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REPRESENTATION OF TEM FIELDS IN GYROTROPIC MEDIA BY SCALAR HERTZ POTENTIALS

1. INTRODUCTION

In isotropic media every two solutions of the Helmholtz equation are scalar Hertz potentials for an electromagnetic field. The field can be found by applying to the potentials the well known differential formulae. It satisfies the homogeneous Maxwell equations. In short, given two scalar Hertz potentials, the corresponding electromagnetic field is determined.

The inverse problem, which consists in finding the scalar Hertz potentials of a given electromagnetic field, has been first properly posed and solved by K.Bochenek /2/ /for isotropic media with real ε and μ /. The special case of TEM fields has also been treated by K.Bochenek /4/. This case is important, because the theorem which solves the problem is based on a similar theorem for TEM fields.

For gyrotropic media scalar Hertz potentials have been introduced by S.Przeździecki and R.A.Hurd /1/. The problem of representation of a given electromagnetic field by scalar Hertz potentials has been investigated by S.Przeździecki and W.Łaprus /3/. In this paper we shall examine the special case of TEM fields.

2. SCALAR HERTZ POTENTIALS

In anisotropic media the Maxwell equations have the form

/2.1/ $$\nabla \times \underline{H} = -i\omega \underline{\varepsilon}\, \underline{E} ,$$

/2.2/ $$\nabla \times \underline{E} = i\omega \underline{\mu}\, \underline{H} ,$$

where ω is real, $\underline{\varepsilon}$ and $\underline{\mu}$ are the permittivity and the permeability tensor with complex components. We assume that the medium is gyrotropic, and we denote by $\underline{a}$ the unit vector which is parallel to the distinguished axis of the medium. Let the z axis of the coordinate system be directed along the vector $\underline{a}$. Then

/2.3/ $$\underline{\varepsilon} = \begin{bmatrix} \varepsilon & -i\varepsilon_g & 0 \\ i\varepsilon_g & \varepsilon & 0 \\ 0 & 0 & \varepsilon_a \end{bmatrix}, \quad \underline{\mu} = \begin{bmatrix} \mu & -i\mu_g & 0 \\ i\mu_g & \mu & 0 \\ 0 & 0 & \mu_a \end{bmatrix}.$$

The functions u and v satisfying in a domain D the system of equations

/2.4/ $$\left(\nabla_t^2 + \frac{\varepsilon_a}{\varepsilon}\frac{\partial^2}{\partial z^2} + k_e^2\right)u + \frac{\varepsilon_a}{\varepsilon}\omega\mu\tau_g\frac{\partial v}{\partial z} = 0,$$

/2.5/ $$-\frac{\mu_a}{\mu}\omega\varepsilon\tau_g\frac{\partial u}{\partial z} + \left(\nabla_t^2 + \frac{\mu_a}{\mu}\frac{\partial^2}{\partial z^2} + k_m^2\right)v = 0$$

are called the scalar Hertz potentials of the electromagnetic field $\underline{E}$, $\underline{H}$ which is given by the formulae

/2.6/ $$E_z = -\frac{\varepsilon_a}{\varepsilon}\nabla_t^2 u , \qquad H_z = -\frac{\mu_a}{\mu}\nabla_t^2 v ,$$

/2.7/ $$\underline{E}_t = \nabla_t\frac{\partial u}{\partial z} + \omega\mu\frac{\varepsilon_g}{\varepsilon}\nabla_t v + i\omega\mu\nabla_t v \times \underline{a} ,$$

/2.8/ $$\underline{H}_t = -\omega\varepsilon\frac{\mu_g}{\mu}\nabla_t u - i\omega\varepsilon\nabla_t u \times \underline{a} + \nabla_t\frac{\partial v}{\partial z} .$$

The following symbols are used:

/2.9/ $\tau_g = \frac{\varepsilon_g}{\varepsilon} + \frac{\mu_g}{\mu}\,,\quad k_e^2 = \omega^2 \varepsilon_a (\mu^2 - \mu_g^2)\mu^{-1}\,,\quad k_m^2 = \omega^2 \mu_a (\varepsilon^2 - \varepsilon_g^2)\varepsilon^{-1}.$

The transversal /with respect to the z axis/ parts of vectors are labelled with the subscript t, in particular, $\nabla_t = (\partial/\partial x, \partial/\partial y, 0)$. The field $\underline{E}$, $\underline{H}$ satisfies /2.1/ - /2.2/ in the domain D.

