6.60 — mechanika cieczy
i gazów
zagadnienia ogólne i varia

Praca habilitacyjna

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2/1990

WARSZAWA 1990

ISSN 0208-5658

Praca wpłynęła do Redakcji dnia 25 lipca 1989 r.

Praca habilitacyjna

recenzent doc.dr hab. Zbigniew Peradzyński

IPPT
PAN

Instytut Podstawowych Problemów Techniki PAN
Nakład 140 egz. Ark.wyd. 8,7 Ark.druk. 11
Oddano do drukarni w lutym 1990 r.
Nr zamówienia 63/90

Warszawska Drukarnia Naukowa, Warszawa
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GRAD'S MOMENT TRUNCATION PROCEDURE VERSUS EXTENDED IRREVERSIBLE THERMODYNAMICS

Summary

In the moment method of Grad's type applied to the Boltzmann equation one considers only such molecular densities f as can be expanded in a series of three-dimensional Hermite polynomials. The expansion coefficients, which are nothing else than the Hermite moments of f, should be considered to be independent gas-state variables satisfying the differential equations of transfer of first order both in the time t and in the place x. Of course, for general problems and collision integrals the practical advantage of the method is based on a somewhat ad hoc truncation procedure. Given the model Boltzmann-Peierls equation, the primary object of this work is to extend the range of validity of Grad's moment procedure to the case of a gas composed of quasiparticles (phonons, magnons, rotons, etc.), more to indicate the universal character of problems involved than aiming at any measure of completeness. Also, the so-called irreducible tensor description of three-dimensional gaseous systems is exploited, and this subject is developed only insofar as it relates to Grad's moment procedure and extended irreversible thermodynamics. Finally, the revised Enskog equation is introduced, and some conceptually and technically difficult problems that stem from it when constructing extended continuum theories of a model hard-sphere fluid are carefully studied. In consequence of applying the "microscopic" expression for the entropy density h, this work provides in essence the only real key in establishing a method which parallels that of Grad for a classical, moderately rarefied, simple, monatomic gas.

NOTE TO THE READER

References are listed numerically at the end of the work. We use the appropriate superscript numeral in parentheses for citation in the text.

Footnotes are numbered using a separate set of numerals and are typed at the bottom of the page to which they refer. We place a line above the footnote, so that it is set off from the text. We use the appropriate superscript numeral without parentheses for citation in the text.

As regards a detailed table of contents, see pp. 171-174.

PROLOGUE

The general problem we deal with is the following. Given a kinetic equation for a one-particle density f describing either a classical or quasiparticle gas,[1,2] is it possible to find an extended thermodynamic theory which is compatible with both continuum mechanics and the kinetic theory? To understand our contribution to this problem, let us briefly recall the usual method(s). First of all, in arriving at traditional hydrodynamics, one tries to determine (exactly or approximately) the so-called normal solutions[1,3] of the kinetic equation depending on position x and time t only through the slow conserved moments of the distribution function f. They deliver, despite from the domains of some operators, the constitutive equations which together with the equations of transfer form the field equations of ordinary thermodynamics. While ordinary thermodynamics has been undeniably useful for the analysis of many phenomena in nonequilibrium macroscopic systems, it presents some very well-known limitations. The first one is that in this theory the response of the system is instantaneous; in fact, the systems have a certain inertia which produces a delay or retardation in their response to any driving force. On the other side, if the mean collision time is not very small compared to the time required for any appreciable macroscopic changes, then the classical hydrodynamic-like behaviour of a gas breaks down at all and Grad´s approach[1,3,4] including the fast nonconserved moments of the distribution function as independent gas-state variables could potentially become a subject of intense interest. Thus in cases where it is no longer permissible to rely on normal solutions, instead of directly treating the kinetic equation, one works occasionally with the infinite hierarchy of the equations of transfer and looks either for those exact solutions which are determined by finitely many moments only[5-7] or, just as in Grad´s approach, for some reasonable truncation schemes.[8]

As we know from the important memoirs on the subject,(1,3,4) if only a few additional fast moments are taken, Grad´s framework is most naturally interpreted as a generalization of ordinary thermodynamics. If many nonconserved gas-state variables are considered, it is more appropriate to regard this method as a systematic, rational procedure for formally approximating or even replacing the kinetic theory by a sequence of theories of continuum mechanics. One way or another, Grad´s approach leads almost automatically to the system of differential equations of transfer of first order in t and x, i.e., equations for which in many cases of interest one can prove the existence of a finite speed of propagation of signals. Also, having in mind all the processes that do not vary slowly in time [and/or space (?)], Grad´s truncation scheme allows the analysis of flow problems well beyond the Chapman-Enskog gradient expansion.(1,3) In the context of the Chapman-Enskog method, we should note that the relation between the ordinary and extended theories can be made precise only by the formal procedure of projecting the mesoscopic level of description of the state of a gaseous system into the space of the physically relevant conserved moments.

In this work we consider Grad´s moment truncation procedure, originally thought limited to those indissolubly linked equations of transfer which follow from Boltzmann´s equation,(1,3,4) as well as its extensions to other kinetic equations: the Boltzmann-Peierls equation(2,9,10) and the revised Enskog equation(11-14). Although we are primarily interested in founding Grad´s approach upon modern continuum theories, such as, for instance, extended irreversible thermodynamics,(15-18) we set forth our postulational basis along with some motivation and some words about the physical circumstances the proposed framework is intended to represent. That Grad´s scheme of "solving" Boltzmann´s equation serves, among other things, the purpose of motivating or justifying extended irreversible thermodynamics, is now a well-known fact. But the crucial aspect of the present contribution is that we attempt to demonstrate the compatibility of Grad´s method with both continuum mechanics and the kinetic theory through the body of explicit calculations which are systematic and exact —— at least formally.(19-22) In addition, all our considerations which culminate in extending the range of potential validity of the intentionally heuristic Grad's moment procedure to quasiparticle gases have never been established before[1] and also should clarify the novelty of this work.

To some extent, it is purely as a matter of convenience and of simplicity of exposition that in Part A we have chosen temporarily to restrict our theory to strongly simplified model systems: one-dimensional classical and quasiparticle gases.(22,23) It should not be forgotten, however, that, despite the considerable number of practically important generalizations, as described in many papers dealing with three-dimensional systems, the elementary formulation suggested in Part A remains very useful in recognizing the nature and origin of the truncation scheme of Grad's type. In some abstract sense, the generalization to three (many) dimensions is straightforward, and comparing the theoretical treatment of Part A with that of Part B on three-dimensional rarefied gases and phonon systems one recognizes a strong methodological agreement. To be sure, from the standpoint of mathematical apparatus this statement can easily be questioned, because the theory of Part B rests heavily, indeed essentially, on well-rounded, though little-known algebraical structure which we conveniently call *irreducible tensor description*.(24-26) While "naive" Cartesian reducible tensor description has proved its efficiency as a mathematical basis for proofs of plausible and practically useful results, it is also no essential building block for our certainly incomplete but systematic investigations in Part C. In this part we introduce the revised Enskog equation and consider some conceptually and technically difficult problems generated by it when constructing extended continuum theories of a model hard-sphere fluid.(20)

We conclude this discussion as follows. The approach taken by the present work is a self-consistent combination of kinetics and phenomenology and should be treated as a

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[1] See, however, Refs. 23-26.

synthesis, at least we hope it to be so, of selected aspects of those of Grad(1,2) and Müller.(15,16) We shall follow and generalize the former in expanding the distribution function f in terms of Hermite or Tchebychef polynomials (functions), but at the same time we shall sharpen the argument to arrive, by the use of Müller's formulation of the axiom of entropy growth, at a derivation of the more elaborate scheme of truncating the infinite hierarchy of equations of transfer. In other words, granted the importance of recognizing a suitably chosen series representation of the distribution function that converges in the *mean*, it is desirable for the moment truncation procedure which we discuss here that this representation should be structured around some organizing principles. Such an organizing principle is provided by continuum mechanics (thermodynamics) itself if its scope is extended by the introduction of supplementary ingredients of a kinetic nature, namely, the explicit functional dependence of the entropy density h and of the entropy flux $\mathbf{s}$ upon f. Of course, this is a break with the traditional notion that this discipline is to be macroscopic, but in the present treatment we do not consider thermodynamics as a theory of measurement.

Here we confine ourselves more to logical structures than to practical purposes.

PART A

ONE-DIMENSIONAL MODELS

CHAPTER 1

A QUASIPARTICLE ANALOGUE OF GRAD'S MOMENT PROCEDURE

No doubt, the key to the stimulating development of Grad's moment truncation scheme lies in those assumptions that are usually not explicitly mentioned. In contrast to the importance here attributed to the problem of specifying general criteria that account for the choice of an appropriate moment representation of the distribution function, this question is rather ignored in most of the traditional formulations. Of course, the relevance of the problem may have been first recognized by Grad, as it appears from the study of the original references.(1,3) However, leaving aside this observation, it seems to us that the characteristic tendency is to discuss in the fullest detail the complications of a purely mathematical aspect of Grad's method, e.g., an expansion of f in terms of Hermite polynomials, evidently at the expense of a deeper understanding of the real status of his propositions. To arrive first at a derivation of the analogue of Grad's moment truncation procedure for one-dimensional quasiparticle gases, where up to now nothing has been done, is not typical. Nevertheless, it is quite conceivable that only through the détour over a traditional presentation, which primarily relies on the notion of a classical rarefied gas, is the original concept of Grad placed on a new universal basis.(22,23)

1.1. Introduction

At low temperature, it is a well-established fact that the thermal properties of a variety of macroscopic bodies such as, for instance, dielectric crystals, ferromagnets, and Helium II can be described adequately with the help of elementary excitations.(27,28) These elementary excitations (phonons, magnons, rotons, etc.) behave in many cases of interest as a gas of interacting quasiparticles obeying the characteristic laws of the kinetic theory.(2,9,10,29,30) For the purposes of this chapter, the subject of the kinetic theory is considered to be coextensive with the theory of the Boltzmann-Peierls equation.(2,9) Of course, the Boltzmann-Peierls equation has somewhat limited validity [see, e.g., Gurevich's monograph(2)], and a more microscopic procedure may be developed in terms of the powerful Green's function techniques.(10,29,30) However, if we are primarily interested in recognizing the nature and origin of fundamental ingredients of continuum thermomechanics, then we can easily find on closer examination that this procedure is of no greater generality than the Boltzmann-Peierls method, as employed for long-wavelength collective quasiparticle behaviour to be discussed here.[2] Explicitly, unlike all other approaches, the Boltzmann-Peierls description provides almost automatically the physically relevant conservation principles of local hydrodynamics and establishes, due to the unique H-theorem, the irreversibility of the evolution of the distribution function toward an equilibrium state of the Bose-Einstein type.

Despite some difficulties first indicated by Jäckle(31) and Buot,(32) Gurevich(2) was right in always stressing that the mean-free-path is the basic length scale and the mean collision time is the basic time scale for the Boltzmann-Peierls equation. Thus, observing that in some problems of quasiparticle gaseous dynamics both these quantities are very small compared to macroscopic sizes or time scales, he invented for quite general quasiparticle (phonon) gases and for a broad class of dispersion relations an intuitive method of estimating a variety of transport coefficients through the Chapman-Enskog scheme. Since in general the mean relaxation (collision) time τ_m is not necessarily very small compared to a representative macroscopic decay time τ_h, a number of attempts (concerning classical gases) have been made to obtain results in the neighbourhood of free flow. Insofar as classical rarefied gase are

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[2] The most significant application of the microscopic theory, being outside the direct scope of these investigations, would seem to be in problems which require some details concerning the precise derivation of the dispersion relations and of the transition rates.

concerned, one possible method is that of Grad. Indeed, within the framework of his procedure, the main object is not only to provide new phenomenological (or pseudo-phenomenological) equations which generalize the usual ones but also to demonstrate that they describe, at least to some extent, the effect of an overlap between the two afore-mentioned time scales.

In this chapter we attempt to extend the range of validity of Grad's ideas to the case of quasiparticle gases, more to indicate the type of problems involved than aiming at any measure of completeness. The choice of a one-dimensional gas serves first of all the purpose of simplicity and of discussing the general and universal scheme of dealing with the hierarchy of equations of transfer. Due to the nasty tensorial problems, such a discussion is practically impossible in three dimensions. Here we proceed as follows. In Section 1.2 we introduce both the moments α_n, $n \in \mathbb{N}_0 := \{0,1,2,...\}$, of the distribution function f and the exact system of equations for them, provided that the dispersion relation,(27) i.e., the precise relationship between the frequency Ω and the wave number k, is given by $\Omega \sim |k|^{\nu}$, $1 < \nu \le 2$. [The $\nu = 1$ case is briefly investigated in Chapter 4; thermodynamics of a low-temperature phonon gas for $\Omega \sim |k|$ is discussed in the fullest detail in Ref. 26.] Yet, in order to render the infinite hierarchy of equations of transfer just mentioned definite, we must necessarily face the difficult problem of evaluating the collision integrals P_n, $n \in \mathbb{N}_0$ (which appear on the r.h.s. of those equations of transfer) as exhibited series in the moments α_n, $n \in \mathbb{N}_0$. Of course, since the quantities P_n are functionally related to f, we cannot calculate them without first having to specify the moment representation of the distribution function. With these statements in mind, Section 1.3 may turn out to be the most important. Indeed, in consequence of the introduction of a suitably chosen Hilbert space, this section provides in essence the only real key for establishing a strict quasiparticle analogue of Grad's expansion of the distribution function f in terms of Hermite polynomials and Hermite coefficients.

As we have already remarked, our approach is a self-consistent combination of kinetics and phenomenology. In Sections 1.4 through 1.6 we use the kinetic theory to derive the entropy principle and the moment equations, which we consider up to an order r (r ≥ 1). Clearly, due to the results of Section 1.3, these equations in general contain all moments, not only those of order 0 to r. To close this gap, we introduce the constitutive equations (1.4.2a) via a pseudo-phenomenological ansatz (1.4.9a) up to quadratic terms in the first r (Tchebychef) moments. Then, in order to determine the unknown expansion coefficients of our ansatz, we make extensive use of the entropy principle in the form of Müller-Liu.(15,16) In actual fact, however, the foregoing pseudo-macroscopic method constitutes the underlying theoretical structure of what Weinert et al.(8) could easily call one of the reasonable but different ways of truncating the indissolubly linked equations of transfer. Above all the phrase "reasonable but different" means *entirely compatible with, and much more subtle and systematic than, the original Grad's truncation scheme.* Since we consider the r ≥ 1 case, the resultant effect of this formalism is a new mesoscopic theory of quasiparticle gases which takes as independent gas-state variables the slow conserved moment α_0 (directly derivable from the energy density e per unit length) and the "fast dissipative fluxes" $\alpha_1,\ldots,\alpha_r$. Section 1.7 achieves the object of explaining, from the kinetic point of view, a phenomenological formula for the entropy production σ_h. In Section 1.8 we derive, by the simple method of applying the normal solution expansion to the 2-moment system of field equations for α_0 and α_1 (r = 1), ordinary quasiparticle hydrodynamics. Section 1.9 is for discussion and conclusion.

1.2. Prolegomena

1.2.A. The Model Boltzmann-Peierls Equation

For the purposes of this chapter, the kinetic theory of a one-dimensional gas of quasiparticles should be regarded as a strongly simplified mathematical model in which the nonequilibrium occupation probability f, aside from its obvious dependence upon the wave number $k \in \mathbb{R} := (-\infty, +\infty)$, is a function of position x and time t satisfying the Boltzmann-Peierls equation of the form(2,9,10)

$$\partial_t f + \nabla_k \Omega \cdot \nabla_x f - \nabla_x \Omega \cdot \nabla_k f = J(f), \tag{1.2.1}$$

where J is the collision operator. Here $\hbar\, \Omega(k,x) = \hbar\, c(x)|k|^{\nu}$, $1 < \nu \leq 2$, represents the energy of a single quasiparticle in the mode k, whereas $2\pi\, \hbar$ and c(x) are Planck's constant and a certain differentiable function of position x, respectively. [The usefulness of the $\nu = 2$ case arises from every attempt to study magnon excitations in a continuum. The "unrealistic" proposition $\Omega \sim |k|^{\nu}$, $1 < \nu < 2$, still retains the crude qualitative physical properties of acoustic excitations in a continuum that might be important ($\Omega \to 0$ for $|k| \to 0$, $\Omega \to +\infty$ for $|k| \to +\infty$).]

Let us consider a collision process, associated with various imperfections in a one-dimensional crystal (and with other possible momentum-destroying scattering events), in which (*a*) one quasiparticle in the mode k spontaneously decays into two k_1 and k_2, or (*b*) two quasiparticles in the modes k_1 and k_2 combine to form one k. Of course, both processes *a* and *b* are subject to the selection rules, although in the described situation the condition of interference[3] cannot be fulfilled. There is, however, a restriction corresponding to conservation of energy. With these statements in mind, we obtain for the collision operator J

$$\begin{aligned} J(f) := \int_{\mathbb{R}} \int_{\mathbb{R}} dk_1 dk_2\, \mathcal{R}(k_1;k,k_2)\, f_1(1+f)(1+f_2) & \\ \times\, \delta(\Omega_1 - \Omega - \Omega_2) & \\ - \int_{\mathbb{R}} \int_{\mathbb{R}} dk_1 dk_2\, \mathcal{R}(k,k_2;k_1)\, f f_2(1+f_1) & \\ \times\, \delta(\Omega_1 - \Omega - \Omega_2) & \end{aligned}$$

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[3] The condition of interference bears a considerable resemblance to the role which was assigned in classical mechanics to the principle of conservation of total momentum of the colliding material particles, but in general it cannot be identified exactly with this principle; for more details, see p. 88 in Ref. 28.

$$+\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dk_1 dk_2\, \mathcal{R}(k_1,k_2;k)\, f_1 f_2 (1+f) \times \delta(\Omega - \Omega_1 - \Omega_2)$$

$$-\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} dk_1 dk_2\, \mathcal{R}(k;k_1,k_2)\, f(1+f_1)(1+f_2) \times \delta(\Omega - \Omega_1 - \Omega_2), \qquad (1.2.2)$$

where δ denotes the Dirac delta and

$$f_m := f(k_m,x,t), \quad \Omega_m := \Omega(k_m,x), \quad m = 1,2. \qquad (1.2.3)$$

With regard to the so-called transition rates $\mathcal{R}(k;k_1,k_2)$ and $\mathcal{R}(k_1,k_2;k)$ for the a-scattering process and the b-scattering process, respectively, we assume that

$$\mathcal{R}(k;k_1,k_2) = \mathcal{R}(k;k_2,k_1) = \mathcal{R}(k_1,k_2;k) > 0. \qquad (1.2.4)$$

Multiplication of $J(f)$ in Eq. (1.2.2) by any function $\mathcal{N}$ of the wave number k and integration throughout the $\mathbb{R}$-space, as well as the subsequent use of the equalities (1.2.4) yield

$$\int_{\mathbb{R}} \mathcal{N}(k) J(f)\, dk = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} dk\, dk_1 dk_2\, \delta(\Omega - \Omega_1 - \Omega_2) \times \mathcal{R}(k;k_1,k_2)\left(\mathcal{N}_1 + \mathcal{N}_2 - \mathcal{N} \right) \times \left[f(1+f_1)(1+f_2) - f_1 f_2 (1+f) \right], \qquad (1.2.5)$$

where

$$\mathcal{N}_m := \mathcal{N}(k_m), \quad m = 1,2.$$

Let us set

$$\mathcal{N} = \Omega, \quad \mathcal{N} = \ln\left(\frac{1+f}{f}\right) \qquad (1.2.6)$$

and

$$f_o(\Omega) := \left(e^{\omega} - 1\right)^{-1}, \quad \omega := \frac{\hbar\,\Omega}{k_B T}, \qquad (1.2.7)$$

where k_B stands for Boltzmann's constant. [The quantity T, as the postulated equality

$$\int_{\mathbb{R}} \Omega f\, dk = \int_{\mathbb{R}} \Omega f_o\, dk \qquad (1.2.8)$$

reveals, is functionally related to f. We call f_o and T the local Bose-Einstein distribution function that corresponds to f and the local absolute temperature associated with f, respectively.] Then, in view of (1.2.5)-(1.2.7), the standard inequality $(X - Y)[\ln(X) - \ln(Y)] \geq 0$ $(X > 0,\ Y > 0)$, and the identity

$$f_o(\Omega_1)[1 + f_o(\Omega)][1 + f_o(\Omega_2)] = f_o(\Omega)f_o(\Omega_2)[1 + f_o(\Omega_1)], \tag{1.2.9}$$

which is valid for $\Omega_1 = \Omega + \Omega_2$, we arrive at

$$\int_{\mathbb{R}} \Omega J(f)\, dk = 0, \tag{1.2.10}$$

$$\int_{\mathbb{R}} J(f) \ln\left(\frac{1 + f}{f}\right) dk \geq 0 , \tag{1.2.11}$$

$$J(f_o) = 0. \tag{1.2.12}$$

Sketchy as the discussion of this section may be, it ought to suffice to justify (1.2.10)-(1.2.12) as a tentative basis for our further investigations.

