1/2008

Raport Badawczy
Research Report

RB/11/2008

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Warszawa 2008

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Abstract. In this paper the long-time behaviour of a unique regular solution to the Cahn-Hilliard system coupled with viscoelasticity is studied. The system arises as a model of phase separation process in a binary deformable alloy.
It is proved that for a sufficiently regular initial data the trajectory of the solution converges to the ω-limit set of these data. Moreover, it is shown that every element of the ω-limit set is a solution of the corresponding stationary problem.
Key words: Cahn-Hilliard, viscoelasticity system, phase separation, long-time behaviour
AMS Subject Classification: 35K50, 35K60, 35L20, 35Q72

1. Introduction

In the continuation of [5] we study in this paper the asymptotic behaviour as $t \to \infty$ of a regular solution to the following Cahn-Hilliard system coupled with viscoelasticity:

$$
(1.1)\qquad
\begin{aligned}
&\boldsymbol{u}_{tt} - \nabla \cdot [W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) + \nu \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t)] = \boldsymbol{b} && \text{in } \Omega^\infty = \Omega \times (0, \infty),\\
&\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1 && \text{in } \Omega,\\
&\boldsymbol{u} = 0 && \text{on } S^\infty = S \times (0, \infty),
\end{aligned}
$$

$$
(1.2)\qquad
\begin{aligned}
&\chi_t - \Delta\mu = 0 && \text{in } \Omega^\infty,\\
&\chi(0) = \chi_0 && \text{in } \Omega,\\
&\boldsymbol{n} \cdot \nabla\mu = 0 && \text{on } S^\infty,
\end{aligned}
$$

$$
(1.3)\qquad
\begin{aligned}
&\mu = -\gamma\Delta\chi + \psi'(\chi) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) && \text{in } \Omega^\infty,\\
&\boldsymbol{n} \cdot \nabla\chi = 0 && \text{on } S^\infty,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary S; the unknowns are the fields $\boldsymbol{u} : \Omega^\infty \to \mathbb{R}^3$, $\chi : \Omega^\infty \to \mathbb{R}$, and $\mu : \Omega^\infty \to \mathbb{R}$, representing respectively the displacement vector, the order parameter and the chemical potential; $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$ is the linearized strain tensor; functions $W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)$ and $\psi(\chi)$ are specified below, ν, γ are positive constants.

The system arises as a model, regularized by a viscous damping, of phase separation process in a deformable two-component $a - b$ alloy cooled below a critical temperature. In the previous paper [5] we have proved the existence and uniqueness of a global in time, regular solution to this system. Moreover, we have shown the existence of an absorbing set. Our objective in the present paper is to study the asymptotic behaviour of the solution as $t \to \infty$.

System (1.1)–(1.3) represents balance laws of linear mementum, mass, and the equation for the chemical potential. The associated free energy density has the Landau-Ginzburg form

$$
(1.4)\qquad f(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi, \nabla\chi) = W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) + \psi(\chi) + \frac{\gamma}{2}|\nabla\chi|^2,
$$

where

$$
(1.5)\qquad W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) = \frac{1}{2}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)) \cdot \boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)),
$$

and

$$
(1.6)\qquad \psi = \frac{1}{4}(1 - \chi^2)^2
$$

represent respectively the elastic energy and the double-well potential; positive constant γ is related to a surface tension.

The order parameter χ characterizes the material phase. In case of a binary alloy it is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components. We shall assume that $\chi = -1$ is identified with the phase a and $\chi = 1$ with the phase b.

The elasticity tensor $\boldsymbol{A} = (A_{ijkl})$ and the eigenstrain tensor $\bar{\boldsymbol{\varepsilon}}(\chi) = (\bar{\varepsilon}_{ij}(\chi))$ are given by

$$(1.7)\qquad \begin{aligned} \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) &= \bar{\lambda} tr \boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{I} + 2\bar{\mu}\boldsymbol{\varepsilon}(\boldsymbol{u}), \\ \bar{\boldsymbol{\varepsilon}}(\chi) &= (1 - z(\chi))\bar{\boldsymbol{\varepsilon}}_a + z(\chi)\bar{\boldsymbol{\varepsilon}}_b, \end{aligned}$$

where $\boldsymbol{I}$ is the identity tensor, $\bar{\lambda}, \bar{\mu}$ are the Lamé constants satisfying $\bar{\mu} > 0$, $3\bar{\lambda} + 2\bar{\mu} > 0$, $\bar{\boldsymbol{\varepsilon}}_a, \bar{\boldsymbol{\varepsilon}}_b$ are constant eigenstrains of phases a, b, and $z : \mathbb{R} \to [0, 1]$ is a sufficiently smooth interpolation function such that

$$(1.8)\qquad z(\chi) = 0 \quad \text{for} \quad \chi \leq -1 \quad \text{and} \quad z(\chi) = 1 \quad \text{for} \quad \chi \geq 1.$$

The term $\nu\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t)$, with $\nu = \text{const} > 0$, represents a viscous stress tensor; ν is a viscosity coefficient. The derivatives of $W(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi)$ with respect to $\boldsymbol{\varepsilon}$ and χ, given by

$$\begin{aligned} W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) &= \boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)), \\ W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) &= -\bar{\boldsymbol{\varepsilon}}'(\chi) \cdot \boldsymbol{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \bar{\boldsymbol{\varepsilon}}(\chi)), \end{aligned}$$

denote respectively the elastic stress tensor and the elastic contribution to the chemical potential. For a detailed description of system (1.1)–(1.3) and the discussion of related literature we refer to [5].

By introducing the linear elasticity operator

$$(1.9)\qquad \boldsymbol{u} \mapsto \boldsymbol{Q}\boldsymbol{u} = \nabla \cdot (\boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \bar{\mu}\Delta\boldsymbol{u} + (\bar{\lambda} + \bar{\mu})\nabla(\nabla \cdot \boldsymbol{u})$$

with the domain $D(\boldsymbol{Q}) = \boldsymbol{H}^2(\Omega)/\boldsymbol{H}_0^1(\Omega)$, and the auxiliary constant quantities

$$(1.10)\qquad \boldsymbol{B} = -\boldsymbol{A}(\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad D = -\boldsymbol{B} \cdot (\bar{\boldsymbol{\varepsilon}}_b - \bar{\boldsymbol{\varepsilon}}_a), \quad E = -\boldsymbol{B} \cdot \bar{\boldsymbol{\varepsilon}}_a,$$

we have

$$(1.11)\qquad \begin{aligned} W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) &= \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{A}\bar{\boldsymbol{\varepsilon}}_a + z(\chi)\boldsymbol{B}, \\ W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) &= z'(\chi)(\boldsymbol{B} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E), \\ \nabla \cdot W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \chi) &= \boldsymbol{Q}\boldsymbol{u} + z'(\chi)\boldsymbol{B}\nabla\chi. \end{aligned}$$

On account of (1.9)–(1.11) it is convenient to recast system (1.1)–(1.3) into the following concised form

$$(1.12)\qquad \begin{aligned} &\boldsymbol{u}_{tt} - \boldsymbol{Q}\boldsymbol{u} - \nu\boldsymbol{Q}\boldsymbol{u}_t = z'(\chi)\boldsymbol{B}\nabla\chi + \boldsymbol{b} && \text{in } \Omega^\infty, \\ &\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0) = \boldsymbol{u}_1 && \text{in } \Omega, \\ &\boldsymbol{u} = 0 && \text{on } S^\infty, \end{aligned}$$

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$$
(1.13)\qquad \begin{aligned} &\chi_t - \Delta\mu = 0 && \text{in } \Omega^\infty,\\ &\chi(0) = \chi_0 && \text{in } \Omega,\\ &\boldsymbol{n}\cdot\nabla\mu = 0 && \text{on } S^\infty, \end{aligned}
$$

$$
(1.14)\qquad \begin{aligned} &\mu = -\gamma\Delta\chi + \psi'(\chi) + z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E) && \text{in } \Omega^\infty,\\ &\boldsymbol{n}\cdot\nabla\chi = 0 && \text{on } S^\infty. \end{aligned}
$$

It has been proved in [5] (see Theorem 2.1 below) that system (1.1)–(1.3) admits the unique global solution $(\boldsymbol{u},\chi,\mu)$ such that

$$
\begin{aligned}
&\boldsymbol{u}\in C^1([0,\infty);\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega))\cap C^2([0,\infty);\boldsymbol{H}^1_0(\Omega)),\\
&\chi\in C([0,\infty);H^2_N(\Omega))\cap C^1([0,\infty);L_2(\Omega)),\\
&\mu\in C([0,\infty);H^2_N(\Omega)),\qquad \fint_\Omega \chi(t)dx = \chi_m := \fint_\Omega \chi_0 dx \qquad \text{for all } t\in[0,\infty),
\end{aligned}
$$

for initial data satisfying

$$
\begin{aligned}
&(\boldsymbol{u}(0),\boldsymbol{u}_t(0),\boldsymbol{u}_{tt}(0),\chi(0),\chi_t(0))\in\mathcal{W}\\
&:=\{(\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega))\times(\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega))\times\boldsymbol{H}^1_0(\Omega)\times H^2_N(\Omega)\times L_2(\Omega)\},
\end{aligned}
$$

where

$$
H^2_N(\Omega)=\{\xi:\xi\in H^2(\Omega),\boldsymbol{n}\cdot\nabla\xi=0 \text{ on } S\}.
$$

Thus, the solution defines the nonlinear, strongly continuous semigroup

$$
\begin{aligned}
S(t):\mathcal{W}\ni(\boldsymbol{u}(0),\boldsymbol{u}_t(0),\boldsymbol{u}_{tt}(0),\chi(0),\chi_t(0))\mapsto\\
(\boldsymbol{u}(t),\boldsymbol{u}_t(t),\boldsymbol{u}_{tt}(t),\chi(t),\chi_t(t))\in\mathcal{W},\quad t\in[0,\infty).
\end{aligned}
$$

In this paper we prove that for any initial data belonging to $\mathcal{W}$ the trajectory of the solution converges as $t\to\infty$ to the ω-limit set of these data.
Moreover, we show that the ω-limit set is compact, connected subset of the space

$$
\mathcal{Z}:=\boldsymbol{H}^1_0(\Omega)\times\boldsymbol{H}^1_0(\Omega)\times\boldsymbol{L}_2(\Omega)\times H^1(\Omega)\times(H^1(\Omega))',
$$

and enjoys the standard properties, namely it is positive invariant under semigroup $S(t)$ defined by the solution and the total energy functional is constant on this set. We prove also that every element of the ω-limit set is a solution of the corresponding stationary problem.

