^{n-m-1}\phi\cos n\phi\,d\phi=\frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}\cos\frac{m\pi}{2},\\&\int_0^{\frac{\pi}{2}}\sin^{m-1}\phi\cos^{n-m-1}\phi\sin n\phi\,d\phi=\frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}\sin\frac{m\pi}{2},\end{aligned}\right\}\begin{aligned}&n>m,\\&1>m>0.\end{aligned}\quad\text{(G)}$$

and taking $n=m+1$,

$$\left.\begin{aligned}&\int_0^{\frac{\pi}{2}}\sin^{n-2}\phi\cos n\phi\,d\phi=\frac{1}{n-1}\sin\frac{n\pi}{2},\\&\int_0^{\frac{\pi}{2}}\sin^{n-2}\phi\sin n\phi\,d\phi=-\frac{1}{n-1}\cos\frac{n\pi}{2},\end{aligned}\right\}(2>n>1).\ \ \ldots\ldots\text{(H)}$$

Replacing ϕ by $\frac{\pi}{2}-\phi$ in formulae (H), we derive

$$\left.\begin{aligned}&\int_0^{\frac{\pi}{2}}\cos^{n-2}\phi\cos n\phi\,d\phi=0,\\&\int_0^{\frac{\pi}{2}}\cos^{n-2}\phi\sin n\phi\,d\phi=\frac{1}{n-1},\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ \text{(I)}$$

that is the formulae (G) still hold good in the limiting case $m=1$.

1168. Since $\Gamma(m)\Gamma(1-m)=\frac{\pi}{\sin m\pi}$ formulae (G) may be written

$$\left.\begin{aligned}&\int_0^{\frac{\pi}{2}}\sin^{m-1}\phi\cos^{n-m-1}\phi\cos n\phi\,d\phi=\frac{\Gamma(n-m)}{\Gamma(n)\,\Gamma(1-m)}\,\frac{\pi}{2\sin\frac{m\pi}{2}},\\&\int_0^{\frac{\pi}{2}}\sin^{m-1}\phi\cos^{n-m-1}\phi\sin n\phi\,d\phi=\frac{\Gamma(n-m)}{\Gamma(n)\,\Gamma(1-m)}\,\frac{\pi}{2\cos\frac{m\pi}{2}},\end{aligned}\right\}\begin{aligned}&(n>m),\\&(1>m>0).\end{aligned}\quad\text{(J)}$$

When m diminishes indefinitely to zero, the limiting form of the first of these integrals is infinite. The second takes the limiting form

$$\int_0^{\frac{\pi}{2}} \cos^{n-1}\phi \frac{\sin n\phi}{\sin\phi} d\phi = \frac{\pi}{2}. \quad \text{..........................(K)}$$

It will be noted that the integral (K) is independent of n.

These results are given by M. Serret, *Calc. Intég.*, pp. 199 to 201.

Differentiating the equations

$$\left.\begin{aligned} \int_0^\infty x^{n-1} e^{-ax} \cos bx\, dx &= \frac{\cos n\theta}{r^n} \Gamma(n), \\ \int_0^\infty x^{n-1} e^{-ax} \sin bx\, dx &= \frac{\sin n\theta}{r^n} \Gamma(n), \end{aligned}\right\} \text{ where } r = \sqrt{a^2+b^2} \text{ and } \theta = \tan^{-1}\frac{b}{a},$$

with respect to n, we have

$$\int_0^\infty x^{n-1} e^{-ax} \cos bx \log x\, dx = \frac{\cos n\theta}{r^n} \frac{d\,\Gamma(n)}{dn} - \left(\frac{\theta \sin n\theta + \cos n\theta \log r}{r^n}\right) \Gamma(n),$$

$$\int_0^\infty x^{n-1} e^{-ax} \sin bx \log x\, dx = \frac{\sin n\theta}{r^n} \frac{d\,\Gamma(n)}{dn} + \left(\frac{\theta \cos n\theta - \sin n\theta \log r}{r^n}\right) \Gamma(n);$$

and eliminating $\dfrac{d\Gamma(n)}{dn}$,

$$\int_0^\infty x^{n-1} e^{-ax} \sin(n\theta - bx) \log\frac{1}{x}\, dx = \frac{\theta}{r^n} \Gamma(n);$$

and if $n=1$, $\quad \displaystyle\int_0^\infty e^{-ax} \sin(\theta - bx) \log\frac{1}{x}\, dx = \frac{\theta}{r}$

where $\quad r = \sqrt{a^2+b^2} \quad$ and $\quad \theta = \tan^{-1}\dfrac{b}{a}$.

Also $\dfrac{d\Gamma(n)}{dn}$ could be approximated to by means of the tables for $\log\Gamma(n)$ if required.

These results are due to Legendre (*Exercices*, p. 369).

1169. Fresnel's Integrals.

We have met the integrals

$$\int_0^\infty \cos\frac{\pi}{2}x^2\, dx = \int_0^\infty \sin\frac{\pi}{2}x^2\, dx = \frac{1}{2},$$

known as Fresnel's Integrals, in an earlier chapter, viz. in the tracing of Cornu's Spiral $ks^2=\psi$ (Art. 560). They are of importance in the Theory of Light. Students interested in the employment of the integrals in Physical Optics are referred to Verdet's *Œuvres*, tom. v., or to Preston's *Theory of Light*, where the various methods adopted in the construction of tables for their values between limits 0 and v will be found explained at length.

Preston gives in the form of examples with hints at solution a very excellent condensation of the chief results arrived at by various investigators—Fresnel, Gilbert, Cauchy, Knockenhauer and Cornu (Preston, *Theory of Light*, pages 220-223).

1170. We may consider shortly some modes of calculation of the more general integral

$$\int_0^v \cos\phi(x)\,dx, \quad \text{where } \phi(x) \equiv A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \dots.$$

Take first two near limits, a and $a+h$, where h is small.

Then $\displaystyle\int_a^{a+h} \cos\phi(x)\,dx = \int_0^h \cos\phi(a+y)\,dy$, by putting $x=a+y$,

$$= \int_0^h \cos\{\phi(a) + y\,\phi'(a)\}\,dy \text{ nearly,}$$

since y lies between 0 and h, and is therefore itself small,

$$= \frac{\sin\{\phi(a)+h\,\phi'(a)\} - \sin\phi(a)}{\phi'(a)} \text{ nearly.}$$

Hence, by taking the limits successively, 0 to h, h to $2h$, $2h$ to $3h$, etc., and adding the results, we may obtain a close approximation to $\displaystyle\int_0^{nh} \cos\phi(x)\,dx$, provided, of course, that $\phi(x)$ is such that $\phi'(x)=0$ has no root between 0 and nh.

1171. A closer approximation may be made as follows:

Since $$F(\mu+y) = F(\mu) + yF'(\mu) + \frac{y^2}{2!}F''(\mu) + \dots,$$

we have, by integration between limits $-\frac{h}{2}$ and $\frac{h}{2}$,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} F(\mu+y)\,dy = hF(\mu) + \frac{1}{3!}\frac{2h^3}{2^3}F''(\mu) + \frac{1}{5!}\frac{2h^5}{2^5}F^{(\text{iv})}(\mu) + \dots,$$

and if $$F(x) \equiv \cos\phi(x), \quad \mu = a + \frac{h}{2} \quad \text{and} \quad x = \mu + y,$$

$$\begin{aligned}\int_a^{a+h} \cos\phi(x)\,dx &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \cos\phi(\mu+y)\,dy \\ &= h\cos\phi(\mu) + \frac{1}{3!}\frac{h^3}{4}\frac{d^2}{d\mu^2}\cos\phi(\mu) + \frac{1}{5!}\frac{h^5}{16}\frac{d^4}{d\mu^4}\cos\phi(\mu) + \dots \\ &= h\cos\phi(\mu) - \frac{h^3}{4!}[\cos\phi(\mu)\,\phi'^2(\mu) + \sin\phi(\mu)\,\phi''(\mu)] + \dots,\end{aligned}$$

from which result we may proceed as before, taking limits 0 to h, h to $2h$, $2h$ to $3h$, etc., and adding the several results.

1172. Fresnel's calculations were based in the manner described above upon a preliminary consideration of the integrals

$$\int_v^{v+h} \cos\frac{\pi x^2}{2}\,dx, \quad \int_v^{v+h} \sin\frac{\pi x^2}{2}\,dx,$$

where the interval h is so small that its square can be rejected.

In this case, putting $x=v+z$,

$$\int_v^{v+h} \cos\frac{\pi x^2}{2}\,dx = \int_0^h \cos\frac{\pi}{2}(v^2+2vz)\,dz = \frac{1}{\pi v}\left[\sin\frac{\pi}{2}(v^2+2vh) - \sin\frac{\pi v^2}{2}\right]$$

and

$$\int_v^{v+h} \sin\frac{\pi x^2}{2}\,dx = \int_0^h \sin\frac{\pi}{2}(v^2+2vz)\,dz = -\frac{1}{\pi v}\left[\cos\frac{\pi}{2}(v^2+2vh) - \cos\frac{\pi v^2}{2}\right].$$

Then taking as intervals $h=\frac{1}{10}$, and making v in succession 0, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, ..., the values of the integrals were approximated to.

1173. The integrals

$$\int_0^v \cos\frac{\pi v^2}{2}\,dv, \quad \int_0^v \sin\frac{\pi v^2}{2}\,dv \quad \text{or} \quad \int_v^\infty \cos\frac{\pi v^2}{2}\,dv, \quad \int_v^\infty \sin\frac{\pi v^2}{2}\,dv$$

may each be expressed in the form $X\cos\frac{\pi v^2}{2} + Y\sin\frac{\pi v^2}{2}$, where X and Y are series of ascending powers of v, in integrating from 0 to v; or descending powers of v when the integration extends from v to infinity. In both cases the integration is performed by "Parts."

In integrating from 0 to v we proceed as follows:

$$\int_0^v \cos\frac{\pi v^2}{2}\,dv = \left[v\cos\frac{\pi v^2}{2}\right]_0^v + \pi\int_0^v v^2\sin\frac{\pi v^2}{2}\,dv,$$

$$\int_0^v v^2\sin\frac{\pi v^2}{2}\,dv = \left[\frac{v^3}{3}\sin\frac{\pi v^2}{2}\right]_0^v - \frac{\pi}{3}\int_0^v v^4\cos\frac{\pi v^2}{2}\,dv,$$

$$\int_0^v v^4\cos\frac{\pi v^2}{2}\,dv = \left[\frac{v^5}{5}\cos\frac{\pi v^2}{2}\right]_0^v + \frac{\pi}{5}\int_0^v v^6\sin\frac{\pi v^2}{2}\,dv,$$

$$\int_0^v v^6\sin\frac{\pi v^2}{2}\,dv = \left[\frac{v^7}{7}\sin\frac{\pi v^2}{2}\right]_0^v - \frac{\pi}{7}\int_0^v v^8\cos\frac{\pi v^2}{2}\,dv,$$

etc.

Hence multiplying by 1, π, $\frac{-\pi^2}{1.3}$, $\frac{-\pi^3}{1.3.5}$, $\frac{\pi^4}{1.3.5.7}$, $\frac{\pi^5}{1.3.5.7.9}$, etc., and adding,

$$\int_0^v \cos\frac{\pi v^2}{2}\,dv = \cos\frac{\pi v^2}{2}\left[\frac{v}{1} - \frac{\pi^2 v^5}{1.3.5} + \frac{\pi^4 v^9}{1.3.5.7.9} - \dots\right]$$

$$+\sin\frac{\pi v^2}{2}\left[\frac{\pi v^3}{1.3} - \frac{\pi^3 v^7}{1.3.5.7} + \frac{\pi^5 v^{11}}{1.3.5.7.9.11} - \dots\right]$$

$$= X\cos\frac{\pi v^2}{2} + Y\sin\frac{\pi v^2}{2}, \text{ say,} \quad \text{.........................(1)}$$

and proceeding in the same way with $\int_0^v \sin\frac{\pi v^2}{2}\,dv$,

$$\int_0^v \sin\frac{\pi v^2}{2}\,dv = -Y\cos\frac{\pi v^2}{2} + X\sin\frac{\pi v^2}{2},$$

and the sum of the squares of the integrals (which gives a measure of the intensity of illumination in a certain case in Physical Optics*) is X^2+Y^2.

It is interesting to note that the series X, Y satisfy the equations

$$\frac{dX}{dv}+\pi vY=1,\quad \frac{dY}{dv}-\pi vX=0,$$

i.e. $$\frac{1}{v}\frac{d}{dv}\left(\frac{1}{v}\frac{d}{dv}\right)X+\pi^2X=-\frac{1}{v^3}\quad\text{and}\quad \frac{1}{v}\frac{d}{dv}\left(\frac{1}{v}\frac{d}{dv}\right)Y+\pi^2Y=\frac{\pi}{v},$$

or $$\left[\left(\frac{d}{dv^2}\right)^2+\frac{\pi^2}{4}\right]X=-\frac{1}{4v^3},\quad \left[\left(\frac{d}{dv^2}\right)^2+\frac{\pi^2}{4}\right]Y=\frac{\pi}{4v}.$$

1174. If it be desired to express the integrals with limits v to ∞ in descending powers of v, the integration by parts must be conducted in the opposite order. Thus

$$\int_v^\infty \cos\frac{\pi v^2}{2}\,dv=\int_v^\infty \frac{1}{\pi v}\left(\pi v\cos\frac{\pi v^2}{2}\right)dv=\left[\frac{1}{\pi v}\sin\frac{\pi v^2}{2}\right]_v^\infty+\int_v^\infty\frac{1}{\pi v^2}\sin\frac{\pi v^2}{2}\,dv,$$

$$\int_v^\infty \frac{1}{\pi v^2}\sin\frac{\pi v^2}{2}\,dv=\int_v^\infty \frac{1}{\pi^2 v^3}\left(\pi v\sin\frac{\pi v^2}{2}\right)dv=\left[-\frac{1}{\pi^2 v^3}\cos\frac{\pi v^2}{2}\right]_v^\infty-3\int_v^\infty\frac{1}{\pi^2 v^4}\cos\frac{\pi v^2}{2}\,dv,$$

$$\int_v^\infty \frac{1}{\pi^2 v^4}\cos\frac{\pi v^2}{2}\,dv=\int_v^\infty \frac{1}{\pi^3 v^5}\left(\pi v\cos\frac{\pi v^2}{2}\right)dv=\left[\frac{1}{\pi^3 v^5}\sin\frac{\pi v^2}{2}\right]_v^\infty+5\int_v^\infty\frac{1}{\pi^3 v^6}\sin\frac{\pi v^2}{2}\,dv,$$

$$\int_v^\infty \frac{1}{\pi^3 v^6}\sin\frac{\pi v^2}{2}\,dv=\int_v^\infty \frac{1}{\pi^4 v^7}\left(\pi v\sin\frac{\pi v^2}{2}\right)dv=\left[-\frac{1}{\pi^4 v^7}\cos\frac{\pi v^2}{2}\right]_v^\infty-7\int_v^\infty\frac{1}{\pi^4 v^8}\cos\frac{\pi v^2}{2}\,dv,$$

etc.