1.2.B. Equations of Transfer and the Entropy Principle

It is tempting to define the moments a_n of f as follows:

$$a_n := \int_{\mathbb{R}} |k|^{\nu} \mathscr{k}^n f\, dk , \quad n \in \mathbb{N}_o , \tag{1.2.13}$$

where

$$\mathscr{k} := |k|^{\nu - 1} g , \quad g := |k|^{-1} k . \tag{1.2.14}$$

If we multiply (1.2.1) by $|k|^{\nu}\mathscr{k}^n$, $n = 0,1,\ldots,$ and integrate the result over $\mathbb{R}$, we obtain the infinite hierarchy of the equations of transfer for the moments of f:

$$\partial_t a_n + \nu c\, \nabla_x a_{n+1} + \nu C^{\nu}_{n+1} W a_{n+1} = P_n , \tag{1.2.15a}$$

$$C^{\nu}_n := \nu^{-1}[\ \nu + 1 + (\nu - 1)n\ ] > 1, \tag{1.2.15b}$$

$$W := \nabla_x c , \tag{1.2.15c}$$

$$P_n := \int_{\mathbb{R}} |k|^{\nu} \mathscr{k}^n J(f)\, dk , \tag{1.2.15d}$$

$$n \in \mathbb{N}_o . \tag{1.2.15e}$$

The quantities P_n, depending on x and t through the distribution function, will be termed collision integrals. From (1.2.5) we see that

$$P_n = \frac{1}{2c} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} dk\,dk_1\,dk_2\, \mathcal{R}(k;k_1,k_2)\, \delta(\Omega - \Omega_1 - \Omega_2)$$

$$\times \left(\Omega_1 \mathcal{A}_1^n + \Omega_2 \mathcal{A}_2^n - \Omega\, \mathcal{A}^n \right)\left(f + ff_1 + ff_2 - f_1 f_2 \right), \qquad (1.2.16)$$

where

$$\mathcal{A}_m := |k_m|^{\nu - 1} g_m , \quad g_m := |k_m|^{-1} k_m , \quad m = 1,2. \qquad (1.2.17)$$

Some scalars and vectors derived from α_n, $n \in \mathbb{N}_o$, are important for their interpretation. Chief among these are the energy density e per unit length and the heat flux q:

$$e := \frac{\hbar c}{2\pi} \alpha_o , \quad q := \frac{\nu \hbar c^2}{2\pi} \alpha_1 . \qquad (1.2.18)$$

Elementary inspection shows that

$$\partial_t e + \nabla_x q = 0. \qquad (1.2.19)$$

The tradition of the kinetic theory of quasiparticle gases chooses the following definition of the entropy density h:

$$h := \frac{k_B}{2\pi} \int_{\mathbb{R}} \left[(1 + f)\ln(1 + f) - f\ln f \right] dk . \qquad (1.2.20)$$

Beginning from (1.2.1) and (1.2.11), we obtain the entropy principle in the form

$$\partial_t h + \nabla_x \Phi = \sigma_h \geq 0, \qquad (1.2.21)$$

where

$$\Phi := \frac{\nu k_B c}{2\pi} \int_{\mathbb{R}} \mathcal{A} \left[(1 + f)\ln(1 + f) - f\ln f \right] dk , \qquad (1.2.22)$$

$$\sigma_h := \frac{k_B}{2\pi} \int_{\mathbb{R}} J(f) \ln\left(\frac{1 + f}{f} \right) dk . \qquad (1.2.23)$$

We may interpret Φ and σ_h as the entropy flux and the entropy production, respectively.

To complete this section, let us introduce the following useful abbreviation:

$$\mathbb{A}_m^n := \left\{ \alpha_{m+1}, \ldots, \alpha_n \right\} . \qquad (1.2.24)$$

1.3. The Moment Representation of the Distribution Function f

1.3.A. A Slightly Modified Lemma of Dijkstra and van leeuwen

Although a rigorous proof is still lacking, it would seem to be impossible to find such a mathematical expression for the transition rate $\mathcal{R}(k;k_1,k_2)$ in Eq. (1.2.16) which could deliver the P_n, $n \in \mathbb{N}_0$, as an exhibited function of $\mathbb{A}^m_{-1}$, $0 \le m < \infty$, without first having to determine f. Thus one should easily agree with the ascertainment that, except for some hypothetical cases, any attempt to study the infinite hierarchy of the equations of transfer for a_n, $n \in \mathbb{N}_0$, immediately runs straight into the nasty problem of both expressing the distribution function in terms of $\mathbb{A}^\infty_{-1}$ and evaluating or estimating the collision integrals P_n, $n \in \mathbb{N}_0$, as *infinite formal series* in the moments.(1,2,4)

In order to overcome [to some extent] the above difficulty, we begin with the definitions:

$$\alpha := \frac{\nu}{\nu - 1}\ , \quad \beta := \frac{\nu + 2}{\nu}\ , \tag{1.3.1a}$$

$$z := \Theta^{\alpha^{-1}} k\ , \quad \Theta := \frac{\hbar c}{k_B T} \quad \left[\Rightarrow \omega = |z|^\alpha \right], \tag{1.3.1b}$$

$$\mathbb{W}_\nu(z) := \frac{\omega^\beta e^\omega}{(e^\omega - 1)^2}\ , \tag{1.3.1c}$$

$$\{\psi\}_o := \int_{\mathbb{R}} \psi(z) \mathbb{W}_\nu(z)\, dz\ , \tag{1.3.1d}$$

$$\{\psi_1 | \psi_2\} := \{\psi_1 \psi_2\}_o\ , \quad ||\psi|| := \left(\{\psi|\psi\}\right)^{1/2}\ . \tag{1.3.1e}$$

Inspection shows that

$$\mathbb{L}^2(\mathbb{R};\, \mathbb{W}_\nu(z) dz) := \left\{ \psi : ||\psi|| < \infty \right\} \tag{1.3.2}$$

is a real Hilbert space with the inner product { | } defined by (1.3.1d) and (1.3.1e).

Lemma. Let $\mathbb{H} : \mathbb{R} \Rightarrow \mathbb{R}^{+} := [0, +\infty)$ be a Lebesgue measurable function with the property that for certain constants $\mathbb{M} > 0$ and $\mathbb{C} > 0$

$$\mathbb{H}(z) \leq \mathbb{M}\, e^{-\mathbb{C}z^2}. \tag{1.3.3}$$

Then the collection of polynomials defined on $\mathbb{R}$ is a dense subset of $\mathbb{L}^2(\mathbb{R}; \mathbb{H}(z)dz)$.

The proof of this slightly modified lemma of Dijkstra and van Leeuwen follows easily from that on pp. 468-469 in Ref. 33 and is based upon Lebesgue's theorem and the fact that the polynomials are dense in $\mathbb{L}^2(\mathbb{R}; \mathbb{M}\, e^{-\mathbb{C}z^2} dz)$.(34,35)

Since $2 \leq 2 + \nu^{-1}(2 - \nu) = \beta$ and $2 \leq (\nu - 1)^{-1}\nu = \alpha$, the weight function $\mathbb{H}_\nu$ satisfies the condition (1.3.3).

Making use of Sansone's reasoning on p. 309 of his monograph,(35) we construct a sequence $\{B_n; n \in \mathbb{N}_o\}$ of the so-called Tchebychef polynomials B_n as follows. Orthogonalizing with respect to the inner product in $\mathbb{L}^2(\mathbb{R}; \mathbb{H}_\nu(z)dz)$ the set $\{z^n; n \in \mathbb{N}_o\}$ of nonnegative powers of $z \in \mathbb{R}$, we obtain a system $\{B_n; n \in \mathbb{N}_o\}$ of polynomials uniquely determined by the two conditions:

(i) $B_n(z)$ is a polynomial of precise degree n in which the coefficient τ_n of z^n is positive;

(ii) the system $\{B_n; n \in \mathbb{N}_o\}$ is orthonormal, i.e.,

$$\{B_n | B_m\} = \{B_n B_m\}_o = \delta_{nm}, \tag{1.3.4}$$

where δ_{nm} is the Kronecker delta.

From the *lemma* we conclude that a family of Tchebychef polynomials $B_n(z)$ forms a basis in $\mathbb{L}^2(\mathbb{R}; \mathbb{H}_\nu(z)dz)$.

Condition I. Let ψ be a certain function defined on $\mathbb{R}$ and suppose that

$$\hat{\psi}(z) := \frac{(e^{\omega} - 1)}{\omega\, e^{\omega}} \frac{[\psi(z) - f_o(\Omega)]}{f_o(\Omega)}. \tag{1.3.5}$$

Then ψ satisfies *Condition I* if $\hat{\psi} \in \mathbb{L}^2(\mathbb{R}; \mathbb{H}_\nu(z)dz)$.

Now, if the distribution function satisfies *Condition I*, then f has a unique expansion

$$f = f_o\left[1 + \frac{\omega\, e^{\omega}}{e^{\omega} - 1} \sum_{n=o}^{\infty} b_n(x,t)\, B_n(z)\right], \qquad (1.3.6)$$

and this expansion converges in the *mean* to the function f.

In virtue of (1.3.4), we find that the Tchebychef expansion coefficients (moments) b_n, $n \in \mathbb{N}_o$, are derivable from $f - f_o$:

$$b_n = \int_{\mathbb{R}} \omega^{2/\nu}\, (f - f_o) B_n(z)\, dz\ . \qquad (1.3.7)$$

Consistent with (1.2.24), we set

$$\mathbb{B}^n_m := \left\{ b_{m+1}, \ldots, b_n \right\}. \qquad (1.3.8)$$

1.3.B. Elementary Recurrence and Conversion Formulae

Let $\tau_n > 0$ be the highest coefficient of B_n, $n \in \mathbb{N}_o$. Then the following recurrence formula(34,35) holds for any three consecutive orthonormal polynomials:

$$B_n(z) = \frac{\tau_n}{\tau_{n-1}}\, zB_{n-1}(z) - \frac{\tau_n \tau_{n-2}}{\tau^2_{n-1}}\, B_{n-2}(z), \qquad (1.3.9a)$$

where

$$\tau_n = \tau^2_{n-1}\left(\ell_n \tau^2_{n-1} - \tau^2_{n-2}\right)^{-1/2}, \qquad (1.3.9b)$$

$$\ell_n := \left\{z^2 B^2_{n-1}\right\}_o, \qquad (1.3.9c)$$

$$n \geq 2\ . \qquad (1.3.9d)$$

On the understanding that $\tau_{-1} := 0$ and $B_{-1} := 0$, Eqs. (1.3.9a)-(1.3.9c) are valid also for n = 1.

Define α_n by

$$\alpha_n := \left(\{z^{2n}\}_o\right)^{-1/2}, \quad n \in \mathbb{N}_o\ . \qquad (1.3.10)$$

Then

$$\{z^{2n}\}_o = 2\alpha^{-1} C^{\nu}_{2n}\, \Gamma(C^{\nu}_{2n})\, \zeta(C^{\nu}_{2n}), \qquad (1.3.11)$$

where Γ and ζ are the Gamma function and the Riemann function, respectively.

Elementary calculus shows that

$$B_0(z) = \tau_0 = \alpha_0 , \quad B_1(z) = \tau_1 z = \alpha_1 z . \tag{1.3.12}$$

Given Eqs. (1.3.9) and (1.3.12), we have no difficulties in principle to evaluate explicitly and for increasing order (n = 3,4,...) as many of the orthonormal Tchebychef polynomials as are needed.

Now, let us introduce the coefficients X_p^n and Y_p^n by setting

$$B_n(z) = \sum_{p=0}^{n} X_p^n z^p , \quad z^n = \sum_{p=0}^{n} Y_p^n B_p(z) . \tag{1.3.13}$$

Due to the fact that for every fixed $n \in \mathbb{N}_0$ the systems $\mathbb{A}^n_{-1}$ and $\mathbb{B}^n_{-1}$ are bijectively related, each member of one of the two sets $\mathbb{A}^n_{-1}$ and $\mathbb{B}^n_{-1}$ is a linear combination of the members of the other. By appealing to Eqs. (1.2.13)-(1.2.14), (1.3.1a)-(1.3.1b), (1.3.7), (1.3.13), and the equality

$$D_n^\nu := \int_{\mathbb{R}} \omega^{2/\nu} f_0(\Omega) z^{2n} \, dz = 1/C^\nu_{2n} \alpha_n^2 , \tag{1.3.14}$$

we obtain for $\mathbb{A}^n_{-1} \leftrightarrow \mathbb{B}^n_{-1}$

$$b_n = \sum_{p=0}^{n} x_p^n \alpha_p - \sum_{p=0}^{[n/2]} D_p^\nu X^n_{2p} , \tag{1.3.15a}$$

$$\alpha_n = \sum_{p=0}^{n} y_p^n b_p + (\nu - 1)^{-1} e^{-C_n^\nu} \sum_{p=0}^{[n/2]} D_p^\nu \delta^n_{2p} , \tag{1.3.15b}$$

where

$$x_p^n := (\nu - 1) \, e^{C_p^\nu} X_p^n , \quad y_p^n := (\nu - 1)^{-1} e^{-C_n^\nu} Y_p^n . \tag{1.3.15c}$$

Here $[u]$ and δ_p^n are the greatest integer ≤ u and the Kronecker delta, respectively.

On recalling (1.2.8) and (1.3.7), we get

$$b_0 = 0 . \tag{1.3.16}$$

By taking 0 for n in Eq. (1.3.15b), it is easy to read off from this equation the relationship between α_0 and e:

$$a_o = \frac{\nu}{(\nu^2 - 1)\alpha_o^2}\,\Theta^{-(\nu + 1)\nu^{-1}} . \qquad (1.3.17a)$$

Of course, the possibility of both inverting (1.3.17a) with respect to Θ and substituting the resulting formula for $\Theta(a_o)$ into f_o allows us to conclude that the Bose-Einstein distribution function depends on x and t only through a_o.

The equation (1.3.15a), when intended to represent the n = 1 case and combined with (1.2.18), expresses no more than

$$b_1 = \frac{2\pi}{\hbar c^2}\,\frac{\alpha_1}{\alpha}\,\Theta^2 q . \qquad (1.3.17b)$$

If we put the Tchebychef expansion (1.3.6) of the distribution function into (1.2.16) and make use of (1.3.15) and (1.3.17a), we are justified in doing the following transformations:

$$P_n = \hat{P}_n\left(a_o, \mathbb{B}_o^{\infty} \right) = \tilde{P}_n\left(\mathbb{A}_{-1}^{\infty} \right), \quad n \in \mathbb{N}_o . \qquad (1.3.18)$$

Thus the important (formal) result of replacing P_n by $\tilde{P}_n$ to express the r.h.s. of (1.2.15a) is, just as in the original Grad´s moment procedure, the infinite system of the indissolubly linked equations of transfer for the expectations a_n, $n \in \mathbb{N}_o$) of $|k|^{\nu}\mathcal{E}^n$, $n \in \mathbb{N}_o$) alone.

1.3.C. Motivation

Insofar as classical gases are concerned, Grad[4] wrote that "by the choice of a Maxwellian, local or absolute, as a starting point, one has almost no alternative to the use of a Hermite polynomial expansion" and that "the choice of Hermite coordinates [Hermite coefficients (moments)] always diagonalizes the entropy." Although in actual fact there are other expansions of comparable if not greater mathematical complexity, Grad´s framework[4] "has been seen to be particularly

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[4] See Ref. 3, pp. 267 and 284.

appropriate in generalizing fluid type equations (near a locally Maxwellian state) and for the linearized Boltzmann equation."

a. In Section 1.3.B our whole discussion has centered around the elementary recurrence and conversion formulae, but *first of all* we need a plausible argument in favour of the usefulness of the very specific postulate (1.3.1c). However, under just this proposition for the weight function, the expansion (1.3.6) of the distribution function constitutes the underlying theoretical structure of what one could recognize as a kinetic background of nonequilibrium thermodynamics of a gas of quasiparticles. Indeed, in taking stock of the connection between kinetics and thermodynamics, one of the most important features of (1.3.6) is the unique way the specific choice $\mathbb{M}_\nu$ affects the approximate dependence of both the entropy density h and the entropy flux Φ upon the Tchebychef expansion coefficients b_p.

To this end, let us set

$$h = \frac{k_B}{2\pi} \int_{\mathbb{R}} \mathbb{F}\, dk \,, \qquad \Phi = \frac{\nu k_B c}{2\pi} \int_{\mathbb{R}} \mathbb{A}\, \mathbb{F}\, dk \,, \tag{1.3.19}$$

$$\mathbb{F} := (1 + f)\ln(1 + f) - f\ln f \,, \tag{1.3.20a}$$

$$\mathbb{F}_o := (1 + f_o)\ln(1 + f_o) - f_o \ln f_o \,, \tag{1.3.20b}$$

and

$$\theta := \sum_{p=1}^{\infty} b_p(x,t)\, B_p(z) \,, \qquad \varepsilon := \frac{\omega\, e^{\omega}}{e^{\omega} - 1}\, \theta \,. \tag{1.3.20c}$$

In the neighbourhood of the state of local equilibrium, if instead of the logarithm $\ln(1 + X)$ $(|X| < 1)$ we use its Taylor expansion $X - \frac{1}{2} X^2 + \frac{1}{3} X^3 + \ldots$, we obtain for $\mathbb{F}$

$$\mathbb{F} = [1 + f_o(1 + \varepsilon)]\ln[(1 + f_o)(1 + e^{-\omega}\varepsilon)]$$

$$- f_o(1 + \varepsilon)\ln[f_o(1 + \varepsilon)] = \mathbb{F}_o + f_o \omega\, \varepsilon$$

$$+ (1 + f_o)(1 + e^{-\omega}\varepsilon)\ln(1 + e^{-\omega}\varepsilon) - f_o(1 + \varepsilon)\ln(1 + \varepsilon)$$

$$= \mathbb{F}_o + f_o\omega\,\varepsilon - \frac{1}{2}\,e^{-\omega}\varepsilon^2 + \frac{1}{6}\,\frac{\left(e^{\omega} + 1\right)}{e^{2\omega}}\,\varepsilon^3 + \dots$$

$$= \underline{\mathbb{F}_o + \omega^{(\nu - 2)\nu^{-1}}\,\mathbb{H}_{\nu}(z)\theta - \frac{1}{2}\,\omega^{(\nu - 2)\nu^{-1}}\,\mathbb{H}_{\nu}(z)\theta^2}$$

$$+ \frac{1}{6}\,\omega^{(\nu - 2)\nu^{-1}}\,F(\omega)\mathbb{H}_{\nu}(z)\theta^3 + \dots, \qquad (1.3.21a)$$

where

$$F(\omega) := \frac{\omega\left(e^{\omega} + 1\right)}{e^{\omega} - 1}\,. \qquad (1.3.21b)$$

Given (1.3.19), only after replacing $\mathbb{F}$ by the underlined expression in (1.3.21a) can the reader see what makes the weight function (1.3.1c) simpler than any other. In so doing, we arrive at

$$h - h_o \cong -\frac{k_B}{4\pi(\nu - 1)}\,\theta^{-1/\nu}\sum_{n=1}^{\infty} b_n\, b_n$$

$$= -\frac{k_B}{4\pi(\nu - 1)}\,\theta^{-1/\nu}\,||\hat{f}||^2 > -\infty\,, \qquad (1.3.22a)$$

$$\Phi - \frac{1}{T}\,q \cong -\frac{\nu k_B c}{2\pi(\nu - 1)}\,\theta^{-1}\sum_{n=1}^{\infty}\frac{\tau_n}{\tau_{n+1}}\,b_n\, b_{n+1}\,, \qquad (1.3.22b)$$

where, of course, f satisfies *Condition I* and

$$h_o := \frac{k_B}{2\pi}\int_{\mathbb{R}} \mathbb{F}_o\, dk = \nu\,\frac{\partial e}{\partial T}\,. \qquad (1.3.22c)$$

Thus, provided that the expansion terms in $\mathbb{F}$ of third (fourth, etc.) order with respect to θ can be considered negligible, it is worthwhile applying the weight function $\mathbb{H}_{\nu}(z)$ as well as the transformation (1.3.15) to "diagonalize" the largest contributions to both $h - h_o$ and $\Phi - (1/T)q$ —— those of interest in linear extended thermodynamics.(15-18,65) In addition, Eq. (1.3.22a) suggests that, among all quasiparticle occupation probabilities f having the same principal moment a_o ($b_o = 0$), the Bose-Einstein distribution function f_o gives h its maximal value. [The correctness of this (to some extent) obvious statement can be demonstrated rigorously.]

b. Notice that Eq. (1.3.22a) contains also complete information about the so-called *second moments* of the equilibrium fluctuations of the "thermodynamic coordinates" a_p and b_p, $p \geq 1$. Indeed, in thermostatics the quantities a_p and b_p are always assumed to have reached their equilibrium values, namely, a_p^o and b_p^o ($b_p^o = 0$). However, it is possible to take fluctuations into consideration and regard a_p^o and b_p^o as random variables ($a_p \to a_p^o \to a_p^o + \delta a_p$, $b_p \to b_p^o \to b_p^o + \delta b_p$, $p \geq 1$). In this way, if we assume that the (relative) probability density $\mathbb{P}$ of fluctuations is given by the Einstein-Smoluchowski formula

$$\mathbb{P} = exp[\, k_B^{-1}\, \mathbb{L}_o(\, h - h_o\,)\,], \tag{1.3.23}$$

$\mathbb{L}_o$ being the length of the macroscopic system, then the approximate expression (1.3.22a) for $h - h_o$ becomes a potential for the fluctuations of b_p^o, or of a_p^o, near the *absolute* equilibrium state; for more details on the subject, see Ref. 17, pp. 474-477.

c. The abstract Hilbert space $\mathbb{L}^2(\mathbb{R};\, \mathbb{H}_\nu(z)dz)$ introduced in Section 1.3.A remains useful in considering the linearized Boltzmann-Peierls equation,(9,10) and the Tchebychef basis $\{B_p;\ p \in \mathbb{N}_o\}$ of $\mathbb{L}^2(\mathbb{R};\, \mathbb{H}_\nu(z)dz)$ is expected to be of interest in estimating the transport coefficient(s) through a sequence of approximations usually made when solving the appropriate Fredholm integral equation. Despite some difficulties first indicated by Jäckle(31) and Buot,(32) research in this direction can be outlined and is under way. For other elementary considerations of a similar nature, see Ref. 2, pp. 124-168 and Ref. 36, pp. 124-129.