In the proof of these results we use arguments similar to that applied in [7] for the single Cahn-Hilliard equation, and the procedure of long-time analysis devised in [1], [2] for phase-field models.

We use the same notation as in [5]. Vectors and tensors are denoted by bold letters. A dot designates the inner product irrespective of the space in question, e.g. for vectors $\boldsymbol{a} = (a_i)$, $\tilde{\boldsymbol{a}} = (\tilde{a}_i)$ and tensors $\boldsymbol{B} = (B_{ij})$, $\tilde{\boldsymbol{B}} = (\tilde{B}_{ij})$ we write $\boldsymbol{a}\cdot\tilde{\boldsymbol{a}} = a_i\tilde{a}_i$, $\boldsymbol{B}\cdot\tilde{\boldsymbol{B}} = B_{ij}\tilde{B}_{ij}$. Here and throughout the summation convention over repeated indices is used.
The symbols ∇ and $\nabla\cdot$ denote the gradient and the divergence operators. For the divergence we use the convention of the contraction over the last index, e.g. $\nabla\cdot\boldsymbol{\varepsilon} = \left(\frac{\partial\varepsilon_{ij}}{\partial x_j}\right)$. For simplicity, the space and time derivatives (material) are denoted by $f_{,i} = \partial f/\partial x_i$, $f_t = \partial f/\partial t$.
Moreover, for $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ we write $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi) = \left(\frac{\partial W(\boldsymbol{\varepsilon},\chi)}{\partial\varepsilon_{ij}}\right)$. We use the standard Sobolev spaces notation. In addition, the spaces of vector- or tensor-valued functions are indicated by bold letters.

2. Main results

First we recall the existence and uniqueness result for (1.1)–(1.3), proved in [5] under the following assumptions:

(A1) $\Omega \subset \mathbb{R}^3$ is a bounded domain with the boundary S of class at least C^2, $T > 0$ is an arbitrary fixed number.
(A2) The Lamé coefficients $\bar{\mu}, \bar{\lambda}$ satisfy

$$\bar{\mu} > 0, \quad 3\bar{\lambda} + 2\bar{\mu} > 0$$

which assures that the elasticity tensor $\boldsymbol{A}$ is coercive and bounded, i.e.,

$$c_*|\boldsymbol{\varepsilon}|^2 \le \boldsymbol{\varepsilon}\cdot\boldsymbol{A}\boldsymbol{\varepsilon} \le c^*|\boldsymbol{\varepsilon}|^2 \tag{2.1}$$

for all symmetric second order tensors $\boldsymbol{\varepsilon}$ in $\mathbb{R}^3$, with positive constants c_* and c^*.
Moreover, due to this condition the operator $\boldsymbol{Q}$ given by (1.9) is strongly elliptic and satisfies

$$\underline{c}_Q\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \le \|\boldsymbol{Q}\boldsymbol{u}\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u}\in D(\boldsymbol{Q}) = \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}^1_0(\Omega) \tag{2.2}$$

with a positive constant $\underline{c}_Q$.
(A3) $W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)$ is given by (1.5); the function $z:\mathbb{R}\to[0,1]$ is of class C^2 satisfying (1.8) and

$$|z'(\chi)| + |z''(\chi)| \le c \quad \text{for all} \quad \chi\in\mathbb{R}.$$

The auxiliary quantities B, D and E are defined in (1.10).
(A4) $\psi(\chi)$ is given by (1.6).
(A5) γ and ν are positive constants.

The next assumption concerns the initial data. In addition to

$$u(0) = u_0, \quad u_t(0) = u_1, \quad \chi(0) = \chi_0 \quad \text{in} \quad \Omega,$$

we introduce in compatibility with (1.12)–(1.14) the initial conditions corresponding to $u_{tt}(0)$ and $\chi_t(0)$:

$$\begin{aligned} u_2 &:= u_{tt}(0) = Qu_0 + \nu Qu_1 + z'(\chi_0)B\nabla\chi_0 + b(0), \\ \chi_1 &:= \chi_t(0) = \Delta\mu(0) \\ &= -\gamma\Delta^2\chi_0 + \Delta[\psi'(\chi_0) + z'(\chi_0)(B\cdot\varepsilon(u_0) + Dz(\chi_0) + E)] \quad \text{in} \quad \Omega. \end{aligned}$$

We assume

(A6) $u_0, u_1 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_2 \in H_0^1(\Omega)$,

$\chi_0 \in H_N^2(\Omega) := \{\xi \in H^2(\Omega) : n\cdot\nabla\xi = 0 \text{ on } S\}$, $\chi_m := \fint_\Omega \chi_0 dx < \infty$, $\chi_1 \in L_2(\Omega)$,

which implies that

$$u_0 \in H^3(\Omega)\cap H_0^1(\Omega), \qquad \chi_0 \in H^4(\Omega)\cap H_N^2(\Omega).$$

As regards the external force, we require

(A7) $b \in L_1(0,\infty; L_2(\Omega)) \cap W_\infty^1(0,\infty; L_2(\Omega))$.

The existence theorem is as follows:

Theorem 2.1. *(see [5], Thm 2.1, 2.3) Let assumptions (A1)–(A7) hold true. Then problem (1.1)–(1.3) (in simplified formulation (1.12)–(1.14)) admits the unique global solution (u, χ, μ) on $[0,\infty)$ such that*

$$\begin{aligned} &u \in C^1([0,\infty); H^2(\Omega)\cap H_0^1(\Omega)) \cap C^2([0,\infty); H_0^1(\Omega)), \\ &\chi \in C([0,\infty); H_N^2(\Omega)) \cap C^1([0,\infty); L_2(\Omega)), \\ &\mu \in C([0,\infty); H_N^2(\Omega)), \quad \fint_\Omega \chi(t)dx = \chi_m \quad \textit{for all} \quad t\in[0,\infty), \\ &u_t \in L_2(0,\infty; H_0^1(\Omega)) \quad \nabla\mu \in L_2(0,\infty; L_2(\Omega)), \end{aligned} \tag{2.3}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad u_{tt}(0) = u_2, \quad \chi(0) = \chi_0, \quad \chi_t(0) = \chi_1, \tag{2.4}$$

and, for any $t\in[0,\infty)$ and any fixed number $T>0$,

$$\begin{aligned} &u_{tt} \in L_2(t,t+T; H^2(\Omega)\cap H_0^1(\Omega)), && u_{ttt} \in L_2(t,t+T; (H_0^1(\Omega))'), \\ &\chi_t \in L_2(t,t+T; H_N^2(\Omega)), && \chi_{tt} \in L_2(t,t+T; (H_N^2(\Omega))'), \\ &\mu \in L_2(t,t+T; H^1(\Omega)), && \mu_t \in L_2(t,t+T; L_2(\Omega)). \end{aligned} \tag{2.5}$$

Furthermore, the solution satisfies the following estimates:

uniform in time

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{C([0,\infty);\boldsymbol{H}_0^1(\Omega))} + \|\boldsymbol{u}_t\|_{C([0,\infty);\boldsymbol{L}_2(\Omega))} \\
(2.6)\qquad &\quad + \|\chi\|_{C([0,\infty);H^1(\Omega))} + \|\boldsymbol{u}_t\|_{L_2(0,\infty;\boldsymbol{H}_0^1(\Omega))} \\
&\quad + \|\nabla\mu\|_{L_2(0,\infty;\boldsymbol{L}_2(\Omega))} \le c_0,
\end{aligned}
$$

$$
\begin{aligned}
&\|\boldsymbol{u}\|_{C^1([0,\infty);\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}_{tt}\|_{C([0,\infty);\boldsymbol{H}_0^1(\Omega))} \\
(2.7)\qquad &\quad + \|\chi\|_{C([0,\infty);H_N^2(\Omega))} + \|\chi_t\|_{C([0,\infty);L_2(\Omega))} \\
&\quad + \|\mu\|_{C([0,\infty);H_N^2(\Omega))} \le c,
\end{aligned}
$$

where

$$c_0 = c_0(\|\boldsymbol{u}_0\|_{\boldsymbol{H}_0^1(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{L}_2(\Omega)}, \|\chi_0\|_{H^1(\Omega)}, \|\boldsymbol{b}\|_{L_1(0,\infty;\boldsymbol{L}_2(\Omega))}),$$
$$c = c(\|\boldsymbol{u}_0\|_{\boldsymbol{H}^2(\Omega)}, \|\boldsymbol{u}_1\|_{\boldsymbol{H}^2(\Omega)}, \|\boldsymbol{u}_2\|_{\boldsymbol{H}^1(\Omega)}, \|\chi_0\|_{H_N^2(\Omega)}, \|\chi_1\|_{L_2(\Omega)}, \|\boldsymbol{b}\|_{W^1_\infty(0,\infty;\boldsymbol{L}_2(\Omega))})$$