In isotropic media the Maxwell equations are of the form

/2.10/ $$\nabla \times \underline{H} = -i\omega\varepsilon \underline{E}\,,$$

/2.11/ $$\nabla \times \underline{E} = i\omega\mu \underline{H}\,,$$

where ε and μ are complex numbers. In this case /2.4/ - /2.5/ reduce to the Helmholtz equations

/2.12/ $$\nabla^2 u + k^2 u = 0\,,$$

/2.13/ $$\nabla^2 v + k^2 v = 0$$

with $k^2 = \omega^2 \varepsilon\mu$, and /2.6/ - /2.8/ reduce to the known formulae

/2.14/ $$E_z = -\nabla_t^2 u\,,\quad H_z = -\nabla_t^2 v\,,$$

/2.15/ $$\underline{E}_t = \nabla_t \frac{\partial u}{\partial z} + i\omega\mu \nabla_t v \times \underline{a}\,,$$

/2.16/ $$\underline{H}_t = -i\omega\varepsilon \nabla_t u \times \underline{a} + \nabla_t \frac{\partial v}{\partial z}\,.$$

The field $\underline{E}$, $\underline{H}$ given by /2.14/ -/2.16/ satisfies the Maxwell equations /2.10/ - /2.11/.

There exists a more compact vectorial form of the formulae /2.6/ -/2.8/ and /2.14/ - /2.16/ in which the z components of the field $\underline{E}$, $\underline{H}$ are written separately /see

S.Przeździecki and R.A.Hurd /1//. This form is independent of the choice of the coordinate system.

It is seen that the formulae /2.14/ - /2.16/ are degenerated with respect to /2.6/ - /2.8/. Namely, the direction of $\underline{a}$ is unimportant.

3. TEM FIELDS IN GYROTROPIC MEDIA

The only possible TEM fields are those for which $\underline{E}\cdot\underline{a}=0$ and $\underline{H}\cdot\underline{a}=0$. Assume the field $\underline{E}$, $\underline{H}$ satisfies the equations /2.1/ - /2.2/ in a domain D of a gyrotropic medium. Putting $E_z=0$ and $H_z=0$ in the equations /2.1/ - /2.2/ and eliminating $\underline{H}_t$ we get

/3.1/ $$\frac{\partial^2 \underline{E}_t}{\partial z^2} + M\underline{E}_t = 0,$$

/3.2/ $$\nabla_t \times \underline{E}_t = 0, \qquad \nabla_t \cdot \underline{E}_t = 0.$$

The matrix $M = \omega^2 \underline{\varepsilon}\,\underline{\mu}$ has the form

/3.3/ $$M = \begin{bmatrix} m & -im_g & 0 \\ im_g & m & 0 \\ 0 & 0 & m_a \end{bmatrix}$$

where

/3.4/ $$m = \omega^2(\varepsilon\mu + \varepsilon_g\mu_g), \quad m_g = \omega^2(\varepsilon\mu_g + \varepsilon_g\mu), \quad m_a = \omega^2\varepsilon_a\mu_a.$$

It is easy to calculate the eigenvalves λ and the eigenvectors $\underline{\gamma}$ of the matrix M:

/3.5/ $$\lambda_1 = m + m_g, \quad \lambda_2 = m - m_g, \quad \lambda_3 = m_a$$

and

/3.6/ $$\underline{r}_1 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}, \qquad \underline{r}_2 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \qquad \underline{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvalue λ_3 and the eigenvector $\underline{r}_3$ are unimportant, since the z component of the vector $\underline{E}_t$ is equal to zero.

If we look for the solution of /3.1/ in the form $\underline{e}(x,y)\exp(ikz)$ then we have

/3.7/ $$(M - k^2 I)\underline{e} = 0,$$

where I is the unit matrix. Hence $k^2 = \lambda$ and $\underline{e} = f(x,y)\underline{r}$ with an arbitrary complex function $f(x,y)$.