Hence multiplying by 1, 1, -1.3, $-1.3.5$, $+1.3.5.7$, etc, and adding,

$$\int_v^\infty \cos\frac{\pi}{2}v^2dv=\sin\frac{\pi v^2}{2}\left(-\frac{1}{\pi v}+\frac{1.3}{\pi^3v^5}-\frac{1.3.5.7}{\pi^5v^9}+\dots\right)$$
$$+\cos\frac{\pi v^2}{2}\left(\frac{1}{\pi^2v^3}-\frac{1.3.5}{\pi^4v^7}+\frac{1.3.5.7.9}{\pi^6v^{11}}-\dots\right)$$
$$=X'\cos\frac{\pi v^2}{2}-Y'\sin\frac{\pi v^2}{2},\text{ say,} \qquad\qquad (2)$$

where $$X'=\frac{1}{\pi^2v^3}-\frac{1.3.5}{\pi^4v^7}+\text{etc.}\quad\text{and}\quad Y'=\frac{1}{\pi v}-\frac{1.3}{\pi^3v^5}+\text{etc.};$$

and similarly $$\int_v^\infty \sin\frac{\pi v^2}{2}\,dv=Y'\cos\frac{\pi v^2}{2}+X'\sin\frac{\pi v^2}{2}.$$

And, as before, the sum of the squares of the integrals is $X'^2+Y'^2$.

Also X', Y' satisfy the differential equations

$$\frac{dX'}{dv}=\pi vY'-1,\quad \frac{dY'}{dv}=-\pi vX',$$

i.e. $$\frac{1}{v}\frac{d}{dv}\frac{1}{v}\frac{dX'}{dv}+\pi^2X'=\frac{1}{v^3},\quad \frac{1}{v}\frac{d}{dv}\frac{1}{v}\frac{d}{dv}Y'+\pi^2Y'=\frac{\pi}{v},$$

or $$\left[\left(\frac{d}{dv^2}\right)^2+\frac{\pi^2}{4}\right]X'=\frac{1}{4v^3},\quad \left[\left(\frac{d}{dv^2}\right)^2+\frac{\pi^2}{4}\right]Y'=\frac{\pi}{4v}.$$

We also obviously have

$$\int_v^\infty \cos\frac{\pi v^2}{2}\,dv=\int_0^\infty\cos\frac{\pi v^2}{2}\,dv-\int_0^v\cos\frac{\pi v^2}{2}\,dv=\frac{1}{2}-X\cos\frac{\pi v^2}{2}-Y\sin\frac{\pi v^2}{2};$$

* See Preston's *Light*.

and similarly

$$\int_v^\infty \sin\frac{\pi v^2}{2}\,dv=\int_0^\infty \sin\frac{\pi v^2}{2}\,dv-\int_0^v \sin\frac{\pi v^2}{2}\,dv=\frac{1}{2}+Y\cos\frac{\pi v^2}{2}-X\sin\frac{\pi v^2}{2}.$$

Also

$$\int_0^v \cos\frac{\pi v^2}{2}\,dv=\int_0^\infty \cos\frac{\pi v^2}{2}\,dv-\int_v^\infty \cos\frac{\pi v^2}{2}\,dv=\frac{1}{2}-X'\cos\frac{\pi v^2}{2}+Y'\sin\frac{\pi v^2}{2},$$

$$\int_0^v \sin\frac{\pi v^2}{2}\,dv=\int_0^\infty \sin\frac{\pi v^2}{2}\,dv-\int_v^\infty \sin\frac{\pi v^2}{2}\,dv=\frac{1}{2}-Y'\cos\frac{\pi v^2}{2}-X'\sin\frac{\pi v^2}{2}.$$

1175. The expansion (1) in ascending powers of v is due to Knockenhauer.* The expansion (2) in descending powers of v is due to Cauchy.†

For the student of the Integral Calculus, perhaps the most interesting of Mr. Preston's quotations is one which expresses Cauchy's series of the last article in the form of definite integrals. These expressions are quoted from the investigations of Gilbert, published in the *Mémoires couronnés de l'Acad. de Bruxelles*, tom. xxxi., p. 1.

Writing $\frac{\pi v^2}{2}=u$, we have

$$\int_0^v \cos\frac{\pi v^2}{2}\,dv=\frac{1}{\sqrt{2\pi}}\int_0^u \frac{\cos u}{\sqrt{u}}\,du,\qquad \int_0^v \sin\frac{\pi v^2}{2}\,dv=\frac{1}{\sqrt{2\pi}}\int_0^u \frac{\sin u}{\sqrt{u}}\,du.$$

Also $$\int_0^\infty x^{-\frac{1}{2}}e^{-ux}\,dx=\frac{\Gamma(\frac{1}{2})}{u^{\frac{1}{2}}}=\frac{\sqrt{\pi}}{\sqrt{u}};$$

$$\therefore\quad \int_0^v \cos\frac{\pi v^2}{2}\,dv=\frac{1}{\sqrt{2\pi}}\int_0^u \cos u\left[\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-ux}}{\sqrt{x}}\,dx\right]du,$$

i.e. $$\frac{1}{\pi\sqrt{2}}\int_0^u\int_0^\infty \frac{e^{-ux}\cos u}{\sqrt{x}}\,du\,dx,$$

or changing the order of integration, which does not alter the limits,

$$=\frac{1}{\pi\sqrt{2}}\int_0^\infty\int_0^u \frac{1}{\sqrt{x}}e^{-ux}\cos u\,dx\,du$$

$$=\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{1}{\sqrt{x}}\left[-e^{-ux}\frac{x\cos u-\sin u}{1+x^2}\right]_0^u dx$$

$$=\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{1}{\sqrt{x}}\left[\frac{x}{1+x^2}-e^{-ux}\frac{x\cos u-\sin u}{1+x^2}\right]du$$

$$=\frac{1}{\pi\sqrt{2}}\left[\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx-\cos u\int_0^\infty \frac{e^{-ux}\sqrt{x}}{1+x^2}\,dx+\sin u\int_0^\infty \frac{e^{-ux}}{\sqrt{x}(1+x^2)}\,dx\right].$$

Now $$\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx=\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\,d\theta,\ \text{by putting } x=\tan\theta,$$

$$=\int_0^{\frac{\pi}{4}} (\sqrt{\tan\theta}+\sqrt{\cot\theta})\,d\theta,$$

$$=\frac{\pi}{\sqrt{2}},\ \text{by Ex. 8, p. 162, Vol I.}$$

* Knockenhauer, *Die Undulationstheorie des Lichts*, p. 36; Preston, *Theory of Light*, p. 220.

† Cauchy, *Comptes Rendus*, tom. xv. 534, 573.

Hence

$$\left.\begin{aligned}\int_0^v \cos\frac{\pi v^2}{2}\,dv &= \frac{1}{2} - \cos\frac{\pi v^2}{2}\cdot\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{x^{\frac{1}{2}}e^{-ux}}{1+x^2}\,dx + \sin\frac{\pi v^2}{2}\,\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{x^{-\frac{1}{2}}e^{-ux}}{1+x^2}\,dx;\\ &\text{and similarly}\\ \int_0^v \sin\frac{\pi v^2}{2}\,dv &= \frac{1}{2} - \cos\frac{\pi v^2}{2}\cdot\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{x^{-\frac{1}{2}}e^{-ux}}{1+x^2}\,dx - \sin\frac{\pi v^2}{2}\,\frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{x^{\frac{1}{2}}e^{-ux}}{1+x^2}\,dx,\end{aligned}\right\}$$

where $u=\dfrac{\pi v^2}{2}$; which express Cauchy's series X', Y' in the respective definite integral forms

$$X' \equiv \frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{x^{\frac{1}{2}}e^{-ux}}{1+x^2}\,dx \quad\text{and}\quad Y' \equiv \frac{1}{\pi\sqrt{2}}\int_0^\infty \frac{x^{-\frac{1}{2}}e^{-ux}}{1+x^2}\,dx.$$

1176. Several other interesting relations amongst these integrals are given by Mr. Preston, to whose book the reader is referred.

A table of the values of Fresnel's integrals, as given by Gilbert, is quoted in Art. 1177 from Mr. Preston's book. The table is carried up to $v=5{\cdot}0$. The oscillatory character of the results is exhibited in the graph of the Cornu Spiral in Art. 560.

1177. Gilbert's Tables of Fresnel's Integrals. Quoted from Preston's *Theory of Light*.

v	$\int_0^v \cos\frac{\pi v^2}{2}\,dv$	$\int_0^v \sin\frac{\pi v^2}{2}\,dv$	v	$\int_0^v \cos\frac{\pi v^2}{2}\,dv$	$\int_0^v \sin\frac{\pi v^2}{2}\,dv$
0·0	0·0000	0·0000	2·6	0·3389	0·5500
0·1	0·0999	0·0005	2·7	0·3926	0·4529
0·2	0·1999	0·0042	2·8	0·4675	0·3915
0·3	0·2994	0·0141	2·9	0·5624	0·4102
0·4	0·3975	0·0334	3·0	0·6057	0·4963
0·5	0·4923	0·0647	3·1	0·5616	0·5818
0·6	0·5811	0·1105	3·2	0·4663	0·5933
0·7	0·6597	0·1721	3·3	0·4057	0·5193
0·8	0·7230	0·2493	3·4	0·4385	0·4297
0·9	0·7648	0·3398	3·5	0·5326	0·4153
1·0	0·7799	0·4383	3·6	0·5880	0·4923
1·1	0·7638	0·5365	3·7	0·5419	0·5750
1·2	0·7154	0·6234	3·8	0·4481	0·5656
1·3	0·6386	0·6863	3·9	0·4223	0·4752
1·4	0·5431	0·7135	4·0	0·4984	0·4205
1·5	0·4453	0·6975	4·1	0·5737	0·4758
1·6	0·3655	0·6383	4·2	0·5417	0·5632
1·7	0·3238	0·5492	4·3	0·4494	0·5540
1·8	0·3363	0·4509	4·4	0·4383	0·4623
1·9	0·3945	0·3734	4·5	0·5258	0·4342
2·0	0·4883	0·3434	4·6	0·5672	0·5162
2·1	0·5814	0·3743	4·7	0·4914	0·5669
2·2	0·6362	0·4556	4·8	0·4338	0·4968
2·3	0·6268	0·5525	4·9	0·5002	0·4351
2·4	0·5550	0·6197	5·0	0·5636	0·4992
2·5	0·4574	0·6192	∞	0·5000	0·5000

1178. Soldner's Function.

The integral $y \equiv \int_0^x \frac{dx}{\log x}$ is known as Soldner's Integral. It is denoted by the symbol $\mathrm{li}(x)$, which is Soldner's original notation. The letters li are suggested by the phrase 'logarithm-integral.'

It is obvious that the integrand has an infinity when $x=1$. Hence, in accordance with the theory of Principal Values (Chapter IX.), when the upper limit is greater than unity, we shall understand this integration to mean

$$Lt_{\epsilon=\eta=0}\left(\int_0^{1-\epsilon} + \int_{1+\eta}^x\right)\frac{dx}{\log x},$$

where ϵ, η are made to diminish indefinitely in a ratio of equality.