are positive constants distinguishing dependence on the data;

for any $t \in [0,\infty)$ and any fixed $T > 0$,

$$(2.8)\qquad \|\chi\|_{L_2(t,t+T;H_N^2(\Omega))} + \|\mu\|_{L_2(t,t+T;H^1(\Omega))} \le c(c_0)(T^{1/2}+1),$$

$$(2.9)\qquad \|\boldsymbol{u}_{tt}\|_{L_2(t,t+T;\boldsymbol{H}^2(\Omega))} + \|\chi_t\|_{L_2(t,t+T;H_N^2(\Omega))} \le c(T^{1/2}+1),$$

$$
\begin{aligned}
(2.10)\qquad &\|\boldsymbol{u}_{ttt}\|_{L_2(t,t+T;(\boldsymbol{H}_0^1(\Omega))')} + \|\chi_{tt}\|_{L_2(t,t+T;(H_N^2(\Omega))')} \\
&\quad + \|\mu_t\|_{L_2(t,t+T;L_2(\Omega))} \le c(T^{1/2}+1)
\end{aligned}
$$

with constants c_0, c as above.

Let us introduce the spaces

$$
\begin{aligned}
(2.11)\qquad &\mathcal{W} := (\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega)) \times (\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega)) \times \boldsymbol{H}_0^1(\Omega) \times H_N^2(\Omega) \times L_2(\Omega), \\
&\mathcal{Z} := \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}_2(\Omega) \times H^1(\Omega) \times (H^1(\Omega))'.
\end{aligned}
$$

In view of (2.7) it is seen that the solution in Theorem 2.1 generates the strongly continuous, nonlinear semigroup

$$(2.12)\qquad S(t) : \zeta_0 \in \mathcal{W} \mapsto \zeta(t) \in \mathcal{W}, \quad t \ge 0,$$

where

$$\zeta_0 := (\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \chi_0, \chi_1), \qquad \zeta(t) := (\boldsymbol{u}(t), \boldsymbol{u}_t(t), \boldsymbol{u}_{tt}(t), \chi(t), \chi_t(t)).$$

Let us introduce the ω-limit set of the initial data $\zeta_0 \in \mathcal{W}$:

$$
\begin{aligned}
&\omega(\zeta_0) := \{\zeta_\infty = (\boldsymbol{u}_\infty, \boldsymbol{u}_{\infty,t}, \boldsymbol{u}_{\infty,tt}, \chi_\infty, \chi_{\infty,t}) \in \mathcal{W} \subset \mathcal{Z} : \\
(2.13)\qquad &\qquad \exists\{t_n\} \subset (0,\infty), \quad t_n \to \infty \quad \text{and} \\
&\qquad \zeta(t_n) = S(t_n)\zeta_0 \to \zeta_\infty \ \text{ strongly in } \mathcal{Z}\}.
\end{aligned}
$$

The main result of this paper is stated in the following

Theorem 2.2. *Assume that (A1)–(A7) hold. Let $S(t): \mathcal{W} \to \mathcal{W}$, $t \geq 0$, be the nonlinear semigroup generated by the unique solution of system (1.1)–(1.3). Then*

(i) *The ω-limit set $\omega(\zeta_0)$ of the initial data $\zeta_0 = (\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \chi_0, \chi_1) \in \mathcal{W} \subset \mathcal{Z}$ is a nonempty, compact and connected subset of the space $\mathcal{Z}$. Furthermore, $\omega(\zeta_0)$ is positive invariant under $S(t)$, i.e.,*

$$S(t)\omega(\zeta_0) \subset \omega(\zeta_0) \quad \textit{for any} \quad t \geq 0;$$

(ii) *If $\boldsymbol{b} = \boldsymbol{0}$ then the map $F_\Omega : \mathcal{W} \to \mathbb{R}$ defined by*

$$F_\Omega(\zeta(t)) = \int\limits_\Omega \left[\frac{1}{2}|\boldsymbol{u}_t(t)|^2 + W(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \chi(t)) + \psi(\chi(t)) + \frac{\gamma}{2}|\nabla\chi(t)|^2\right] dx, \tag{2.14}$$

is the Lyapunov functional for the semigroup $S(t)$, i.e.,

$$F_\Omega(S(t)\zeta_0) \leq F_\Omega(\zeta_0) \quad \textit{for any} \quad \zeta_0 \in \mathcal{W}, \quad t \geq 0;$$

F_Ω is constant on the ω-limit set $\omega(\zeta_0)$.

(iii) *Every element $\zeta_\infty = (\boldsymbol{u}_\infty, \boldsymbol{u}_{\infty,t}, \boldsymbol{u}_{\infty,tt}, \chi_\infty, \chi_{\infty,t})$ of the ω-limit set $\omega(\zeta_0)$ is characterized by*

$$\zeta_\infty \equiv (\boldsymbol{u}_\infty, \boldsymbol{0}, \boldsymbol{0}, \chi_\infty, 0) \tag{2.15}$$

with functions $\boldsymbol{u}_\infty, \chi_\infty$ independent of time, solving the stationary problem corresponding to (1.1)–(1.3):

$$\begin{aligned} &-\nabla \cdot W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty), \chi_\infty) = 0 \quad && \textit{a.e. in} \quad \Omega, \\ &\boldsymbol{u}_\infty = \boldsymbol{0} && \textit{a.e. on} \quad S, \end{aligned} \tag{2.16}$$

$$\begin{aligned} &-\gamma\Delta\chi_\infty + \psi'(\chi_\infty) + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty), \chi_\infty) = \bar{\mu} \quad && \textit{a.e. in} \quad \Omega, \\ &\boldsymbol{n} \cdot \nabla\chi_\infty = 0 && \textit{a.e. on} \quad S, \\ &\fint\limits_\Omega \chi_\infty dx = \chi_m := \fint\limits_\Omega \chi_0 dx, \end{aligned} \tag{2.17}$$

where $\bar{\mu}$ is a constant to be determined along with functions $\boldsymbol{u}_\infty, \chi_\infty$.

In the proof of Theorem 2.2 the crucial role play uniform in time estimates (2.6) and (2.7). In particular, the estimates on $\boldsymbol{u}_t$ and $\nabla\mu$ in L_2-norms on the infinite time interval $(0, \infty)$ (which are due to the mechanical and diffusive dissipation) assure that $\boldsymbol{u}_t$ and $\nabla\mu$ vanish at the limit $t \to \infty$.

3. Outline of the existence proof. Basic estimates

In this section we present the main ideas of the proof of Theorem 2.1 (see [5] for details) with complementary estimates needed in the study of the asymptotic behaviour. The proof consists in prolonging the local solution on the intervals $[kT,(k+1)T]$, $T>0$, $k\in\mathbb{N}\cup\{0\}$, up to $k=\infty$. The existence of a local solution is obtained by implementing a Galerkin method and passing to the limit with the approximation. The crucial role in prolonging the local solution play absorbing type estimates with the property of exponentially time-decreasing influence of the initial data. We use two kinds of such estimates: the energy and the regularity ones. The energy estimates are derived on the basis of the original form (1.1)–(1.3) of the system whereas the regularity estimates on the basis of its time-differentiated form.

3.1. Energy estimates

3.1.1. Energy identity

A characteristic property of system (1.1)–(1.3) is the mass conservation

$$\frac{d}{dt}\int_\Omega \chi(t)dx = 0 \quad \text{for} \quad t>0,$$

which follows from $(1.2)_1$ and $(1.2)_3$, and shows that the mean value of χ is preserved, i.e.,

$$\fint_\Omega \chi(t)dx = \fint_\Omega \chi_0 dx =: \chi_m \quad \text{for} \quad t>0. \tag{3.1}$$

Another property is the energy identity

$$\begin{aligned}&\frac{d}{dt}F(t)+\nu\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}_t(t))\cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t))dx+\int_\Omega |\nabla\mu(t)|^2dx\\ &=\int_\Omega \boldsymbol{b}(t)\cdot\boldsymbol{u}_t(t)dx \quad \text{for} \quad t>0,\end{aligned} \tag{3.2}$$

where function $F:[0,\infty)\to[0,\infty)$, given by

$$F(t)=\int_\Omega\left[\frac{1}{2}|\boldsymbol{u}_t(t)|^2+W(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\chi(t))+\psi(\chi(t))+\frac{\gamma}{2}|\nabla\chi(t)|^2\right]dx$$

corresponds to the total energy of the system. The two nonnegative integrals on the left-hand side of (3.2) correspond to the mechanical and diffusive dissipation.

Formally, (3.2) results by testing $(1.1)_1$ by $\boldsymbol{u}_t(t)$, $(1.2)_1$ by $\mu(t)$ and $(1.3)_1$ by $-\chi_t(t)$, integrating over Ω and by parts, and summing up the resulting identities.