Let $V \subset D$ be a circular cylinder whose axis is parallel to the z axis, and which is bounded by two planes z = const. Denote by V_0 the projection of V onto the xy plane. The solution of the equations /3.1/ - /3.2/ in V is the sum

/3.8/ $$\underline{E}_t = \sum_{\varrho} (\underline{e}_{\varrho}^{+} e^{ik_{\varrho}z} + \underline{e}_{\varrho}^{-} e^{-ik_{\varrho}z}),$$

where $k_{\varrho}^2 = \lambda_{\varrho}$ and

/3.9/ $$\underline{e}_{\varrho}^{\pm} = f_{\varrho}^{\pm}(x,y)\underline{r}_{\varrho}$$

with $\varrho = 1,2$. The complex vector functions $\underline{e}_{\varrho}^{\pm}$ are defined in the domain V_0 and satisfy there the equations /3.2/. Because of the particular form of these functions, their real and imaginary parts are related to each other, viz.,

/3.10/ $$\mathrm{Im}\,\underline{e}_1^{\pm} = \underline{a} \times \mathrm{Re}\,\underline{e}_1^{\pm} \quad \text{and} \quad \mathrm{Im}\,\underline{e}_2^{\pm} = -\underline{a} \times \mathrm{Re}\,\underline{e}_2^{\pm}.$$

Therefore, we may confine ourselves to considering the real parts.

The fields $\mathrm{Re}\,\underline{e}_\varrho^{\pm}$ are planar vector fields satisfying the equations /3.2/ in the domain V_o. To such fields we can apply the results of the appendix. Thus, by /A.4/,

$$\text{/3.11/} \qquad \mathrm{Re}\, e^{\pm}_{\varrho x} = \mathcal{I}m\, F^{\pm}_{\varrho} \qquad \text{and} \qquad \mathrm{Re}\, e^{\pm}_{\varrho y} = \mathcal{R}e\, F^{\pm}_{\varrho}$$

where $F_\varrho^{\pm}$ are analityc functions of the complex variable $\zeta = x + jy$, and the symbols $\mathcal{R}e$ and $\mathcal{I}m$ denote the real and the imaginary part with respect to the imaginary unit j /this new unit is introduced to avoid confusion/. The functions $F_\varrho^{\pm}(\zeta)$ are defined in V_o.

Our next step is to find the solution of the equations /3.1/ - /3.2/ in the whole domain D. We assume that D has the following property. For every two points $P, Q \in D$ there exists a finite sequence ofcylinders $V^1, \ldots, V^n \subset D$ such that $P \in V^1$ and $Q \in V^n$, and that every two subsequent cylinders have in common a domain. The domain is assumed to be three-dimensional.

Now, the solutions of /3.1/ - /3.2/ in two subsequent cylinders coincide in the common domain of the two cylinders. Let $V_o^1, \ldots, V_o^n$ be the projections of $V^1, \ldots, V^n$ onto the xy plane. Then the functions $F_\varrho^{\pm}$ defined in V_o^n are the analytic continuations of the functions $F_\varrho^{\pm}$ defined in V_o^1. By using such analytic continuations we can define the functions $F_\varrho^{\pm}$ analytic in the domain D_o which is the projection of D onto the xy plane.

It follows from the above consideration that there exists a one-to-one correspondence between the field $\underline{E}_t$ defined in D and the four analytic functions $F_\zeta^\pm$ defined in D

It may happen that there exists a subdomain B_o of D_o such that every straight line parallel to the z axis and crossing the domain B_o has more than one section in common with D. In other words, there exists more than one domain $B \subset D$ whose projection onto the xy plane is B_o. Hence, it is possible for $F_\zeta^\pm$ to have different values in different domains B with the same projection B_o. In that case the Riemann surface of $F_\zeta^\pm$ may have more than one sheet in B_o.

In general, the properties of the Riemann surface of $F_\zeta^\pm$ in D_o are related to the form of the domain D. This is explained by the following self-evident lemmas.

LEMMA1. If D_o is simply connected, then the functions $F_\zeta^\pm$ are single valued in D_o.

LEMMA 2. If B_o does not exist, then the functions $F_\zeta^\pm$ are single valued in D_o /even for multiply connected D_o/.

The first lemma follows from properties of analytic functions, the second follows from the uniqueness of solutions of the Maxwell equations.