1179. Properties of the Function.

It follows that $\frac{d}{dx}\mathrm{li}(x) = \frac{1}{\log x}$. Hence

$$\frac{d}{dx}\mathrm{li}(x^{m+1}) = \frac{(m+1)x^m}{\log x^{m+1}} = \frac{x^m}{\log x}, \qquad \frac{d}{dx}\mathrm{li}(a+bx) = \frac{b}{\log(a+bx)},$$

$$\frac{d}{dx}\mathrm{li}(e^x) = \frac{e^x}{\log e^x} = \frac{e^x}{x}, \qquad \frac{d}{dx}\mathrm{li}(e^{-x}) = \frac{-e^{-x}}{\log e^{-x}} = \frac{e^{-x}}{x},$$

$$\frac{d}{dx}\mathrm{li}(e^{a+x}) = \frac{e^a e^x}{\log e^{a+x}} = \frac{e^a e^x}{a+x}, \qquad \frac{d}{dx}\mathrm{li}(\sin x) = \frac{\cos x}{\log \sin x}, \text{ etc.}$$

Hence conversely we may express certain integrals in terms of a Soldner's function, viz.

$$\int \frac{x^m}{\log x}dx = \mathrm{li}(x^{m+1}) + C, \quad \text{or between limits } \int_b^a \frac{x^m}{\log x}dx = \mathrm{li}(a^{m+1}) - \mathrm{li}(b^{m+1}),$$

$$\int \frac{dx}{\log(a+bx)} = \frac{\mathrm{li}(a+bx)}{b} + C, \quad \text{or between limits } \int_{p_2}^{p_1} \frac{dx}{\log(a+bx)} = \frac{\mathrm{li}(a+bp_1) - \mathrm{li}(a+bp}{b}$$

$$\int \frac{e^x}{x}dx = \mathrm{li}(e^x) + C, \quad i.e. \int_b^a \frac{e^x}{x}dx = \mathrm{li}(e^a) - \mathrm{li}(e^b), \text{ and so on.}$$

1180. To enable the arithmetical calculations of such results to be made, Soldner constructed a table of the values of $\mathrm{li}(x)$ to seven decimal places for values of x, from $x = {\cdot}00$ to $x = 1{\cdot}00$, at the latter of which the function is infinite, the values being negative; and a further table of the values of $\mathrm{li}\,x$, giving the values to seven places, for $x = 1, 1{\cdot}1, 1{\cdot}2, 1{\cdot}3, 1{\cdot}4$, which are negative, and $1{\cdot}5, 1{\cdot}6, \ldots, 2, 2{\cdot}5, 3, 4, 5, \ldots, 20$, which are positive, and at certain intervals from 22 to 1220, all taken to eight significant figures.

It is unnecessary to give the tables here. They will be found reproduced in De Morgan's *Diff. and Int. Calculus*, pages 662 and 663. A few extracts from these tables will indicate the shape of the graph :

x	li (x) (–)
·00	·000
·05	·013
·10	·032
·15	·056
·20	·085
·25	·119
·30	·157
·40	·253
·50	·379

x	li (x) (–)
·60	·547
·70	·781
·80	1·134
·90	1·776
·95	2·444
·98	3·345
·99	4·033
1·00	∞

x	li (x) (–)
1·0	∞
1·1	1·676
1·2	0·934
1·3	0·480
1·4	0·145

x	li (x) (+)
1·5	0·125
1·6	0·354
1·8	0·733
2·0	1·045
2·5	1·667
3·0	2·164
4·0	2·968
5·0	3·635
10·0	6·166

x	li (x) (+)
20·0	9·905
30·0	13·023
40·0	15·840
100·0	30·126
200·0	50·192
400·0	85·4
600	117·6
1040	183·4
1220	217·4

The march of the function can then be seen to be as represented by the accompanying graph.

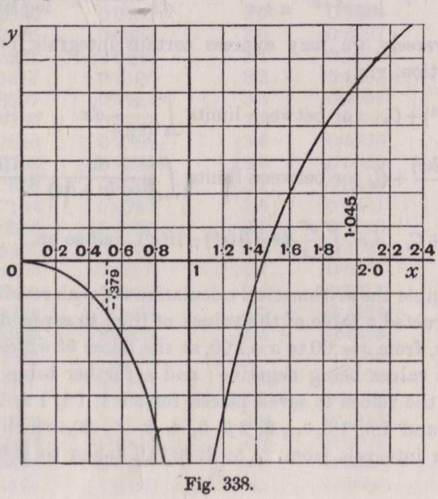

Fig. 338.

1181. Method of Computation.

We proceed to show how these values were computed.

It will be seen that, by putting $x=e^{-y}$ or $x=e^{y}$, the integral $\int_0^a \frac{dx}{\log x}$ can be thrown into the forms $-\int_{-\log a}^{\infty} e^{-y}\frac{dy}{y}$ or $\int_{-\infty}^{\log a} e^{y}\frac{dy}{y}$.

Now, so long as n is greater than zero, we have by expansion

$$\int_v^\infty x^{n-1}e^{-x}\,dx=\int_v^\infty x^{n-1}\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots\right)dx$$

$$=C-\frac{v^n}{n}+\frac{v^{n+1}}{(n+1)1!}-\frac{v^{n+2}}{(n+2)2!}+\frac{v^{n+3}}{(n+3)3!}-\dots,$$

where C is to be found. The series is convergent for all positive values of v and does not become infinite with v. Also, when $v=0$, the value of the integral is $\Gamma(n)$. Hence $C=\Gamma(n)$.

Hence $\int_v^\infty x^{n-1}e^{-x}\,dx=\Gamma(n)-\frac{v^n}{n}+\frac{v^{n+1}}{(n+1)1!}-\frac{v^{n+2}}{(n+2)2!}+\dots.$

This may be arranged as

$$\int_v^\infty x^{n-1}e^{-x}\,dx=\Gamma(n)-\frac{1}{n}-\frac{v^n-1}{n}+\frac{v^{n+1}}{(n+1)1!}-\frac{v^{n+2}}{(n+2)2!}+\dots$$

$$=\frac{\Gamma(n+1)-1}{n}-\frac{v^n-1}{n}+\frac{v^{n+1}}{(n+1)1!}-, \text{ etc.}$$

Now, if we make n diminish indefinitely, $Lt\,\frac{v^n-1}{n}=\log v$, and $Lt\,\frac{\Gamma(n+1)-1}{n}$ is the limit, when $n=0$, of $\frac{\Gamma(x+n)-\Gamma(x)}{n}$ for the value $x=1$, *i.e.*

$$\left[\frac{d}{dx}\Gamma(x)\right]_{x=1} \quad\text{or}\quad \Gamma'(1),$$

or as $\Gamma(1)=1$, this is the same as $\left[\frac{d}{dx}\log\Gamma(x)\right]_{x=1}$, *i.e.* $-\gamma$, where γ is Euler's Constant.

Hence $\int_v^\infty \frac{e^{-x}}{x}\,dx=-\gamma-\log v+\frac{v}{1.1!}-\frac{v^2}{2.2!}+\frac{v^3}{3.3!}-\dots,$ ....(A)

Hence we have, putting $v=\log a$,

$$\operatorname{li}\left(\frac{1}{a}\right)=-\int_{\log a}^\infty \frac{e^{-x}}{x}\,dx=\gamma+\log(\log a)-\frac{\log a}{1.1!}+\frac{(\log a)^2}{2.2!}-\frac{(\log a)^3}{3.3!}+\dots \quad (a>1), \tag{B}$$

Again, by expansion,

$$-\int_{-\log a}^{-\epsilon}\frac{e^{-x}}{x}\,dx=\log\left(\frac{\log a}{\epsilon}\right)+\frac{\log a-\epsilon}{1.1!}+\frac{(\log a)^2-\epsilon^2}{2.2!}+\dots \quad (a>1)$$

and $-\int_\eta^\infty \frac{e^{-x}}{x}\,dx \quad =\gamma+\log\eta-\frac{\eta}{1.1!}+\frac{\eta^2}{2.2!}-\dots,$

and upon addition, diminishing ϵ and η indefinitely in a ratio of equality, the Principal Value of $\operatorname{li}(a)$ is given by

$$\operatorname{li}(a)=-\int_{-\log a}^\infty \frac{e^{-x}}{x}\,dx=-Lt\left(\int_{-\log a}^{-\epsilon}+\int_\eta^\infty\right)\frac{e^{-x}}{x}\,dx, \text{ where } \epsilon=\eta=0,$$

$$=\gamma+\log(\log a)+\frac{\log a}{1.1!}+\frac{(\log a)^2}{2.2!}+\frac{(\log a)^3}{3.3!}+\dots \quad (a>1). \text{ ....(C)}$$

As there is manifest discontinuity when $a=1$, and the Principal Value is taken in integrating over the discontinuity in the second case, formula (C) will not be derivable from formula (B) by putting $\frac{1}{a}$ for a in the former. It will be observed, however, that the two series then only differ by $\log(-1)$, which is the effect of the discontinuity.

By means of the expansion of

$$\frac{1}{\log(1+x)}=\frac{1}{x}\frac{1}{1-\frac{x}{2}+\frac{x^2}{3}-\dots}$$
$$=x^{-1}+K_1+K_2x+K_3x^2+K_4x^3+\dots,$$

where the coefficients may be calculated either by actual division or by multiplying up by $x\left(1-\frac{x}{2}+\dots\right)$ and equating coefficients, giving

$$K_1=\tfrac{1}{2},\quad K_2=-\tfrac{1}{12},\quad K_3=\tfrac{1}{24},\quad K_4=-\tfrac{19}{720},\quad K_5=\tfrac{3}{160},\quad K_6=-\tfrac{863}{60480},\text{ etc.,}$$

we have, $a<1$,

$$\text{li}(1-a)=\int_0^{1-a}\frac{dz}{\log z}=\int_{-1}^{-a}\frac{dx}{\log(1+x)}=\log a-K_1(a-1)+\tfrac{1}{2}K_2(a^2-1)-\text{etc.};$$

and by Art. 944, putting $e^{-\beta}=v$,

$$\gamma=Lt_{b=1}\left\{\int_0^b\frac{dv}{1-v}+\int_0^b\frac{dv}{\log v}\right\}$$
$$=Lt_{b=1}\{\text{li}\,b-\log(1-b)\}=Lt_{a=0}\{\text{li}(1-a)-\log a\}=K_1-\frac{K_2}{2}+\frac{K_3}{3}-\dots,$$

whence $$\text{li}(1-a)=\gamma+\log a-K_1a+\frac{K_2}{2}a^2-\dots.\qquad\text{(D)}$$

Again $\text{li}(1+a)=\text{Prin. Val. of}\int_0^{1+a}\frac{dz}{\log z}=\text{P.V. of}\int_{-1}^{a}\frac{dz}{\log(1+z)}$

$$=Lt_{\epsilon=0}\left(\int_{-1}^{-\epsilon}+\int_{\epsilon}^{a}\right)\frac{dz}{\log(1+z)}$$
$$=Lt_{\epsilon=0}\,[(\gamma+\log\epsilon-K_1\epsilon+\tfrac{1}{2}K_2\epsilon^2-\dots)$$
$$+\{\log a-\log\epsilon+K_1(a-\epsilon)+\tfrac{1}{2}K_2(a^2-\epsilon^2)+\dots\}];$$
$$\therefore\ \text{li}(1+a)=\gamma+\log a+K_1a+K_2\frac{a^2}{2}+\dots.\qquad\text{(E)}$$

Also, by Taylor's Theorem,

$$\text{li}(a+x)=\text{li}(a)+x(\log a)^{-1}+\frac{d}{da}(\log a)^{-1}\frac{x^2}{2!}+\frac{d^2}{da^2}(\log a)^{-1}\frac{x^3}{3!}+\dots.$$

Other results will be found in De Morgan's *Differential and Int. Calc.*, pages 660 to 664. By aid of these series Soldner calculated the numerical values of the table for the function $\text{li}(a)\equiv\int_0^a\frac{dx}{\log x}$.

We may therefore now regard such functions as

$$\frac{1}{\log x},\quad\frac{x^m}{\log x},\quad\frac{e^x}{x},\quad\frac{\cosh x}{x},\quad\frac{e^x}{x+a},\text{ etc.,}$$

as integrable in terms of Soldner's function, and therefore their integrals calculable by means of his table, for assigned values of the limits.

1182. Frullani's Theorem: Elliott's and Leudesdorf's Extensions.

Suppose $F(xy)$ a function of the product xy of the coordinates of a point in the plane of x, y lying in the region bounded by the y-axis, an ordinate at infinity and the two straight lines $y=a$ and $y=b$ parallel to the x-axis. Let a and b be supposed of the same sign. Let $F(z)$ and $F'(z)$, where $z=xy$, be finite and continuous functions for all points in this region and also along the boundaries.

Suppose also that $F(xy)$ takes definite finite values at $x=0$ and at $x=\infty$ from the value $y=b$ to $y=a$ inclusive, and

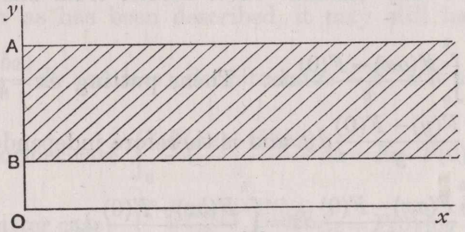

Fig. 339.

denote them by $F(0)$ and $F(\infty)$ respectively. Consider the surface integral of $F'(xy)$ over this region. This is expressed by $\int_0^\infty \int_b^a F'(xy)\,dx\,dy$, or, what is the same thing, $\int_b^a \int_0^\infty F'(xy)\,dy\,dx$.

The first form of the integral is

$$=\int_0^\infty \frac{\Big[F(xy)\Big]_{y=b}^{y=a}}{x}\,dx=\int_0^\infty \frac{F(ax)-F(bx)}{x}\,dx.$$

The second form of the integral is

$$=\int_b^a \frac{\Big[F(xy)\Big]_{x=0}^{x=\infty}}{y}\,dy=[F(\infty)-F(0)]\int_b^a \frac{dy}{y}$$

$$=[F(\infty)-F(0)]\log\frac{a}{b}.$$

Hence it appears that

$$\int_0^\infty \frac{F(ax)-F(bx)}{x}\,dx=[F(\infty)-F(0)]\log\frac{a}{b}. \quad \text{.........(1)}$$

Similarly, if we integrate over the region bounded by

$$x=-\infty,\quad x=0,\quad y=a,\quad y=b,$$

we obtain in the same manner

$$\int_{-\infty}^{0}\frac{F(ax)-F(bx)}{x}dx=[F(0)-F(-\infty)]\log\frac{a}{b},\ \ldots\ldots\ldots(2)$$

provided $F(xy)$ takes a definite value $F(-\infty)$ at $x=-\infty$.

In cases where $F(\infty)=0$ or $F(0)=0$ the theorem takes the simpler forms $\int_0^\infty\frac{F(ax)-F(bx)}{x}dx=F(0)\log\frac{b}{a}$ or $F(\infty)\log\frac{a}{b}$ respectively.