From (3.2) we infer the Lyapunov property, namely if $\boldsymbol{b} = \boldsymbol{0}$ then

$$\frac{d}{dt}F(t) \leq 0, \tag{3.3}$$

which shows that F is nonincreasing on solutions paths, i.e.,

$$F(t) \leq F(0) \quad \text{for} \quad t \geq 0.$$

On account of structure assumptions (A3)–(A5),

$$F(t) \geq c_F(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^1_0(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\chi(t)\|^2_{H^1(\Omega)}) - c'_F \tag{3.4}$$

with some explicitly computed positive constants c_F and c'_F. Hence, $F(t)$ provides estimates for $(\boldsymbol{u}(t), \boldsymbol{u}_t(t), \chi(t))$ in energy norms $\boldsymbol{H}^1_0(\Omega) \times \boldsymbol{L}_2(\Omega) \times H^1(\Omega)$. Integrating (3.2) with respect to time from $t = 0$ to $t \in (0, \infty)$, we get

$$\begin{aligned} &\|\boldsymbol{u}\|_{L_\infty(0,\infty;\boldsymbol{H}^1_0(\Omega))} + \|\boldsymbol{u}_t\|_{L_\infty(0,\infty;\boldsymbol{L}_2(\Omega))} + \|\chi\|_{L_\infty(0,\infty;H^1(\Omega))} \\ &\quad + \|\boldsymbol{u}_t\|_{L_2(0,\infty;\boldsymbol{H}^1_0(\Omega))} + \|\nabla\mu\|_{L_2(0,\infty;\boldsymbol{L}_2(\Omega))} \leq c_0 \end{aligned} \tag{3.5}$$

with constant $c_0 = c(\|(\boldsymbol{u}_0, \boldsymbol{u}_1, \chi_0)\|_{\boldsymbol{H}^1_0(\Omega)\times\boldsymbol{L}_2(\Omega)\times H^1(\Omega)}, \|\boldsymbol{b}\|_{L_1(0,\infty;\boldsymbol{L}_2(\Omega))})$. Since $F(\cdot)$ is continuous on $[0, \infty)$ this shows estimate (2.6).

3.1.2. Additional estimates

From (1.14) it follows, on account of (3.5), that

$$\left|\int_\Omega \mu dx\right| \leq c \int_\Omega (|\chi|^3 + |\varepsilon(\boldsymbol{u})| + 1)dx \leq c(c_0) \quad \text{for} \quad t \geq 0. \tag{3.6}$$

Hence, by the Poincaré inequality, estimates (3.5) and (3.6) imply that for any $t \geq 0$ and any fixed $T > 0$,

$$\begin{aligned} \|\mu\|^2_{L_2(t,t+T;L_2(\Omega))} &\leq c \int_t^{t+T} \left(\|\nabla\mu\|^2_{\boldsymbol{L}_2(\Omega)} + \left|\int_\Omega \mu dx\right|^2\right)dt \\ &\leq c\|\nabla\mu\|^2_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))} + cT \sup_{t\in[t,t+T]} \left|\int_\Omega \mu dx\right|^2 \\ &\leq c(c_0)(T+1). \end{aligned} \tag{3.7}$$

Thus,

$$\|\mu\|^2_{L_2(t,t+T;H^1(\Omega))} \le c(c_0)(T+1)$$

which shows the second estimate in (2.8).
The first estimate in (2.8) follows by testing (1.14) by $\Delta\chi$ and using the Cauchy-Schwarz equality, which on account of (3.7) and (3.5) yields

$$\begin{aligned}\gamma\|\Delta\chi\|_{L_2(t,t+T;L_2(\Omega))} &\le \|\mu\|_{L_2(t,t+T;L_2(\Omega))} + \|\chi^3-\chi\|_{L_2(t,t+T;L_2(\Omega))}\\ &+ c(\|\varepsilon(\boldsymbol{u})\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))}+1) \le c(c_0)(T^{1/2}+1).\end{aligned}$$

This together with (3.1), by the ellipticity property of the Laplace operator, shows $(2.8)_1$.

3.2. Energy estimates of absorbing type

3.2.1. A differential inequality for a modified energy function

Let $G:[0,\infty)\to[0,\infty)$ be the function defined by

$$G(t) = F(t) + \frac{\nu c_* d_1}{2}\int\limits_\Omega \Big[\boldsymbol{u}_t(t)\cdot\boldsymbol{u}(t) + \frac{\nu}{2}\varepsilon(\boldsymbol{u}(t))\cdot \boldsymbol{A}\varepsilon(\boldsymbol{u}(t))\Big]\,dx \tag{3.8}$$

with constant $c_*>0$ given in coercivity condition (2.1) and $d_1>0$ denoting constant from the Korn inequality

$$d_1^{1/2}\|\boldsymbol{u}\|_{\boldsymbol{H}^1_0(\Omega)} \le \|\varepsilon(\boldsymbol{u})\|_{\boldsymbol{L}_2(\Omega)} \quad \text{for} \quad \boldsymbol{u}\in\boldsymbol{H}^1_0(\Omega).$$

By definition of $G(t)$, it holds

$$G(t) \ge F(t) - \frac{1}{4}\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}.$$

Hence, similarly to $F(t)$, the function $G(t)$ provides estimates on $(\boldsymbol{u},\boldsymbol{u}_t,\chi)$ in energy norms $\boldsymbol{H}^1_0(\Omega)\times\boldsymbol{L}_2(\Omega)\times H^1(\Omega)$.

It has been proved (see [5], Lemma 3.3) that solutions of (1.1)–(1.3) satisfy the differential inequality

$$\begin{aligned}&\frac{d}{dt}G(t) + \beta_1 G(t) + \frac{\nu c_* d_1}{8}\|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^1_0(\Omega)} + \frac{1}{2}\|\nabla\mu(t)\|^2_{L_2(\Omega)}\\ &\le \Lambda_1\|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \Lambda_2 \quad \text{for} \quad t>0,\end{aligned} \tag{3.9}$$

with some explicitly computed positive constants $\beta_1,\Lambda_1,\Lambda_2$.

The proof of (3.9) is based on the three identities: the energy identity (3.2), the identity

$$\gamma \int_\Omega |\nabla\chi|^2 dx + \int_\Omega [\psi'(\chi)\chi + W_{,\chi}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\chi] dx = \int_\Omega \mu\chi dx,$$

resulting from testing equation $(1.3)_1$ by $\chi(t)$, and the identity

$$\frac{d}{dt}\int_\Omega \boldsymbol{u}_t \cdot \boldsymbol{u} dx + \int_\Omega W_{,\varepsilon}(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) dx + \nu \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot \boldsymbol{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_t) dx$$
$$= \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u} dx + \int_\Omega |\boldsymbol{u}_t|^2 dx,$$

following by testing $(1.1)_1$ by $\boldsymbol{u}(t)$. An appropriate technical construction based on structure assumptions on $W(\boldsymbol{\varepsilon}(\boldsymbol{u}),\chi)$ and $\psi(\chi)$ and using straightforward calculations allows to deduce (3.9) from the mentioned above identities.

3.2.2. Absorbing estimate for $G(t)$

From (3.9) it follows that

$$G(t) \le A_1(1-e^{-\beta_1 t}) + G(0)e^{-\beta_1 t}, \quad t \ge 0, \tag{3.10}$$

where

$$A_1 = \frac{1}{\beta_1}(\Lambda_1 \sup_{t\in(0,\infty)} \|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \Lambda_2).$$

This estimate is of key importance for prolonging the solution step by step on $[kT,(k+1)T]$, $k\in\mathbb{N}\cup\{0\}$, up to $k=\infty$. In particular, it provides the estimate

$$\begin{aligned} &\|\boldsymbol{u}(t)\|_{\boldsymbol{H}^1_0(\Omega)} + \|\boldsymbol{u}_t(t)\|_{\boldsymbol{L}_2(\Omega)} + \|\chi(t)\|_{H^1(\Omega)} \le G(t) + c'_F \\ &\le A_1 + G(0) + c'_F \equiv c_1 \end{aligned} \tag{3.11}$$

on each time interval $[kT,(k+1)T]$ with constant c_1 independent of k.

3.2.3. Absorbing set in energy norms

Inequality (3.10) implies that

$$\limsup_{t\to\infty} G(t) < A_1.$$

Thus, for any positive number A_1' satisfying $A_1' > A_1$, there exists a time moment $t_1 = t_1(G(0),A_1')$ given by

$$t_1 = \frac{1}{\beta_1}\log\frac{G(0)}{A_1'-A_1},$$

such that $G(t) < A_1'$ for all $t \geq t_1$, hence

$$\text{(3.12)} \qquad \|\boldsymbol{u}(t)\|_{\boldsymbol{H}_0^1(\Omega)} + \|\boldsymbol{u}_t(t)\|_{\boldsymbol{L}_2(\Omega)} + \|\chi(t)\|_{H^1(\Omega)} < A_1' + c_F' \equiv c_{1a} \quad \text{for all} \quad t \geq t_1.$$

This shows the absorbing set for $(\boldsymbol{u}, \boldsymbol{u}_t, \chi)$ in energy norms $\boldsymbol{H}_0^1(\Omega) \times \boldsymbol{L}_2(\Omega) \times H^1(\Omega)$.