4. SCALAR HERTZ POTENTIALS

First, we shall find the general form of the functions u and v. We put $E_z = 0$ and $H_z = 0$ in the formulae /2.6/ and obtain

/4.1/ $$\nabla_t^2 u = 0 , \qquad \nabla_t^2 v = 0 .$$

Taking this into account we write /2.4/ - /2.5/ in the form

/4.2/ $$\left(\frac{\partial^2}{\partial z^2} + \frac{\varepsilon}{\varepsilon_a} k_e^2\right) u + \omega\mu\tau_g \frac{\partial v}{\partial z} = 0 ,$$

/4.3/
$$-\omega\varepsilon\tau_g \frac{\partial u}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + \frac{\mu}{\mu_a} k_m^2\right) u = 0.$$

The solution of the equations /4.1/ - /4.3/ in a cylinder $V \subset D$ is equal to

/4.4/
$$u = \sum_{\varsigma} \left(U_{\varsigma}^{+} e^{ik_{\varsigma}z} + U_{\varsigma}^{-} e^{-ik_{\varsigma}z}\right),$$

and

/4.5/
$$v = \sum_{\varsigma} \left(V_{\varsigma}^{+} e^{ik_{\varsigma}z} + V_{\varsigma}^{-} e^{-ik_{\varsigma}z}\right),$$

where the functions $U_{\varsigma}^{\pm}(x,y)$ and $V_{\varsigma}^{\pm}(x,y)$ are solutions of the Laplace equations /4.1/ in the domain V_o. In addition,

/4.6/
$$V_{\varsigma}^{\pm} = \pm i Y_{\varsigma} U_{\varsigma}^{\pm}$$

with

/4.7/
$$Y_1 = -\frac{\varepsilon}{\mu}\sqrt{\frac{\mu+\mu_g}{\varepsilon+\varepsilon_g}} \quad \text{and} \quad Y_2 = \frac{\varepsilon}{\mu}\sqrt{\frac{\mu-\mu_g}{\varepsilon-\varepsilon_g}},$$

so that it suffices to determine $U_{\varsigma}^{\pm}$

The real and the imaginary parts of $U_{\varsigma}^{\pm}$ are harmonic functions in V_o. By applying arguments similar to those of the section 3 we can define the four harmonic functions Re $U_{\varsigma}^{\pm}$ and the four harmonic functions Im $U_{\varsigma}^{\pm}$ in the whole domain D_o. It is worth emphasizing that, in general, the functions $U_{\varsigma}^{\pm}$ are not analytic functions of the variable $\zeta = x + iy$.

Now we proceed to determining the scalar Hertz potentials u and v for a given TEM field.

THEOREM. Every TEM field satisfying the Maxwell equations /2.1/ - /2.2/ in a domain D has scalar Hertz potentials in D.

<u>The proof</u>. We insert /3.8/ and /4.4/ - /4.5/ into /2.7/. Equating the coefficients of the functions $\exp(\pm ik_{\varsigma}z)$ we obtain

/4.8/
$$\underline{e}_{\varsigma}^{\pm} = i\alpha_{\varsigma}^{\pm} \nabla_t U_{\varsigma}^{\pm} + \beta_{\varsigma}^{\pm} \nabla_t \times \underline{a}\, U_{\varsigma}^{\pm},$$

where

/4.9/ $\alpha_\varsigma^{\pm} = \pm(k_\varsigma + Y_\varsigma \omega\mu \frac{\varepsilon_z}{\varepsilon})$ and $\beta_\varsigma^{\pm} = \mp Y_\varsigma \omega\mu$.

The real part of /4.8/ has the form

/4.10/ $$\mathrm{Re}\, \underline{e}_\varsigma^{\pm} = -\nabla_t \mathrm{Im}\, \alpha_\varsigma^{\pm} U_\varsigma^{\pm} + \nabla_t \times \underline{a}\, \mathrm{Re}\, \beta_\varsigma^{\pm} U_\varsigma^{\pm}.$$

By the lemma of the appendix /cf. the formula /A.9//, for the planar vector field $\mathcal{F} = \mathrm{Re}\, \underline{e}_\varsigma^{\pm}$ the potentials

/4.11/ $\varphi = \mathrm{Im}\, \alpha_\varsigma^{\pm} U_\varsigma^{\pm}$ and $A_z = \mathrm{Re}\, \beta_\varsigma^{\pm} U_\varsigma^{\pm}$

do exist and are given by the integrals /A.14/ and /A.15/ in the domain D_0. That is

/4.12/ $$\mathrm{Im}\, \alpha_\varsigma^{\pm} U_\varsigma^{\pm} = -\mathrm{Im} \int_{C_0} (F_\varsigma^{\pm} - \tilde{F}_\varsigma^{\pm})\, d\zeta,$$

and

/4.13/ $$\mathrm{Re}\, \beta_\varsigma^{\pm} U_\varsigma^{\pm} = -\mathrm{Re} \int_{C_0} \tilde{F}_\varsigma^{\pm}\, d\zeta,$$

where $F_\varsigma^{\pm}$ are the analytic functions corresponding to the fields $\mathrm{Re}\, \underline{e}_\varsigma^{\pm}$ in D_0, and $\tilde{F}_\varsigma^{\pm}$ are the analytic functions in D_0 which are chosen according to the considerations of the appendix.