1183. We may examine these results from another point of view.

Let $u\equiv\int_0^{\frac{h}{a}}\frac{F(ax)-F(0)}{x}dx$. Then, putting $ax=y$, $\frac{dx}{x}=\frac{dy}{y}$,

and $u=\int_0^h\frac{F(y)-F(0)}{y}dy$, and is therefore independent of a.

Hence $\int_0^{\frac{h}{a}}\frac{F(ax)-F(0)}{x}dx=\int_0^{\frac{h}{b}}\frac{F(bx)-F(0)}{x}dx$

$$=\int_0^{\frac{h}{a}}\frac{F(bx)}{x}dx-\int_{\frac{h}{b}}^{\frac{h}{a}}\frac{F(bx)}{x}dx-\int_0^{\frac{h}{b}}\frac{F(0)}{x}dx.$$

Therefore $\int_0^{\frac{h}{a}}\frac{F(ax)-F(bx)}{x}dx+\int_{\frac{h}{b}}^{\frac{h}{a}}\frac{F(bx)}{x}dx=F(0)\int_{\frac{h}{b}}^{\frac{h}{a}}\frac{dx}{x}$

$$=F(0)\log\frac{b}{a}.$$

Now, in the second integral, viz. $\int_{\frac{h}{b}}^{\frac{h}{a}}\frac{F(bx)}{x}dx$, both limits become infinite, when h is indefinitely increased, but they are separated by an infinite interval $\frac{h}{a}-\frac{h}{b}=\frac{b-a}{ab}h$. Hence it cannot be assumed that this integral vanishes, and it must be investigated in each case.

If, however, $F(bx)$ tends to take a definite finite value $F(\infty)$ when x is increased indefinitely, let its value between the limits $\frac{h}{b}$ and $\frac{h}{a}$ be called $F(\infty)+\epsilon$, where ϵ is ultimately an infinitesimal, and let ϵ_1 and ϵ_2 be the greatest and least values of ϵ for values of x between $\frac{h}{b}$ and $\frac{h}{a}$. Thus $\int_{\frac{h}{b}}^{\frac{h}{a}} \frac{F(bx)}{x}dx$ lies between

$$(F(\infty)+\epsilon_1)\log\frac{b}{a} \quad\text{and}\quad (F(\infty)+\epsilon_2)\log\frac{b}{a},$$

and therefore in the limit becomes $F(\infty)\log\frac{b}{a}$, and the theorem becomes

$$\int_0^\infty \frac{F(ax)-F(bx)}{x}dx=[F(\infty)-F(0)]\log\frac{a}{b}.$$

But supposing $F(bx)$ not to take up a definite limiting value such as has been described, it may still happen that $Lt_{h=\infty}\int_{\frac{h}{b}}^{\frac{h}{a}}\frac{F(bx)}{x}dx$ assumes a definite value $-K$, or it may vanish.

In the former case $\int_0^\infty \frac{F(ax)-F(bx)}{x}dx=K-F(0)\log\frac{a}{b}$.

In the latter case $\int_0^\infty \frac{F(ax)-F(bx)}{x}dx=F(0)\log\frac{b}{a}$.

The formula $\int_0^\infty \frac{F(ax)-F(bx)}{x}dx=F(0)\log\frac{b}{a}$ is known as Frullani's Theorem. According to Dr. Williamson it was communicated by Frullani to Plana in 1821, and subsequently published in *Mem. del. Soc. Ital.*, 1828.

The more general form

$$\int_0^\infty \frac{F(ax)-F(bx)}{x}dx=[F(\infty)-F(0)]\log\frac{a}{b}$$

is due to Prof. E. B. Elliott (*Educational Times*, 1875).*

1184. As examples we may take

1. $\int_0^\infty \frac{\tan^{-1}ax-\tan^{-1}bx}{x}dx=(\tan^{-1}\infty-\tan^{-1}0)\log\frac{a}{b}=\frac{\pi}{2}\log\frac{a}{b}$.

2. $\int_0^\infty \log\frac{p+qe^{-ax}}{p+qe^{-bx}}\frac{dx}{x}=\{\log p-\log(p+q)\}\log\frac{a}{b}=\log\left(1+\frac{q}{p}\right)\log\frac{b}{a}$.

These two examples are given by Bertrand, but arrived at in a different manner.

* Both references are due to Prof. Williamson, pages xi and 156, *Int. Calc.*

Now, consider the integral $[ac]-[bc]-[ad]+[bd]$, or, as it may be written for short, $[(a-b)(c-d)]$.

By two applications of the above theorem this becomes

$$\int_0^\infty \int_0^\infty [S(ax, cy)-S(bx, cy)-S(ax, dy)+S(bx, dy)]\frac{dx\,dy}{xy}$$

$$=\int_0^\infty (\alpha-\beta)[S(\infty, cy)-S(0, cy)]\frac{dy}{y} - \int_0^\infty (\alpha-\beta)[S(\infty, dy)-S(0, dy)]\frac{dy}{y}$$

$$=(\alpha-\beta)\int_0^\infty [S(\infty, cy)-S(\infty, dy)]\frac{dy}{y}-(\alpha-\beta)\int_0^\infty [S(0, cy)-S(0, dy)]\frac{dy}{y}$$

$$=(\alpha-\beta)(\gamma-\delta)[S(\infty, \infty)-S(\infty, 0)] - (\alpha-\beta)(\gamma-\delta)[S(0, \infty)-S(0, 0)],$$

and as S is a symmetric function $S(\infty, 0)=S(0, \infty)$.

Hence, we obtain

$$(\alpha-\beta)(\gamma-\delta)[S(\infty, \infty)-2S(\infty, 0)+S(0, 0)],$$

which, for short, may be written $(\alpha-\beta)(\gamma-\delta)S(\infty-0)^2$.

Hence, the extension to a double integral may be written

$$[(a-b)(c-d)]=S(\infty-0)^2(\alpha-\beta)(\gamma-\delta).$$

In the papers cited, the result is extended to multiple integrals of a higher order. The student should have no difficulty in doing this for himself.

1189. **On the Transition from Real Constants to Complex Constants in Results of Differentiation and Integration.**

Let us premise that, in the remarks following, the variable is a real one, viz. x, that the path of integration is along a portion of the x-axis, that the limits of any integrals occurring are real quantities, and that the constants occurring are independent of the limits; also that the functions dealt with are finite and continuous, and such as to possess differential coefficients.

1190. **Lemma I.**

Let u_1 and u_2 be two real functions of x which continually approach to and ultimately differ by less than any assignable quantities from definite limiting values v_1 and v_2 respectively as x continually approaches a definite value a. We may then put $u_1=v_1+\epsilon_1$ and $u_2=v_2+\epsilon_2$, where ϵ_1 and ϵ_2 are quantities which ultimately vanish when x approaches indefinitely closely to a, so that $\epsilon_1+\iota\epsilon_2$ also ultimately vanishes, where ι stands for $\sqrt{-1}$.

Then $$u_1+\iota u_2=v_1+\iota v_2+\epsilon_1+\iota\epsilon_2$$

and $$Lt(u_1+\iota u_2)=v_1+\iota v_2+Lt(\epsilon_1+\iota\epsilon_2)=v_1+\iota v_2=Lt\,u_1+\iota\,Lt\,u_2.$$

1191. **Lemma II.**

If, upon putting $x+h$ for x, u_1 and u_2 take the values U_1 and U_2 respectively, it follows that $u_1+\iota u_2$ takes the value $U_1+\iota U_2$, and therefore

$$Lt_{h=0}\frac{(U_1+\iota U_2)-(u_1+\iota u_2)}{h}=Lt\frac{U_1-u_1}{h}+\iota\,Lt\frac{U_2-u_2}{h},$$

i.e. $$\frac{d}{dx}(u_1+\iota u_2)=\frac{du_1}{dx}+\iota\frac{du_2}{dx}.$$

Hence, when a function of x containing a complex constant $p+\iota q$, but no other unreal quantity, can be separated into its real and imaginary parts as

$$F(x, p+\iota q)=F_1(x, p, q)+\iota F_2(x, p, q),$$

then $$\frac{d}{dx}F(x, p+\iota q)=\frac{d}{dx}F_1(x, p, q)+\iota\frac{d}{dx}F_2(x, p, q).$$

1192. It has been desirable to consider these results in detail, though they might be thought obvious. For in our idea of a limit we have had constantly in mind some real quantitative arithmetical or algebraical result from which the function under consideration could be made to differ by less than any assignable real quantity by making the variable approach nearer and nearer to its assigned value; and it has not hitherto been necessary to consider the case where the function involves unreal constants.

1193. It is well known that the separation of a complex function into its real and imaginary parts can be effected in all the ordinary cases when the function is of algebraic, exponential, logarithmic, circular or hyperbolic or inverse circular or inverse hyperbolic form, such as

$$(p+\iota q)^n,\ (p+\iota q)^{a+\iota b},\ a^{p+\iota q},\ \log(p+\iota q),\ \sin(p+\iota q),\ \tan^{-1}(p+\iota q),\text{ etc.},$$

as well as in any combination of such functions.

Lemma III. If $F(z)$ be any function of z expressible as a power series with real coefficients, viz. $F(z)\equiv\Sigma A_n z^n$, with radius of convergency ρ, then

$$F(p+\iota q)=\Sigma A_n(p+\iota q)^n=\Sigma A_n r^n e^{n\iota\theta},\quad\text{where } r=\sqrt{p^2+q^2}<\rho,\quad \theta=\tan^{-1}q/p$$
$$=X+\iota Y,\text{ say,}$$

where $X=\Sigma A_n r^n\cos n\theta$, $Y=\Sigma A_n r^n\sin n\theta$, and both these series are convergent if $\Sigma A_n r^n$ be convergent, and then $X+\iota Y$ is convergent.

We then have $X-\iota Y=\Sigma A_n r^n e^{-n\iota\theta}=\Sigma A_n(p-\iota q)^n=F(p-\iota q)$.

The separation into real and imaginary parts is then effected by addition and subtraction of the equations

$$X+\iota Y=F(p+\iota q),\quad X-\iota Y=F(p-\iota q),$$

giving $$2X=F(p+\iota q)+F(p-\iota q),\quad 2\iota Y=F(p+\iota q)-F(p-\iota q).$$

1194. **Lemma IV.**

When $F(x, p+\iota q)$ can be thus separated into real and unreal parts, as

$$F(x, p+\iota q)=F_1(x, p, q)+\iota F_2(x, p, q),$$

F_1 and F_2, besides containing x, may be regarded as conjugate functions of p and q, and therefore

$$\frac{\partial F_1}{\partial p}=\frac{\partial F_2}{\partial q},\quad \frac{\partial F_1}{\partial q}=-\frac{\partial F_2}{\partial p};$$

and differentiating with regard to x,

$$\frac{\partial}{\partial p}\left(\frac{dF_1}{dx}\right)=\frac{\partial}{\partial q}\left(\frac{dF_2}{dx}\right),\quad \frac{\partial}{\partial q}\left(\frac{dF_1}{dx}\right)=-\frac{\partial}{\partial p}\left(\frac{dF_2}{dx}\right),$$

i.e. $\frac{dF_1}{dx}$ and $\frac{dF_2}{dx}$ are also conjugate functions of p and q;

Now, consider the integral $[ac]-[bc]-[ad]+[bd]$, or, as it may be written for short, $[(a-b)(c-d)]$.

By two applications of the above theorem this becomes

$$\int_0^\infty \int_0^\infty [S(ax, cy)-S(bx, cy)-S(ax, dy)+S(bx, dy)]\frac{dx\,dy}{xy}$$

$$=\int_0^\infty (\alpha-\beta)[S(\infty, cy)-S(0, cy)]\frac{dy}{y} - \int_0^\infty (\alpha-\beta)[S(\infty, dy)-S(0, dy)]\frac{dy}{y}$$

$$=(\alpha-\beta)\int_0^\infty [S(\infty, cy)-S(\infty, dy)]\frac{dy}{y}-(\alpha-\beta)\int_0^\infty [S(0, cy)-S(0, dy)]\frac{dy}{y}$$

$$=(\alpha-\beta)(\gamma-\delta)[S(\infty, \infty)-S(\infty, 0)] - (\alpha-\beta)(\gamma-\delta)[S(0, \infty)-S(0, 0)],$$

and as S is a symmetric function $S(\infty, 0)=S(0, \infty)$.

Hence, we obtain

$$(\alpha-\beta)(\gamma-\delta)[S(\infty, \infty)-2S(\infty, 0)+S(0, 0)],$$

which, for short, may be written $(\alpha-\beta)(\gamma-\delta)S(\infty-0)^2$.

Hence, the extension to a double integral may be written

$$[(a-b)(c-d)]=S(\infty-0)^2(\alpha-\beta)(\gamma-\delta).$$

In the papers cited, the result is extended to multiple integrals of a higher order. The student should have no difficulty in doing this for himself.