In summary, the important properties of (1.3.6) just established should cast light on the far reaching similarity between Grad's method and the one we are presently developing so as to derive and generalize all results that his approach, if it could be specialized by him to the case of a one-dimensional gas of quasiparticles, would deliver. No doubt, we may regard (1.3.6) as a strict quasiparticle analogue of Grad's expansion of the distribution function in terms of Hermite polynomials and call the Tchebychef moments b_p analogues of Hermite coefficients.

1.4. Survey of the Truncation Procedures

1.4.A. Description of the Original Grad's Method

Due to the intricacy of (1.3.18), it would be difficult to use (1.2.15) to qualify the α_n as functions of x and t. Thus, in order to get a determined system of field equations for $\mathbb{A}^r_{-1}$ ($r \geq 1$), we must decide to approximate (1.3.6) by a finite sum of Tchebychef polynomials, say, up to B_r. In the system of equations up to $\partial_t \alpha^r$ we set all higher order Tchebychef coefficients equal to zero:

$$b_p = 0, \quad p > r , \tag{1.4.1a}$$

$$P_n = \hat{P}_n(\alpha_0 , \mathbb{B}^\infty_0) \rightarrow \hat{P}_n(\alpha_0 , \mathbb{B}^r_0) = \tilde{P}_n(\mathbb{A}^r_{-1}) , \tag{1.4.1b}$$

$$n = 0, \ldots , r . \tag{1.4.1c}$$

On the l.h.s. of (1.2.15a), only the final equation for α_r itself is altered; in this context, combine $b_{r+1} = 0$ with Eq. (1.3.15a) in which $n = r + 1$.

Henceforth we shall refer to (1.4.1a) as Grad's ansatz. In Refs. 1, 3, and 4 there is no reasoning at all in favour of the *correctness* of the ascertainment that Grad's ansatz must be extracted, for deciding whether it is valid or not, from either Boltzmann's equation or at least the infinite hierarchy of the equations of transfer. Rather, after a very careful analysis of the contents of these memoirs, one should attempt to demonstrate that Grad's ansatz is essentially independent of the kinetic theory itself. Since, however, making use of $b_p = 0$

for $p > r$ automatically yields Grad's r-moment system of field equations, Grad proposes to understand the *compatibility* of his truncation procedure with Boltzmann's equation in the following sense It is expected that the r-moment system will describe adequately the approximate behaviour of the first r' moments a_n of the exact solution f of Boltzmann's equation ($r \geq r'$, r' stated once and for all), the approximation improving as r tends to infinity. [Today we can phrase Grad's proposition more precisely only in the restrictive (but still nontrivial) case of a spatially uniform gas. Indeed, Grad's r-moment system of equations of transfer, each of which is of first order in the time, suggests that in order to obtain a unique solution we should have to prescribe the initial values of $a_n(x,t)$, $0 \leq n \leq r$, where x lies in the part of space we wish to consider. At the same time, in a problem with no space variation the kinetic approach lends itself to formulation of the initial-value condition regarding $f(k,x,t)$ for a precise solution of the Boltzmann-Peierls equation. This *single condition* must in turn lead to a theory of the (uniquely generated) solutions of *all* Grad's systems of equations of transfer, and hence we conclude that his proposition is not empty and may be verified rigorously for spatially uniform gases —— at least in principle.]

Following and slightly generalizing Grad's line of thought, Weinert et al. stated on p. 2267 in Ref. 8 that, insofar as a spatially uniform gas is concerned, "*it is easy to think of several reasonable but different ways of truncating the equations* [of transfer]. *However, if they are consistent and convergent, all truncation schemes should approach the same results in the infinite-order approximation.*" [In other words, the relaxation of the lowest moments $a_n \in \mathbb{A}^{r'}_{-1}$ ($r \geq r'$, $r \to \infty$) should be *insensitive* to our particular truncation procedure, and should be dominated (in all but the exceptional initial-value problems) by the linear relaxation mode(s).]

1.4.B. The Pseudo-Macroscopic Method

1.4.B.a. The Entropy Principle in the Liu-Müller Spirit

Since higher order Tchebychef coefficients are not of interest in themselves but rather are parameters which spring up in the r-moment system as a nuisance and have to be eliminated, we introduce, instead of the "primitive" ansatz (1.4.1a), the pseudo-phenomenological relations of the form

$$b_p = \tilde{b}_p\left(\mathbb{A}^r_{-1};c \right) , \quad p > r . \qquad (1.4.2a)$$

[Of course, to obtain $a_{r+1} = \tilde{a}_{r+1}$, we must use $b_{r+1} = \tilde{b}_{r+1}$ and Eq. (1.3.15a) for $n = r + 1$.] The same view applies to auxiliary constitutive functions for the entropy density h and the entropy flux Φ:(15,22)

$$h = \tilde{h}\left(\mathbb{A}^r_{-1};c \right), \quad \Phi = \tilde{\Phi}\left(\mathbb{A}^r_{-1};c \right) . \qquad (1.4.2b)$$

The simplest possible constitutive assumption, which we adopt here in order to have the hyperbolic rather than parabolic system of differential equations of transfer, results from letting (1.4.2) at one point and time depend upon the values of $\mathbb{A}^r_{-1}$ and c at that point and time.

Due to the fact that, by definition, every thermodynamic process (in extended irreversible thermodynamics of "degree" r) gives rise to $\mathbb{A}^r_{-1}$ and $\mathbb{B}^\infty_r$ such as to satisfy both the r-moment system and (1.4.2a), the entropy principle

$$\partial_t h + \nabla_x \Phi \geq 0 \qquad \left(h = \tilde{h} \ , \ \Phi = \tilde{\Phi} \right) \qquad (1.4.3)$$

can prove nothing about fields which are not so related. But, according to Liu's theorem,(37) the new [with respect to (1.4.3)] inequality

$$\partial_t h + \nabla_x \Phi$$

$$- \sum_{n=0}^{r} \Delta^r_n \left[\partial_t a_n + \nu c \nabla_x a_{n+1} + \nu C^{\nu}_{n+1} W a_{n+1} - P_n \right]$$

$$= \sum_{n=0}^{r} \left\{ \left[\frac{\partial h}{\partial a_n} - \Delta^r_n \right] \partial_t a_n + \left[\frac{\partial \Phi}{\partial a_n} - \nu c \left(\Delta^r_{n-1} + \Delta^r_r \frac{\partial a_{r+1}}{\partial a_n} \right) \right] \nabla_x a_n + \Delta^r_n P_n \right\} + \left[\frac{\partial \Phi}{\partial c} - \nu c \Delta^r_r \frac{\partial a_{r+1}}{\partial c} - \nu \sum_{p=0}^{r+1} C^{\nu}_p \Delta^r_{p-1} a_p \right] \nabla_x c \geq 0 , \quad \Delta^r_{-1} := 0 \qquad (1.4.4)$$

holds for *all*[5] fields $\mathbb{A}^r_{-1}(x,t)$ and $c(x)$. The constraint multipliers Δ^r_n, $0 \leq n \leq r$, just introduced depend *locally* upon $\mathbb{A}^r_{-1}$ and c:

$$\Delta^r_n = \tilde{\Delta}^r_n(\mathbb{A}^r_{-1} ; c), \quad 0 \leq n \leq r ; \qquad (1.4.5)$$

we call Δ^r_n Lagrange multipliers. Since the time-space derivatives $\partial_t a_n$, $\nabla_x a_n$, and $\nabla_x c$ can be chosen *arbitrarily* and *independently* of any other term in (1.4.4), it follows that the quantities in square brackets [•••] must vanish lest the inequality (1.4.4) be violated by some such choices and, in view of these observations, we arrive at

$$\frac{\partial h}{\partial a_n} - \Delta^r_n = 0, \qquad (1.4.6a)$$

$$\frac{\partial \Phi}{\partial a_n} - \nu c \left[\Delta^r_{n-1} + \Delta^r_r \frac{\partial a_{r+1}}{\partial a_n} \right] = 0, \qquad (1.4.6b)$$

$$\frac{\partial \Phi}{\partial c} - \nu c \Delta^r_r \frac{\partial a_{r+1}}{\partial c} - \nu \sum_{p=0}^{r+1} C^{\nu}_p \Delta^r_{p-1} a_p = 0. \qquad (1.4.6c)$$

The entropy inequality (1.4.4) reduces to the residual one:

$$\sum_{n=0}^{r} \Delta^r_n P_n = \sum_{n=1}^{r} \Delta^r_n P_n \geq 0. \qquad (1.4.7)$$

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[5] Here the c(x) is regarded as assignable arbitrarily in principle, while in any particular application it should be specified uniquely as a part of the definition of the problem.

1.4.B.b. Constitutive Relations in the Second-Order Theory

Without the danger of falling into errors, rather than consider $\mathbb{A}^r_{-1}$ we choose to examine α_0 and $\mathbb{B}^r_0$ as independent gas-state variables:

$$E_0 = \tilde{E}_0(\, \mathbb{A}^r_{-1};c\,) = \hat{E}_0(\, \alpha_0,\mathbb{B}^r_0;c\,) \ , \qquad (1.4.8a)$$

$$E_0 = b_p,\ h,\ \Phi,\ \Delta^r_n \ , \qquad (1.4.8b)$$

$$1 \le r < p \ , \quad 0 \le n \le r \ . \qquad (1.4.8c)$$

Indeed, from (1.3.15) we see that $\mathbb{A}^r_0$ and $\mathbb{B}^r_0$ can be expressed as linear combinations of each other. Let us suppose that the quasiparticle system departs only slightly from local equilibrium. Since the nonequilibrium variables $\mathbb{B}^r_0$ all vanish at local equilibrium, it makes sense to expand $\hat{E}_0$ in powers of b_p, $1 \le p \le r$. Dropping those resulting terms which are of higher than second (third) order in the first r Tchebychef moments, we obtain for $\hat{E}_0$

$$b_n = b^{r|n}_0 + \sum_{p=1}^{r} b^{r|n}_p\, b_p + \sum_{p=1}^{r}\ \sum_{s=1}^{r} b^{r|n}_{ps}\, b_p b_s \ , \ r < n \ , \qquad (1.4.9a)$$

$$h = h^r_0 + \sum_{p=1}^{r} h^r_p\, b_p + \sum_{p=1}^{r}\ \sum_{s=1}^{r} h^r_{ps}\, b_p b_s + \sum_{p=1}^{r}\ \sum_{s=1}^{r}\ \sum_{m=1}^{r} h^r_{psm}\, b_p b_s b_m \ , \qquad (1.4.9b)$$

$$\Phi = \Phi^r_0 + \sum_{p=1}^{r} \Phi^r_p\, b_p + \sum_{p=1}^{r}\ \sum_{s=1}^{r} \Phi^r_{ps}\, b_p b_s + \sum_{p=1}^{r}\ \sum_{s=1}^{r}\ \sum_{m=1}^{r} \Phi^r_{psm}\, b_p b_s b_m \ , \qquad (1.4.9c)$$

$$\Delta_n^r = \Delta_o^{r|n} + \sum_{p=1}^{r} \Delta_p^{r|n} b_p + \sum_{p=1}^{r} \sum_{s=1}^{r} \Delta_{ps}^{r|n} b_p b_s \,, \quad 0 \le n \le r, \tag{1.4.9d}$$

where the expansion coefficients $b_o^{r|n}$ through $\Delta_{ps}^{r|n}$ may depend on α_o (or Θ) and c. Of course, the higher order Tchebychef moments $\mathbb{B}_r^\infty$ and the entropy flux Φ vanish also in the state of local equilibrium, and hence we conclude that

$$\Phi_o^r = 0 \,, \quad b_o^{r|n} = 0 \quad (r < n). \tag{1.4.10}$$

Let us for the moment assume that there exists no quasiparticle counterpart to Grad's method and, in particular, that there are no Tchebychef analogues of Hermite coefficients. Then, insofar as the Liu-Müller scheme(15) is concerned, we could equally well have applied in this chapter functionals of f other than b_n, provided that they are linearly and bijectively related to b_n. The crucial postulate is not Eq. (1.4.9a), which bears a considerable resemblance to the role which was assigned to the excessive flux moment α_{r+1} in the work by Liu and Müller;(15) rather, it is the following proposition valid *only* for Tchebychef (Hermite) coefficients:(22)

$$b_p^{r|n} = 0, \quad 1 \le p \le r < n < \infty \,. \tag{1.4.11}$$

This ansatz is motivated by its undeniable *consistency* with that of Grad. When it comes to suggesting $b_p^{r|n} \neq 0$, apparent discrepancies with assumptions equivalent to those on which Grad's method is founded could not be eliminated. Indeed, in Grad's truncation procedure b_n for $n > r$ are considered to be negligible. Thus in Section 1.5.B we seek the coefficients $b_{ps}^{r|n}$ appearing in Eq. (1.4.9a) under the natural assumption (1.4.11), which puts our pseudo-phenomenological approach in harmony with the older one.(1,3) [Within the framework set up here, the higher order Tchebychef coefficients $\mathbb{B}_r^\infty$ are equal to zero in the simplified theory which does not appeal to nonlinear constitutive relations.] In justifying (1.4.9a) and (1.4.11), we reason also as follows: If P_n, $1 \le n \le r$, can be evaluated approximately by both substituting (1.3.6) into (1.2.16) and dropping from the formally exact expansion of (1.2.16) all products $b_p b_s$ in which $p > r$ and/or $s > r$, then it could be unreasonable to expect all terms which are linear in the fields b_n, $n > r$) to be significantly larger than the remaining products $b_p b_s$ $(1 \leq p \leq r,\ 1 \leq s \leq r)$.

Given the pseudo-macroscopic method, the discussion of Section 1.4.A is of general import and may be repeated essentially word for word with only slight changes. By the very nature of any truncation procedure of Grad's type, the kinetic theory cannot deliver (1.4.9a) and (1.4.11) although it may —— and does —— deliver (1.4.9b) and (1.4.9c), as we shall soon see [cf. Section 1.5.A], of course after the constitutive coefficients $b_{ps}^{r|n}$ have been calculated on grounds outside the kinetic argument itself [cf Section 1.5.B].

1.4.B.c. Compatibility Conditions

In extended irreversible thermodynamics of degree n - 1 $(r + 1 \leq n)$, the Tchebychef coefficient b_n can be thought of as being the "first" excessive unknown

$$b_n = \sum_{p=1}^{n-1} \sum_{s=1}^{n-1} b_{ps}^{n-1|n}\, b_p b_s = \sum_{p=1}^{r} \sum_{s=1}^{r} b_{ps}^{n-1|n}\, b_p b_s$$

$$+ \ 2 \underline{\sum_{p=1}^{r} \sum_{s=r+1}^{n-1} b_{ps}^{n-1|n}\, b_p b_s}$$

$$+ \underline{\sum_{p=r+1}^{n-1} \sum_{s=r+1}^{n-1} b_{ps}^{n-1|n}\, b_p b_s}\ , \qquad (1.4.12)$$

where the underlined expressions on the r.h.s. of Eq. (1.4.12) will only be important for r + 1 < n, eventually giving rise to what we have previously called extended irreversible thermodynamics of degree r.[σ] Consistent with this observation, we attempt to arrive at (1.4.9a) supplemented by (1.4.10) and (1.4.11) once more by regarding in the second step b_{n-1} as a constitutive quantity rather than an independent gas-state variable,

$$b_{n-1} = \sum_{p=1}^{n-2} \sum_{s=1}^{n-2} b_{ps}^{n-2|n-1} b_p b_s , \qquad (1.4.13)$$

according to the tenets of extended irreversible thermodynamics of degree n - 2. Obviously, we do this on the understanding that Eq. (1.4.13) —— when substituted into (1.4.12) —— does not contribute to

$$b_n = \sum_{p=1}^{n-2} \sum_{s=1}^{n-2} b_{ps}^{n-2|n} b_p b_s , \qquad r + 1 < n , \qquad (1.4.14)$$

because potentially important terms appearing then in the underlined expressions are of third (fourth) order with respect to $b_p \in \mathbb{B}_o^{n-2}$. In the last n - r step, we write

$$b_{r+1} = \sum_{p=1}^{r} \sum_{s=1}^{r} b_{ps}^{r|r+1} b_p b_s , \qquad (1.4.15)$$

but this equality, due to arguments that in essence are exactly the same as those just formulated, does not contribute to

$$b_n = \sum_{p=1}^{r} \sum_{s=1}^{r} b_{ps}^{r|n} b_p b_s , \qquad r + 1 < n \qquad (1.4.16)$$

as well. Finally, comparing (1.4.16) with the first expression on the r.h.s. of (1.4.12), we obtain the important compatibility conditions

$$b_{ps}^{r|n} = b_{ps}^{n-1|n} , \quad 1 \le p \le r, \quad 1 \le s \le r, \quad r < n \qquad (1.4.17)$$

between extended irreversible thermodynamics of degree r and

[σ] In obtaining the first underlined expression, we assume with no loss of generality that $b_{ps}^{n-1|n} = b_{sp}^{n-1|n}$.

that of degree n - 1.

1.5. Evaluation of Constitutive Coefficients

1.5.A. Solutions for h *and* Φ

The Tchebychef coefficients b_p in (1.3.6) are functions of position x and time t, i.e., fields in the ordinary sense of continuum mechanics. In order for the entropy density h and the entropy flux Φ, which are defined in the kinetic theory, *practically with no loss of generality*, by (1.2.20), (1.2.22), and (1.3.6), to satisfy the extra requirements (1.4.2b), it is sufficient that higher order Tchebychef moments be uniquely determined from $\mathbb{A}^r_{-1}$ and c through the constitutive relations (1.4.2a). Since we know something about $\mathbb{B}^\infty_r$, let us take (1.4.9a) to eliminate $b_p \in \mathbb{B}^\infty_r$ from the expression (1.3.20c) for Θ. Then, recalling the definition (1.3.20a) and putting the formula (1.3.21a) into (1.3.19), we obtain for h and Φ:

Entropy

$$h : \quad h^r_o = \frac{\nu k_B}{2\pi(\nu - 1)\alpha^2_o}\,\Theta^{-1/\nu}, \qquad h^r_p = 0, \tag{1.5.1a}$$

$$h^r_{ps} = -\frac{k_B}{4\pi(\nu - 1)}\,\Theta^{-1/\nu}\,\delta^s_p, \tag{1.5.1b}$$

$$h^r_{psm} = \frac{k_B}{12\pi(\nu - 1)}\,\Theta^{-1/\nu}\left\{FB_pB_sB_m\right\}_o; \tag{1.5.1c}$$

Entropy Flux

$$\Phi : \quad \Phi^r_p = \frac{\nu k_B c}{2\pi(\nu - 1)\alpha_1}\,\Theta^{-1}\,\delta^p_1, \tag{1.5.2a}$$

$$\Phi^r_{ps} = -\frac{\nu k_B c}{4\pi(\nu - 1)}\,\Theta^{-1}\left(\frac{\tau_p}{\tau_{p+1}}\,\delta^{p+1}_s + \frac{\tau_{p-1}}{\tau_p}\,\delta^{p-1}_s\right], \tag{1.5.2b}$$

$$\Phi^{r}_{psm} = - \frac{\nu k_{B} c}{2\pi(\nu - 1)} \Theta^{-1} \frac{\tau_{r}}{\tau_{r+1}} \delta^{r}_{(p} b^{r|r+1}_{sm)}$$

$$+ \frac{\nu k_{B} c}{12\pi(\nu - 1)} \Theta^{-1} \left\{ F H_{p} B_{s} B_{m} \right\}_{o} , \qquad (1.5.2c)$$

where

$$H_{p} := \frac{\tau_{p-1}}{\tau_{p}} B_{p-1} + \frac{\tau_{p}}{\tau_{p+1}} B_{p+1} \qquad (1.5.2d)$$

and δ^{s}_{p} is, as we know, the Kronecker delta. The round brackets around a set of n subscripts indicate the sum over the n! permutations of the indices, divided by n!. Concerning the precise definitions of F and $\{\psi\}_{o}$, see (1.3.21b) and (1.3.1d), respectively.