After simple calculations we obtain from /4.12/ and /4.13/ the functions $\mathrm{Re}\, U_\varsigma^{\pm}$ and $\mathrm{Im}\, U_\varsigma^{\pm}$ in the form of the linear combinations of the integrals /4.12/ and /4.13/. Obviously, these functions are harmonic in D_0, since the integrands are analytic in D_0. This remark concludes the proof of the theorem.

5. TEM FIELDS IN ISOTROPIC MEDIA

We consider the field $\underline{E}$, $\underline{H}$ satisfying the Maxwell equations /2.10/ - /2.11/ in a domain D of an isotropic medium such that $\underline{E}\cdot\underline{a} = 0$ and $\underline{H}\cdot\underline{a} = 0$. Putting $E_z = 0$ and $H_z = 0$ in /2.10/ - /2.11/ and eliminating $\underline{H}_t$ we get

/5.1/ $$\frac{\partial^2 \underline{E}_t}{\partial z^2} + k^2 \underline{E}_t = 0,$$

/5.2/ $$\nabla_t \times \underline{E}_t = 0, \qquad \nabla_t \cdot \underline{E}_t = 0.$$

In a cylinder $V \subset D$ the solution of /5.1/ - /5.2/ is the sum

/5.3/ $$\underline{E}_t = \underline{e}^+ e^{ikz} + \underline{e}^- e^{-ikz}$$

where $k^2 = \omega^2 \varepsilon \mu$ and $\underline{e}^{\pm}(x,y)$ are complex vector functions satisfying the equations /3.2/ in the domain V_o.

The fields $Re\, \underline{e}^{\pm}$ and $Im\, \underline{e}^{\pm}$ are planar vector fields in V_o satisfying /3.2/. Hence

/5.4/ $$Re\, e_x^{\pm} = Im\, F_r^{\pm}, \qquad Re\, e_y^{\pm} = Re\, F_r^{\pm},$$

and

/5.5/ $$Im\, e_x^{\pm} = Im\, F_i^{\pm}, \qquad Im\, e_y^{\pm} = Re\, F_i^{\pm},$$

where $F_r^{\pm}$ and $F_i^{\pm}$ are analytic function of the complex variable $\zeta = x + jy$ defined in V_o. By means of analytic continuation we can define the four functions $F_r^{\pm}$ and $F_i^{\pm}$ in the whole domain D_o.

We put $E_z = 0$ and $H_z = 0$ in the formulae /2.6/ and get

/5.6/ $$\nabla_t^2 u = 0, \qquad \nabla_t^2 v = 0.$$

Therefore, we can write /2.12/ - /2.13/ in the form

/5.7/ $$\frac{\partial^2 u}{\partial z^2} + k^2 u = 0,$$

/5.8/ $$\frac{\partial^2 v}{\partial z^2} + k^2 v = 0.$$

The solution of the equations /5.6/ - /5.8/ in a cylinder $V \subset D$ is given by

/5.9/ $$u = U^+ e^{ikz} + U^- e^{-ikz},$$

and

/5.10/ $$v = V^+ e^{ikz} + V^- e^{-ikz},$$

where the functions $U^{\pm}(x,y)$ and $V^{\pm}(x,y)$ are solutions of the

Laplace equations /5.6/ in the domain V_o.

The real and the imaginary parts of $U^{\pm}$ and $V^{\pm}$ are harmonic functions in V_o. As in the section 4, we can easily define four harmonic functions $\mathrm{Re}\, U^{\pm}$, $\mathrm{Im}\, U^{\pm}$ and four harmonic functions $\mathrm{Re}\, V^{\pm}$, $\mathrm{Im}\, V^{\pm}$ in the whole domain D_o.