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Then $$u_1+\iota u_2=v_1+\iota v_2+\epsilon_1+\iota\epsilon_2$$

and $$Lt(u_1+\iota u_2)=v_1+\iota v_2+Lt(\epsilon_1+\iota\epsilon_2)=v_1+\iota v_2=Lt\,u_1+\iota\,Lt\,u_2.$$

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$$Lt_{h=0}\frac{(U_1+\iota U_2)-(u_1+\iota u_2)}{h}=Lt\frac{U_1-u_1}{h}+\iota\,Lt\frac{U_2-u_2}{h},$$

i.e. $$\frac{d}{dx}(u_1+\iota u_2)=\frac{du_1}{dx}+\iota\frac{du_2}{dx}.$$

1196. As examples of these facts, let us consider

(1) the differentiation of $x^{p+\iota q}$, where p and q are here, as always, real. We have

$$\frac{d}{dx}x^{p+\iota q}=\frac{d}{dx}(x^p e^{\iota q\log x})=\frac{d}{dx}[x^p\{\cos(q\log x)+\iota\sin(q\log x)\}]$$

$$=\frac{d}{dx}[x^p\{\cos(q\log x)\}]+\iota\frac{d}{dx}[x^p\{\sin(q\log x)\}], \text{ by Lemma II.,}$$

$$=\left[px^{p-1}\cos(q\log x)+x^p\left(-\frac{q}{x}\right)\sin(q\log x)\right]$$

$$+\iota\left[px^{p-1}\sin(q\log x)+x^p\left(\frac{q}{x}\right)\cos(q\log x)\right]$$

$$=(p+\iota q)x^{p-1}[\cos(q\log x)+\iota\sin(q\log x)]=(p+\iota q)x^{p-1}e^{\iota q\log x}$$

$$=(p+\iota q)x^{p+\iota q-1},$$

as might be expected from the principle of permanence of form stated above.

Hence the rule $\frac{d}{dx}x^n=nx^{n-1}$ holds whether n be real or complex.

Conversely, $\int x^{p+\iota q-1}\,dx=\frac{x^{p+\iota q}}{p+\iota q}$,

and therefore the rule for integration, viz. $\int x^{n-1}\,dx=\frac{x^n}{n}$, also holds whether the index n be real or complex.

(2) Consider $\frac{d}{dx}a^{(p+\iota q)x}$.

This is $\frac{d}{dx}e^{px\log a}[\cos(qx\log a)+\iota\sin(qx\log a)]$

$$=\frac{d}{dx}e^{px\log a}\cos(qx\log a)+\iota\frac{d}{dx}e^{px\log a}\sin(qx\log a)$$

$$=(p+\iota q)\log a\,e^{px\log a}[\cos(qx\log a)+\iota\sin(qx\log a)]$$

$$=(p+\iota q)\log a\,.\,a^{(p+\iota q)x},$$

which is the ordinary rule for differentiating a^{nx} when n is real.

Hence $\frac{d}{dx}a^{nx}=n\log a\,.\,a^{nx}$ whether n be real or complex, and conversely $\int a^{nx}\,dx=\frac{a^{nx}}{n\log a}$ whether n be real or complex.

(3) Consider $\frac{d}{dx}\log_{p+\iota q}x$,

i.e. $$\frac{d}{dx}\frac{\log_e x}{\log_e(p+\iota q)}=\frac{1}{\log_e(p+\iota q)}\frac{d}{dx}\log x=\frac{1}{x}\cdot\frac{1}{\log_e(p+\iota q)},$$

which is again the ordinary rule for $\frac{d}{dx}\log_a x$, viz. $\frac{1}{x}\cdot\frac{1}{\log_e a}$.

(4) Consider $\frac{d}{dx}\tan^{-1}\frac{x}{p+\iota q}$.

Let $\tan^{-1}\frac{x}{p+\iota q}=X-\iota Y$, and therefore $\tan^{-1}\frac{x}{p-\iota q}=X+\iota Y$.

i.e. $\dfrac{dF}{dx}$, which is equal to $\dfrac{dF_1}{dx}+\iota\dfrac{dF_2}{dx}$, besides involving x, involves p and q as a function of $p+\iota q$, and $\equiv\phi(x, p+\iota q)$, say.

It might be said that this also is a self-evident fact arising from the principle that the process of differentiation with regard to x takes no cognisance of the particular values of any constants involved. But as our experience of this fact is based upon the behaviour of functions containing only real constants, it is desirable at this stage to make this point also clear and to establish it explicitly.

We have then $\dfrac{d}{dx}F(x, p+\iota q)$ of the form $\phi(x, p+\iota q)$ for all real values of x, p and q, and we have to identify the form of this function ϕ.

Now the *form* of a function is merely a means of defining the *particular manner in which the several variables and constants are involved* in its construction, and is independent of any particular values assignable to those variables and constants.

Suppose then that it has been discovered in the case of a real constant p that $\dfrac{d}{dx}F(x, p)$ takes the form $f(x, p)$, a known form say, for all values of x and p; then since, when $q=0$ we also have $\dfrac{d}{dx}F(x, p)=\phi(x, p)$ for all values of x and p, we must have $\phi(x, p)\equiv f(x, p)$; that is, the form of the function ϕ is identified as being the same functional form as that obtained in the differentiation of $F(x, p)$ for a real value of p.

1195. It is assumed in what precedes that we are dealing with a function $F(x, p)$ which is continuous and finite for the whole of some range of values of x within which x lies, whatever real value p may have, and that the differentiation of F with regard to x is a possible operation; and that these suppositions will not be affected if we change p to $p+\iota q$. Further, that F_1 and F_2 are continuous and finite functions of x for the same range, and that differentiation with regard to x, p or q is a possible operation. Under these circumstances we may infer that if

$$\frac{d}{dx}F(x, p)=f(x, p),$$

where p is a real constant, we shall also have a result of the same form when p is a complex constant.

If then it be distinctly understood that the *definition of integration* used is that it is the *reversal of the operation of differentiation*, *i.e.* the discovery of a function $F(x, p+\iota q)$, which upon differentiation with regard to x shall give rise to a stated result $f(x, p+\iota q)$, it will follow *under the limitations stated above*, that if $\int f(x, p)\,dx=F(x, p)$, where p is a real constant, we shall also have $\int f(x, p+\iota q)\,dx=F(x, p+\iota q)$, where $p+\iota q$ is a complex constant, and the integrals being indefinite a real arbitrary constant C may be supposed added in the first case, and a complex arbitrary constant $C_1+\iota C_2$ in the second.

In the general theory of Definite Integrals, *i.e.* of those integrals between certain specified limits whose values may be sometimes found, as has been seen in the last three chapters, without any knowledge of the function which forms the indefinite integral, the indefinite integral is an unknown function of x, generally not capable of expression in finite terms by means of any of the known ordinary Algebraic, Exponential or Logarithmic, Circular, Hyperbolic or Inverse Functions.

1198. If then $f(x, c)$ be the known or unknown function of x, whose differential coefficient with regard to x is $F(x, c)$, we have

$$\int_b^a F(x, c)\,dx = \Big[f(x, c)\Big]_b^a = f(a, c) - f(b, c) = \chi(a, b, c) \text{ say,}$$

and the two definitions, viz. that of inverse differentiation and that of summation, agree except in the case where $F(x, c)$ assumes an infinite value or becomes discontinuous between the limits $x=a$ and $x=b$, and this will hold when c is changed to any other value, say c', so long as such change does not make $F(x, c')$ become infinite or discontinuous for any value of x lying between $x=a$ and $x=b$, or at either limit.

It will follow that *whichever definition may have been used* in obtaining a specific result such as

$$\int_b^a F(x, c)\,dx = \chi(a, b, c),$$

where c is real, that result will still hold *under certain conditions* when a complex $p+\iota q$ is substituted for c, that is,

$$\int_b^a F(x, p+\iota q)\,dx = \chi(a, b, p+\iota q),$$

that is, *provided that none of the stipulations with regard to F and χ have been violated by the transformation.*

This entails that $F(x, c)$ shall be *finite and continuous* for all values of x from $x=b$ to $x=a$ inclusive.

That $F(x, p+\iota q)$ shall be *separable into real and imaginary parts* as

$$F_1(x, p, q) + \iota F_2(x, p, q).$$

That when this separation has been effected both $F_1(x, p, q)$ and $F_2(x, p, q)$ shall be *finite and continuous* functions of x for all values of x from $x=b$ to $x=a$ inclusive.

That $\chi(a, b, p+\iota q)$ is likewise *separable into real and imaginary parts* $\chi_1(a, b, p, q)$ and $\chi_2(a, b, p, q)$.

That when any convergent infinite series has been used, or its use in any way implied in the establishment of the primary result

$$\int_b^a F(x, c)\,dx = \chi(a, b, c),$$

or in the separation of $F(x, p+\iota q)$, $\chi(a, b, p+\iota q)$ into their respective real and imaginary parts, the convergency shall remain unaffected by the substitution of $p+\iota q$ for the real constant c for all values of x from $x=b$ to $x=a$ inclusive; and further, that when this convergency holds only

Then $$2X=\tan^{-1}\frac{2px}{p^2+q^2-x^2},\quad 2Y=\tanh^{-1}\frac{2qx}{p^2+q^2+x^2},$$

and $$\frac{dX}{dx}-\iota\frac{dY}{dx}=p\frac{p^2+q^2+x^2}{(p^2+q^2-x^2)^2+4p^2x^2}-\iota q\frac{p^2+q^2-x^2}{(p^2+q^2+x^2)^2-4q^2x^2}.$$

But since
$$(p^2+q^2-x^2)^2+4p^2x^2=(p^2+q^2+x^2)^2-4q^2x^2=(p^2-q^2+x^2)^2+4p^2q^2,$$

we have $$\frac{d}{dx}\tan^{-1}\frac{x}{p+\iota q}=\frac{p(p^2+q^2+x^2)-\iota q(p^2+q^2-x^2)}{(p^2-q^2+x^2)^2+4p^2q^2}$$
$$=\frac{(p+\iota q)[(p-\iota q)^2+x^2]}{[(p-\iota q)^2+x^2][(p+\iota q)^2+x^2]}=\frac{p+\iota q}{(p+\iota q)^2+x^2}.$$

That is, the ordinary rule for differentiating
$$\tan^{-1}\frac{x}{a},\quad\text{viz. } \frac{d}{dx}\tan^{-1}\frac{x}{a}=\frac{a}{a^2+x^2},$$
holds whether a be real or complex.

It also follows that $\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$ holds whether a be real or complex.

(5) Similarly, we might go on to discuss the other standard cases. The student may verify these for himself.

1197. **Essential Difference in the Two Definitions of Integration.** Now the *summation* definition of integration loses its meaning when the integrand becomes infinite or discontinuous between or at the limits of integration. Let $x=c$ be a value of x at which the integrand becomes infinite or discontinuous. Then, if the integrand be regarded as the differential coefficient of some function of x, say y, there is a discontinuity in the value of dy/dx for the value $x=c$. And to interpret the summation definition it has been seen in Chapter IX. how Cauchy has given a new summation definition of $\int_b^a(\quad)dx$, viz. the limit of the summation
$$\int_b^{c-\epsilon}(\quad)dx+\int_{c+\eta}^a(\quad)dx,$$
where ϵ and η are to be diminished indefinitely in a ratio of equality, obtaining what Cauchy calls the Principal Value of the Integral. In this way the *discontinuity itself is avoided.* It is approached indefinitely closely from opposite sides, but the discontinuous element is omitted. Thus a geometrical meaning is given to the symbol $\int_b^a(\quad)dx$, which, from the summation definition, would be otherwise meaningless. But regarding the integrand as the differential coefficient of the function y, the discontinuity itself *is an essential characteristic of that function.* Hence the two definitions do not agree if such points as the one under consideration occur within the range of integration. But it has been seen earlier that in the absence of such cases occurring between the limits of integration, there is agreement between the two definitions.

In the general theory of Definite Integrals, *i.e.* of those integrals between certain specified limits whose values may be sometimes found, as has been seen in the last three chapters, without any knowledge of the function which forms the indefinite integral, the indefinite integral is an unknown function of x, generally not capable of expression in finite terms by means of any of the known ordinary Algebraic, Exponential or Logarithmic, Circular, Hyperbolic or Inverse Functions.

1198. If then $f(x, c)$ be the known or unknown function of x, whose differential coefficient with regard to x is $F(x, c)$, we have

$$\int_b^a F(x, c)\,dx = \Big[f(x, c)\Big]_b^a = f(a, c) - f(b, c) = \chi(a, b, c) \text{ say,}$$

and the two definitions, viz. that of inverse differentiation and that of summation, agree except in the case where $F(x, c)$ assumes an infinite value or becomes discontinuous between the limits $x=a$ and $x=b$, and this will hold when c is changed to any other value, say c', so long as such change does not make $F(x, c')$ become infinite or discontinuous for any value of x lying between $x=a$ and $x=b$, or at either limit.

It will follow that *whichever definition may have been used* in obtaining a specific result such as

$$\int_b^a F(x, c)\,dx = \chi(a, b, c),$$

where c is real, that result will still hold *under certain conditions* when a complex $p+\iota q$ is substituted for c, that is,

$$\int_b^a F(x, p+\iota q)\,dx = \chi(a, b, p+\iota q),$$

that is, *provided that none of the stipulations with regard to* F *and* χ *have been violated by the transformation.*

This entails that $F(x, c)$ shall be *finite and continuous* for all values of x from $x=b$ to $x=a$ inclusive.

That $F(x, p+\iota q)$ shall be *separable into real and imaginary parts* as

$$F_1(x, p, q) + \iota F_2(x, p, q).$$

That when this separation has been effected both $F_1(x, p, q)$ and $F_2(x, p, q)$ shall be *finite and continuous* functions of x for all values of x from $x=b$ to $x=a$ inclusive.

That $\chi(a, b, p+\iota q)$ is likewise *separable into real and imaginary parts* $\chi_1(a, b, p, q)$ and $\chi_2(a, b, p, q)$.

That when any convergent infinite series has been used, or its use in any way implied in the establishment of the primary result

$$\int_b^a F(x, c)\,dx = \chi(a, b, c),$$

or in the separation of $F(x, p+\iota q)$, $\chi(a, b, p+\iota q)$ into their respective real and imaginary parts, the convergency shall remain unaffected by the substitution of $p+\iota q$ for the real constant c for all values of x from $x=b$ to $x=a$ inclusive; and further, that when this convergency holds only within definite limits of the values of p and q, the truth of the permanence of form of the result can only be inferred between such limits.