3.3. Regularity estimates of absorbing type

3.3.1. A differential inequality in higher norms

Let $N : [0, \infty) \to [0, \infty)$ be the function constructed on a regular solution, defined as a linear combination with appropriately chosen coefficients (depending on the constant c_1 in (3.11)) of the modified energy $G(t)$ and the norms

$$\|\boldsymbol{Q}\boldsymbol{u}(t)\|^2_{\boldsymbol{L}_2(\Omega)}, \quad \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}, \quad \|\boldsymbol{Q}\boldsymbol{u}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)}, \quad \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{L}_2(\Omega)},$$
$$\|\chi(t)\|^2_{L_2(\Omega)}, \qquad \|\Delta\chi(t)\|^2_{L_2(\Omega)}, \qquad \|\chi_t(t)\|^2_{L_2(\Omega)}.$$

Here $\boldsymbol{Q}^{1/2}$ stansds for the fractional power of the operator $\boldsymbol{Q}$ with the domain $D(\boldsymbol{Q}^{1/2}) = \boldsymbol{H}_0^1(\Omega)$, satisfying

$$\|\boldsymbol{Q}^{1/2}\boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)} = (-\boldsymbol{Q}\boldsymbol{u}, \boldsymbol{u})_{\boldsymbol{L}_2(\Omega)} = \bar{\mu}\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}_2(\Omega)} + (\bar{\lambda} + \bar{\mu})\|\nabla \cdot \boldsymbol{u}\|^2_{L_2(\Omega)} \quad \text{for} \quad \boldsymbol{u} \in D(\boldsymbol{Q}).$$

By the construction, function $N(t)$ satisfies the bound

$$\text{(3.13)} \qquad \begin{aligned} N(t) \geq c_N (&\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}_0^1(\Omega)} \\ &+ \|\chi(t)\|^2_{H^2_{\mathcal{N}}(\Omega)} + \|\chi_t(t)\|^2_{L_2(\Omega)}) - c_N' \end{aligned}$$

with explicitly computed, positive constants c_N and c_N' dependent on c_1. Thus, $N(t)$ provides estimates for $(\boldsymbol{u}(t), \boldsymbol{u}_t(t), \boldsymbol{u}_{tt}(t), \chi(t), \chi_t(t))$ in the norms of $\boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^2(\Omega) \times \boldsymbol{H}_0^1(\Omega) \times H^2_{\mathcal{N}}(\Omega) \times L_2(\Omega)$.

It has been proved in [5], Lemma 4.5, that solutions of (1.1)-(1.3) satisfy the differential inequality

$$\text{(3.14)} \quad \frac{d}{dt}N(t) + \beta_5 N(t) + \tilde{\beta}_5 \tilde{N}(t) \leq \Lambda_3\|\boldsymbol{b}(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \Lambda_4\|\boldsymbol{b}_t(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \Lambda_5 \quad \text{for} \quad t > 0,$$

where

$$\tilde{N}(t) = \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\chi_t(t)\|^2_{H^2(\Omega)},$$

and β_5, $\tilde{\beta}_5$, Λ_3, Λ_4, Λ_5 are explicitly computed positive constants depending on c_1.

The derivation of such inequality is based on differentiating system (1.12)–(1.14) with respect to time. A straightforward but technical procedure consists of the following main steps. In the first step we derive a differential inequality corresponding to the elasticity system (1.12):

$$
\begin{aligned}
&\frac{d}{dt}H(t) + \beta_2 H(t) + \tilde\beta_2 \tilde H(t)\\
&\le c_H(\|\nabla\chi(t)\|^2_{L_2(\Omega)} + \|\chi_t(t)\nabla\chi(t)\|^2_{L_2(\Omega)} + \|\nabla\chi_t(t)\|^2_{L_2(\Omega)} + \|\boldsymbol{b}(t)\|^2_{L_2(\Omega)}\\
&\quad + \|\boldsymbol{b}_t(t)\|^2_{L_2(\Omega)}) \quad \text{for} \quad t > 0,
\end{aligned} \tag{3.15}
$$

where $H : [0,\infty) \to [0,\infty)$ is a linear combination of the norms

$$
\|\boldsymbol{Q}\boldsymbol{u}(t)\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_t(t)\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{Q}\boldsymbol{u}_t(t)\|^2_{L_2(\Omega)}, \quad \|\boldsymbol{Q}^{1/2}\boldsymbol{u}_{tt}(t)\|^2_{L_2(\Omega)},
$$

and

$$
\tilde H(t) = \|\boldsymbol{u}_{tt}(t)\|^2_{L_2(\Omega)} + \|\boldsymbol{Q}\boldsymbol{u}_{tt}(t)\|^2_{L_2(\Omega)},
$$

β_2, $\tilde\beta_2$, c_H are explicitly computed positive constants.

In the second step we derive a differential inequality corresponding to system (1.13), (1.14) which allows to handle the terms on the right-hand side of (3.15). The inequality has the form

$$
\frac{d}{dt}J(t) + \beta_3 J(t) + \tilde\beta_3 \tilde J(t) \le c_J(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t))\|^2_{L_2(\Omega)} + \chi^2_m + 1) \quad \text{for} \quad t > 0, \tag{3.16}
$$

where $J : [0,\infty) \to [0,\infty)$ is a linear combination of the norms

$$
\|\chi(t)\|^2_{L_2(\Omega)}, \quad \|\Delta\chi(t)\|^2_{L_2(\Omega)}, \quad \|\chi_t(t)\|^2_{L_2(\Omega)};
$$

and

$$
\tilde J(t) = \|\chi_t(t)\|^2_{H^2(\Omega)},
$$

$\beta_3, \tilde\beta_3, c_J$ are positive constants.

In the third step we combine (3.15) and (3.16) to conclude the differential inequality

$$
\begin{aligned}
&\frac{d}{dt}K(t) + \beta_4 K(t) + \tilde\beta_4 \tilde K(t)\\
&\le c_K(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t))\|^2_{L_2(\Omega)} + \|\boldsymbol{b}(t)\|^2_{L_2(\Omega)} + \|\boldsymbol{b}_t(t)\|^2_{L_2(\Omega)} + 1) \quad \text{for} \quad t > 0,
\end{aligned} \tag{3.17}
$$

where $K : [0,\infty) \to [0,\infty)$ is a linear combination of the terms of $H(t)$ and $J(t)$;

$$
\tilde K(t) = \tilde H(t) + \tilde J(t),
$$

and β_4, $\tilde\beta_4$, c_K are positive constants.

Finally, combining inequalities (3.17) and (3.9) allows to absorb the term $\|\boldsymbol{\varepsilon}(\boldsymbol{u}_t(t)\|^2_{L_2(\Omega)}$ on the right-hand side of (3.17) and thereby conclude (3.14).

3.3.2. Absorbing estimate for $N(t)$

On account of (3.14) we have

$$N(t) \le A_2(1 - e^{-\beta_5 t}) + N(0)e^{-\beta_5 t} \tag{3.18}$$

where

$$A_2 = \frac{1}{\beta_5}\big(\Lambda_3 \sup_{t\in(0,\infty)} \|b(t)\|^2_{L_2(\Omega)} + \Lambda_4 \sup_{t\in(0,\infty)} \|b_t(t)\|^2_{L_2(\Omega)} + \Lambda_5\big).$$

Estimate (3.18) allows to prolong a regular solution step by step on the intervals $[kT,(k+1)T]$, $k \in \mathbb{N}$. It provides the following uniform in k bound

$$\sup_{k\in\mathbb{N}\cup\{0\}} \max_{t\in[kT,(k+1)T]} N(t) \le A_2 + N(0). \tag{3.19}$$

Moreover, by integrating (3.14) with respect to time, it follows that

$$\sup_{k\in\mathbb{N}\cup\{0\}} \int\limits_{kT}^{(k+1)T} \tilde{\beta}_5 \tilde{N}(t)dt \le TA_2\beta_5 + A_2 + N(0). \tag{3.20}$$

In view of definitions of $N(t)$ and $\tilde{N}(t)$, estimates (3.19) and (3.20) imply the corresponding bounds on $\boldsymbol{u}$ and χ in (2.7)–(2.9).
Furthermore, testing $(1.2)_1$ by $\Delta\mu$ and using estimate (2.7) on χ_t gives

$$\|\Delta\mu\|_{C([0,\infty);L_2(\Omega))} \le c.$$

Hence, recalling (3.6), the elliptic property of the Laplace operator with homogeneous boundary condition implies that

$$\|\mu\|_{C([0,\infty);H^2_N(\Omega))} \le c\bigg((\|\Delta\mu\|_{C([0,\infty);L_2(\Omega))} + \sup_{t\in[0,\infty)} \bigg|\int\limits_{\Omega} \mu dx\bigg|\bigg) \le c$$

which shows estimate on μ in (2.7).