By inserting /5.3/ and /5.9/ - /5.10/ into the formula /2.15/ we obtain

/5.11/ $$\underline{e}^{\pm} = \pm ik \nabla_t U^{\pm} + i\omega\mu \nabla_t \times \underline{a} V^{\pm},$$

and then

/5.12/ $$\mathrm{Re}\, \underline{e}^{\pm} = \mp \nabla_t \mathrm{Im}\, kU^{\pm} - \nabla_t \times \underline{a}\, \mathrm{Im}\, \omega\mu V^{\pm},$$

/5.13/ $$\mathrm{Im}\, \underline{e}^{\pm} = \pm \nabla_t \mathrm{Re}\, kU^{\pm} + \nabla_t \times \underline{a}\, \mathrm{Re}\, \omega\mu V^{\pm}.$$

Applying the lemma of the appendix we find

/5.14/ $$\pm \mathrm{Im}\, kU^{\pm} = -\mathrm{Im} \int_{C_o} (F_r^{\pm} - \tilde{F}_r^{\pm})\, d\zeta,$$

/5.15/ $$-\mathrm{Im}\, \omega\mu V^{\pm} = -\mathrm{Re} \int_{C_o} \tilde{F}_r^{\pm}\, d\zeta,$$

/5.16/ $$\pm \mathrm{Re}\, kU^{\pm} = -\mathrm{Im} \int_{C_o} (F_i^{\pm} - \tilde{F}_i^{\pm})\, d\zeta,$$

/5.17/ $$\mathrm{Re}\, \omega\mu V^{\pm} = -\mathrm{Re} \int_{C_o} \tilde{F}_i^{\pm}\, d\zeta,$$

where $\tilde{F}_r^{\pm}$ and $\tilde{F}_i^{\pm}$ are the auxiliary functions correspodning to $F_r^{\pm}$ and $F_i^{\pm}$. From /5.14/ - /5.17/ we can obtain the harmonic functions $\mathrm{Re}\, U^{\pm}$, $\mathrm{Im}\, U^{\pm}$ and $\mathrm{Re}\, V^{\pm}$, $\mathrm{Im}\, V^{\pm}$ in D_o.

It is interesting to note that the real and the imaginary part of the function $\omega\mu V^{\pm}$, given by /5.17/ and /5.15/, are determined by the real parts of two different analytic functions. Hence, in general, the function $\omega\mu V^{\pm}$ cannot be an analytic function of the complex variable $\zeta = x + iy$ /or of the variable $\zeta^* = x - iy$/. A similar remark applies to the function $kU^{\pm}$.

6. CONCLUSIONS

Every TEM field in a domain D of a gyrotropic medium can be expressed in terms of four analytic functions defined in D_0, which is the projection of D onto the xy plane /the distinguished axis of the medium coincides with the z axis/. D_0 may be multiply connected. Moreover, every TEM field in D has two scalar Hertz potentials u and v in D. The functions u and v are expressed in terms of the four analytic functions mentioned above.

The correspondence between a TEM field and the appropriate four analytic functions is unique. However, the scalar Hertz potentials are defined nonuniquely: there exist more than one pair of Hertz potentials corresponding to a given TEM field.

APPENDIX

We shall consider a vector field $\underline{\mathcal{F}}$ which is transversal with respect to the z axis and independent of z. Such a field will be called planar vector field. We assume that $\underline{\mathcal{F}}$ is a real vector function defined in a domain D_o of the xy plane, and that the first derivatives of $\underline{\mathcal{F}}$ are continuous. The domain D_o is assumed to be multiply connected.

Let $\underline{\mathcal{F}}$ be a solution of the equations

/A.1/ $$\nabla \times \underline{\mathcal{F}} = 0, \qquad \nabla \cdot \underline{\mathcal{F}} = 0$$

in the domain D_o. Rewriting /A.1/ we get the Cauchy-Riemann equations for $\mathcal{F}_x$ and $\mathcal{F}_y$, viz.,

/A.2/ $$\frac{\partial \mathcal{F}_y}{\partial x} - \frac{\partial \mathcal{F}_x}{\partial y} = 0, \quad \frac{\partial \mathcal{F}_x}{\partial x} + \frac{\partial \mathcal{F}_y}{\partial y} = 0.$$

Hence, the function

/A.3/ $$F(\zeta) = \mathcal{F}_y(x,y) + i\,\mathcal{F}_x(x,y)$$

of the complex variable ζ = x+iy is analytic in D_o. In other words, to every planar field $\underline{\mathcal{F}}$ satisfying /A.1/ corresponds an analytic function F given by /A.3/. And conversely, to every, analytic function F in D_o corresponds a planar field $\underline{\mathcal{F}}$ in D_o, given by the formulae

/A.4/ $$\mathcal{F}_x = Im\,F \qquad \text{and} \qquad \mathcal{F}_y = Re\,F\,,$$

which satisfies /A.1/.