1.5.B. Lagrange Multipliers and Basic Constitutive Functions

In the n = r + 1 case, Eq. (1.3.15a) reduces to

$$a_{r+1} = - \frac{1}{x^{r+1}_{r+1}} \sum_{p=0}^{r} x^{r+1}_{p} a_{p}$$

$$+ \frac{1}{x^{r+1}_{r+1}} \sum_{p=0}^{[(r+1)/2]} D^{\nu}_{p} X^{r+1}_{2p} + \frac{1}{x^{r+1}_{r+1}} b_{r+1} . \qquad (1.5.3)$$

Hence, combining (1.5.3) with (1.4.9a) for n = r + 1 and transforming [with the help of (1.3.15a)] $\mathbb{B}^{r}_{o}$ into $\mathbb{A}^{r}_{o}$ $(\mathbb{B}^{r}_{o} \to \mathbb{A}^{r}_{o})$, we arrive at the precise specification of $a_{r+1} = \tilde{a}_{r+1}(\mathbb{A}^{r}_{-1};c)$ within the framework of the second-order constitutive theory. The same view applies to (1.4.9b)-(1.4.9d). We insert (1.5.3) and (1.4.9b)-(1.4.9d) into (1.4.6) and drop there, of course after replacing $\mathbb{A}^{r}_{o}$ by $\mathbb{B}^{r}_{o}$ $(\mathbb{A}^{r}_{o} \to \mathbb{B}^{r}_{o})$, all terms of higher than second order in $b_{p} \in \mathbb{B}^{r}_{o}$. [Since (1.4.9a) and (1.4.9d) are all only given here to within second-order terms inclusive, higher order terms would not be reliable.] In the result we obtain certain polynomials in $b_p \in \mathbb{B}^r_o$ of degree 2, and we conclude from their vanishing that the coefficients of the constitutive representations for Δ^r_n, $0 \le n \le r$, and b_{r+1} must obey the following equalities:

Lagrange Multipliers

$$\Delta^r_n : \quad \Delta^{r|o}_o = \frac{k_B}{2\pi}\,\Theta\ , \quad \Delta^{r|n}_o = 0, \quad 1 \le n \le r\ , \tag{1.5.4a}$$

$$\Delta^{r|n}_p = -\frac{k_B}{2\pi}\,\Theta C^{\nu}_n - \nu^{-1}\sum_{q=n}^{r} X^{a}_n\,\delta_{qp}\ , \tag{1.5.4b}$$

$$\Delta^{r|n}_{ps} = \frac{k_B}{4\pi}\,\Theta C^{\nu}_n - \nu^{-1}\sum_{q=n}^{r} X^{a}_n \left\{FB_qB_pB_s\right\}_o\ ; \tag{1.5.4c}$$

The Excessive Flux Moment $(a_{r+1} \leftrightarrow b_{r+1})$

$$b_{r+1} : \quad b^{r|r+1}_{ps} = \frac{1}{2}\left\{FB_{r+1}B_pB_s\right\}_o\ , \tag{1.5.5}$$

where δ_{qp} is the Kronecker symbol.

If we replace r by n - 1 in (1.5.5), then (1.4.17) gives us for the

Higher Order Tchebychef Moments $(b_n \in \mathbb{B}^{\infty}_r)$

$$b_n : \quad b^{r|n}_{ps} = \frac{1}{2}\left\{FB_nB_pB_s\right\}_o\ . \tag{1.5.6}$$

Analyzing the results (1.5.1)-(1.5.6), two final remarks are of interest. First of all, when it is tacitly supposed that the general nonlinear representations (1.4.8a) exist, at least for certain definite choices of the physically relevant values of $b_n \in \mathbb{B}^r_o$, and are exact solutions of the Liu-Müller equations (1.4.6), the *Taylor replacements* (1.4.9) of these practically unknown constitutive representations satisfy (1.4.6) only approximately, i.e., up to terms which are not of higher than second order in the first r Tchebychef moments. Secondly, insofar as the existence of the Liu-Müller theory can follow from the variational approach based upon the entropy maximum principle,(20,21,26,39,66) in the neighbourhood of the state of local equilibrium the arguments of (1.4.8a) should probably be restricted to those satisfying not only the obvious conditions $|b_n| \ll 1$, $1 \le n \le r$, but also, curiously enough, the inequality $b_r \le 0$, where $r = 2,4,6,\ldots$; for more details of general import, see Ref. 68 and Section 2.3 of Chapter 2. Thus, while the functions on the r.h.s. of (1.4.9) make sense for arbitrary $\mathbb{B}^r_o$, in particular, for $|b_n| \ll 1$, $1 \le n \le r$, and $b_r > 0$, the variational theory gives no status to that sense. Although Eqs. (1.4.9) interpreted as applying to $|b_n| \ll 1$, $1 \le n \le r$ $(r = 1,2,3,\ldots)$ are, strictly speaking, unjustifiable extensions of the statements in the variational theory, we may —— and should —— regard (1.4.9a) as an independent truncation postulate, which, if supplemented by (1.4.10) and (1.4.11), is compatible with Grad's ansatz and is also motivated by our pseudo-macroscopic method [of precise sense only in the severely restrictive, artificial domain of its mathematical validity]. Of course, these observations and the serious difficulties faced in interpreting or even accepting the entropy maximum principle in extended irreversible thermodynamics reinforce rather than diminish the potential importance of (1.4.9a) for $|b_n| \ll 1$, $1 \le n \le r$ $(r = 1,2,3,\ldots)$. Indeed, that importance, as displayed by the discussion of Sections 1.4.A and 1.4.B.b, rests primarily on the informal ideas of Grad's type altogether different from those Dreyer(39) and Banach(20) adduced to suggest their starting point, namely, the entropy maximum principle.

We summarize this discussion as follows. Several fundamental reasons, some of them reasons of expediency and efficiency and one of them a reason based upon the desired but still unfounded reliable interpretation of the entropy maximum principle in extended irreversible thermodynamics, show why no reference to both Dreyer's variational approach and artificial conditions that Eqs. (1.4.8a) exist and satisfy (1.4.6) exactly is very useful in directly generalizing Grad's moment truncation procedure via Banach's pseudo-macroscopic method. It is only if we *necessarily* wish to harmonize the *independent* pseudo-macroscopic method(22) with that of Dreyer(30) that the existence of (1.4.8a), or of (1.4.2), becomes a real, almost unsolvable problem of the Liu-Müller framework.

1.6. The "Distribution Functions" Determined by Finitely Many Moments

In regard to the task of obtaining a completely determined system of field equations for $\mathbb{A}^r_{-1}$, Grad reasons that if f is represented ("approximated" fairly closely) by

$$f = f_o\left[1 + \frac{\omega\, e^\omega}{e^\omega - 1} \sum_{n=o}^{r} b_n B_n\right], \qquad (1.6.1)$$

then it is reasonable to expect the higher order Tchebychef coefficients $\mathbb{B}^\infty_r$ to be significantly smaller than those in the list $\mathbb{B}^r_o$. [That Grad treats only a three-dimensional, classical, moderately rarefied, simple, monatomic gas and makes use of Hermite polynomials and Hermite coefficients, is inessential in the present connection.] The constitutive postulates (1.4.9a) for b_n, $n > r$ $\left(b^{r|n}_o = 0,\ b^{r|n}_p = 0\right)$ are both consistent with Grad's universal method of truncating a series representation of the distribution function and, in the course of linking the proposition (1.6.1), *which suggests itself*, with its natural generalization, demonstrably effective —— at least formally.

Indeed, if we replace $b_n \in \mathbb{B}^\infty_r$ in (1.3.6) by their values in terms of $b_p \in \mathbb{B}^r_o$, resulting from taking (1.4.9a), (1.4.10), (1.4.11), and (1.5.6), we may write the distribution function in the more explicit form as follows:

$$f = f_o\left[1 + \frac{\omega\, e^\omega}{e^\omega - 1}\left(\sum_{n=o}^{r} b_n B_n\right.\right.$$

$$+ \underline{\frac{1}{2} \sum_{p=r+1}^{\infty} \sum_{n=0}^{r} \sum_{m=0}^{r} \left\{FB_nB_mB_p\right\}_o B_p b_n b_m} \Big]\Big] . \qquad (1.6.2)$$

A *considerable simplification*[7] of the underlined expression in (1.6.2) can easily be achieved by observing that the series

$$\sum_{p=0}^{\infty} \left\{FB_nB_mB_p\right\}_o B_p$$

converges in the *mean* [cf. Section 1.3.A] to the function FB_nB_m. Hence

$$f = f_o \left[1 + \frac{\omega\ e^{\omega}}{e^{\omega} - 1} \left(\sum_{n=0}^{r} b_n^* B_n + \frac{1}{2} F \sum_{n=0}^{r} \sum_{m=0}^{r} b_n b_m B_n B_m \right]\right] , \qquad (1.6.3a)$$

where

$$b_n^* := b_n - \frac{1}{2} \sum_{p=0}^{r} \sum_{s=0}^{r} \left\{FB_nB_pB_s\right\}_o b_p b_s . \qquad (1.6.3b)$$

In terms of continuum thermodynamics, as described, for instance, in Chapter I of Ref. 4, it is possible to show that the r-moment system of field equations generated by (1.6.1) or (1.6.3) represents a continuum theory of fluids of the rate type satisfying the axiom of entropy growth (*in the sense made precise by a second-order perturbation theory*), as it should be. According to the original Truesdell-Muncaster proposition,(4) a sequence of such theories amounts in turn to *formally approximating (replacing)* the kinetic equation with a suitable H-theorem.

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[7] As Eqs. (1.6.3) explicitly demonstrate, we are faced with a solvable problem of calculating the first r Tchebychef polynomials and the elements $\{FB_nB_pB_s\}_o$ generated by them.

1.7. The Liu-Müller Formula for the Entropy Production σ_h

Our truncation procedure gives rise to some important features not available in other approaches, especially when it comes to explaining the origin of the Liu-Müller formula⁽¹⁵⁾

$$\sigma_h = \sum_{n=0}^{r} \Delta_n^r \, P_n \tag{1.7.1}$$

for the entropy production σ_h; in this context, see also the entropy inequality (1.4.7).

To derive (1.7.1) directly from (1.2.23), let us first observe that if we adopt (1.3.6), (1.3.20c), $ln(1 + X) = X - \frac{1}{2} X^2 + \ldots$ ($|X| < 1$), (1.3.21b), (1.4.9a), (1.4.10)-(1.4.11), (1.5.6), (1.3.13), (1.3.1a)-(1.3.1b), (1.5.4), and, finally, (1.4.9d), then we are justified in doing the following sequence of transformations:

$$ln\left(\frac{1 + f}{f}\right) = \omega + ln\left[1 + \frac{\omega}{e^{\omega} - 1}\,\theta\right] - ln\left[1 + \frac{\omega\, e^{\omega}}{e^{\omega} - 1}\,\theta\right]$$

$$\rightarrow \omega - \omega\,\theta + \frac{1}{2}\,\omega\,F\,\theta^2 \rightarrow \omega - \omega \sum_{q=0}^{r} b_q\, B_q$$

$$+ \frac{1}{2}\,\omega \sum_{q=0}^{r} \sum_{p=1}^{r} \sum_{s=1}^{r} \left\{F B_q B_p B_s\right\}_o \, b_p b_s B_q$$

$$= \omega - \omega \sum_{n=0}^{r} z^n \sum_{p=1}^{r} b_p \sum_{q=n}^{r} X_n^q\, \delta_{qp}$$

$$+ \frac{1}{2}\,\omega \sum_{n=0}^{r} z^n \sum_{p=1}^{r} \sum_{s=1}^{r} b_p b_s \sum_{q=n}^{r} X_n^q \left\{F B_q B_p B_s\right\}_o$$

$$= \frac{2\pi}{k_B} \sum_{n=0}^{r} |k|^{\nu} \hbar^n \left[\Delta_o^{r|n} + \sum_{p=1}^{r} \Delta_p^{r|n}\, b_p + \sum_{p=1}^{r} \sum_{s=1}^{r} \Delta_{ps}^{r|n}\, b_p b_s\right]. \tag{1.7.2}$$

Hence, with the aid of (1.2.23), (1.4.9d), and the definition (1.2.15d), we arrive at (1.7.1). Were this not the case, the derived moment representation (1.6.3) would not be correct; for more details of general import on the Liu-Müller residual inequality, see Section 2.2.B in the next chapter.

1.8. Transition to Ordinary Quasiparticle Hydrodynamics

If we recall (1.4.1a) and (1.6.1) for r = 1 and make use of (1.2.15), (1.2.16), (1.3.15b) [in which n = 2 $\{ b_o = 0, y_1^2 = 0 \}$], and (1.3.17a), then Grad's 2-moment system of field equations for Θ and a_1 $\{ \Theta \sim 1/T, a_1 \sim q \}$ can be shown to take the following form:

$$\partial_t a_o + \nu c \nabla_x a_1 + 2\nu W a_1 = 0, \tag{1.8.1a}$$

$$\partial_t a_1 + \nu c \nabla_x a_2 + (3\nu - 1) W a_2 = - \lambda_o a_1, \tag{1.8.1b}$$

where

$$a_o = \frac{\nu}{(\nu^2 - 1)\alpha_o^2} \Theta^{-(\nu + 1)\nu^{-1}}, \tag{1.8.1c}$$

$$a_2 = \frac{\nu}{(\nu - 1)(3\nu - 1)\alpha_1^2} \Theta^{-(3\nu - 1)\nu^{-1}}, \tag{1.8.1d}$$

and

$$\lambda_o := \frac{1}{2} \alpha_1^2 (\nu - 1) \Theta^{(2\nu - 1)\nu^{-1}}$$
$$\times \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} dk dk_1 dk_2 \, \mathscr{R}(k_1;k,k_2) \, f_o f_{o2}(1 + f_{o1})$$
$$\times \left(\omega_1 k_1 - \omega k - \omega_2 k_2\right)^2 \delta(\Omega_1 - \Omega - \Omega_2) > 0, \tag{1.8.1e}$$

$$f_{om} := f_o(\Omega_m), \quad \omega_m := \frac{\hbar \Omega_m}{k_B T}, \quad m = 1,2. \tag{1.8.1f}$$

The balance equation (1.8.1a) remains valid in ordinary quasiparticle hydrodynamics, but (1.8.1b) formulates a constitutive relation for $a_1 \sim q$ of Fourier's type. To see this, let us consider a flow problem in which the *mean* (k-independent) relaxation time $\tau_R := 1/\lambda_o$ is short compared to the time τ_h required for any appreciable macroscopic changes of Θ [then, since $v := \nu c\Theta^{-(\nu - 1)\nu^{-1}}$ is the representative group velocity, $\ell_h := v\tau_h$ denotes some typical macroscopic length characterizing the spatial variation of Θ]. Write Eq. (1.8.1b) in the form

$$\partial_t a_1 + \mathbb{A}_1 + \lambda_o a_1 = 0, \tag{1.8.2}$$

where

$$\mathbb{A}_1 := \nu c \, \nabla_x a_2 + \underline{(3\nu - 1)\; W\, a_2} \tag{1.8.3}$$

is, by $\tau_R \ll \tau_h$, almost constant in a time comparable to τ_R. Over such a short period of time, we obtain, except for a very small error, the solution of the differential equation (1.8.2) as consisting of a transient plus a steady state, i.e.,

$$a_1(t) = \mathbb{G}\, exp\left(-\lambda_o t\right) - \frac{1}{\lambda_o}\, \mathbb{A}_1, \tag{1.8.4}$$

$\mathbb{G}$ being a certain quasi-constant. But the important scale of $\bar{t} := t/\tau_R$ is of order unity, and, no matter what value a_1 has initially, within a short time $t \cong \tau_R$ the a_1 approaches its quasi-equilibrium value. Thus (1.8.4) becomes

$$a_1 = -\frac{1}{\lambda_o}\, \mathbb{A}_1 \;. \tag{1.8.5}$$

Now, if the variation of c over ℓ_h is very small [$c^{-1}\ell_h|W| = c^{-1}\ell_h|\nabla_x c| \ll 1$] —— as is almost always the case —— then we can neglect the underlined expression in (1.8.3). In the result we arrive at

$$q = \frac{\nu\hbar c^2}{2\pi}\, a_1 = -\,\varkappa^{(1)}\, \nabla_x T \;, \tag{1.8.6}$$

where the value

$$\varkappa^{(1)} = \frac{\nu^2 c^2 k_B}{2\pi a_1^2 (\nu - 1)\lambda_o}\; \Theta^{-(2\nu - 1)\nu^{-1}} \tag{1.8.7}$$

of the *thermal conductivity* $\varkappa$ given here is what Chapman and

Cowling(36) could call the first approximation to its true value; in this context, see our comments regarding the Fredholm integral equation at the end of Section 1.3.C. [The reasoning of Gurevich on pp. 124-168 in Ref. 2 to justify his extensive use of the Chapman-Enskog method of solving the Boltzmann-Peierls equation does not refer to the spectral properties(31,32) of the linearized Boltzmann-Peierls collision operator but is much like that Chapman and Cowling have described on p. 117 in Ref. 36: the fact that the solution of the kinetic equation which is obtained by Enskog's method depends on no parameters other than the value of θ throughout the gas appears to be in accordance with *experiment*; see also Beck et al., pp. 354, 367-368 in Ref. 29 and pp. 40 and 58 in Ref. 10. In addition, it is possible to show that the nonexistence of a gap in the eigenvalue spectrum of the linearized Boltzmann-Peierls collision operator(31,32) does not culminate in real mathematical difficulties (divergencies) concerning the precise calculation of a variety of transport coefficients through the Chapman-Enskog scheme. In other words, this procedure is *effective* in the sense that in all but the exceptional cases the first iterate to which it primarily refers actually exists.]

Since in general τ_R is not necessarily very small compared to a representative time τ_h, one would expect (1.8.1) to demonstrate, at least to some extent, the effect of an overlap between the two time scales, viz. the *mean* collision time τ_R and the macroscopic decay time τ_h. Also, Grad's 2-moment system does not predict that any perturbation propagates instantaneously in space, as is always the case if one works with (1.8.6). In summary, if the condition $\tau_R \ll \tau_h$ is not rigorously fulfilled, then Grad's framework differs considerably from Fourier's description fashioned by (1.8.6) and should give better results.

1.9. Conclusions and Final Remarks

As we have already remarked, it is purely as a matter of convenience and of simplicity of exposition that we have chosen temporarily to restrict our theory to a strongly simplified model system: a one-dimensional gas consisting of quasiparticles, each characterized by $\Omega \sim |k|^{\nu}$, $1 < \nu \leq 2$. Indeed, part of the usefulness of the formalism is that, insofar as the exact equations of transfer and the Tchebychef representation of f are concerned, it can be extended to hold not only for the dispersion relations $\Omega(k,x)$ other than those given here but also for three-dimensional (quasiparticle) gases. However, despite the considerable number of practically important generalizations as described in Refs. 25 and 26, the present theory remains useful in establishing the *structural features* of the truncation scheme of Grad's type.

While some readers take the statement just mentioned as self-evident, others might reject it as insufficient and raise the following objection: "Even if one accepts the ansatz (1.4.9a), the range of validity of the resulting theory is rather obscure. It is clearly not impossible to find that it has only trivial applications." This is almost true, but not quite so. First, in Section 1.3.C there are serious indications that our scheme has also interesting potentialities of practical application not yet exploited, e.g., the usefulness of Tchebychef polynomials in calculating the transport coefficients, equilibrium fluctuations, etc. Secondly, the method of the transition to ordinary quasiparticle hydrodynamics, described here very sketchily, has been slightly modified in practice *and yieldied valuable information concerning the nonlinear constitutive regime not otherwise attainable*; for more details, cf. Ref. 26, Section VI B.

Leaving aside these observations, we have tried to demonstrate that our theory does not differ considerably from Grad's approach(1,3) and that it has in fact the same range of potential validity.

CHAPTER 2

COMMENTS ON THE MOMENT TRUNCATION PROCEDURES FOR CLASSICAL GASES DESCRIBED BY MA'S KINETIC EQUATION

In this chapter we restrict our attention to one-dimensional, classical gases described by Ma's kinetic equation for a one-particle density f.(38) All problems concerning the moment truncation procedure of Grad's type can be formulated in such a step-by-step analogy to those of Chapter 1 that in Section 2.1 we confine ourselves to giving the final answer with a few explanatory remarks. The crucial difference between the present and the previous points of view comes to sharp focus in the discussion of Section 2.2 on a gas of "one-dimensional Maxwellian molecules". By no more than a straightforward reasoning, in this section we deliver the collision integrals P_n, $n \geq 3$, as exhibited functions of Hermite moments of f, and then, using the auxiliary formulae for Lagrange multipliers Δ^r_n, $0 \leq n \leq r$, we test the Liu-Müller residual inequality. With regard to the recent publications by Dreyer(39) and Banach(20), who have utilized the entropy maximum principle to obtain a definite system of field equations for the moments $\alpha_n \in \mathbb{A}^r_{-1}$, $r \geq 3$, of the distribution function,[8] Section 2.3 considers some difficulties(68) faced in the interpretation of this principle.