3.3.3. Additional estimates

On the basis of (2.6)–(2.9) we can deduce additional estimates (2.10) which are used in the long-time analysis. The estimate on μ_t in (2.10) follows from the identity (resulting by differentiating $(1.14)_1$ with respect to time)

$$\mu_t = -\gamma\Delta\chi_t + \varphi''(\chi)\chi_t + z''(\chi)\chi_t(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + Dz(\chi) + E) + z'(\chi)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_t) + Dz'(\chi)\chi_t)$$

by testing it by μ_t and applying the Cauchy-Schwarz inequality. Then

$$
\begin{aligned}
\|\mu_t\|_{L_2(t,t+T;L_2(\Omega))} &\leq c(\|\Delta\chi_t\|_{L_2(t,t+T;L_2(\Omega))} \\
&\quad + \|\chi^2\chi_t\|_{L_2(t,t+T;L_2(\Omega)} + \|\chi_t\|_{L_2(t,t+T;L_2(\Omega))} \\
&\quad + \|\chi_t\varepsilon(\boldsymbol{u})\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))} + \|\varepsilon(\boldsymbol{u}_t)\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))}) \\
&\leq c(T^{1/2}+1),
\end{aligned}
\tag{3.21}
$$

where we used (2.7) and (2.9), in particular the bounds

$$
\|\chi\|_{L_2(t,t+T;L_\infty(\Omega))} \leq cT^{1/2}, \quad \|\chi_t\|_{L_2(t,t+T;L_\infty(\Omega))} \leq c(T^{1/2}+1).
$$

Estimate on $\boldsymbol{u}_{ttt}$ in (2.10) follows from equation $(1.12)_1$ differentiated with respect to t. Then for any test function $\boldsymbol{\eta} \in L_2(t,t+T;\boldsymbol{H}^1(\Omega))$,

$$
\begin{aligned}
&\left|\int_t^{t+T} (\boldsymbol{u}_{ttt},\boldsymbol{\eta})_{L_2(\Omega)}dt'\right| = \left|\int_t^{t+T} [-(\boldsymbol{A}\varepsilon(\boldsymbol{u}_t) + \nu\boldsymbol{A}\varepsilon(\boldsymbol{u}_{tt}),\varepsilon(\boldsymbol{\eta}))_{\boldsymbol{L}_2(\Omega)}\right. \\
&\qquad \left. + (z''(\chi)\chi_t\boldsymbol{B}\nabla\chi + z'(\chi)\boldsymbol{B}\nabla\chi_t,\boldsymbol{\eta})_{L_2(\Omega)} + (\boldsymbol{b}_t,\boldsymbol{\eta})_{L_2(\Omega)}]dt'\right| \\
&\leq c(\|\varepsilon(\boldsymbol{u}_t)\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))} + \|\varepsilon(\boldsymbol{u}_{tt})\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))})\|\nabla\boldsymbol{\eta}\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))} \\
&\quad + c(\|\chi_t\nabla\chi\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))} + \|\nabla\chi_t\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega)} \\
&\quad + \|\boldsymbol{b}_t\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))})\|\boldsymbol{\eta}\|_{L_2(t,t+T;\boldsymbol{L}_2(\Omega))} \\
&\leq c(T^{1/2}+1)\|\boldsymbol{\eta}\|_{L_2(t,t+T;\boldsymbol{H}^1(\Omega))},
\end{aligned}
$$

where we used (2.7) and (2.9). This shows $(2.10)_1$.

Similarly, by testing equation $(1.13)_1$ differentiated with respect to time by a function $\xi \in L_2(0,T;H^2_N(\Omega))$, and using (3.21) we get

$$
\begin{aligned}
\left|\int_t^{t+T} (\chi_{tt},\xi)_{L_2(\Omega)}dt'\right| &= \left|\int_t^{t+T} (\mu_t,\Delta\xi)_{L_2(\Omega)}dt'\right| \\
&\leq \|\mu_t\|_{L_2(t,t+T;L_2(\Omega))}\|\xi\|_{L_2(t,t+T;H^2_N(\Omega))} \\
&\leq c(T^{1/2}+1)\|\xi\|_{L_2(t,t+T;H^2_N(\Omega))}.
\end{aligned}
$$

This shows $(2.10)_2$.

3.3.4. Absorbing set in stronger norms

For completness we recall also (see [5], Thm 2.2) the absorbing set in the norms induced by the function $N(t)$.
On account of absorbing estimate (3.12) in energy norms we can infer from inequality (3.18) that for all $t \geq t_1$,

$$N(t) \leq A_{2a}(1 - e^{-\beta_{5a}t}) + N(0)e^{-\beta_{5a}t} \tag{3.22}$$

where β_{5a} and A_{2a} are positive constants independent of the initial condition $N(0)$, obtained by replacing in corresponding expressions constant c_1 from (3.11) by c_{1a} from (3.12). From (3.22) it follows that

$$\limsup_{t\to\infty} N(t) < A_{2a}.$$

⊥ ≤

Thus, for any positive number A_2' satisfying $A_2' > A_{2a}$, there exists a time moment $t_2 = t_2(N(0), A_2')$,

$$t_2 = \frac{1}{\beta_{5a}} \log \frac{N(0)}{A_2' - A_{2a}},$$

such that $N(t) < A_2'$ for all $t \geq t_* = \max\{t_1, t_2\}$. Hence, on account of (3.13),

$$\begin{aligned} c_{Na}(\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_t(t)\|^2_{\boldsymbol{H}^2(\Omega)} + \|\boldsymbol{u}_{tt}(t)\|^2_{\boldsymbol{H}^1_0(\Omega)} + \|\chi(t)\|^2_{H^2_N(\Omega)} \\ + \|\chi_t(t)\|^2_{L_2(\Omega)}) < A_2' + c'_{Na} \quad \text{for all} \quad t \geq t_* \end{aligned}$$

where c_{Na} and c'_{Na} are positive numbers independent of $N(0)$. This shows the absorbing set for $(\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{u}_{tt}, \chi, \chi_t)$ in $\boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^2(\Omega) \times \boldsymbol{H}^1_0(\Omega) \times H^2_N(\Omega) \times L_2(\Omega)$.

4. Proof of Theorem 2.2

(i) Due to estimate (2.7) the orbit $\bigcup_{t\geq 0} S(t)\zeta_0$ starting at $\zeta_0 = (\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \chi_0, \chi_1)$ is bounded in the space $\mathcal{W}$, thus is relatively compact in $\mathcal{Z}$. Hence, the ω-limit set $\omega(\zeta_0)$ is a nonempty and compact subset of $\mathcal{Z}$. Moreover, since by (2.3),

$$(\boldsymbol{u}, \boldsymbol{u}_t, \boldsymbol{u}_{tt}, \chi, \chi_t) \in C([0,\infty); \mathcal{W}) \subset C([0,\infty); \mathcal{Z}),$$

the known results of the theory of dynamical systems (see e.g. [3], Prop. 2,1) show that this set is connected in $\mathcal{Z}$, and positive invariant under $S(t)$. Indeed, if $\zeta \in \omega(\zeta_0)$, say $\zeta = \lim_{n\to\infty} S(t_n)\zeta_0$, then

$$S(t)\zeta = \lim_{n\to\infty} S(t)S(t_n)\zeta_0 = \lim_{n\to\infty} S(t+t_n)\zeta_0 \in \omega(\zeta_0).$$

(ii) The map F_Ω given by (2.14) coincides with the function $F(t)$ defined in energy identity (3.2). Thus the Lyapunov property of F_Ω results immediately from (3.3). The claim that F_Ω is constant on $\omega(\zeta_0)$ follows from a general result in [3], Proposition 2.2, due to the continuity of F_Ω. Indeed, let $F_{\Omega\infty} := \lim_{t\to\infty} F_\Omega(S(t)\zeta_0)$. Choosing any $\zeta \in \omega(\zeta_0)$, say $\zeta = \lim_{n\to\infty} S(t_n)\zeta_0$, we deduce by the continuity of F_Ω that

$$F_\Omega(\zeta) = F_\Omega(\lim_{n\to\infty} S(t_n)\zeta_0) = \lim_{n\to\infty} F_\Omega(S(t_n)\zeta_0) = F_{\Omega\infty}$$

which shows that F_Ω is constant on $\omega(\zeta_0)$.

(iii) We shall characterize the elements of the ω-limit set.

Let $(\boldsymbol{u}_\infty, \boldsymbol{u}_{\infty,t}, \boldsymbol{u}_{\infty,tt}, \chi_\infty, \chi_{\infty,t}) \in \omega(\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \chi_0, \chi_1)$, and t_n be a sequence of positive numbers such that $t_n \to \infty$ and

$$(4.1) \qquad \begin{aligned} &(\boldsymbol{u}(t_n), \boldsymbol{u}_t(t_n), \boldsymbol{u}_{tt}(t_n), \chi(t_n), \chi_t(t_n)) \to \\ &\quad (\boldsymbol{u}_\infty, \boldsymbol{u}_{\infty,t}, \boldsymbol{u}_{\infty,tt}, \chi_\infty, \chi_{\infty,t}) \quad \text{strongly in } \mathcal{Z}. \end{aligned}$$

For a fixed number $T > 0$ and $t \in [0,T]$ we define functions

$$(4.2) \qquad \begin{aligned} &\boldsymbol{u}_n(t) := \boldsymbol{u}(t_n + t), \quad \boldsymbol{u}_{n,t}(t) := \boldsymbol{u}_t(t_n + t), \quad \boldsymbol{u}_{n,tt}(t) := \boldsymbol{u}_{tt}(t_n + t), \\ &\chi_n(t) := \chi(t_n + t), \quad \chi_{n,t}(t) := \chi_t(t_n + t), \quad \mu_n(t) := \mu(t_n + t), \\ &\boldsymbol{b}_n(t) := \boldsymbol{b}(t_n + t), \end{aligned}$$

where $(\boldsymbol{u}, \chi, \mu)$ is the solution of (1.1)–(1.3) on $[0,\infty)$. Thus, $(\boldsymbol{u}_n, \chi_n, \mu_n)$ solve the system

$$(4.3) \qquad \begin{aligned} &\boldsymbol{u}_{n,tt} - \boldsymbol{Q}\boldsymbol{u}_n - \nu \boldsymbol{Q}\boldsymbol{u}_{n,t} = z'(\chi_n)\boldsymbol{B}\nabla\chi_n + \boldsymbol{b}_n && \text{in } \Omega^T = \Omega \times (0,T), \\ &\boldsymbol{u}_n(0) = \boldsymbol{u}(t_n), \quad \boldsymbol{u}_{n,t}(0) = \boldsymbol{u}_t(t_n) && \text{in } \Omega, \\ &\boldsymbol{u}_n = 0 && \text{on } S^T = S \times (0,T), \end{aligned}$$

$$(4.4) \qquad \begin{aligned} &\chi_{n,t} - \Delta\mu_n = 0 && \text{in } \Omega^T, \\ &\chi_n(0) = \chi(t_n) && \text{in } \Omega, \\ &\boldsymbol{n}\cdot\nabla\mu_n = 0 && \text{on } S^T. \end{aligned}$$

$$(4.5) \qquad \begin{aligned} &\mu_n = -\gamma\Delta\chi_n + \psi'(\chi_n) + z'(\chi_n)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + Dz(\chi_n) + E) && \text{in } \Omega^T, \\ &\boldsymbol{n}\cdot\nabla\chi_n = 0 && \text{on } S^T. \end{aligned}$$