The equations /A.1/ imply existence of a scalar potential φ and of a vector potential $\underline{A}$ /with $A_x=0$ and $A_y=0$/ in the following sense. If a function $\varphi(x,y)$ defined in D_o satisfies there the Laplace equation $\Delta\varphi = 0$, then the field

/A.5/ $$\underline{\mathcal{F}} = -\nabla\varphi$$

satisfies /A.1/ in D_o. Similarly, if a function $A_z(x,y)$ defined in D_o satisfies there the Laplace equation $\Delta A_z = 0$, then the field

/A.6/ $$\underline{\mathcal{F}} = \nabla \times \underline{A}$$

satisfies the equations /A.1/ in D_o. Thus, given a scalar potential or a vector potential in D_o, we easily find the corresponding planar field. The inverse problem of finding the scalar potential and/or the vector potential for a given field in D_o is more subtle.

Consider the integral

/A.7/ $$\varphi = -\int_{C_o} (\mathcal{F}_x, \mathcal{F}_y)\cdot \underline{dl}$$

which is obtained by integrating /A.5/ along a curve $C_o \subset D_o$ from a point P_o to a point Q_o. $\underline{dl}$ denotes the vector (dx,dy). The integration constant has been deleted as unimportant for the further consideration. Suppose the point P_o is fixed and the point Q_o is variable. In order to define properly a function $\varphi(Q_o)$ the integral /A.7/ should be independent of the path C_o between P_o and Q_o, or, in other words, it should vanish for every closed curve $C_o \subset D_o$. For a simply connected domain D_o this condition implies vanishing of the curl of $\underline{\mathcal{F}}$ in D_o, and hence, it is obviously fulfilled. However, in the case of a multiply connected domain D_o some of the closed curves $C_o \subset D_o$ enclose the "holes" of D_o, i.e., domains /of the xy plane/ which are not contained in D_o. So we come to the conclusion that the integral /A.7/ may not vanish for some closed curves $C_o \subset D_o$, and consequently, the scalar potential φ may not exist.

A similar conclusion is valid for the integral

/A.8/ $$A_z = \int_{C_o} (-\mathcal{F}_y, \mathcal{F}_x) \cdot d\underline{l} ,$$

which results from the integration of /A.6/. In a simply connected domain D_o the integral /A.8/ vanishes for every closed curve $C_o \subset D_o$, since the divergence of $\underline{\mathcal{F}}$ vanishes in D_o. But in a multiply connected domain D_o /A.8/ may be different from zero for some closed curves $C_o \subset D_o$, and the vector potential $\underline{A}$ may not exist.

Thus we see that, in general, the field $\underline{\mathcal{F}}$ cannot be expressed by means of /A.5/ or /A.6/ in the whole domain D_o which is multiply connected /this is of course possible in simply connected subdomains of D_o/. However, $\underline{\mathcal{F}}$ can always be expressed by using a pair of potentials φ and $\underline{A}$. This is explained by the following LEMMA. For every planar field satisfying the equations /A.1/ in D_o there exist a scalar potential φ and a vector potential $\underline{A}$ such that

/A.9/ $$\underline{\mathcal{F}} = -\nabla\varphi + \nabla \times \underline{A}$$

in the domain D_o.

T h e   p r o o f. It can be carried out either in terms of vector analysis or in terms of analytic functions. We choose the latter way.