2.1. The Pseudo-Macroscopic Method of Truncation

2.1.A. Ma's Collision Operator

Let us consider a one-dimensional, classical gas for which each "molecule" of unit mass is subject to an external body force K; K may be a function of position x and time t, but not

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[8] Strictly speaking, we think about a closed system of equations of transfer for the macroscopic velocity field u and the direct, internal moments $\alpha_n \in \mathbb{A}^r_{-1}$ $(\alpha_1 = 0,\ r \geq 3)$ of the distribution function.

of momentum λ. We start with a kinetic description of the gas by an equation of change

$$\partial_t f + \lambda \circ \nabla_x f + K \circ \nabla_\lambda f = J(f) \qquad (2.1.1)$$

for the one-point distribution function $f(\lambda,x,t)$, where J is Ma's collision operator(38,22,68) defined as follows:

$$J(f) := \int\int\int\int\int d\lambda' d\lambda'' d\omega\, d\omega' d\omega''\ \mathcal{R}(\lambda,\lambda',\lambda''|\omega,\omega',\omega'')$$

$$\times \left[f(\omega)f(\omega')f(\omega'') - f(\lambda)f(\lambda')f(\lambda'')\right], \qquad (2.1.2a)$$

$$\mathcal{R}(\lambda,\lambda',\lambda''|\omega,\omega',\omega'') = \hat{C}_o\, \delta(\lambda + \lambda' + \lambda'' - \omega - \omega' - \omega'')$$

$$\times\, \delta\left[\frac{1}{2}\left(\lambda^2 + \lambda'^2 + \lambda''^2\right) - \frac{1}{2}\left(\omega^2 + \omega'^2 + \omega''^2\right)\right]. \qquad (2.1.2b)$$

Again, as before,(38,22,68) and in this case we assume that, wherever the distribution function in (2.1.2a) occurs, it must be evaluated at the same space-time point (x,t). Here $\mathcal{R}$ and $\hat{C}_o$, $\hat{C}_o > 0$, are the transition rates of the collision process $(\lambda,\lambda',\lambda'') \leftrightarrow (\omega,\omega',\omega'')$, which, as (2.1.2b) explicitly indicates, satisfies the momentum-energy conservation law. Generally, the $\hat{C}_o$ depends on λ, λ', λ'', ω, ω', and ω''. The condition of indistinguishability of the "particles" implies that the factor $\hat{C}_o$ in (2.1.2b) remains unaltered on all possible permutations of $(\lambda,\lambda',\lambda'')$ and $(\omega,\omega',\omega'')$, respectively, and, in addition, on interchange $(\lambda,\lambda',\lambda'') \leftrightarrow (\omega,\omega',\omega'')$.

In a fictitious three-dimensional space of vectors $(\lambda,\lambda',\lambda'')$, the principle of conservation of momentum

$$\lambda + \lambda' + \lambda'' = \mathcal{P} \qquad (2.1.3a)$$

determines a plane and the principle of conservation of energy

$$\lambda^2 + \lambda'^2 + \lambda''^2 = 2\mathcal{E} \qquad (2.1.3b)$$

defines a spherical surface. The plane and the spherical surface have an intersection, which is a circle of radius

$$\chi = \left(2\mathcal{E} - \mathcal{P}^2/3\right)^{1/2}. \qquad (2.1.4)$$

The $\mathcal{P}/\sqrt{3}$ is the distance from the plane to the origin and χ can be interpreted as the energy in the center-of-mass frame. The momentum-energy conservation law demands that $(\lambda,\lambda',\lambda'')$ and $(\omega,\omega',\omega'')$ be on the same circle characterized by (2.1.4).

In other words, the two vectors $(\lambda,\lambda',\lambda'')$ and $(\omega,\omega',\omega'')$ will uniquely be specified by $(\mathcal{P},\chi)$ and the angles φ and ψ, respectively; φ and ψ are measured along the circle. Now, draw a line from the center of the circle to the intersection of the plane and the λ axis. Following Ma,(38) we choose this line as $\varphi = 0$ $(\psi = 0)$. Then, a little algebra gives

$$\lambda = \mathcal{P}/3 + \chi \left(2/3\right)^{1/2} \cos(\varphi) \ , \qquad (2.1.5a)$$

$$\lambda' = \mathcal{P}/3 - \chi \left(1/\sqrt{6}\right) \cos(\varphi) - \chi \left(1/\sqrt{2}\right) \sin(\varphi) \ , \qquad (2.1.5b)$$

$$\lambda'' = \mathcal{P}/3 - \chi \left(1/\sqrt{6}\right) \cos(\varphi) + \chi \left(1/\sqrt{2}\right) \sin(\varphi) \ . \qquad (2.1.5c)$$

Replacing φ by ψ in (2.1.5), we immediately arrive at the transformation rule for $(\omega,\omega',\omega'')$.

Given Ma's collision operator J, elementary inspection shows that

$$\int d\lambda\ J(f) \ln\left(\mathbb{C}f\right) \le 0 \ , \qquad (2.1.6a)$$

$$\int d\lambda\ \mathcal{N}(\lambda) J(f) = 0 \quad \text{for} \quad \mathcal{N}(\lambda) = 1,\ \lambda,\ \lambda^2 \ , \qquad (2.1.6b)$$

$$J(f) = 0 \ \leftrightarrow\ f = f_o \ , \qquad (2.1.6c)$$

where $\mathbb{C}$ is a certain constant of no greater importance in our further considerations and f_o is a locally Maxwellian distribution. Obviously, Eqs. (2.1.6) lead to the familiar local entropy inequality

$$\partial_t h + \nabla_x \left(h\, u + \Phi \right) = \sigma_h \ge 0 \qquad (2.1.7)$$

with the standard kinetic expressions

$$h := - k_B \int d\lambda\ f \ln\left(\mathbb{C}f\right) \ , \qquad (2.1.8a)$$

$$\Phi := - k_B \int d\lambda\ \bar{\lambda} \ln\left(\mathbb{C}f\right) \ , \qquad \bar{\lambda} := \lambda - u \ , \qquad (2.1.8b)$$

$$\sigma_h := - k_B \int d\lambda\ J(f) \ln\left(\mathbb{C}f\right) \ , \qquad (2.1.8c)$$

$$u := \int d\lambda\ \lambda f \Big/ \int d\lambda\ f \qquad (2.1.8d)$$

for the entropy density h, the entropy flux Φ, the entropy production σ_h, and the macroscopic velocity u.

2.1.B. Equations of Transfer

The best of several starting points in forming the various mesoscopic theories framed by the kinetic equation (2.1.1) is that initiated by defining the direct internal moments of the distribution function:

$$a_n(x,t) := \int d\lambda\, \bar{\lambda}^n f(\lambda,x,t), \quad n \in \mathbb{N}_0. \tag{2.1.9}$$

On the basis of (2.1.1) and (2.1.9), we easily arrive at the system

$$\partial_t a_n + \nabla_x \left(a_n u + a_{n+1} \right) + n a_n \nabla_x u - \frac{n a_{n-1}}{a_0} \nabla_x a_2 = P_n(f), \tag{2.1.10a}$$

$$\partial_t u + u \nabla_x u + \frac{1}{a_0} \nabla_x a_2 - K = 0, \tag{2.1.10b}$$

$$a_{-1} := 0, \quad a_1 = 0, \quad 0 \le n \le r, \quad 3 \le r \tag{2.1.10c}$$

of balance equations for $a_n \in \mathbb{A}^r_{-1}$ and u, where the quantities

$$P_n(f) := \int d\lambda\, \bar{\lambda}^n J(f) \tag{2.1.10d}$$

(depending on x and t through the distribution function) represent collision integrals in the nonconserved moment equations; due to (2.1.6b), the necessary conditions

$$P_n = 0 \quad \text{if} \quad n = 0,1,2 \tag{2.1.11}$$

are automatically fulfilled.

A guide for eliminating f from the r.h.s. of (2.1.10a) and allowing a treatment in the finite hierarchy (of the first r equations of transfer) of the fields that depend on space and time only, as traditionally in continuum physics, was first sketched by Grad.(1,2) His method, which we adopt in this chapter, is based from the outset upon an expansion

$$f(\lambda,x,t) = f_0(\lambda,x,t) \left[1 + \sum_{n=3}^{\infty} \frac{1}{n!} b_n(x,t)\, B_n(\hat{\lambda}) \right] \tag{2.1.12}$$

of the distribution function around a local Maxwellian

$$f_o := (2\pi)^{-1/2}\, \theta\, a_o \exp\left(- \frac{1}{2} \hat{\lambda}^2\right) , \qquad (2.1.13a)$$

$$\theta := (a_o/a_2)^{1/2} , \quad \hat{\lambda} := \theta\, \bar{\lambda} \qquad (2.1.13b)$$

in terms of the orthonormal Hermite polynomials(34,35)

$$B_n(\lambda) := (\lambda - \nabla_\lambda)^n \cdot 1 \qquad (2.1.14)$$

that, according to Eq. (2.1.12), are multiplied by the so-called Hermite expansion coefficients

$$b_n = \frac{1}{a_o} \int d\lambda\, f(\lambda,x,t) B_n(\hat{\lambda}) . \qquad (2.1.15)$$

We now consider the two standard equalities in the theory of one-dimensional Hermite polynomials, namely,

$$B_n(\lambda) = \sum_{k=0}^{n} X^n_k \lambda^k , \quad \lambda^n = \sum_{k=0}^{n} Y^n_k B_k(\lambda) , \qquad (2.1.16a)$$

$$X^n_k,\ Y^n_k = \begin{cases} 0,\ 0 & \text{for } n - k \in 2\mathbb{N}_o + 1 \\ \dfrac{(-1)^{(n-k)/2} n!}{k!(n-k)!!} ,\ \dfrac{n!}{k!(n-k)!!} & \text{for } n - k \in 2\mathbb{N}_o \end{cases} , \qquad (2.1.16b)$$

where

$$(2n)!! := 2 \cdot 4 \cdot \ldots \cdot 2n , \qquad (2.1.17a)$$

$$2\mathbb{N}_o := \{0,2,\ldots\} , \quad 2\mathbb{N}_o + 1 := \{1,3,\ldots\} . \qquad (2.1.17b)$$

Substituting (2.1.16a) in (2.1.15) and (2.1.9), one obtains for $b_n \in \mathbb{B}^\infty_{-1}$ and $a_n \in \mathbb{A}^\infty_{-1}$ ($b_o = 1,\ b_1 = b_2 = 0$)

$$b_n = \sum_{k=0}^{n} x^n_k a_k , \quad a_n = \sum_{k=0}^{n} y^n_k b_k , \qquad (2.1.18)$$

where

$$x^n_k := \frac{\theta^k}{a_o} X^n_k , \quad y^n_k := \frac{a_o}{\theta^n} Y^n_k . \qquad (2.1.19)$$

2.1.C. Basic Results According to the Pseudo-Macroscopic Method

In neither case can the system (2.1.10) of balance equations serve as field equations for the internal state variables $\mathbb{A}^r_{-1}$ ($r \geq 3$) and u, because excessive quantities have appeared, namely, the flux moment a_{r+1} in the *rth* hierarchy equation and, as the r.h.s. of (2.1.10a) and the transformation rules (2.1.12) and (2.1.18) indicate, the higher order Hermite coefficients $b_m \in \mathbb{B}^\infty_r$. Consequently, these unknowns must be considered as constitutive quantities:

$$a_{r+1} = \tilde{a}_{r+1}(\mathbb{A}^r_{-1}) , \quad b_m = \tilde{b}_m(\mathbb{A}^r_{-1}) , \quad m > r. \qquad (2.1.20)$$

The local constitutive assumption that we adopt in (2.1.20) implies that a_{r+1} and $b_m \in \mathbb{B}^\infty_r$ depend at one point and time upon the values of $\mathbb{A}^r_{-1}$ at that point and time, generally in a nonlinear way. In passing, observe that u drops out of (2.1.20), because of the local constitutive assumption and the principle of material frame indifference.(40,62) If, however, we have additional information that the first n Hermite coefficients b_m can be transformed into the first n direct moments a_m by a nonsingular linear transformation (2.1.18), the specific choice $\mathbb{A}^r_{-1}$ of the internal state variables loses its absolute importance, since then dependences of a_{r+1} and $b_m \in \mathbb{B}^\infty_r$ on a_0, a_2, and $b_m \in \mathbb{B}^r_2$ will interest us as well:

$$a_{r+1} = \hat{a}_{r+1}(a_0 , a_2 , \mathbb{B}^r_2), \quad b_m = \hat{b}_m(a_0 , a_2 , \mathbb{B}^r_2), \quad m > r. \qquad (2.1.21)$$

It is, of course, possible to conceive of states of the gas slightly removed from local equilibrium. When this is done, the general constitutive representations (2.1.21) can be expanded in powers of the coefficients $b_m \in \mathbb{B}^r_2$, which, by their very nature, vanish in the state of local equilibrium. The second-order representations can be written down easily and we obtain

$$b_n = \sum_{k=3}^{r} \sum_{m=3}^{r} b^{r|n}_{km} b_k b_m , \quad n > r , \qquad (2.1.22)$$

where we shall think of the coefficients $b_{km}^{r|n}$ as parameters depending in general on both a_o and a_2. [Denoting by ρ and e the mass density and the internal energy density (per unit length), respectively, we arrive at $a_o = \rho$ and $a_2 = 2e$. Due to these equalities, a_o and a_2 are sometimes called *slow conserved moments* of f.$^{(17)}$] Obviously, as a result of (2.1.18) and (2.1.22) for n = r + 1, the constitutive function for the flux moment a_{r+1} can be formulated explicitly, so that we obtain

$$a_{r+1} = - \frac{1}{x_{r+1}^{r+1}} \sum_{k=0}^{r} x_k^{r+1} a_k + \frac{1}{x_{r+1}^{r+1}} b_{r+1} . \tag{2.1.23}$$

Upon adopting the pseudo-macroscopic method of calculating $b_{km}^{r|n}$, as described in Chapter 1 or in Ref. 22, we find that

$$b_{km}^{r|n} = \frac{1}{2k!m!} \left\{B_n B_k B_m\right\}_o , \quad r < n, \quad 3 \le k \le r, \quad 3 \le m \le r, \tag{2.1.24a}$$

where, by definition,

$$\left\{B_k B_m \cdots B_p\right\}_o := (2\pi)^{-1/2} \int d\lambda \; exp\left(- \frac{1}{2} \lambda^2\right)$$
$$\times B_k(\lambda) B_m(\lambda) \cdots B_p(\lambda) . \tag{2.1.24b}$$

The remaining task is, then, to determine the values of $\left\{B_k B_m B_p\right\}_o$:

$$\left\{B_k B_m B_p\right\}_o = \frac{k!m!p!}{[(k+m-p)/2]![(m+p-k)/2]![(p+k-m)/2]!} \tag{2.1.25a}$$

for $k + m + p \in 2\mathbb{N}_o$, $k + m - p \ge 0$, $m + p - k \ge 0$, $p + k - m \ge$ ≥ 0 and

$$\left\{B_k B_m B_p\right\}_o = 0 \tag{2.1.25b}$$

for other choices of the indices.

Comparing Eqs. (2.1.24a) with Eqs. (1.5.6), one recognizes a strong structural similarity. But "classical" solutions for $b_{km}^{r|n}$ are considerably distinguished from "non-classical" ones both by having a quite different mathematical interpretation of the symbols B_n and $\left\{B_nB_kB_m\right\}_o$ and by having the function $F(\omega)$ simply equal to 1; concerning the definition of the element $F(\omega)$ [which appears in (1.5.6) but not in (2.1.24a)], see Eq. (1.3.21b). In addition, let us observe that if $F = 1$ $(F \neq 1)$, then the expansion coefficients $b_{km}^{r|n}$ *are* (*are not*) equal to zero for $n > k + m$. As a consequence of this [at first sight insignificant] fact, in the third-order constitutive theory, which we are considering only as useful illustration, the Hermite (Tchebychef) moments b_n, $n > 2r$, *can* (*cannot*) be expected to be small compared to those in the list $\mathbb{B}_r^{2r}$.

With Eqs. (2.1.22), (2.1.24a), and (2.1.12), one obtains

$$f = f_o\Bigg[1 + \underbrace{\sum_{k=3}^{r} \frac{1}{k!} b_k \; B_k(\hat{\lambda})}_{\varepsilon_1^r}$$

$$\underbrace{+ \frac{1}{2} \sum_{k=3}^{r} \sum_{m=3}^{r} \sum_{n=r+1}^{2r} \frac{1}{k!m!n!} \left\{B_nB_kB_m\right\}_o b_kb_m \; B_n(\hat{\lambda})\Bigg]}_{\varepsilon_2^r} .$$

(2.1.26)

Hence the quantity ε_2^r can be considered negligible, as in Grad's approach,$^{(1,3)}$ only in the simplest case of linear extended irreversible thermodynamics of degree r.

On replacing $\left\{B_nB_kB_m\right\}_o$ by the identities (2.1.25), we get the explicit result for f

$$f = f_o\Bigg[1 + \sum_{k=3}^{r} \frac{1}{k!} b_k \; B_k$$

$$+ \sum_{n=p_1}^{2r} \sum_{k=p_2}^{[n/2]} \sum_{m=0}^{v'} \frac{b_kb_{n-k}B_{n-2m}}{m!(k-m)!(n-m-k)!}$$

$$- \frac{1}{2} \sum_{k = p_3}^{r} \sum_{m = 0}^{w''} \frac{b_k b_k B_{2k - 2m}}{m!(k - m)!(k - m)!} \Bigg] , \qquad (2.1.27a)$$

where

$$p_1 := \mathit{max}(\ 6,\ r + 1\ ), \quad w' := [(n - r - 1)/2],$$

$$p_2 := \mathit{max}(\ 3,\ n - r\ ), \quad w'' := [(2k - r - 1)/2], \qquad (2.1.27b)$$

$$p_3 := \mathit{max}(\ 3,\ [(r + 2)/2]\ )\ .$$

By definition, the symbol [n/2] denotes the greatest integer ≤ n/2.

In regard to the task of extracting from (2.1.10) a definite system of field equations for $\mathbb{A}^r_{-1}$ and u, therefore, Eq. (2.1.27a) plays exactly the same role as Eq. (1.6.3a) does in the theory of quasiparticle gases. However, to examine the difficult problem of evaluating the collision integrals P_n, $3 \le n \le r$, directly in terms of $\mathbb{A}^r_{-1}$, or of a_0, a_2, and $\mathbb{B}^r_2$, let us look ahead to Section 2.2.A, in which we take up the constitutive functions $P_n = \hat{P}_n(\ a_0, a_2, \mathbb{B}^r_2\ )$ for a gas of "Maxwellian molecules" [$\hat{\mathbb{C}}_o$ = constant; see Eq. (2.1.2b)].

2.2. A Gas of "One-Dimensional Maxwellian Molecules"

2.2.A. Evaluation of Collision Integrals

If the transition rate $\hat{\mathbb{C}}_o$ appearing in (2.1.2) is explicitly determined from $(\lambda, \lambda', \lambda'', \omega, \omega', \omega'')$, or from (χ, φ, ψ),[o] then, since (2.1.27) allows the evaluation of $\hat{P}_n(\ a_0, a_2, \mathbb{B}^r_2\ )$ within the framework of the second-order constitutive theory, one obtains constitutive relations for the collision integrals P_n in the following form:

---

[o] Of course, the transition rate $\hat{\mathbb{C}}_o$ cannot depend on total momentum $\mathcal{P}$ of the colliding molecules.

$$P_n = \sum_{k=3}^{r} P_k^{r|n}\, b_k + \sum_{k=3}^{r} \sum_{m=3}^{r} P_{km}^{r|n}\, b_k b_m , \quad 3 \le n \le r , \tag{2.2.1}$$

the expansion coefficients $P_k^{r|n}$ and $P_{km}^{r|n}$ being certain functions of a_0 and a_2.

Although, strictly speaking, $\hat{C}_o = \tilde{C}_o(\chi, \varphi, \psi)$, we shall make the simplifying assumption characterized by

$$\hat{C}_o = \frac{1}{4\pi^2 (6\pi)^{1/2}}\, C_o , \quad C_o = \text{const} > 0 . \tag{2.2.2}$$

Due to the fact that the kinetic equation for a one-dimensional gas should serve some pedagogical purpose, as originally in the work by Ma,(38) we do not inquire as to whether there exists the interaction potential between particles producing (corresponding to) (2.2.2).