By virtue of (2.6) and (2.7) the following estimates hold true independently of T and n:

$$(4.6) \qquad \begin{aligned} &\|\boldsymbol{u}_n\|_{L_\infty(0,T;\boldsymbol{H}_0^1(\Omega))} + \|\boldsymbol{u}_{n,t}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\chi_n\|_{L_\infty(0,T;H^1(\Omega))} \\ &\quad + \|\boldsymbol{u}_{n,t}\|_{L_2(0,T;\boldsymbol{H}_0^1(\Omega))} + \|\nabla\mu\|_{L_2(0,T;L_2(\Omega))} \le c_0. \end{aligned}$$

$$
(4.7)\qquad \begin{aligned}&\|\boldsymbol{u}_n\|_{W^1_\infty(0,T;\boldsymbol{H}^2(\Omega))} + \|\boldsymbol{u}_{n,tt}\|_{L_\infty(0,T;\boldsymbol{H}^1_0(\Omega))} + \|\chi_n\|_{L_\infty(0,T;H^2_N(\Omega))}\\ &\quad + \|\chi_{n,t}\|_{L_\infty(0,T;L_2(\Omega))} + \|\mu_n\|_{L_\infty(0,T;H^2_N(\Omega))} \le c.\end{aligned}
$$

Moreover, by (2.8)–(2.10),

$$
(4.8)\qquad \|\chi_n\|_{L_2(0,T;H^3_N(\Omega))} + \|\mu_n\|_{L_2(0,T;H^4(\Omega))} \le c(T),
$$

$$
(4.9)\qquad \|\boldsymbol{u}_{n,tt}\|_{L_2(0,T;\boldsymbol{H}^2(\Omega))} + \|\chi_{n,t}\|_{L_2(0,T;H^2_N(\Omega))} \le c(T),
$$

$$
(4.10)\qquad \begin{aligned}&\|\boldsymbol{u}_{n,ttt}\|_{L_2(0,T;(\boldsymbol{H}^1_0(\Omega))')} + \|\chi_{n,tt}\|_{L_2(0,T;(H^2_N(\Omega))')}\\ &\quad + \|\mu_{n,t}\|_{L_2(0,T;L_2(\Omega))} \le c(T)\end{aligned}
$$

with constant $c(T)$ depending on T but not on n.

The above estimates allow to pass to the weak limit $n \to \infty$ in (4.3)–(4.5). In fact, it follows from (4.6)–(4.10) that there exist functions $(\bar{\boldsymbol{u}}, \bar\chi, \bar\mu)$ with

(4.11)

$$
\begin{aligned}
&\bar{\boldsymbol{u}} \in W^1_\infty(0,T;\boldsymbol{H}^2(\Omega)\cap \boldsymbol{H}^1_0(\Omega)), && \bar{\boldsymbol{u}}_{tt} \in L_\infty(0,T;\boldsymbol{H}^1_0(\Omega)) \cap L_2(0,T;\boldsymbol{H}^2(\Omega)),\\
&\bar{\boldsymbol{u}}_{ttt} \in L_2(0,T;(\boldsymbol{H}^1_0(\Omega))'), && \bar\chi \in L_\infty(0,T;H^2_N(\Omega)),\\
&\bar\chi_t \in L_\infty(0,T;L_2(\Omega)) \cap L_2(0,T;H^2_N(\Omega)), && \bar\chi_{tt} \in L_2(0,T;(H^2_N(\Omega))'),\\
&\bar\mu \in L_\infty(0,T;H^2_N(\Omega)), && \bar\mu_t \in L_2(0,T;L_2(\Omega)),
\end{aligned}
$$

and subsequences of $(\boldsymbol{u}_n, \chi_n, \mu_n)$ (which we still denote by the same indices) such that as $n \to \infty$,

$$
(4.12)\qquad \begin{aligned}
&\boldsymbol{u}_n \to \bar{\boldsymbol{u}} && \text{weakly} - * \text{ in } W^1_\infty(0,T;\boldsymbol{H}^2(\Omega)),\\
&\boldsymbol{u}_{n,tt} \to \bar{\boldsymbol{u}}_{tt} && \text{weakly} - * \text{ in } L_\infty(0,T;\boldsymbol{H}^1_0(\Omega)) \quad \text{and}\\
& && \text{weakly in } L_2(0,T;\boldsymbol{H}^2(\Omega)),\\
&\boldsymbol{u}_{n,ttt} \to \bar{\boldsymbol{u}}_{ttt} && \text{weakly in } L_2(0,T;(\boldsymbol{H}^1_0(\Omega))'),
\end{aligned}
$$

$$
(4.13)\qquad \begin{aligned}
&\chi_n \to \bar\chi && \text{weakly} - * \text{ in } L_\infty(0,T;H^2_N(\Omega)),\\
&\chi_{n,t} \to \bar\chi_t && \text{weakly} - * \text{ in } L_\infty(0,T;L_2(\Omega)) \quad \text{and}\\
& && \text{weakly in } L_2(0,T;H^2_N(\Omega)),\\
&\chi_{n,tt} \to \chi_{tt} && \text{weakly in } L_2(0,T;(H^2_N(\Omega))'),
\end{aligned}
$$

$$
(4.14)\qquad \begin{aligned}
&\mu_n \to \mu && \text{weakly} - * \text{ in } L_\infty(0,T;H^2_N(\Omega)),\\
&\mu_{n,t} \to \bar\mu_t && \text{weakly in } L_2(0,T;L_2(\Omega)).
\end{aligned}
$$

Hence, by the standard compactness results (see e.g. [6]) it follows in particular that for $n \to \infty$,

$$
\begin{aligned}
&\boldsymbol{u}_n \to \boldsymbol{u}, \quad \boldsymbol{u}_{n,t} \to \bar{\boldsymbol{u}}_t \qquad \text{strongly in } C([0,T]; \boldsymbol{H}^1_0(\Omega)) \quad \text{and a.e. in } \Omega^T, \\
&\boldsymbol{u}_{n,tt} \to \bar{\boldsymbol{u}}_{tt} \quad \text{strongly in } L_2(0,T; \boldsymbol{H}^1_0(\Omega)) \cap C([0,T]; \boldsymbol{L}_2(\Omega)) \text{ and a.e. in } \Omega^T,
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
&\chi_n \to \chi \qquad \text{strongly in } C([0,T]; H^1(\Omega)) \quad \text{and a.e. in } \Omega^T, \\
&\chi_{n,t} \to \bar{\chi}_t \quad \text{strongly in } L_2(0,T; H^1(\Omega)) \cap C([0,T]; (H^1(\Omega))') \quad \text{and a.e. in } \Omega^T,
\end{aligned} \tag{4.16}
$$

$$
\mu_n \to \mu \quad \text{strongly in } C([0,T]; H^1(\Omega)) \quad \text{and a.e. in } \Omega^T. \tag{4.17}
$$

Moreover, due to the bounds on the dissipative terms (see (2.6))

$$
\|\boldsymbol{u}_t\|_{L_2(0,\infty;\boldsymbol{H}^1_0(\Omega))} + \|\nabla\mu\|_{L_2(0,\infty;\boldsymbol{L}_2(\Omega))} \le c_0,
$$

we deduce that as $n \to \infty$ ($t_n \to \infty$)

$$
\boldsymbol{u}_{n,t}(\cdot) = \boldsymbol{u}_{,t}(t_n + \cdot) \to \boldsymbol{0} \quad \text{strongly in } L_2(0,\infty; \boldsymbol{H}^1_0(\Omega)) \tag{4.18}
$$

and

$$
\nabla\mu_n(\cdot) = \nabla\mu(t_n + \cdot) \to \boldsymbol{0} \quad \text{strongly in } L_2(0,\infty; \boldsymbol{L}_2(\Omega)).
$$

Hence, in view of (4.15), (4.17),

$$
\bar{\boldsymbol{u}}_t = \boldsymbol{0} \quad \text{and} \quad \nabla\bar{\mu} = \boldsymbol{0}. \tag{4.19}
$$

This implies that $\bar{\boldsymbol{u}}$ does not depend on time and $\bar{\mu}$ does not depend on space variables. Consequently, by (4.15),

$$
\begin{aligned}
&\boldsymbol{u}_n \to \bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}(0) \quad \text{strongly in } C([0,T]; \boldsymbol{H}^1_0(\Omega)), \\
&\boldsymbol{u}_{n,t} \to \bar{\boldsymbol{u}}_t = \boldsymbol{0} \qquad \text{strongly in } C([0,T]; \boldsymbol{H}^1_0(\Omega)), \\
&\boldsymbol{u}_{n,tt} \to \bar{\boldsymbol{u}}_{tt} = \boldsymbol{0} \quad \text{strongly in } C([0,T]; \boldsymbol{L}_2(\Omega)),
\end{aligned} \tag{4.20}
$$

and by $(4.14)_1$,

$$
\Delta\mu_n \to \Delta\bar{\mu} = 0 \quad \text{weakly} - * \text{ in } L_\infty(0,T; L_2(\Omega)). \tag{4.21}
$$