First, taking into account /A.4/ and observing that

/A.10/
$$F(\zeta)\,d\zeta = (\mathrm{Re}\,F, -\mathrm{Im}\,F)\cdot \underline{dl} + i\,(\mathrm{Im}\,F, \mathrm{Re}\,F)\cdot \underline{dl}\ ,$$

we rewrite the integrals /A.7/ and /A.8/ in the form

/A.11/
$$\varphi = -\mathrm{Im}\int_{C_0} F\,d\zeta\ ,$$

and

/A.12/
$$A_z = -\mathrm{Re}\int_{C_0} F\,d\zeta\ .$$

We introduce an auxiliary planar field $\tilde{\underline{\mathcal{F}}}$ in D_0 satisfying there /A.1/, and we denote by $\tilde{F}(\zeta)$ the corresponding analytic function in D_0. Assume $\tilde{F}(\zeta)$ is such that

/A.13/
$$\mathrm{Re}\int_{C_0} \tilde{F}\,d\zeta = 0 \quad \text{and} \quad \mathrm{Im}\int_{C_0} \tilde{F}\,d\zeta = \mathrm{Im}\int_{C_0} F\,d\zeta$$

for every closed curve $C_0 \subset D_0$. Then the integral

/A.14/
$$\varphi = -\mathrm{Im}\int_{C_0} (F-\tilde{F})\,d\zeta$$

vanishes for every closed curve $C_0 \subset D_0$. By the assumption /A.13/ the integral

/A.15/
$$A_z = -\mathrm{Re}\int_{C_0} F\,d\zeta$$

has the same property. Thus, /A.14/ and /A.15/ define functions φ and A_z in the domain D_0, provided the curve C_0 links a fixed point P_0 and a variable point Q_0. The definitions /A.14/ and /A.15/ should not be confused with the previous definitions /A.11/ and /A.12/. /A.14/ and /A.15/ define a p a i r of potentials φ and A_z, whereas /A.11/ and /A.12/ define e i t h e r φ o r A_z for a field $\underline{\mathcal{F}}$.

A function $\tilde{F}(\zeta)$ satisfying /A.13/ is easy to find. Denote by D_o^* the simply connected domain whose boundary coincides with the outer boundary of D_o, i.e., the domain obtained by "filling up" the holes of D_o. Then every function of complex variable defined in D_o^*, analytic in D_o, and having suitable poles in $D_o^* - D_o$, can serve as $\tilde{F}(\zeta)$. The simplest choice is

/A.16/ $$\tilde{F}(\zeta) = \sum_{\nu=1}^{n} c_\nu / (\zeta - \zeta_\nu) ,$$

where ζ_ν corresponds to a pole located in the ν-th hole of D_o, and c_ν the complex number satisfying /A.13/.

Evidently, the functions given by the formulae /A.14/ and /A.15/ satisfy /A.9/, since

/A.17/ $$-\nabla\varphi = \underline{\mathcal{F}} - \underline{\tilde{\mathcal{F}}} \quad and \quad \nabla \times \underline{A} = \underline{\tilde{\mathcal{F}}} .$$

This remark concludes the proof of the lemma.

It is worth noting that for a given $\underline{\mathcal{F}}$ there exists more than one pair of potentials φ and A_z. Consequently, the decomposition of $\underline{\mathcal{F}}$ into the irrotational part and the solenoidal part, as given by /A.17/, is not unique. Indeed, if $\overset{\circ}{F}$ is an analytic function in D_o such that

/A.18/ $$\int_{C_o} \overset{\circ}{F} d\zeta = 0$$

for every closed curve $C_o \subset D_o$, then the auxiliary function $\tilde{F}$ can be replaced by $\tilde{F} + \overset{\circ}{F}$ without affecting the equations /A.13/. The condition /A.18/ is satisfied by every analytic function defined in D_o^*. In general, $\overset{\circ}{F}$ may have poles in $D_o^* - D_o$ provided the sum of the residua is equal to zero.

The potentials φ^N and A_z^N corresponding to the field $\underline{\mathcal{F}} = 0$ in D_o are called null potentials. They are given by

/A.19/ $$\varphi^{N} = \mathrm{Im}\int_{C_0} \overset{\circ}{F} d\zeta \quad \text{and} \quad A_z^{N} = -\mathrm{Re}\int_{C_1} \overset{\circ}{F} d\zeta .$$

/In this case the auxiliary function $\tilde{F}$ has the property /A.18/ therefore, it is identified with $\overset{\circ}{F}$./ Note that the null potentials have nothing to do with the potentials $\overset{\bullet}{\varphi}$ and $\overset{\circ}{A}_z$ corresponding to $\overset{\bullet}{F}$. The existence of null potentials is obviously related to the fact that potentials are not unique: if φ and A_z correspond to a field $\underline{\mathcal{F}}$, then $\varphi + \varphi^N$ and $A_z + A_z^N$ also correspond to $\underline{\mathcal{F}}$.

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