On using Ma´s collision operator (2.1.2) [with $\hat{C}_o$ calculated from (2.2.2)] and the definition (2.1.10d) regarding the collision integral P_n, as well as the series representation (2.1.12) [supplemented by Eqs. (2.1.22) and (2.1.24a)], we find for the coefficients $P_k^{r|n}$ and $P_{km}^{r|n}$ in (2.2.1)

$$P_k^{r|n} = \frac{a_o^3}{3(2\pi)^{3/2}}\, C_o\, e^{-n} \sum_{p=3}^{n} Y_p^n\, \mathbb{R}_k^p , \tag{2.2.3a}$$

$$P_{km}^{r|n} = \frac{a_o^3}{3(2\pi)^{3/2}}\, C_o\, e^{-n} \sum_{p=3}^{n} Y_p^n \left[\mathbb{R}_{km}^p + \underline{\sum_{q=r+1}^{2r} \mathbb{R}_q^p\, b_{km}^{r|q}} \right] , \tag{2.2.3b}$$

where the dimensionless numbers $\mathbb{R}_k^p$ and $\mathbb{R}_{km}^p$ are given by

$$\mathbb{R}_k^p := \frac{1}{4\pi^2 (6\pi)^{1/2} k!} \int d\mathcal{P}\, d\chi\, d\varphi\, d\psi\, \chi \exp\left[-\frac{1}{2}\left\{ \chi^2 + \frac{\mathcal{P}^2}{3} \right\} \right]$$
$$\times B_p(\lambda) [3B_k(\omega) - B_k(\lambda) - 2B_k(\lambda')] , \tag{2.2.4a}$$

$$\mathbb{R}^{p}_{km} := \frac{1}{4\pi^{2}(6\pi)^{1/2}k!m!} \int d\mathcal{P}\, d\chi\, d\varphi\, d\psi\, \chi\, \exp\left[-\frac{1}{2}\left(\chi^{2} + \frac{\mathcal{P}^{2}}{3}\right)\right]$$

$$\times B_{p}(\lambda)\left[3B_{k}(\omega)B_{m}(\omega') - B_{k}(\lambda')B_{m}(\lambda'')\right.$$

$$\left. - B_{k}(\lambda)B_{m}(\lambda') - B_{k}(\lambda')B_{m}(\lambda)\right] . \qquad (2.2.4b)$$

For the convenience of the reader, the abbreviated integral

$$\int d\mathcal{P}\, d\chi\, d\varphi\, d\psi\, \mathcal{N}(\mathcal{P},\chi,\varphi,\psi)$$

is understood to be the multiple integral

$$\int_{-\infty}^{+\infty} \int_{0}^{+\infty} \int_{0}^{2\pi} \int_{0}^{2\pi} d\mathcal{P}\, d\chi\, d\varphi\, d\psi\, \mathcal{N}(\mathcal{P},\chi,\varphi,\psi) .$$

The procedure for obtaining (2.2.4) is compounded of two structurally different parts, one of which arises from transforming to $(\mathcal{P},\chi,\varphi,\psi)$ in place of $(\lambda,\lambda',\lambda'',\omega,\omega',\omega'')$ as variables of integration [this change of variables of integration is described in the work by Ma(38)], and the second from replacing in the resulting integral χ by $(1/\Theta)\chi$, $\mathcal{P} - 3u$ by $(1/\Theta)\mathcal{P}$, $\bar{\lambda}$ by $(1/\Theta)\lambda$, etc. Making the above scale transformation, we find that the dependence of the presently dimensionless velocities $(\lambda,\lambda',\lambda'')$, which occur in (2.2.4) through the Hermite polynomials B_m, upon $(\mathcal{P},\chi,\varphi)$ is identical with that in Eqs. (2.1.5), except that the variables $\mathcal{P}$ and χ are now dimensionless quantities. When the angle measured along the circle is chosen to be ψ, an analogous argument obviously applies to the velocities $(\omega,\omega',\omega'')$.

For many applications, the Hermite polynomials are derived by means of their generating function:(34,35)

$$s(\lambda|\tau) := \exp\left(\lambda\tau - \frac{1}{2}\tau^{2}\right) = \sum_{p=0}^{\infty} \frac{\tau^{p}}{p!} B_{p}(\lambda) . \qquad (2.2.5)$$

Equation (2.2.5) implies that $\mathbb{R}^{p}_{k}$ divided by p! and $\mathbb{R}^{p}_{km}$ divided by p! are the coefficients of $\tau^{p}\tau'^{k}$ and $\tau^{p}\tau'^{k}\tau''^{m}$, respectively, in the power series representations of

$$\hat{\mathbb{R}}(\tau,\tau') := \frac{1}{4\pi^2(6\pi)^{1/2}} \int d\mathcal{P}\, d\chi\, d\varphi\, d\psi\, \chi \exp\left[-\frac{1}{2}\left(\chi^2 + \frac{\mathcal{P}^2}{3}\right)\right]$$

$$\times\, s(\lambda|\tau)\left[3s(\omega|\tau') - 2s(\lambda'|\tau') - s(\lambda|\tau')\right] \qquad (2.2.6a)$$

and

$$\hat{\mathbb{R}}(\tau,\tau',\tau'') := \frac{1}{4\pi^2(6\pi)^{1/2}} \int d\mathcal{P}\, d\chi\, d\varphi\, d\psi$$

$$\times\, \chi \exp\left[-\frac{1}{2}\left(\chi^2 + \frac{\mathcal{P}^2}{3}\right)\right] s(\lambda|\tau)\left[3s(\omega|\tau')s(\omega'|\tau'')\right.$$

$$\left.- s(\lambda'|\tau')s(\lambda''|\tau'') - s(\lambda|\tau')s(\lambda'|\tau'') - s(\lambda'|\tau')s(\lambda|\tau'')\right] . \qquad (2.2.6b)$$

The direct method of performing the integrations over $d\mathcal{P}$, $d\chi$, $d\varphi$, and $d\psi$ in (2.2.6) rests essentially upon the following values of the definite integrals:

$$\int_{-\infty}^{+\infty} d\xi\, \exp\left(-p\xi^2 - q\xi\right) = \left(\pi/p\right)^{1/2} \exp\left(\frac{q^2}{4p}\right) , \qquad (2.2.7a)$$

$$\int_{0}^{+\infty} d\xi\, \xi \exp\left(-p\xi^2\right) I_0(\delta\xi) = \frac{1}{2p} \exp\left(\frac{\delta^2}{4p}\right) , \qquad (2.2.7b)$$

$$\int_{0}^{+\infty} d\xi\, \xi \exp\left(-p\xi^2\right) I_0(\delta\xi) I_0(\varepsilon\xi) = \frac{1}{2p} \exp\left(\frac{\delta^2 + \varepsilon^2}{4p}\right) I_0\left(\frac{\delta\varepsilon}{2p}\right), \qquad (2.2.7c)$$

$$\int_{0}^{2\pi} d\xi\, \exp\left(x_1 \cos\xi + x_2 \sin\xi\right) = 2\pi\, I_0\left[\left(x_1^2 + x_2^2\right)^{1/2}\right] , \qquad (2.2.7d)$$

where

$p > 0$, $x_1^2 + x_2^2 \geq 0$, and

$$I_0(\xi) := \sum_{k=0}^{\infty} \frac{1}{k!k!} \left(\frac{\xi}{2}\right)^{2k} \qquad (2.2.8)$$

is the modified Bessel function of order zero. From Eqs. (2.2.6) and (2.2.7) we have the results for $\hat{\mathbb{R}}(\tau,\tau')$ and $\hat{\mathbb{R}}(\tau,\tau',\tau'')$:

$$\hat{\mathbb{R}}(\tau,\tau') = 3\, exp\left(\frac{1}{3}\tau\tau'\right) I_o\left(\frac{2}{3}\tau\tau'\right) - exp(\tau\tau') - 2\,, \qquad (2.2.9a)$$

$$\hat{\mathbb{R}}(\tau,\tau',\tau'') = 3\, exp\left[\frac{1}{3}(\tau\tau' + \tau\tau'')\right] I_o\left[\frac{2}{3}\tau\left(\tau'^2 + \tau''^2 - \tau'\tau''\right)^{1/2}\right] - exp(\tau\tau') - exp(\tau\tau'') - 1\,. \qquad (2.2.9b)$$

Since $\hat{\mathbb{R}}(\tau,\tau')$ is a function of $\tau\tau'$ alone, all terms in the power series representation of (2.2.9a) have equal power in τ and τ' and a matrix of the coefficients in front of $\tau^p\tau'^k$ must be diagonal:

$$\mathbb{R}^p_k = 0 \quad \text{for} \quad p \neq k\,. \qquad (2.2.10a)$$

We can also deduce from (2.2.9b) that

$$\mathbb{R}^p_{km} = 0 \quad \text{unless} \quad p = k + m\,. \qquad (2.2.10b)$$

Consequently, in extended irreversible thermodynamics of degree r, $r \geq 3$, the underlined expression occurring in (2.2.3b) vanishes altogether, and, in spite of the result (2.1.24a) potentially giving rise to nonlinear effects, the nonlinear constitutive representations (2.1.22) for $b_n \in \mathbb{B}^{2r}_r$ are of no importance in determining $\hat{P}_n(\alpha_o, \alpha_2, \mathbb{B}^r_2)$ from (2.2.1). Of course, this will not be the case with $\hat{\mathbb{C}}_o \neq$ const, i.e., with "molecules" other than "Maxwellian ones".

2.2.B. Illustration of the Sense of the Residual Inequality

This section is devoted to the detailed evaluation of the Liu-Müller formula

$$\sigma_h = \sum_{n=3}^{r} \Delta^r_n P_n \qquad (2.2.11)$$

for the entropy production σ_h. [Of course, the method of Section 1.7 regarding the (approximate) derivation of (1.4.7) via (1.2.23) and (1.6.3) applies to a one-dimensional, classical gas as well.(22)] In taking the cumulant expansion technique as described in Refs. 41 and 42, the calculation of the entropy production σ_h from both (2.1.8c) and the exponential proposition(41,43,67,68)

$$f = \frac{1}{\mathbb{C}}\, exp\left[- k_B^{-1} \sum_{n=0}^{r} \Lambda_n^r\, \overline{\lambda}^{n}\right] \qquad (2.2.12)$$

for the distribution function in place of (2.1.27) was until recently(43,67) not considered to be first of all an *unsolved* (!!) problem involving the precise nonlinear dependence of Λ_n^r on the gas-state variables:(68)

$$\Lambda_n^r = \tilde{\Lambda}_n^r\left(\mathbb{A}_{-1}^r \right) = \hat{\Lambda}_n^r\left(a_0, a_2, \mathbb{B}_2^r \right) . \qquad (2.2.13)$$

However, in view of the ambiguity in the value of Λ_n^r derived from $a_p \in \mathbb{A}_{-1}^r$, the operational use(41,43,67) in Eq. (2.2.12) of $\Lambda_n^r \sim b_p$ $(3 \le n \le r,\ 3 \le p \le r)$ in analyzing the shape $\sigma_h = \hat{\sigma}_h\left(a_0, a_2, \mathbb{B}_2^r \right)$, *for such a situation in which the* $|b_p|$, $3 \le p \le r$, *are by no means small compared to* 1, is rapidly losing our acceptance.(68) No doubt, if we formally set $\Lambda_n^r = \Delta_n^r$, $3 \le n \le r$, then the ansatz (2.2.12) harmonizes with (2.2.11). Despite this observation, the objective of the "exponential" approach to the mesoscopic theory of strongly nonlinear gas flows is mostly unattainable,(68) not only from any possible defect in the method of substituting the considerably simplified version of (2.2.13) into (2.2.12), but also from inherent properties of (2.2.12) itself. Indeed, irrespective of whether r belongs to $2\mathbb{N}_0 + 1$ or not, Eq. (2.2.12) leads to *divergent integrals*,(68) at least for some physically relevant values of $b_n \in \mathbb{B}_0^r$ [cf., e.g., Section 2.3], and use of the polynomial "replacement" of the exponential form of f may seem at present unavoidable.(20)

Ignoring for the moment (2.2.12), the important work by Liu and Müller(15) indicates in turn that $b_p \in \mathbb{B}_2^r$ should (?) be small anyway in extended irreversible thermodynamics of degree r, so that we are entirely content with the polynomial constitutive representations for Lagrange multipliers:

$$\Delta_n^r = \Delta_0^{r|n} + \sum_{k=3}^{r} \Delta_k^{r|n}\, b_k + \sum_{k=3}^{r} \sum_{m=3}^{r} \Delta_{km}^{r|n}\, b_k b_m , \qquad (2.2.14)$$

where, for instance,

$$\Delta_0^{r|n} = 0 \,, \qquad (2.2.15a)$$

$$\Delta_k^{r|n} = - \frac{k_B}{k!} \theta^n \sum_{p=n}^{r} X_n^p \delta_{pk} \,, \qquad (2.2.15b)$$

$$3 \leq n \leq r \,. \qquad (2.2.15c)$$

Substituting (2.2.1) and (2.2.14) in (2.2.11) and rejecting those resulting terms which are of higher than third order in $b_p \in \mathbb{B}_2^r$,[10] we obtain for Eq. (2.2.11)

$$\sigma_h = \sigma_h^L + \sigma_h^N \,, \qquad (2.2.16)$$

where

$$\sigma_h^L := \sum_{n=3}^{r} \sum_{k=3}^{r} \sum_{m=3}^{r} \Delta_k^{r|n} P_m^{r|n} b_k b_m \,, \qquad (2.2.17a)$$

$$\sigma_h^N \sim b_k b_m b_p \,. \qquad (2.2.17b)$$

The part σ_h^L of σ_h is required to be nonnegative for all possible choices of the values of $b_p \in \mathbb{B}_2^r$ (otherwise linear extended irreversible thermodynamics of degree r breaks down at all), but the local entropy inequality $\sigma_h = \sigma_h^L + \sigma_h^N \geq 0$ may formally cease to be true when $b_p \in \mathbb{B}_2^r$ are suitably chosen. Fortunately, because the one-dimensional gas departs only slightly from the state of local equilibrium, as normally in the constitutive theory of second order —— and indeed of arbitrary order ——, the contribution σ_h^N to the entropy production σ_h is (in all but the exceptional cases) very small compared with that due to linear "effects". In this sense the inequality $\sigma_h \geq 0$ does not lose its importance. Obvious as these facts are, they are sometimes disregarded in formal extensions of the range of validity of extended irreversible thermodynamics to far-from-equilibrium situations. [In this context, see Refs. 20, 67, and 68 (and the debate over the present status of extended theories).]

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[10] In the second-order constitutive theory, σ_h can be evaluated to within the third-order terms in $b_p \in \mathbb{B}_2^r$ inclusive.

We are now in a position to test the appropriateness of our intuitive observation concerning the sign of $\sigma_h{}^L$ in (2.2.17a) and, accordingly, that of σ_h in (2.2.16). In order to evaluate the r.h.s. of (2.2.17a), we return to Eqs. (2.2.3a), (2.2.15b), (2.1.16b), and (2.2.10a). Then it follows immediately that

$$\sigma_h{}^L = - \frac{k_B a_o^3}{3(2\pi)^{3/2}} \mathbb{C}_o \sum_{p=3}^{r} \frac{1}{p!} \mathbb{R}_p^p \, b_p b_p > \qquad (2.2.18)$$

and, in virtue of $a_o > 0$ and $\mathbb{C}_o > 0$, the main point is to demonstrate that $\mathbb{R}_p^p < 0$ for $p \geq 3$. To this end, by representing the generating function $\hat{\mathbb{R}}(\tau,\tau')$ in Eq. (2.2.9a) as a series in the variables τ and τ', we arrive at

$$\mathbb{R}_p^p = - \frac{1}{p!} + \frac{1}{3^{p-1}} \sum_{k=0}^{[p/2]} \frac{1}{k!k!(p-2k)!} , \quad p \geq 3. \qquad (2.2.19)$$

Putting

$$k! = \frac{1}{2^k} (2k)!! , \quad (2k)!! = \frac{(2k)!}{(2k-1)!!} , \qquad (2.2.20a)$$

$$(2k-1)!! := 1 \cdot 3 \cdot \ldots \cdot (2k-1) < \frac{1}{2} (2k)!! , \qquad k \geq 1 \qquad (2.2.20b)$$

into (2.2.19) and rearranging, we obtain

$$\mathbb{R}_p^p = - \frac{1}{p!} + \frac{1}{2 \cdot 3^{p-1} p!} + \left\{ \frac{1}{2 \cdot 3^{p-1} p!} + \right.$$

$$\left. \frac{1}{3^{p-1}} \sum_{k=1}^{[p/2]} \frac{2^{2k}}{(2k)!!(2k)!!(p-2k)!} \right\} < - \frac{1}{p!} + \frac{1}{2 \cdot 3^{p-1} p!}$$

$$+ \frac{1}{2 \cdot 3^{p-1}} \sum_{k=0}^{[p/2]} \frac{2^{2k}}{(2k)!(p-2k)!} = - \frac{1}{p!} + \frac{1}{2 \cdot 3^{p-1} p!}$$

$$+ \frac{1}{4 \cdot 3^{p-1} p!} \left[3^p + (-1)^p \right] \leq \frac{9}{4 \cdot 3^p p!} \left[1 - 3^{p-2} \right]. \qquad (2.2.21)$$

Hence the proof of the implication $(p \geq 3) \Rightarrow (\mathbb{R}^p_p < 0)$ is complete.

2.3. In Search of Other Truncation Schemes. Difficulties Faced in Interpretation of the Entropy Maximum Principle

According to the tenets of *information theory*, in extended irreversible thermodynamics of degree r, $r \geq 3$, the estimate of the unknown moments a_n, $n > r$, results from f which gives the known moments a_n, $0 \leq n \leq r$, *correctly* and also maximizes the following entropy functional of the distribution function:

$$H(f) := - k_B \int d\lambda \; f \, ln(\mathbb{C}f) + \sum_{n=0}^{r} \left[\Lambda^r_n - k_B \delta_{no}\right]\left[a_n - \int d\lambda \; \bar{\lambda}^n f\right], \tag{2.3.1}$$

where the Λ-multipliers are associated with the constraints (2.1.9) for $0 \leq n \leq r$. Due to the formula

$$\delta H = - k_B \int d\lambda \left[ln(\mathbb{C}f) + k_B^{-1} \sum_{n=0}^{r} \Lambda^r_n \bar{\lambda}^n \right] \delta f \; , \tag{2.3.2}$$

one obtains (2.2.12) for f that makes (2.3.1) maximal. If

$$\left| \int d\lambda \; \bar{\lambda}^n \, exp\left[- k_B^{-1} \sum_{n=0}^{r} \Lambda^r_n \bar{\lambda}^n \right] \right| < \infty \quad , \tag{2.3.3}$$

then inserting (2.2.12) into (2.1.9) yields the explicit dependence of a_n, $0 \leq n \leq r$, upon Λ^r_n, $0 \leq n \leq r$, practically unmanageable although invertible in principle for the Λ-multipliers as nonlinear functions of $\mathbb{A}^r_{-1}$.

Condition II. Let f_o be a local Maxwellian density that corresponds to f:

$$\int d\lambda \; \bar{\lambda}^n (f - f_o) = 0, \quad n = 0,1,2.$$

Then f satisfies *Condition II* if $|f_o^{-1}(f - f_o)|$ is bounded by a power of $\bar{\lambda}$ [or, more generally, by a function of $\bar{\lambda}$ which is square-integrable over $\mathbb{R} = (-\infty,+\infty)$ with weight f_o].