That the path of the original integration for values of x from a point $x=b$ to a point $x=a$ along the x-axis shall not have been altered in any way by the proposed change from a real constant c to a complex constant $p+\iota q$.

With such stipulations, we therefore have

$$\int_b^a \{F_1(x, p, q)+\iota F_2(x, p, q)\}\, dx=\chi_1(a, b, p, q)+\iota\chi_2(a, b, p, q),$$

whence $\int_b^a F_1(x, p, q)\, dx=\chi_1(a, b, p, q)$; $\int_b^a F_2(x, p, q)\, dx=\chi_2(a, b, p, q)$.

1199. If $F(x, c)$ and $\chi(a, b, c)$ be such that $\int_b^a F(x, c)\, dx=\chi(a, b, c)$ for all real values of c, and that $F(x, c)$ is developable as a series of positive integral powers of c uniformly and unconditionally convergent between specific values of c, for all values of x from b to a, so that $\int_b^a F(x, c)\, dx$ is capable of term by term integration, and is also developable in a like convergent series, and if $\chi(a, b, c)$ be also developable in a series of positive integral powers of c convergent for a specific range of values of c, the coefficients of like powers of c in $\int_b^a F(x, c)\, dx$ and $\chi(a, b, c)$ are equal for all values of c for which each series is convergent. And provided that this convergency remains in both series when we substitute a complex value $p+\iota q$ for c, the equality of $\int_b^a F(x, p+\iota q)\, dx$ and $\chi(a, b, p+\iota q)$ will still hold good for such values of p and q as do not disturb that convergency and do not cause F to assume an infinite or discontinuous value for any value of x between b and a.

If it be proposed to conduct the transition from c to $p+\iota q$ by a preliminary change to $p+q$, we have $\int_b^a F(x, p+q)\, dx=\chi(a, b, p+q)$; and if expansions of $F(x, p+q)$ and $\chi(a, b, p+q)$ be possible in series of integral powers of q, each uniformly convergent between specific limits of q, the coefficients of like powers of q in the expansions of $\int_b^a F(x, p+q)\, dx$ and $\chi(a, b, p+q)$ will be equal, and therefore, provided the convergency of these series be maintained when a change from q to ιq is made in them, and provided also that such changes have not caused F to assume an infinite or discontinuous value for any value of x between $x=b$ and $x=a$, we may infer that the transition to the complex $p+\iota q$ is legitimate.

1200. In the use of the method the precautions necessary before the results obtained can be accepted as rigorously established, are somewhat irksome, and this has caused mathematicians to look askance at the process. In fact it has become usual to regard it as a method of suggestion of new integrals to be verified by other methods rather than as a mode of investigation. For instance, De Morgan remarks: "It is a matter of some difficulty to say how far this practice may be carried, it being most certain that there is an extensive class of cases in which it is allowable, and as extensive a class in which either the transformation, or neglect of some essential modification incident to the manner of doing it, leads to positive error. It is also certain that the line which separates the first and second class has not been distinctly drawn."

De Morgan, after citing several instances of the success of the method, gives as one of failure, the case of $\int_0^\infty \frac{dx}{1+x^2}=[\tan^{-1}x]=\frac{\pi}{2}$.

By putting $y\sqrt{-1}$ in place of x, he obtains $\int_0^\infty \frac{dx}{1+x^2}=\sqrt{-1}\int_0^\infty \frac{dy}{1-y^2}$, and remarks concerning this that it is "an equation which we cannot either affirm or deny, since the subject of integration in the second side becomes infinite between the limits."

We may, however, note with regard to this, that it apparently escaped De Morgan that having put $x=\sqrt{-1}\,y$, the range of values of y over which the integration is assumed to be conducted is not a *range of real values*, as was the case in the integration for the range of real values of x from 0 to ∞. In fact y ranges from $\frac{0}{\sqrt{-1}}$ to $\frac{\infty}{\sqrt{-1}}$, corresponding to the real range of x from 0 to ∞; and all the values through which y passes in this range are imaginaries, so that y never passes through the value 1 at all, and therefore the subject of integration never becomes infinite as De Morgan asserts. As a matter of fact, if we write $\frac{k}{\sqrt{-1}}$, for the upper limit,

$$\int_0^{\frac{k}{\sqrt{-1}}} \frac{dy}{1-y^2}=\frac{1}{2}\int_0^{\frac{k}{\sqrt{-1}}}\left(\frac{1}{1-y}+\frac{1}{1+y}\right)dy=\frac{1}{2}\left[\log\frac{1+y}{1-y}\right]_0^{\frac{k}{\sqrt{-1}}}$$

$$=\frac{1}{2}\log\frac{1+\frac{k}{\sqrt{-1}}}{1-\frac{k}{\sqrt{-1}}}=\frac{1}{2}\log\left(\frac{\frac{1}{k}+\frac{1}{\sqrt{-1}}}{\frac{1}{k}-\frac{1}{\sqrt{-1}}}\right), \text{ and when } k \text{ is } \infty$$

$$=\frac{1}{2}\log(-1)=\frac{1}{2}\log[\cos(2n-1)\pi+\iota\sin(2n-1)\pi]$$

$$=\frac{1}{2}\log e^{\iota(2n-1)\pi}=\frac{(2n-1)\pi\iota}{2},$$

where n is an integer.

Hence $\sqrt{-1}\int_0^{\frac{k}{\sqrt{-1}}}\frac{dy}{1-y^2}$ has one of the values of $-(2n-1)\frac{\pi}{2}$, where n is an integer. The value $n=0$ gives the particular value $\frac{\pi}{2}$, which we have assigned to the left side, viz. $\int_0^\infty \frac{dx}{1+x^2}$.

But if in the formula $\int \frac{dx}{1+c^2x^2}=\frac{1}{c}\tan^{-1}cx$, c be replaced by ιc, we have $\int \frac{dx}{1-c^2x^2}=\frac{1}{c}\tanh^{-1}cx$. Both the right-hand side and the integrand become ∞ at $x=c^{-1}$ during the march of x from 0 to ∞. Therefore, with those limits, the change proposed is inadmissible. We defer the consideration of the use of a complex variable to the next chapter. And it is to be understood in all the remarks made in course of this discussion, that the march of the variable between its limits is not to be interfered with by the substitution of a complex constant for a real one, *i.e.* that the change of c to $p+\iota q$ is not supposed to be one which can be brought about by a change in the *variable*, as is done in the case cited.

Illustrations.

1201. (1) Taking $\int x^{n-1}dx=\frac{x^n}{n}$, write $n=a+\iota b$.

Then $$\int x^{a-1}x^{\iota b}\,dx=x^{a+\iota b}/(a+\iota b) \quad \text{[Art. 1196 (1)]},$$

i.e. $$\int x^{a-1}\{\cos(b\log x)+\iota\sin(b\log x)\}\,dx$$
$$=[x^a\cos(b\log x)+\iota x^a\sin(b\log x)](a-\iota b)/(a^2+b^2);$$
whence, writing $x=e^\theta$,
$$\int e^{a\theta}\cos b\theta\,d\theta=e^{a\theta}\frac{a\cos b\theta+b\sin b\theta}{a^2+b^2},\quad \int e^{a\theta}\sin b\theta\,d\theta=e^{a\theta}\frac{a\sin b\theta-b\cos b\theta}{a^2+b^2},$$
which are the well-known results proved elsewhere without the use of complex values.

(2) In the integral $I\equiv\int_a^b\frac{dx}{x+c}=\Big[\log(x+c)\Big]_a^b=\log\frac{b+c}{a+c}$, put $c=qe^{\iota\alpha}$.

Then $$\frac{1}{x+c}=\frac{1}{x+qe^{\iota\alpha}}=\frac{x+qe^{-\iota\alpha}}{x^2+2qx\cos\alpha+q^2}=\frac{x+q\cos\alpha-\iota q\sin\alpha}{x^2+2qx\cos\alpha+q^2},$$
and
$$\log\frac{b+qe^{\iota\alpha}}{a+qe^{\iota\alpha}}=\frac{1}{2}\log\frac{b^2+2bq\cos\alpha+q^2}{a^2+2aq\cos\alpha+q^2}+\iota\left(\tan^{-1}\frac{q\sin\alpha}{b+q\cos\alpha}-\tan^{-1}\frac{q\sin\alpha}{a+q\cos\alpha}\right).$$
Therefore
$$\left.\begin{aligned}&\int_a^b\frac{x+q\cos\alpha}{x^2+2qx\cos\alpha+q^2}dx=\frac{1}{2}\log\frac{b^2+2bq\cos\alpha+q^2}{a^2+2aq\cos\alpha+q^2}\\ \text{and}\quad&\int_a^b\frac{q\sin\alpha\,dx}{x^2+2qx\cos\alpha+q^2}=\tan^{-1}\frac{b+q\cos\alpha}{q\sin\alpha}-\tan^{-1}\frac{a+q\cos\alpha}{q\sin\alpha},\end{aligned}\right\}$$
results which are obviously true otherwise.

The process is valid, for all the conditions laid down in Art. 1198 are fulfilled.

(3) In $I\equiv\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$, write $a=ce^{\iota\alpha}$,
(a, c both $+^{\text{ve}}$; α, acute),
$$\int_0^\infty e^{-cx\cos\alpha}\,e^{-\iota cx\sin\alpha}\cos bx\,dx=\frac{c(b^2e^{\iota\alpha}+c^2e^{-\iota\alpha})}{b^4+2b^2c^2\cos2\alpha+c^4}$$

Equating real and unreal parts,

$$I_1=\int_0^\infty e^{-cx\cos\alpha}\cos bx\cos(cx\sin\alpha)\,dx=\frac{c(b^2+c^2)\cos\alpha}{b^4+2b^2c^2\cos 2\alpha+c^4},$$

$$I_2=\int_0^\infty e^{-cx\cos\alpha}\cos bx\sin(cx\sin\alpha)\,dx=\frac{c(c^2-b^2)\sin\alpha}{b^4+2b^2c^2\cos 2\alpha+c^4}.$$

The change from a to $ce^{\iota\alpha}$ does not affect the path of integration with regard to x from 0 to ∞; the integrands remain finite and continuous throughout the range, and though the upper limit is infinite both integrands are zero when x is infinite, and the conditions of the validity of the process are all satisfied. Hence it will be fair to assume the results correct. They may be readily verified otherwise.

(4) In $I\equiv\int_0^\infty e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{2a}$, write $a=ce^{\iota\alpha}$, (a and $c+^{\text{ve}}$; α, acute).

Then $$\int_0^\infty e^{-c^2x^2(\cos 2\alpha+\iota\sin 2\alpha)}\,dx=\frac{\sqrt{\pi}}{2c}e^{-\iota\alpha}.$$

Therefore $$\left.\begin{aligned}\int_0^\infty e^{-c^2x^2\cos 2\alpha}\cos(c^2x^2\sin 2\alpha)\,dx&=\frac{\sqrt{\pi}}{2c}\cos\alpha,\\ \int_0^\infty e^{-c^2x^2\cos 2\alpha}\sin(c^2x^2\sin 2\alpha)\,dx&=\frac{\sqrt{\pi}}{2c}\sin\alpha.\end{aligned}\right\}$$

The new integrands satisfy the conditions under which the transition is permissible.

Putting $\alpha=\frac{\pi}{4}$, we have Fresnel's integrals of Art. 1163, viz.

$$\int_0^\infty \cos c^2x^2\,dx=\frac{\sqrt{\pi}}{2c\sqrt{2}},$$

$$\int_0^\infty \sin c^2x^2\,dx=\frac{\sqrt{\pi}}{2c\sqrt{2}}.$$

(5) In $I\equiv\int_{-\infty}^\infty e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{a}$, write $a=m(1+\alpha)$, (α, $+^{\text{ve}}$).

Then $$\int_{-\infty}^\infty e^{-m^2(1+\alpha)^2x^2}\,dx=\frac{\sqrt{\pi}}{m(1+\alpha)}.$$

Both sides are capable of expansion in powers of α, convergent for values of α which lie between -1 and $+1$. And both series remain convergent when we replace α by an unreal quantity with modulus <1. Hence, writing $\beta\sqrt{-1}$ for α, where $\beta<1$, we obtain

$$\int_{-\infty}^\infty e^{-m^2(1-\beta^2)x^2}(\cos 2m^2\beta x^2-\iota\sin 2m^2\beta x^2)\,dx=\frac{\sqrt{\pi}}{m}\,\frac{1}{1+\iota\beta}=\frac{\sqrt{\pi}}{m}\,\frac{1-\iota\beta}{1+\beta^2}$$

whence $(\beta<1)$;

$$\left.\begin{aligned}\int_{-\infty}^\infty e^{-m^2(1-\beta^2)x^2}\cos 2m^2\beta x^2\,dx&=\frac{\sqrt{\pi}}{m}\,\frac{1}{1+\beta^2},\\ \int_{-\infty}^\infty e^{-m^2(1-\beta^2)x^2}\sin 2m^2\beta x^2\,dx&=\frac{\sqrt{\pi}}{m}\,\frac{\beta}{1+\beta^2}\end{aligned}\right\}(\beta<1).$$

[SERRET, *Calc. Int.*, p. 140.]

(6) Taking the integral

$$I \equiv \int_{-\infty}^{\infty} e^{-p^2x^2} \cosh 2qx\, dx = \frac{\sqrt{\pi}}{p} e^{\frac{q^2}{p^2}}$$

we observe that $\cosh 2qx$ and $e^{\frac{q^2}{p^2}}$ can both be developed in ascending powers of q which are both convergent series, and that if we write ιq for q, the convergence will not be affected.