Since $\chi_{n,t} = \Delta\mu_n$ in Ω^T, $(4.16)_2$ and (4.21) imply that

$$
\chi_{n,t} \to \bar{\chi}_t = \Delta\bar{\mu} = 0 \quad \text{strongly in } C([0,T]; (H^1(\Omega))'). \tag{4.22}
$$

Hence, $\bar{\chi}$ does not depend on time, and in accord with $(4.16)_1$,

$$\chi_n \to \bar{\chi} = \bar{\chi}(0) \quad \text{strongly in} \quad C([0,T]; H^1(\Omega)).$$

Now, owing to (4.20)–(4.23) and recalling assumption (4.1), we deduce that

$$\bar{\boldsymbol{u}}(t) = \bar{\boldsymbol{u}}(0) = \lim_{n\to\infty} \boldsymbol{u}_n(0) = \lim_{n\to\infty} \boldsymbol{u}(t_n) = \boldsymbol{u}_\infty, \tag{4.24}$$

$$0 = \bar{\boldsymbol{u}}_t(t) = \bar{\boldsymbol{u}}_t(0) = \lim_{n\to\infty} \boldsymbol{u}_{n,t}(0) = \lim_{n\to\infty} \boldsymbol{u}_t(t_n) = \boldsymbol{u}_{\infty,t},$$

$$0 = \bar{\boldsymbol{u}}_{tt}(t) = \bar{\boldsymbol{u}}_{tt}(0) = \lim_{n\to\infty} \boldsymbol{u}_{n,tt}(0) = \lim_{n\to\infty} \boldsymbol{u}_{tt}(t_n) = \boldsymbol{u}_{\infty,tt},$$

and

$$\begin{aligned} &\bar{\chi}(t) = \bar{\chi}(0) = \lim_{n\to\infty} \chi_n(0) = \lim_{n\to\infty} \chi(t_n) = \chi_\infty, \\ &0 = \bar{\chi}_t(t) = \bar{\chi}_t(0) = \lim_{n\to\infty} \chi_{n,t}(0) = \lim_{n\to\infty} \chi_t(t_n) = \chi_{\infty,t}. \end{aligned} \tag{4.25}$$

This shows that any element $\zeta_\infty = (\boldsymbol{u}_\infty, \boldsymbol{u}_{\infty,t}, \boldsymbol{u}_{\infty,tt}, \chi_\infty, \chi_{\infty,t}) \in \omega(\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \chi_0, \chi_1)$ satisfies (2.15).

It remains to prove (2.16), (2.17). To this end we pass to the limit $n \to \infty$ in the weak formulation of (4.3)–(4.5):

$$\begin{aligned} &\int_0^T (\boldsymbol{u}_{n,tt} - \boldsymbol{Q}\boldsymbol{u}_n - \nu \boldsymbol{Q}\boldsymbol{u}_{n,t}, \boldsymbol{\eta})dt = \int_0^T (z'(\chi_n)B\nabla\chi_n + \boldsymbol{b}_n, \boldsymbol{\eta})dt \\ &\qquad \forall \boldsymbol{\eta} \in L_2(0,T; \boldsymbol{L}_2(\Omega)), \\ &\int_0^T (\chi_{n,t}, \xi)dt = \int_0^T (\Delta\mu_n, \xi)dt \quad \forall \xi \in L_2(0,T; L_2(\Omega)), \\ &\int_0^T (\mu_n, \varsigma)dt = \int_0^T (-\gamma\Delta\chi_n + \psi'(\chi_n) + z'(\chi_n)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + Dz(\chi_n) + E, \varsigma)dt \\ &\qquad \forall \varsigma \in L_2(0,T; L_2(\Omega)) \end{aligned} \tag{4.26}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L_2(\Omega)$.
Clearly, by virtue of the weak convergences (4.12)–(4.14) the linear terms in (4.26) converge to the corresponding limits. The convergence of the nonlinear terms can be concluded with the help of the standard nonlinear convergence lemma (see [4], Chapter 1, Lemma 1.3).
In fact, recalling assumptions (A3), (A4) on $z(\cdot)$ and $\psi(\cdot)$ and using estimates (4.6) we have

$$\begin{aligned} &\|\psi'(\chi_n)\|_{L_\infty(0,T;L_2(\Omega))} \le c(\|\chi_n\|^3_{L_\infty(0,T;L_6(\Omega))} + 1) \le c(c_0), \\ &\|z'(\chi_n)B\nabla\chi_n\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} \le c\|\nabla\chi_n\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} \le c(c_0), \\ &\|z'(\chi_n)(\boldsymbol{B}\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + Dz(\chi_n) + E)\|_{L_\infty(0,T;L_2(\Omega))} \\ &\le c(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_n)\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + 1) \le c(c_0). \end{aligned} \tag{4.27}$$

Thanks to these uniform estimates and the pointwise convergences (see (4.15), (4.16), (4.24), (4.25))

$$\boldsymbol{u}_n \to \hat{\boldsymbol{u}} = \boldsymbol{u}_\infty, \quad \chi_n \to \hat{\chi} = \chi_\infty \quad \text{a.e. in } \ \Omega^T,$$

the nonlinear convergence lemma implies that

$$(4.28)\quad \begin{aligned} &\psi'(\chi_n) = \chi_n^3 - \chi_n \to \chi_\infty^3 - \chi_\infty = \psi'(\chi_\infty) && \text{weakly} - * \text{ in } \ L_\infty(0,T;L_2(\Omega)),\\ &z'(\chi_n)B\nabla\chi_n \to z'(\chi_\infty)B\nabla\chi_\infty && \text{weakly} - * \text{ in } \ L_\infty(0,T;\boldsymbol{L}_2(\Omega)),\\ &z'(\chi_n)(B\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_n) + Dz(\chi_n) + E) \to \\ &\quad \to z'(\chi_\infty)(B\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty) + Dz(\chi_\infty) + E) && \text{weakly} - * \text{ in } \ L_\infty(0,T;L_2(\Omega)). \end{aligned}$$

Moreover, since by assumption (A7), $\boldsymbol{b} \in L_1(\mathbb{R}_+;\boldsymbol{L}_2(\Omega))$, we have

$$\boldsymbol{b}_n(\cdot) = \boldsymbol{b}(t_n + \cdot) \to \boldsymbol{0} \quad \text{strongly in} \quad L_1(0,\infty;\boldsymbol{L}_2(\Omega)).$$

Consequently, passing to the limit $n \to \infty$ in (4.26) yields

$$(4.29)\quad \begin{aligned} &-\int_0^T (Q\boldsymbol{u}_\infty, \boldsymbol{\eta})dt = \int_0^T (z'(\chi_\infty)B\nabla\chi_\infty, \boldsymbol{\eta})dt \quad \forall \boldsymbol{\eta} \in L_2(0,T;\boldsymbol{L}_2(\Omega)),\\ &\int_0^T \bar{\mu}(1,\varsigma)dt = \int_0^T (-\gamma\Delta\chi_\infty + \psi'(\chi_\infty) + z'(\chi_\infty)(B\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty) + Dz(\chi_\infty + E), \varsigma)dt\\ &\quad \forall \varsigma \in L_2(0,T;L_2(\Omega)). \end{aligned}$$

Since $\boldsymbol{u}_\infty$ and χ_∞ do not depend on time, it follows from $(4.29)_2$ that $\bar{\mu}$ does not depend on time as well, thus $\bar{\mu} = \text{const}$. Moreover, the above identities reduce to

$$(4.30)\quad \begin{aligned} &-(Q\boldsymbol{u}_\infty, \bar{\boldsymbol{\eta}}) = (z'(\chi_\infty)B\nabla\chi_\infty, \bar{\boldsymbol{\eta}}) \quad \forall \bar{\boldsymbol{\eta}} \in \boldsymbol{L}_2(\Omega),\\ &(\bar{\mu}, \bar{\varsigma}) = (-\gamma\Delta\chi_\infty + \psi'(\chi_\infty) + z'(\chi_\infty)(B\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty)\\ &\qquad + Dz(\chi_\infty) + E), \varsigma) \quad \forall \bar{\varsigma} \in L_2(\Omega). \end{aligned}$$

Hence, recalling that (see (4.11), (4.24), (4.25)) $\boldsymbol{u}_\infty \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{H}_0^1(\Omega)$, $\chi_\infty \in H_N^2(\Omega)$, it follows that $\boldsymbol{u}_\infty$, χ_∞ satisfy the following system

$$(4.31)\quad \begin{aligned} -Q\boldsymbol{u}_\infty &= z'(\chi_\infty)B\nabla\chi_\infty && \text{in } \ \Omega,\\ \boldsymbol{u}_\infty &= \boldsymbol{0} && \text{on } \ S, \end{aligned}$$

$$(4.32)\quad \begin{aligned} &-\gamma\Delta\chi_\infty + \psi'(\chi_\infty) + z'(\chi_\infty)(B\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_\infty) + Dz(\chi_\infty) + E) = \bar{\mu} && \text{in } \ \Omega,\\ &\boldsymbol{n}\cdot\nabla\chi_\infty = 0 && \text{on } \ S,\\ &\int_\Omega \chi_\infty dx = \chi_m, \end{aligned}$$

where $\bar{\mu}$ is a constant.

Clearly, in view of (1.11) the above system is equivalent to (2.16), (2.17). Thereby the proof of Theorem 2.2 is completed. □

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