In extended irreversible thermodynamics of gaseous systems it frequently happens that the Lagrange and Λ-multipliers are closely related to one another. The matter is described in detail by Dreyer,(30) who, using (*without proof*) *Condition II* with respect to the exponential function (2.2.12), was able to show not only that (2.2.12) ensures the existence and mathematical realization of the Liu-Müller theory but also that

$$\Lambda^r_n = \Delta^r_n \quad \text{for} \quad 3 \le n \le r \, . \tag{2.3.4}$$

While we have not searched every one of the available papers, in our study of some of the most important contributions we have not found any claim in favour of the validity of (2.3.3) for $r \ge 3$ and of

$$\left| \frac{1}{f_o \mathbb{C}} \, exp\left[- k_B^{-1} \sum_{n=0}^{r} \Lambda^r_n \, \bar{\lambda}^n \right] - 1 \right| \le \mathbb{C}_1 |\bar{\lambda}|^m \tag{2.3.5}$$

for $\mathbb{C}_1 = \text{const} > 0$, $m \in \mathbb{N}_0$, and $r \ge 3$. Since, of course, we are not justified in choosing for r here 3,5,7,..., let us assume that r = 4,6,8,... . Then the exact nonlinear constitutive relations (2.1.20), *if they exist at all*, should be determined both from the "microscopic" definitions of the moments of f and from Eq. (2.2.12), and the starting point for proof of such relations must be nothing else than the *unquestionable demonstration* of the existence of the inequalities (2.3.3) and (2.3.5) for *all* possible values of $a_n \in \mathbb{A}^r_{-1}$ [and/or $b_n \in \mathbb{B}^r_2$]. In the neighbourhood of the state of local equilibrium, a very good approximation to $\Lambda^r_n = \hat{\Lambda}^r_n (a_0, a_2, \mathbb{B}^r_2)$ may be obtained by utilizing the first two terms in the polynomial expansion of $\hat{\Lambda}^r_n$. From (2.3.4), (2.2.14), (2.2.15), and (2.1.16b) it is not difficult to see that

$$\Lambda^r_r \cong - \frac{k_B}{r!} \, \theta^r \, b_r \tag{2.3.6}$$

for $|b_n| \ll 1$, $3 \le n \le r$, and that

$$- k_B^{-1} \sum_{n=0}^{r} \Lambda^r_n \, \bar{\lambda}^n \sim (\bar{\lambda})^r \to +\infty \tag{2.3.7}$$

for $0 < b_r \ll 1$ and $|\bar{\lambda}| \to +\infty$. Hence Eq. (2.2.12) must be viewed as a rather untractable consequence of the entropy maximum principle that prevents us(68) from pushing the explicit calculations very far. Thus, due to the nonexistence(68) of convergent integrals if $0 < b_r \ll 1$, it is not impossible that at least some of the readers will find the pseudo-macroscopic method, which is specific in statement, *compatible with Grad's ideas*, formally rigorous in proof based upon convergent integrals, independent of Dreyer's approach,(39,66) to be perhaps the more appealing proposition.(22)

We summarize our discussion as follows. So as both to allow the case in which (2.2.12) does not lead to divergent integrals and to demonstrate the existence of the general constitutive functions (2.1.20) that satisfy *exactly* the entropy principle in the form of Müller-Liu, *in all probability* it is sufficient to assume that $b_r \leq 0$. Although from the physical point of view the extra requirement that $b_r \leq 0$ *is only admissible (8) but not very plausible* and seems to cause difficulties in interpreting or even accepting the entropy maximum principle, one can use (2.2.12) in order to *motivate* some aspects of the postulational basis of the pseudo-macroscopic method and consequently that of Müller-Liu.(15,26) However, it is essential to point out that the potential importance of the final results of our theory does not depend on this kind of motivation and is associated with Grad's ideas rather than the axioms of extended irreversible thermodynamics; in particular, there is no necessity to choose $r = 2s$ $(s = 2,3,\dots)$ and $b_r \leq 0$, insofar as Eqs. (2.1.22) and (2.1.24a) themselves are concerned. But we have explained already these problems and of general interest in this regard is our discussion in Section 1.5.B of Chapter 1. [Cf. also the supplementary comments, as offered in Sections V A and V B of Ref. 26 and in Section 7 of Ref. 68, on the advantages (disadvantages) of applying the entropy maximum principle in extended irreversible thermodynamics.]

PART B

IRREDUCIBLE TENSOR DESCRIPTION

The conclusions we have reached so far concerning Grad's method are quite specific in that they apply to one-dimensional (classical or quasiparticle) gases. In many cases of interest, the treatment of certain aspects of the kinetic theory of "actual" gases requires, as a prerequisite, a comprehensive discussion of some complicated tensorial problems. In this part we deal with the so-called irreducible tensor description of three-dimensional gaseous systems, developing this subject only insofar as it relates to Grad's moment procedure and to those *universal* questions which we have already formulated in Chapters 1 and 2.

CHAPTER 3

EQUATIONS OF TRANSFER FOR THE SYMMETRIC TRACELESS MOMENTS: CLASSICAL RAREFIED GASES

3.1. Introduction

Consider as a starting point of this chapter Boltzmann's equation describing a classical, moderately rarefied, simple, monatomic gas. One of the basic problems before us ,then, is to derive from it, by means of the elementary operations of multilinear algebra,(44) i.e., without the necessity of handling spherical tensors,[11] the infinite hierarchy of the equations of transfer for the *relative irreducible symmetric traceless moments* of the one-point distribution function f.

Working in a somewhat different direction, Coope and Snider(45) proposed a very general formulation of the method of reduction of the tensors, one whose beginnings may be found in the papers by Grad,(1,3) Ikenberry,(46,47) Ikenberry and Truesdell,(48) as well as in Hamermesh's book.(49) With respect to the relative symmetric traceless moments of the distribution function, however, the theoretical results for the irreducible equations of transfer outlined earlier, especially those in the careful texts on Maxwellian iteration(48,50) and in the pioneering work by Johnston,(50) are the strongest yet obtained, being in fact the *only* ones presently available; for more details, see also the important Truesdell-Muncaster monograph.(4)

Although Truesdell et al.(4,48,50) have no difficulties in principle to calculate explicitly and for increasing order as many of the irreducible equations of transfer as are needed,

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[11] The relation of Ikenberry's tensorial harmonics to the properties of three-dimensional space is more apparent than with spherical tensors, where one axis is arbitrarily distinguished, and, furthermore, they can often be more easily handled, i.e., without the necessity of tables of numerical coefficients.

they do not provide in their articles the *compact* and *single* expression for an infinite hierarchy of the tensorial symmetric traceless balance equations. To the best of our knowledge, Johnston(69) was the first to invent the general method of deriving it. But for reasons that will become clear later, he did not solve the problem completely and the fundamental result (12) written down on p. 1457 in Ref. 69, although of course compatible with the relatively simple proposition of Section 3.3, is not equivalent or similar in every respect to our equation (3.3.21). We shall have more to say about this in Section 3.4.A, especially due to the fact that, as elementary inspection reveals, we arrive at the counterpart to Johnston's framework differently and by the method of straightforward interest also for other gaseous systems and kinetic equations.

In order to account for the potential applicability of the irreducible equations of transfer [in the solution scheme regarding Boltzmann's equation, for instance], one must necessarily exhibit, in terms of the relative symmetric traceless moments, an explicit formula for the irreducible collision integrals —— they appear on the r.h.s. of those equations of transfer and depend functionally upon the distribution function f. To this end, apart from a few special cases, *e.g.*, Maxwellian molecules, the very hard particle model, etc., for which, if required, the alternative methods may be developed, a large amount of theoretical work(5-8,51-54,70-73) has been done, treating only such molecular densities f as can be expanded in a series of the complete set of Hermite or Laguerre polynomials.(34,35,55)

Obviously, the expansion for the distribution function f that fits in *together* and *with* the irreducible equations of transfer is now of great interest. It has long been known what the choice of Laguerre polynomials(34,35) for classical gases makes simpler than any other. The new outcome of the present treatment, at least we hope it to be so, is a direct demonstration of the crucial aspect of this fact in consequence of applying the ordinary transformations of multilinear algebra(44) and, unlike many previous efforts, without the necessity of referring to spherical harmonics.

Although the symmetric traceless tensors have been considered from the mathematical point of view by Coope and Snider,(45) insofar as we have seen they have not been appreciated and systematically used in the literature on the kinetic theory, except for the relevant Truesdell-Ikenberry-Muncaster framework(4,48,50) and the interesting Johnston's approach(60) tending toward a general formulation. It is, then, no surprise that a serious discussion of the above-mentioned problems must be delayed until Section 3.3, in order that the essential ideas of the irreducible tensor description for a classical gas may be presented unaccompanied by any marginal arguments of purely technical importance.

Here we proceed as follows. Section 3.2.A deals with the simplest properties of the symmetric traceless tensors, whereas Section 3.2.B introduces the notion of the ∇-operator as well as of its various modifications. Our presentation of some of the auxiliary concepts is necessarily very brief, being a subject of Appendices 3.A and 3.B only for completeness. In Section 3.3 and Appendices 3.C and 3.D the algebraic difficulties in calculating the irreducible equations of transfer are faced once more.(69) Section 3.4.A achieves the object of transforming Grad's expansion of the distribution function f in terms of three-dimensional Hermite polynomials(55) into its irreducible counterpart fashioned by mathematical apparatus such as Laguerre polynomials(34,35) and Ikenberry's tensorial harmonics.(46,47) In the older view the irreducible moment representation of the distribution function f was postulated rather than logically deduced, and Ikenberry's harmonics were replaced by their spherical analogues; see, however, Ref. 74 and the literature quoted there. Section 3.4.B discusses, after demonstrating that the Laguerre-Ikenberry expansion coefficients of f show up in a very particular way in the approximate expression for the entropy density h, the irreducible variant of Grad's moment truncation procedure. Given a gas of Maxwellian molecules, Section 3.5.A offers some useful but elementary comments regarding the general structure of the symmetric traceless collision integrals. In Section 3.5.B our attention will be directed more particularly to the problem of establishing the explicit relationship between extended and ordinary thermodynamics. We conclude this chapter with final remarks of Section 3.6.

Whether the direct notation for tensors is preferable over the Cartesian one must be decided individually for each problem. In this chapter, except for Appendices 3.A, 3.B, and 3.C, we make no use of the latter.

3.2. Prolegomena

The symmetrizer Π, the trace operator with respect to the pair (β,ν), denoted by $\underset{(\beta,\nu)}{\mathrm{Tr}}$, the inner product $M^{\alpha} \bullet M^{\beta}$ of the tensors M^{α} and M^{β}, the so-called contravariant metric (unit) tensor of a three-dimensional Euclidean vector space $\mathbb{E}$, denoted by I, and the ∇-operator are precisely defined in Appendix 3.A.

3.2.A. Symmetric Traceless Tensors

a. Let $\mathbb{E}$ be a three-dimensional Euclidean vector space and consider for each $\alpha \geq 2$ the *αth* tensorial power $\mathbb{E}^{\alpha} := \otimes^{\alpha}\mathbb{E}$ of $\mathbb{E}$. We extend the definition of $\mathbb{E}^{\alpha}$ to the cases $\alpha = 1$ and $\alpha = 0$ by setting $\mathbb{E}^{1} := \mathbb{E}$ and $\mathbb{E}^{0} := \mathbb{R}$, where $\mathbb{R}$ stands for the set of real numbers. The image space $\Pi\mathbb{E}^{\alpha}$ $(\alpha \geq 0)$ of the symmetrizer Π in $\mathbb{E}^{\alpha}$ will be denoted by $\mathbb{E}^{\alpha}_{s}$. Elementary calculus shows that(44,45)

$$\dim \mathbb{E}^{\alpha} = 3^{\alpha}\ , \quad \dim \mathbb{E}^{\alpha}_{s} = \frac{1}{2}(\alpha + 1)(\alpha + 2), \quad \alpha \geq 2. \qquad (3.2.1)$$

b. Suppose that M^{α} and M^{β} are the tensors of degrees α and β, respectively $\left(M^{\alpha} \in \mathbb{E}^{\alpha},\ M^{\beta} \in \mathbb{E}^{\beta} \right)$. Then the equality

$$M^{\alpha} \vee M^{\beta} := \Pi\left(M^{\alpha} \otimes M^{\beta}\right) \in \mathbb{E}^{\alpha+\beta}_{s}\ , \quad \alpha \geq 0,\ \beta \geq 0 \qquad (3.2.2)$$

defines the *symmetric tensor product* of M^{α} and M^{β} $\left(M^{0} \otimes M^{\alpha} := M^{\alpha} \otimes M^{0} := M^{0}M^{\alpha},\ M^{0} \in \mathbb{R} \right)$.

c. The trace operator with respect to the pair (β,ν), denoted by $\underset{(\beta,\nu)}{\mathrm{Tr}}$, determines a linear map:

$$\underset{(\beta,\nu)}{\mathrm{Tr}} : \mathbb{E}^{\alpha} \rightarrow \mathbb{E}^{\alpha-2}, \quad \alpha \geq 2,\ \alpha \geq \beta \geq 1,\ \alpha \geq \nu \geq 1,\ \beta \neq \nu .$$

The trace operator Tr is the restriction of a linear map $\underset{(\beta,\nu)}{\mathrm{Tr}}$ with the arbitrarily chosen pair (β,ν) to $\mathbb{E}^{\alpha}_{s} \subset \mathbb{E}^{\alpha}$ $(\alpha \geq 2)$. Moreover, we define Tr to be the identity on $\mathbb{E}^{1}_{s} = \mathbb{E}$ and $\mathbb{E}^{0}_{s} = \mathbb{R}$: $\mathrm{Tr}M^{1} := M^{1}$, $\mathrm{Tr}M^{0} := M^{0}$.

d. The kernel of Tr in $\mathbb{E}^{\alpha}_{s}$, denoted by $\mathrm{Ker}_{\alpha}\mathrm{Tr}$ $(\alpha \geq 2)$, is the subset of tensors $M^{\alpha} \in \mathbb{E}^{\alpha}_{s}$ such that $\mathrm{Tr}M^{\alpha} = 0$. We extend the definition of $\mathrm{Ker}_{\alpha}\mathrm{Tr}$ to the cases $\alpha = 1$ and $\alpha = 0$ by setting $\mathrm{Ker}_{1}\mathrm{Tr} := \mathbb{E}^{1}_{s} = \mathbb{E}$ and $\mathrm{Ker}_{0}\mathrm{Tr} := \mathbb{E}^{0}_{s} = \mathbb{R}$. The elements of $\mathrm{Ker}_{\alpha}\mathrm{Tr}$ will be called *symmetric traceless tensors of degree* α. The image space of Tr in $\mathbb{E}^{\alpha}_{s}$, denoted by $\mathrm{Im}_{\alpha}\mathrm{Tr}$ $(\alpha \geq 2)$, is the set of tensors $M^{\alpha-2} \in \mathbb{E}^{\alpha-2}_{s}$ of the form $M^{\alpha-2} = \mathrm{Tr}M^{\alpha}$ for some $M^{\alpha} \in \mathbb{E}^{\alpha}_{s}$. Adopting the standard theorem associated with the notion of the rank of a linear mapping of finite-dimensional vector spaces, we find that

$$\mathit{dim}\ \mathrm{Ker}_{\alpha}\mathrm{Tr} + \mathit{dim}\ \mathrm{Im}_{\alpha}\mathrm{Tr} = \mathit{dim}\ \mathbb{E}^{\alpha}_{s}, \quad \alpha \geq 2 . \qquad (3.2.3)$$

e. Given $M^{\alpha} \in \mathbb{E}^{\alpha}_{s}$ and $M^{\alpha+2\beta} \in \mathbb{E}^{\alpha+2\beta}_{s}$, it is useful to introduce the following tensors:

$$\hat{M}^{\alpha[\beta]} := \mathrm{Tr}^{\beta}M^{\alpha}, \quad \hat{M}^{\alpha|\beta} := \mathrm{Tr}^{\beta}M^{\alpha+2\beta}, \qquad (3.2.4)$$

where $\mathrm{Tr}^{\beta}M^{\alpha} \in \mathbb{E}^{\alpha-2\beta}_{s}$ and $\mathrm{Tr}^{\beta}M^{\alpha+2\beta} \in \mathbb{E}^{\alpha}_{s}$ are the results of the β-fold successive application of the Tr-operator to the tensors M^{α} and $M^{\alpha+2\beta}$, respectively $(\mathrm{Tr}^{0}M^{\alpha} := M^{\alpha})$.

f. The action of $\cup$ on $M^{\alpha} \in \mathbb{E}^{\alpha}$, and $M^{\beta} \in \mathbb{E}^{\beta}$ $(\alpha + \beta \geq 2)$ is characterized by

$$M^{\alpha} \cup M^{\beta} := \Pi \underset{(1,\alpha+1)}{\mathrm{Tr}} (M^{\alpha} \otimes M^{\beta}) \in \mathbb{E}^{\alpha+\beta-2}_{s} . \qquad (3.2.5)$$

We call $M^{\alpha} \cup M^{\beta} = M^{\beta} \cup M^{\alpha}$ the *contracted symmetric tensor product* of M^{α} and M^{β}.

g. Let us suppose that $\nu := min(\alpha,\beta)$. Then in contracting $M^{\alpha} \in \mathbb{E}^{\alpha}$ with $M^{\beta} \in \mathbb{E}^{\beta}$ the ν-fold contraction(45) is denoted by $\bullet$:

$$M^{\alpha} \bullet M^{\beta} = M^{\beta} \bullet M^{\alpha} \in \mathbb{E}^{\alpha + \beta - 2\nu} , \tag{3.2.6}$$

$$M^{0} \bullet M^{\alpha} = M^{\alpha} \bullet M^{0} = M^{0}M^{\alpha} . \tag{3.2.7}$$

However, some convention, as to which of the 2ν indices are to be contracted, must be followed when doing the contraction; for more details, see Appendix 3.A. The tensor $M^{\alpha} \bullet M^{\beta}$ will be termed the *inner product* of M^{α} and M^{β}.

h. The norm $|M^{\alpha}|$ of a tensor $M^{\alpha} \in \mathbb{E}^{\alpha}$ is defined as the positive square-root of $M^{\alpha} \bullet M^{\alpha}$:

$$|M^{\alpha}| := \left(M^{\alpha} \bullet M^{\alpha}\right)^{1/2} . \tag{3.2.8}$$

i. The abbreviated symbols I and I^{α} $(\alpha \geq 2)$ stand for the so-called contravariant metric (unit) tensor of $\mathbb{E}$ $\left(I \in \mathbb{E}^{2}_{s}\right)$ and the tensor product $\otimes^{\alpha} I \in \mathbb{E}^{2\alpha}$, respectively. We extend the definition of I^{α} to the cases $\alpha = 1$ and $\alpha = 0$ by setting $I^{1} := I$ and $I^{0} := 1 \in \mathbb{E}^{0} = \mathbb{R}$.

j. Assume that

$$M^{\alpha} * I^{\beta} := \ell(\alpha + 2\beta,\beta)\, M^{\alpha} \smile I^{\beta} \in \mathbb{E}^{\alpha + 2\beta}_{s}, \tag{3.2.9}$$

where

$$\ell(\alpha,\beta) := \frac{\alpha!}{2^{\beta}\beta!(\alpha - 2\beta)!} , \quad \alpha \geq 2\beta. \tag{3.2.10}$$

Then certain useful properties of the Tr-operator are included in the statement of the following

Theorem

I. Suppose that $\alpha \geq 2$, $\beta \geq 1$, and $M^{\alpha} \in \mathbb{E}^{\alpha}_{s}$. Then we have

$$\mathrm{Tr}\left(M^{\alpha} * I^{\beta}\right) = \hat{M}^{\alpha[1]} * I^{\beta} + (2\alpha + 2\beta + 1)\, M^{\alpha} * I^{\beta - 1} . \tag{3.2.11}$$

II. Let $\alpha = 0,1$ and $\beta \geq 1$. Then we obtain

$$\mathrm{Tr}\left(M^{\alpha} * I^{\beta}\right) = (2\alpha + 2\beta + 1)\, M^{\alpha} * I^{\beta - 1} . \tag{3.2.12}$$

III. Assume that $M^{\alpha} \in \mathrm{Ker}_{\alpha}\mathrm{Tr}$ ($\alpha \geq 2$) and consider the quantity

$$k(\alpha,\beta) := \frac{(2\alpha + 1)!!}{(2\alpha + 2\beta + 1)!!} . \tag{3.2.13}$$

Then the equality

$$\mathrm{Tr}^{\beta}\left(M^{\alpha} * I^{\omega}\right) = \left[k(\alpha + \omega - \beta,\beta)\right]^{-1} M^{\alpha} * I^{\omega - \beta}, \quad \omega \geq \beta \tag{3.2.14}$$

holds; Eq. (3.2.14) holds also for $\alpha = 0,1$.

The equality (3.2.14) can easily be derived from (3.2.11) and (3.2.12). Insofar as Eqs. (3.2.11) and (3.2.12) are concerned, the main ingredients involved in the algebraically difficult and very tedious calculation of $\mathrm{Tr}\left(M^{\alpha} * I^{\beta}\right)$ have been satisfactorily presented by Grad on pp. 393-395 in Ref. 1, although for the special case in which $\alpha = 0$. Thus the uninteresting proof of (3.2.11) and (3.2.12) will not be given.

k. The linear operator *Appr* : $\mathbb{E}_s^{\alpha} \to \mathbb{E}_s^{\alpha + 2}$ ($\alpha \geq 0$) is determined by

$$\mathit{Appr}M^{\alpha} := \sum_{\beta = 0}^{[\alpha/2]} (-1)^{\beta} k(\alpha - \beta,\beta + 1)\, \hat{M}^{\alpha[\beta]} * I^{\beta + 1} , \tag{3.2.15}$$

where

$[u]$:= the greatest integer $\leq u$.