Hence, we may safely infer that

$$\int_{-\infty}^{\infty} e^{-p^2x^2} \cos 2qx\, dx = \frac{\sqrt{\pi}}{p} e^{-\frac{q^2}{p^2}},$$

and as the integrands in these integrals are not affected by changing the sign of x in either case, either integral may be taken from 0 to ∞, and the results are still true, provided in that case the right-hand sides be halved.

(7) In $I \equiv \int_0^{\infty} e^{-c^2\left(x^2+\frac{a^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2c} e^{-2c^2a}$, write $c = ke^{\iota\alpha}$.

Then $$\int_0^{\infty} e^{-k^2e^{2\iota\alpha}\left(x^2+\frac{a^2}{x^2}\right)} dx = \frac{\sqrt{\pi}}{2k} e^{-\iota\alpha} e^{-2ak^2e^{2\iota\alpha}};$$

$$\left.\begin{aligned}
\therefore \int_0^{\infty} e^{-k^2\left(x^2+\frac{a^2}{x^2}\right)\cos 2\alpha} \cos\left\{k^2\left(x^2+\frac{a^2}{x^2}\right)\sin 2\alpha\right\} dx \\
= \frac{\sqrt{\pi}}{2k} e^{-2ak^2\cos 2\alpha} \cos(\alpha + 2ak^2 \sin 2\alpha), \\
\int_0^{\infty} e^{-k^2\left(x^2+\frac{a^2}{x^2}\right)\cos 2\alpha} \sin\left\{k^2\left(x^2+\frac{a^2}{x^2}\right)\sin 2\alpha\right\} dx \\
= \frac{\sqrt{\pi}}{2k} e^{-2ak^2\cos 2\alpha} \sin(\alpha + 2ak^2 \sin 2\alpha).
\end{aligned}\right\}$$

[Cf. Cauchy, *Mém. des Sav. Étrangers*, i., p. 638.]

(8) Taking Laplace's integral $\int_0^{\infty} e^{-a^2x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{b^2}{a^2}}$, write $a = ce^{\frac{\iota\pi}{4}}$; then $a^2 = c^2e^{\frac{\iota\pi}{2}} = \iota c^2$ and $e^{-a^2x^2} = e^{-\iota c^2x^2} = \cos c^2x^2 - \iota \sin c^2x^2$.

Therefore $\int_0^{\infty} (\cos c^2x^2 - \iota \sin c^2x^2) \cos 2bx\, dx = \frac{\sqrt{\pi}}{2c} e^{-\iota\left(\frac{\pi}{4}-\frac{b^2}{c^2}\right)};$

whence $$\int_0^{\infty} \cos c^2x^2 \cos 2bx\, dx = \frac{\sqrt{\pi}}{2c} \cos\left(\frac{\pi}{4} - \frac{b^2}{c^2}\right),$$

$$\int_0^{\infty} \sin c^2x^2 \sin 2bx\, dx = \frac{\sqrt{\pi}}{2c} \sin\left(\frac{\pi}{4} - \frac{b^2}{c^2}\right),$$

results due to Fourier.*

* *Traité de la Chaleur*, p. 533; Gregory, *D.C.*, p. 485.

PROBLEMS.

1. Show that $\displaystyle\int_0^\pi \cos^n x \cos nx\, dx = \frac{\pi}{2^n}$. [COLLEGES, 1892.]

Show also that $\displaystyle\int_0^\pi \cos^n \theta \cos (n-2r)\theta\, d\theta = {}^nC_r \frac{\pi}{2^n}$.

2. Evaluate $\displaystyle\int_0^k \left(1-\frac{x}{k}\right)^k x^{n-1}\, dx$, where n is positive and k a positive integer. [ST. JOHN'S, 1892.]

3. Prove that $\displaystyle\frac{2}{\pi}\int_0^\pi e^{c\cos x} \sin (c \sin x) \sin nx\, dx = \frac{c^n}{n!}$. [MATH. TRIPOS., 1872.]

4. If m be a positive integer, prove that

$$\int_0^{\frac{\pi}{2}} (2\cos x)^{m-1} x \sin (m+1)x\, dx = \frac{\pi}{4m}.$$

[COLLEGES ϵ, 1883.]

5. If n be positive and less than unity, show that

$$\int_0^\infty \frac{\cos rx}{x^n}\, dx = \frac{r^{n-1}}{\Gamma(n)} \frac{\pi}{2} \sec \frac{n\pi}{2}.$$

[COLLEGES β, 1889.]

6. Show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 2s\psi \cos p\psi}{\cos^p \psi}\, d\psi = \pi(-1)^s 2^{p-2} \frac{p(p+1)\dots(p+s-1)}{s!},$$

where p is any negative quantity or any positive proper fraction. [COLLEGES γ, 1888.]

7. Establish the result

$$\int_0^1 \cosh (p \log x) \log (1+x) \frac{dx}{x} = \frac{1}{2p}\left(\frac{\pi}{\sin p\pi} - \frac{1}{p}\right) \quad (p<1).$$

[COLLEGES δ, 1883.]

8. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{1 - 2\sin\alpha \sin\theta + \sin^2\theta}$. [COLLEGES β, 1890.]

9. Show that the product of the two integrals

$$\int_0^\infty e^{-x^2} x^{2n-1}\, dx \quad\text{and}\quad \int_0^\infty e^{-x^2} x^{1-2n}\, dx \quad\text{is}\quad \frac{\pi}{4\sin n\pi}.$$

[COLLEGES α, 1890.]

10. If $u = \displaystyle\int_0^h e^{x^2}\, dx$, show that $u^2 = \displaystyle\int_0^{\frac{\pi}{4}} (e^{h^2 \sec^2\theta} - 1)\, d\theta$. [COLLEGES α, 1890.]

11. Show that $\displaystyle\int_0^\infty \frac{\log_e \sin\theta}{a^2+\theta^2}\, d\theta = \frac{\pi}{2a} \log\{\tfrac{1}{2}(1-e^{-2a})\}$. [COLLEGES, 1892, etc.]

12. Show that

$$\int_0^\pi \tan^{-1}\frac{\alpha\sin x}{1+\alpha\cos x}\,dx = 2\left(\alpha+\frac{\alpha^3}{3^2}+\frac{\alpha^5}{5^2}+\dots\right) \text{ if } \alpha<1.$$

[MATH. TRIPOS, 1882.]

13. Prove that

$$\int_0^{2\pi}\cos nc\theta\log(1+2m\cos c\theta+m^2)\,d\theta = -\frac{2\pi}{n}m^n \text{ or } \frac{2\pi}{n}m^n,$$

according as n is even or odd $(1>m>0)$. [R. P.]

14. Find the value of $\int_0^\pi \sin n\theta\tan^{-1}\frac{a\sin\theta}{1-a\cos\theta}\,d\theta$, where $-1<a<1$ and n is an integer. [OXFORD II. P., 1900.]

15. If m, n being each less than unity, and $\sin x = n\sin(x+y)$, show that

$$\int_0^\pi \frac{x\sin y\,dy}{1-2m\cos y+m^2} = \frac{\pi}{2m}\log\frac{1}{1-mn}.$$

[ST. JOHN'S, 1891.]

16. Show that

$$\int_0^\infty \frac{x^{2m}}{(x^{2n}+a^{2n})^{k+1}}\,dx = Qa^{-2(k+1)n+2m+1}\frac{\pi}{2n}\operatorname{cosec}\frac{2m+1}{2n}\pi,$$

where m, n and k are all positive integers and $m<n$, and Q is the coefficient of c^k in the expansion of $(1-c)^{\frac{2m+1-2n}{2n}}$ in ascending powers of c. [COLLEGES α, 1887.]

17. Prove that

$$\int_0^\infty \frac{dx}{(1+x^2)(1-2a\cos x+a^2)} = \frac{\pi}{2(1-a^2)}\frac{e+a}{e-a} \quad (0<a<1).$$

[COLLEGES γ, 1888.]

18. Prove that $\int_0^\pi \frac{\cos n\theta}{a-\cos\theta}\,d\theta = \frac{(a-\sqrt{a^2-1})^n}{(a^2-1)^{\frac{1}{2}}}\pi$, where $a>1$.

[ST. JOHN'S, 1881.]

19. Prove that

$$\int_0^\pi\left\{\frac{e+\cos\theta}{1+2e\cos\theta+e^2}\right\}^2 d\theta = \frac{\pi}{2(1-e^2)} \quad\text{or}\quad \frac{\pi}{2}\,\frac{2e^2-1}{e^2(e^2-1)},$$

according as $e<1$ or $e>1$. [R. P.]

20. Show that $\int_0^\infty \frac{x^n\,dx}{1+2x\cos\alpha+x^2} = \frac{\pi}{\sin n\pi}\frac{\sin n\alpha}{\sin\alpha}$, where n is not an integer and $\pi>\alpha>0$. [ST. JOHN'S, 1891.]

21. Show that $\int_0^\infty \frac{dx}{1+x^n}=\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}$, if $n>1$, and thence show that if n be positive,

$$\int_0^\infty \log x \log\left(1+\frac{a^n}{x^n}\right)dx=\pi a \operatorname{cosec}\frac{\pi}{n}\left(\log a-\frac{\pi}{n}\cot\frac{\pi}{n}-1\right).$$

[MATH. TRIPOS, 1883.]

22. Expand the definite integral

$$\int_0^1 dx\frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1-ux)^\gamma},$$

in the form of a series of ascending powers of u; and thence or otherwise find the relations which must subsist between α, β, γ and the indices α', β', γ' of a like integral, in order that the two integrals may be to each other in a ratio independent of u.

[SMITH'S PRIZE, 1875.]

23. Prove that

$$\int_0^\pi \frac{\sin^2 x\,dx}{(1-2a\cos x+a^2)(1-2b\cos x+b^2)}=\frac{\pi}{2(1-ab)}\quad \begin{Bmatrix} a<1 \\ b<1 \end{Bmatrix}.$$

[COLLEGES γ, 1893.]

24. Point out the fallacy in the following train of reasoning. By putting $ax=y$, we have

$$\int_0^\infty \frac{e^{-ax}}{x}dx=\int_0^\infty \frac{e^{-y}}{y}dy;\quad \therefore \int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx=\int_0^\infty \frac{e^{-y}}{y}dy-\int_0^\infty \frac{e^{-y}}{y}dy=0.$$

Show that the value of the latter integral is $\log\frac{b}{a}$.

[TRINITY COLLEGE, 1882.]

25. Deduce from the expansion of $\log(1+x)$ that if $x \not> 1$

$$\frac{x^2}{1^2}+\frac{x^4}{2^2}+\frac{x^6}{3^2}+\frac{x^8}{4^2}+\ldots=\frac{1}{2\pi}\int_0^\pi [\log(1+2x\cos\theta+x^2)]^2 d\theta.$$

Deduce Euler's series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}.$$

26. Show that if $I_r=\int_0^\pi \sin r\theta\cot\frac{\theta}{2}\,d\theta$, then $I_r=I_{r-1}$.

Hence show that $I_r=\pi$.

27. By differentiating $u=\int_0^{\frac{h}{a}}\frac{\phi(ax)}{x^2}dx$ with regard to a, show that

$$\frac{du}{da}=\int_0^h \frac{\phi'(x)}{x}dx-\phi'(0)\log a-\frac{\phi(h)}{h}.$$

Hence deduce

$$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x^2}\,dx=(a-b)\int_0^\infty \frac{\phi'(x)}{x}\,dx-\phi'(0)[a\log a-b\log b-a+b]$$
$$-(a-b)\,Lt_{h=\infty}\frac{\phi(h)}{h},$$

on the supposition that ϕ is such that $Lt_{h=\infty}\int_{\frac{h}{b}}^{\frac{h}{a}}\frac{\phi(bx)}{x^2}\,dx$ vanishes.

Apply this to show that $\int_0^\infty \frac{\cos ax-\cos bx}{x^2}\,dx=\frac{\pi}{2}(b-a)$.

[BERTRAND, *Calc. Int.*, p. 225.]

28. Prove that if m, n are positive integers whose H.C.F. is r, and $m=r\mu$, $n=r\nu$, and p, q numerically less than unity, then will

$$\int_0^\pi \frac{dx}{(1-2p\cos mx+p^2)(1-2q\cos nx+q^2)}=\frac{\pi}{(1-p^2)(1-q^2)}\,\frac{1+p^\nu q^\mu}{1-p^\nu q^\mu}.$$

29. Show that $\int_0^\pi \frac{\cos rx}{1+e\cos x}\,dx=\frac{\pi}{\sqrt{1-e^2}}\left[\frac{\sqrt{1-e^2}-1}{e}\right]^r$.

[COLLEGES δ, 1884.]

30. Evaluate $$\int_0^{\frac{\pi}{2}}\sin^2 x\log\tan x\,dx.$$

31. Prove that if n be a positive integer,

(i) $\int_0^{\frac{\pi}{2}}\cos 2n\theta\log(\sin\theta)\,d\theta=-\frac{\pi}{4n}$, (ii) $\int_0^{\frac{\pi}{2}}\cos nx\,(\cos x)^n dx=\frac{\pi}{2^{n+1}}$.

32. Prove that, n being a positive integer,

(i) $\int_0^{\frac{\pi}{2}}\cos 2n\theta\log\sin\theta\,d\theta=-\frac{\pi}{4n}$;

(ii) $\int_0^{\frac{\pi}{2}}\cos 2n\theta\{\log(2\sin\theta)\}^2 d\theta=\pi A_n/2n$;

(iii) $\int_0^{\frac{\pi}{2}}\{\log(2\sin\theta)\}^4 d\theta=\pi^5/288+\sum_1^\infty \pi A_n{}^2/n^2$,

where $A_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n-1}+\frac{1}{2n}$.

[ST. JOHN'S, 1891.]

33. Evaluate $\int_0^\pi \frac{x\sin x}{(1-a\cos x)^2}\,dx \quad (a<1)$.

[COLLEGES, 1890.]

34. Prove that if n be a positive integer and $\pi/2>\alpha>0$, then

$$\int_0^\infty \frac{dx}{x}\,\frac{\sin^{2n-1}x}{(1-\sin^2\alpha\sin^2 x)^n}=\frac{\pi}{2^n}\,\frac{1\,.\,3\,.\,5\dots(2n-3)}{(n-1)!}\sec^{2n-1}\alpha.$$

[ST. JOHN'S, 1887.]

35. Show that $2\int_0^{\frac{\pi}{2}} \sec x \log(1+\sin\alpha\cos x)\,dx = \pi\alpha - \alpha^2$.

Hence deduce $\int_0^1 \frac{\log\{2/(1+x^2)\}}{1-x^2}\,dx.$ [TRINITY, 1884.]

36. Prove that if $x<1$,

$$\int_0^{\pi} \log\frac{1+x\cos\theta}{1-x\cos\theta}\,\frac{d\theta}{\cos\theta} = 2\pi\sin^{-1}x.$$ [COLLEGES α, 1891.]

37. If $u+v=4$, $u-v=2\sin\theta$, show that

$$\int_0^{\infty} \log\frac{u^v}{v^u}\cdot\frac{d\theta}{\theta} = \frac{\pi^2}{3} - \pi\log\left(2\cos^2\frac{\pi}{12}\right).$$

38. If m and n are positive integers, prove that

$$\int_0^{\infty} \frac{\cos(2m+1)x - \cos(2n+1)x}{x\sin x}\,dx = (n-m)\pi.$$

[OXFORD II., 1890.]

39. Prove that

$$\int_0^{\frac{\pi}{2}} \{\tan^{-1}(a\tan x) - \tan^{-1}(b\tan x)\}(\tan x+\cot x)\,dx = \frac{\pi}{2}\log\frac{a}{b},$$

where a and b are both positive. [OXFORD II., 1886.]

40. Show that $\int_0^{\infty} \frac{e^{-bx}\sin\beta x - e^{-ax}\sin\alpha x}{x}\,dx = 0$ if $\frac{a}{\alpha} - \frac{b}{\beta} = 0$, and a and b be positive. [CLARE, CAIUS AND KING'S, 1885.]

41. Prove that $\iiint e^{\frac{lx+my+nz}{\alpha}}\,dx\,dy\,dz$ extended over the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is equal to $4\pi abc/e$, α being equal to $\sqrt{a^2l^2+b^2m^2+c^2n^2}$ and l, m, n being direction cosines.

[COLLEGES, 1886.]

42. Show that

$$\int_0^{\infty} \left\{\frac{e^{-ax}-e^{-bx}}{x^2} + (a-b)\frac{e^{-bx}}{x}\right\}dx = b - a - a\log\frac{b}{a},$$

where a and b are positive quantities. [TRINITY, 1892.]

43. Prove that $\int_0^{\frac{\pi}{2}} \frac{\{\theta - \tan^{-1}(n\tan\theta)\}\sin 2\theta\,d\theta}{1+2n\cos 2\theta+n^2} = \frac{\pi}{4n}\log\frac{1+n}{1+n^2}$ if n be less than unity.

Determine also the value of the same integral when n is greater than unity. [ST. JOHN'S, 1891.]

44. Prove that, for any value of n, provided α be between 0 and π,

$$\int_0^\infty \frac{dx}{(1+x^n)(1+2x\cos\alpha+x^2)} = \frac{\alpha}{2\sin\alpha}$$

and
$$\int_0^\infty \frac{(1+x^2)dx}{(1+x^n)(1-2x^2\cos 2\alpha+x^4)} = \frac{\pi}{4\sin\alpha}.$$

[ST. JOHN'S COLL., 1881.]

45. Prove that if c be positive and less than unity,

$$\int_0^\pi \sin 2n\phi \int_0^\infty e^{-xc^2\sin^2\phi}\cos\{cx\sin^2\phi(1-c\cos\phi)\}\,dx\,d\phi = 2\pi c\frac{1-c^{2n}}{(1-c^2)^2}.$$

[MATH. TRIPOS, 1886.]

46. Prove that

$$\int_0^c \int_0^{\sqrt{\frac{c^2-z^2}{1+m^2}}} \int_{my}^{\sqrt{c^2-y^2-z^2}} \frac{x(x^2+y^2+z^2)}{(x^2+y^2+z^2+4c^2)^4}\,dz\,dy\,dx = \frac{\pi}{12000c^2\sqrt{1+m^2}}.$$

[ST. JOHN'S, 1885.]

47. Show that

$$\int_0^\pi \int_0^{2\pi} f(m\cos\theta+n\sin\theta\sin\phi+p\sin\theta\cos\phi)\sin\theta\,d\theta\,d\phi$$
$$= 2\pi\int_{-1}^{+1} f\{x\sqrt{m^2+n^2+p^2}\}\,dx.$$

[POISSON.]

48. Prove that if n be a positive integer,

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin^{2n+2}x\{\sin^{2n+2}y-\sin^{2n+2}x\}}{\sin^2 y-\sin^2 x}\,dy\,dx = \frac{\pi^2}{8}.$$

[ST. JOHN'S, 1888.]

49. Prove that

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (1-\sin^2\omega\sin^2\theta)^{\frac{m}{2}}\sin^{n+1}\omega\,d\theta\,d\omega$$

is a symmetric function of m and n. [MATH. TRIP., 1895.]

50. Prove that

$$\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-(x^4+2x^2y^2\cos 2\alpha+y^4)}\,dx\,dy = \sqrt{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}}.$$

[OX. II. PUB., 1902.]

51. Prove that

$$\int_{-\infty}^\infty e^{ax}\frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)dx = (-1)^n\sqrt{2\pi}\,a^n e^{\frac{a^2}{2}}.$$

52. If $u=(ab'-a'b)x^2+(ac'-a'c)xy+(bc'-b'c)y^2$, prove that

$$\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-u^2}u\,dx\,dy = \frac{\pi}{\sqrt{E}},$$

where $E=4(ab'-a'b)(bc'-b'c)-(ca'-c'a)^2$, provided

$4(b^2-ac)(b'^2-a'c')>(2bb'-ac'-a'c)^2$. [ST. JOHN'S, 1886.]

53. Show that

$$\int_0^\infty\int_0^\infty e^{-ax^2-2cxy-by^2}\,dx\,dy=\frac{1}{2\sqrt{ab-c^2}}\cos^{-1}\frac{c}{\sqrt{ab}}$$

if $a>0$ and $ab-c^2>0$. [I. C. S., 1897.]

54. Show that

$$\int_0^\infty\int_0^\infty y\cosh 2cxy\,e^{-ax^2-by^2}\,dx\,dy=\frac{\sqrt{\pi a}}{4(ab-c^2)}$$

if a, b, c are positive quantities and $ab-c^2>0$. [I. C. S., 1897.]

55. Show that

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} F(1-\sin\theta\cos\phi)\sin\theta\,d\theta\,d\phi=\tfrac{1}{2}\pi\int_0^1 F(u)\,du.$$

[St. John's, 1891.]

56. Prove that

$$\int_0^\infty\int_0^\infty \phi(a^2x^2+b^2y^2)\,dx\,dy=\frac{\pi}{4ab}\int_0^\infty\phi(x)\,dx.$$

57. Calculate the value of $\iint\frac{dx\,dy}{r_1r_2}$ taken throughout the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where r_1 and r_2 are the distances of the point x, y from the foci.

[Colleges α, 1889.]

58. If $V\equiv\sin p_1\theta\sin p_2\theta\ \sin p_3\theta\ldots\sin p_{2n+1}\theta$, where $p_1, p_2, \ldots p_{2n+1}$ are any positive integers whose sum is odd, prove that

$$\int_0^\infty\frac{V\,d\theta}{\theta}=\int_0^{\frac{\pi}{2}}\frac{V\,d\theta}{\sin\theta}.$$

[St. John's, 1892.]

59. Show, by means of Landen's Transformation

$$\tan(\theta-\phi)=\frac{a-b}{a+b}\tan\theta,$$

that $$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a^2\cos^2\theta+b^2\sin^2\theta)^{\frac{1}{2}}}=\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(a_1{}^2\cos^2\phi+b_1{}^2\sin^2\phi)^{\frac{1}{2}}},$$

where a_1 and b_1 are respectively the arithmetic and the geometric means between a and b.

Point out the value of this result in the calculation of the numerical value of the definite integral. [Math. Tripos, 1889.]

60. If p be the length of the perpendicular from the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, on an element dS of the surface, prove that

$$\iint\frac{dS}{p}=2\pi a^2b^2c^2\left\{\frac{d}{d(a^2)}+\frac{d}{d(b^2)}+\frac{d}{d(c^2)}\right\}^2\int_0^\infty\frac{dx}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}.$$

[COLLEGES γ, 1901.]

61. Show that $\displaystyle\int_0^\infty\frac{\sin r\theta\sin n\theta}{\theta\sin\theta}d\theta=n\frac{\pi}{2}$,

provided n is an integer and r any quantity $>n-1$.

[MATH. TRIP., 1873.]

62. Prove that $$\int_0^4\frac{\log x}{\sqrt{4x-x^2}}dx=0.$$

[CLARE, CAIUS, KING'S, 1886.]

63. Prove that $2\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\log(1+\sin 2\theta)\,d\theta+\pi\log 2=0.$

Hence, or otherwise, find the value of

$$\frac{1}{2^2}+\frac{1\,.\,3}{2\,.\,4^2}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6^2}+\dots.$$

[OX. I. P., 1900.]

64. If u, u' are essentially positive quadratic functions of x; Δ, Δ' their discriminants and H the invariant intermediate to Δ and Δ', prove that

$$\int_{-\infty}^{\infty}\log\frac{u'}{u}\cdot\frac{dx}{u}=\frac{\pi}{\sqrt{\Delta}}\log\frac{H+2\sqrt{\Delta\Delta'}}{4\Delta}.$$

[NANSON, *E.T.*, 13406.]

65. If $\displaystyle\sum_{n=0}a_nx^n=\phi(x)$ and $\displaystyle\sum_{n=0}b_nx^n=\psi(x)$,

show that

$$\sum_{n=0}a_nb_nx^n=\frac{1}{2\pi}\int_0^\pi\{\phi(xe^{\iota\theta})+\phi(xe^{-\iota\theta})\}\{\psi(e^{\iota\theta})+\psi(e^{-\iota\theta})\}\,d\theta-a_0b_0.$$

If also $\displaystyle\sum_{n=0}c_nx^n=\chi(x)$, show how to express $\displaystyle\sum_{n=0}a_nb_nc_nx^n$ by means of a double integral.

[SMAASEN.]

66. Prove that

$$1+\frac{\mu x}{1!\,2!}+\frac{\mu^2x^2}{2!\,4!}+\frac{\mu^3x^3}{3!\,6!}+\dots$$
$$=\frac{2}{\pi}\int_0^\pi e^{\mu\cos\theta}\cosh\left(\sqrt{x}\cos\frac{\theta}{2}\right)\cos(\mu\sin\theta)\cos\left(\sqrt{x}\sin\frac{\theta}{2}\right)d\theta-1.$$

[W. H. L. RUSSELL.]

67. Show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ax}\cos^{2n}x\,dx = \frac{(2n)!}{(a^2+2^2)(a^2+4^2)\ldots\{a^2+(2n)^2\}}\cdot\frac{2\sinh\frac{a\pi}{2}}{a}.$$

Hence prove that

$$1+\frac{x}{a^2+2^2}+\frac{x^2}{(a^2+2^2)(a^2+4^2)}+\frac{x^3}{(a^2+2^2)(a^2+4^2)(a^2+6^2)}+\ldots$$

$$=\frac{a}{2}\operatorname{cosech}\frac{a\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{a\theta}\cosh(\sqrt{x}\cos\theta)\,d\theta.$$

[W. H. L. RUSSELL.]

68. Show that $\int_{-\infty}^{\infty} e^{-\frac{ax^2}{4}}(e^x-\cos x)\,dx = 4\sqrt{\frac{\pi}{a}}\sinh\frac{1}{a}.$

[W. H. L. RUSSELL.]

69. Establish the results

$$\text{(i)} \int_0^\infty f\left(x+\frac{1}{x}\right)\log x\,\frac{dx}{x} = 0.$$

$$\text{(ii)} \int_0^\infty f\left(x+\frac{1}{x}\right)\tan^{-1}x\,\frac{dx}{x} = \frac{\pi}{4}\int_0^\infty f\left(x+\frac{1}{x}\right)\frac{dx}{x}.$$

[LIOUVILLE.]

70. Establish the results

$$\text{(i)} \int_0^\infty f\left(x+\frac{1}{x}\right)\frac{1}{1+x^n}\,\frac{dx}{x} = \frac{1}{2}\int_0^\infty f\left(x+\frac{1}{x}\right)\frac{dx}{x}.$$

$$\text{(ii)} \int_0^\infty \frac{dx}{(1+x^2)(1+x^n)} = \frac{\pi}{4}.$$

$$\text{(iii)} \int_0^{\frac{\pi}{2}} \frac{F(\sin 2\theta)}{1+\tan^n\theta}\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} F(\sin\theta)\,d\theta.$$

[GLAISHER, *Messenger of Math.*, No. 70.]

71. If $J_n(x)$ be Bessel's function, show that

$$\int_0^\infty \frac{J_n(ax)}{x^{n-m}}\,dx = \frac{x^n}{\sqrt{\pi}\,2^n\Gamma(n+\frac{1}{2})}\,\frac{\Pi\left(\frac{m-1}{2}\right)}{\Pi\left(n-\frac{m+1}{2}\right)}.\ (2n+1>0>m>-1).$$

[MATH. TRIP., 1898.]