In view of the *theorem* just formulated, the foregoing definition yields $\mathrm{Tr}\mathit{Appr}M^{\alpha} = M^{\alpha}$. With this observation in mind, and by no more than a direct reasoning, we arrive at

$$\mathrm{Im}_{\alpha}\mathrm{Tr} = \mathbb{E}_s^{\alpha - 2} , \quad \alpha \geq 2 . \tag{3.2.16}$$

Carrying our argumentation a step further, we see from (3.2.1), (3.2.3), and (3.2.16) that *dim* $\mathrm{Ker}_{\alpha}\mathrm{Tr} = 2\alpha + 1$.

l. Let $M^{\alpha} \in \mathbb{E}_s^{\alpha}$ ($\alpha \geq 2$). Then the symmetric traceless tensor $\langle M^{\alpha} \rangle$ is defined by

$$\langle M^{\alpha} \rangle := M^{\alpha} - \mathit{Appr}\hat{M}^{\alpha[1]} \in \mathrm{Ker}_{\alpha}\mathrm{Tr} . \tag{3.2.17}$$

Moreover, we postulate that

$$\langle M^{1} \rangle := M^{1}, \quad \langle M^{0} \rangle := M^{0} . \tag{3.2.18}$$

We will show in Appendix 3.B that

$$\langle M^{\alpha} * I^{\beta}\rangle := \ell(\alpha + 2\beta,\beta)\ \langle M^{\alpha} \vee I^{\beta}\rangle = 0 \qquad \text{for} \quad \beta > 0. \qquad (3.2.19)$$

m. Construct $M^{\alpha[\beta]}$ and $M^{\alpha|\beta}$:

$$M^{\alpha[\beta]} := \langle \hat{M}^{\alpha[\beta]}\rangle \ , \qquad M^{\alpha|\beta} := \langle \hat{M}^{\alpha|\beta}\rangle \ . \qquad (3.2.20)$$

Then, as the *important* equality(60,51)

$$M^{\alpha} = \sum_{\beta = 0}^{[\alpha/2]} k(\alpha - 2\beta,\beta)\ M^{\alpha[\beta]} * I^{\beta} \ , \qquad \alpha \geq 0 \qquad (3.2.21)$$

reveals, each tensor $M^{\alpha} \in \mathbb{E}_{s}^{\alpha}$ $(\alpha \geq 0)$ can directly be decomposed in terms of $M^{\alpha[\beta]} \in \mathrm{Ker}_{\alpha - 2\beta}\mathrm{Tr}$.

n. Given $M^{\alpha} \in \mathbb{E}^{\alpha}$ and $M^{\beta} \in \mathbb{E}^{\beta}$, we introduce the *symmetric traceless tensor product*:

$$M^{\alpha} \wedge M^{\beta} := \langle M^{\alpha} \vee M^{\beta}\rangle \in \mathrm{Ker}_{\alpha + \beta}\mathrm{Tr} \qquad (3.2.22)$$

and the *contracted symmetric traceless tensor product*:

$$M^{\alpha} \cap M^{\beta} := \langle M^{\alpha} \cup M^{\beta}\rangle \in \mathrm{Ker}_{\alpha + \beta - 2}\mathrm{Tr} \ . \qquad (3.2.23)$$

o. The symmetric traceless tensors constructed from a decomposable tensor and, in particular, those from a *single unit vector* g $\left(\otimes^{\alpha} g;\ \otimes^{1} g := g,\ \otimes^{0} g := 1 \in \mathbb{E}^{0} = \mathbb{R} \right)$,

$$Y^{\alpha}(g) := \langle \otimes^{\alpha} g\rangle \in \mathrm{Ker}_{\alpha}\mathrm{Tr} \ , \qquad \alpha \geq 0 \ , \qquad |g| = 1 \ , \qquad (3.2.24)$$

are used quite widely and have been discussed by several authors.(4,46-48,50) We call $Y^{\alpha}(g)$ *Ikenberry's tensorial harmonics*. Weinert(72) was able to prove that $Y^{\alpha}(g)$ are nothing else than Maxwell's multipole representations of spherical harmonics.(35)

p. The projection of $\mathbb{E}^{\alpha}$ onto the irreducible subspace $\mathrm{Ker}_{\alpha}\mathrm{Tr}$ of symmetric traceless tensors of degree α will be denoted by $E(\alpha|\alpha)$. The *natural projection* $E(\alpha|\alpha) \in \mathbb{E}^{2\alpha}$, for which we have

$$E(\alpha|\alpha) \circ M^{\alpha} := \langle \Pi M^{\alpha}\rangle \ , \qquad (3.2.25)$$

is separately symmetric traceless in both sets of α indices, because, by definition, it is symmetric traceless in one set; in this context, see the work by Coope and Snider.(45)

3.2.B The ∇-Operator and its Various Modifications

Let M^α be a *differentiable* tensor field in $\mathbb{E}$ and ∇ be a gradient with respect to $x \in \mathbb{E}$. Given the ∇-operator, for the most part we shall deal with a variety of other tensorial operators associated with ∇:

a. The action of $\nabla \circ$ on a tensor field M^α of degree α, $\alpha \geq 1$, is defined by

$$\nabla \circ M^\alpha := \underset{(1,2)}{\mathrm{Tr}} \; \nabla M^\alpha \in \mathbb{E}^{\alpha - 1} . \tag{3.2.26}$$

b. In accord with the Meyer-Schröter work,(5-7) the action of the symmetric operator $\nabla \vee$ on a tensor field M^α is defined by

$$\nabla \vee M^\alpha := \Pi \nabla M^\alpha \in \mathbb{E}_s^{\alpha + 1} . \tag{3.2.27}$$

c. Finally, the effect of the symmetric traceless operator $\nabla \wedge$ on a tensor field M^α is given by

$$\nabla \wedge M^\alpha := \langle \nabla \vee M^\alpha \rangle \in \mathrm{Ker}_{\alpha + 1}\mathrm{Tr} . \tag{3.2.28}$$

3.3. Equations of Transfer for the Moments

Given a three-dimensional, classical, moderately rarefied, simple, monatomic gas for which each molecule of unit mass is subject to an external body force K_1 [K_1 being a function of position x and time t but not of momentum (velocity) λ], we start our discussion with the following equation of change for the one-particle density $f(\lambda,x,t)$:

$$\partial_t f + \lambda \circ \nabla_x f + a \circ \nabla_\lambda f = J(f), \tag{3.3.1}$$

where a is the molecular acceleration and J denotes the Boltzmann collision operator.(1,3,4) The molecular acceleration a is tied to the external body force K_1 through *Newton's Second Law* which takes the form(19,62)

$$a = K_1 + K_2 - \lambda \bullet L_2 \tag{3.3.2}$$

in any arbitrary non-inertial frame. Here K_2 and L_2 are, respectively, the velocity-independent part of the inertial force and the skew-symmetric tensor associated with the angular velocity of the reference frame relative to an inertial framing.

For immediate use we introduce the *relative reducible moment* $M^\alpha \in \mathbb{E}_s^\alpha$ of the form

$$M^\alpha := \int_{\mathbb{E}} (\otimes^\alpha \bar{\lambda}) f \, d\lambda \, , \quad \alpha \geq 0 \, , \tag{3.3.3}$$

where

$$\bar{\lambda} := \lambda - u \, , \quad u := \int_{\mathbb{E}} \lambda f \, d\lambda \Big/ \int_{\mathbb{E}} f \, d\lambda \, . \tag{3.3.4}$$

Of course, the quantity u can be interpreted as a macroscopic velocity field.

By (3.3.1)-(3.3.4) we easily show that

$$\partial_t M^\alpha + \nabla \cdot \left(u \otimes M^\alpha + M^{\alpha+1} \right) + \alpha \, L \cup M^\alpha$$
$$+ \alpha \left(\partial_t u + u \cdot L - K \right) \vee M^{\alpha-1} = P^\alpha \, , \quad \alpha \geq 0 \, , \tag{3.3.5a}$$

$$K := K_1 + K_2 \, , \quad L := L_1 + L_2 \, , \tag{3.3.5b}$$

$$L_1 := \nabla u \, , \quad M^{-1} := 0 \, , \tag{3.3.5c}$$

where the collision integrals P^α are defined by

$$P^\alpha := \int_{\mathbb{E}} \otimes^\alpha \bar{\lambda} \, J(f) \, d\lambda \in \mathbb{E}_s^\alpha \, , \quad \alpha \geq 0 \, . \tag{3.3.6}$$

[To simplify our notation, in Eq. (3.3.5a) the ∇_x-operator is replaced by the operator ∇.] From the balance of linear momentum, we see at once that the evolution of M^α is unaffected by both external body forces and translational accelerations of the non-inertial frame and is affected by rotations of the reference frame only through the molecular Coriolis force represented by $- \lambda \bullet L_2$. Indeed, combining

$$\partial_t u + u \cdot L + R = K \, , \quad R := \frac{1}{M^0} \nabla \cdot M^2 \tag{3.3.7}$$

with (3.3.5a), we arrive at

$$\partial_t M^{\alpha} + \nabla \cdot \left(u \otimes M^{\alpha} + M^{\alpha + 1} \right) + \alpha\left(L \cup M^{\alpha} - R \vee M^{\alpha - 1} \right) = P^{\alpha}, \qquad \alpha \ge 0. \qquad (3.3.8)$$

We now wish to obtain the balance equations for $\hat{M}^{\alpha[\beta]}$. To this end we appeal to the

Lemma. Let $M^{\alpha} \in \mathbb{E}^{\alpha}_{s}$, $L \in \mathbb{E}^{2}$, and $R \in \mathbb{E}$ be three arbitrary tensors. Then the action of the operator $\alpha\, \mathrm{Tr}^{\beta}$ on $L \cup M^{\alpha}$ and $R \vee M^{\alpha - 1}$ is given by

$$\alpha\, \mathrm{Tr}^{\beta}\left(L \cup M^{\alpha}\right) = (\alpha - 2\beta)\, L \cup \hat{M}^{\alpha[\beta]} + 2\beta\, L \cdot \hat{M}^{\alpha[\beta - 1]}, \qquad (3.3.9a)$$

$$\alpha\, \mathrm{Tr}^{\beta}\left(R \vee M^{\alpha - 1}\right) = (\alpha - 2\beta)\, R \vee \hat{M}^{\alpha - 1[\beta]} + 2\beta\, R \cdot \hat{M}^{\alpha - 1[\beta - 1]}, \qquad (3.3.9b)$$

$$\alpha \ge 2\beta \ge 0. \qquad (3.3.9c)$$

Proof. If $\beta = 1$, the *lemma* follows from the argumentation in Appendix 3.C. Now assume that (3.3.9a) is true for $\beta - 1$ ($\beta \ge 2$). Then we have

$$\alpha\, \mathrm{Tr}^{\beta}\left(L \cup M^{\alpha}\right) = \mathrm{Tr}\left[\alpha\, \mathrm{Tr}^{\beta - 1}\left(L \cup M^{\alpha}\right)\right]$$

$$= (\alpha - 2\beta + 2)\, \mathrm{Tr}\left(L \cup \hat{M}^{\alpha[\beta - 1]}\right) + 2(\beta - 1)\, L \cdot \hat{M}^{\alpha[\beta - 1]}.$$

Applying (3.3.9a) for $\beta = 1$ and $\alpha = \alpha' - 2\beta + 2$ ($\alpha' \to \alpha$), we obtain

$$\alpha\, \mathrm{Tr}^{\beta}\left(L \cup M^{\alpha}\right) = (\alpha - 2\beta)\, L \cup \hat{M}^{\alpha[\beta]} + 2L \cdot \hat{M}^{\alpha[\beta - 1]} + 2(\beta - 1)\, L \cdot \hat{M}^{\alpha[\beta - 1]},$$

and so the induction is closed. In exactly the same way one can show that Eq. (3.3.9b) holds also. This completes the proof of (3.3.9). ■

Due to the fact that for certain admissible choices of the pair (α,β) the contracted moments

$$\mathcal{M}^{\alpha}_{\beta} := \left\{\hat{M}^{\alpha[\beta - 1]}, \hat{M}^{\alpha - 1[\beta]}, \hat{M}^{\alpha - 1[\beta - 1]}\right\}$$

appearing on the r.h.s. of (3.3.9a) and (3.3.9b) are not *well* *defined*, we extend the range of validity of Eqs. (3.3.9) to these cases by setting then $\mathcal{M}^{\alpha}_{\beta} = 0$.

In view of (3.3.8) and (3.3.9), an equation of change for $\hat{M}^{\alpha[\beta]}$ $(\alpha \geq 2\beta \geq 0)$ is easily seen to be

$$\partial_t \hat{M}^{\alpha[\beta]} + \nabla \cdot \left(u \otimes \hat{M}^{\alpha[\beta]} + \hat{M}^{\alpha+1[\beta]} \right)$$

$$+ (\alpha - 2\beta)\left(L \cup \hat{M}^{\alpha[\beta]} - R \vee \hat{M}^{\alpha-1[\beta]} \right)$$

$$+ 2\beta\left(L \cdot \hat{M}^{\alpha[\beta-1]} - R \cdot \hat{M}^{\alpha-1[\beta-1]} \right) = \hat{P}^{\alpha[\beta]} , \qquad (3.3.10)$$

where, according to our notation [see Section 3.2.A, Eqs. (3.2.4)],

$$\hat{P}^{\alpha[\beta]} = \mathrm{Tr}^{\beta} P^{\alpha} . \qquad (3.3.11)$$

To proceed further in the calculation of the irreducible equations of transfer, we have to evaluate the action of the natural projection $E(\alpha - 2\beta | \alpha - 2\beta)$ upon each tensorial expression in Eq. (3.3.10) directly in terms of $M^{\alpha[\beta]}$ or of $M^{\alpha|\beta}$; we call $M^{\alpha[\beta]} \in \mathrm{Ker}_{\alpha - 2\beta}\mathrm{Tr}$ and $M^{\alpha|\beta} \in \mathrm{Ker}_{\alpha}\mathrm{Tr}$ the *relative symmetric traceless moments* of the one-point distribution function f. This decisive step is the only one which is algebraically difficult, and so for the moment we simply assert that

$$\langle \partial_t \hat{M}^{\alpha[\beta]} \rangle = \partial_t M^{\alpha[\beta]} , \qquad (3.3.12)$$

$$\langle \nabla \cdot \left(u \otimes \hat{M}^{\alpha[\beta]} \right) \rangle = \nabla \cdot \left(u \otimes M^{\alpha[\beta]} \right) , \qquad (3.3.13)$$

$$\langle \nabla \cdot \hat{M}^{\alpha+1[\beta]} \rangle = \frac{\alpha - 2\beta}{(2\alpha - 4\beta + 1)} \nabla \wedge M^{\alpha+1[\beta+1]}$$

$$+ \nabla \cdot M^{\alpha+1[\beta]} , \qquad (3.3.14)$$

$$(\alpha - 2\beta) \langle L \cup \hat{M}^{\alpha[\beta]} \rangle = (\alpha - 2\beta) \frac{\alpha - 2\beta - 1}{2\alpha - 4\beta - 1} L \wedge M^{\alpha[\beta+1]}$$

$$+ (\alpha - 2\beta) L \cap M^{\alpha[\beta]} , \qquad (3.3.15)$$

$$(\alpha - 2\beta)\langle R \vee \hat{M}^{\alpha - 1[\beta]}\rangle = (\alpha - 2\beta)\, R \wedge M^{\alpha - 1[\beta]}, \qquad (3.3.16)$$

$$2\beta\langle L \bullet \hat{M}^{\alpha[\beta - 1]}\rangle = \frac{2\beta}{2\alpha - 4\beta + 3}\, S\, M^{\alpha[\beta]}$$

$$+ \frac{2\beta(\alpha - 2\beta)(\alpha - 2\beta - 1)}{(2\alpha - 4\beta - 1)(2\alpha - 4\beta + 1)}\, L \wedge M^{\alpha[\beta + 1]}$$

$$+ \frac{4\beta(\alpha - 2\beta)}{2\alpha - 4\beta + 3}\, U \cap M^{\alpha[\beta]} + 2\beta\, L \bullet M^{\alpha[\beta - 1]}, \qquad (3.3.17)$$

$$2\beta\langle R \bullet \hat{M}^{\alpha - 1[\beta - 1]}\rangle = \frac{2\beta(\alpha - 2\beta)}{2\alpha - 4\beta + 1}\, R \wedge M^{\alpha - 1[\beta]}$$

$$+ 2\beta\, R \bullet M^{\alpha - 1[\beta - 1]}, \qquad (3.3.18)$$

$$\langle \hat{P}^{\alpha[\beta]}\rangle = P^{\alpha[\beta]}, \qquad (3.3.19)$$

where

$$U := \Pi L, \qquad S := \mathrm{Tr}\Pi L. \qquad (3.3.20)$$

For completeness, however, we shall present in Appendix 3.D a sketch of the proof of the equality (3.3.17). Then the derivation of Eqs. (3.3.12)-(3.3.16), (3.3.18), and (3.3.19) can be repeated essentially word for word with only slight technical changes in the method.

A glance at Eqs. (3.3.10), (3.3.12)-(3.3.19), and the transformation rule $M^{\alpha + 2\beta[\beta]} = M^{\alpha|\beta}$ shows that

$$\partial_t M^{\alpha|\beta} + \nabla \bullet \left[u \otimes M^{\alpha|\beta} + M^{\alpha + 1|\beta}\right] + \frac{\alpha}{2\alpha + 1}\, \nabla \wedge M^{\alpha - 1|\beta + 1}$$

$$+ \frac{1}{2\alpha + 3}\left[2\beta\, S M^{\alpha|\beta} + 4\alpha\beta\, U \cap M^{\alpha|\beta} + \alpha(2\alpha + 3)\, L \cap M^{\alpha|\beta}\right]$$

$$+ \frac{\alpha(2\alpha + 2\beta + 1)}{2\alpha + 1}\left[\frac{\alpha - 1}{2\alpha - 1}\, L \wedge M^{\alpha - 2|\beta + 1} - R \wedge M^{\alpha - 1|\beta}\right]$$

$$+ 2\beta\left[L \bullet M^{\alpha + 2|\beta - 1} - R \bullet M^{\alpha + 1|\beta - 1}\right] = P^{\alpha|\beta} \qquad (3.3.21)$$

for $\alpha \geq 0$ and $\beta \geq 0$, where, by definition,

$$M^{\alpha|-1} := 0 \ , \quad M^{-1|\beta} := 0 \ , \quad M^{-2|\beta} := 0 \tag{3.3.22}$$

and

$$p^{\alpha|\beta} := p^{\alpha + 2\beta[\beta]} \ . \tag{3.3.23}$$

As elementary inspection reveals, the differential equations just obtained do not separate; in general, not only $M^{\nu|\gamma}$ with $\nu + 2\gamma = \alpha + 2\beta$ and $\gamma = \beta - 1 < \beta$ occurs in Eq. (3.3.21), but also the symmetric traceless moments $M^{\nu|\gamma}$ with $\nu + 2\gamma = \alpha + 2\beta + 1 > \alpha + 2\beta$ are found in the irreducible equation of transfer for $M^{\alpha|\beta}$, at least insofar as the l.h.s. of that equation is concerned. Even if one ignores a very difficult problem concerning the form of the explicit and exact dependence of $p^{\alpha|\beta}$ upon $M^{\nu|\gamma}$, these simple observations demonstrate the so-called forward coupling of the equations of moments.

In conclusion, we rest content to point out that the special cases of Eq. (3.3.21) were derived and extensively applied, as one of many segments of the whole procedure called *Maxwellian iteration*, by Truesdell and Ikenberry in the middle 1950s, but, to the best of our knowledge, they have not to this day published the *single* and *compact* expression (3.3.21). Johnston(69) was the first to solve some of the nasty tensorial problems of this work and to invent the general method of deriving the irreducible equations of transfer for Ikenberry's tensorial expansion coefficients $f_\alpha \in \mathrm{Ker}_\alpha \mathrm{Tr}$ of f. On the other side, since the irreducible "moments" f_α are not fields in the ordinary sense of continuum mechanics [aside from the dependence upon position x and time t, they are functions of $|\bar{\lambda}| \in \mathbb{R}^+ := [0,\infty)$], it is hoped that the present theory of the equations of moments constitutes partly an extension and partly a supplement to that in Refs. 48, 69, and 4.

3.4. Illustration of the Idea of Irreducible Tensor Description, Applied to the Moment Procedure of Grad's Type

3.4.A. Reducible and Irreducible Representations of the Distribution Function

In the paper in which Grad(1) developed his thirteen moment approximation to the velocity distribution function f for a rarefied gas he utilized certain three-dimensional Hermite polynomials, whose properties he considered in a companion paper.(55) In terms of a dimensionless vector $\hat{\lambda}$, the Hermite polynomials, which are components of tensors $B^{\alpha}(\hat{\lambda}) \in \mathbb{E}^{\alpha}_{s}$ defined by

$$B^{\alpha}(\hat{\lambda}) := (-1)^{\alpha}\, \mathbb{H}_{o}^{-1} \left(\partial/\partial\hat{\lambda}\right)^{\alpha} \mathbb{H}_{o} \,, \tag{3.4.1a}$$

$$\mathbb{H}_{o}(\hat{\lambda}) := \left(2\pi\right)^{-3/2} exp\left(-\frac{1}{2}|\hat{\lambda}|^{2}\right) \tag{3.4.1b}$$

are of total degree α in the three components of $\hat{\lambda} \in \mathbb{E}$, as is readily seen from (3.4.1). [In Eq. (3.4.1a) $\left(\partial/\partial\hat{\lambda}\right)^{\alpha} \mathbb{H}_{o}$ is the result of the α-fold successive application of the differential operator $\partial/\partial\hat{\lambda} := \nabla_{\lambda|\lambda = \hat{\lambda}}$ to the scalar function $\mathbb{H}_{o}$.] Inspection shows that the Hermite polynomials are orthogonal with respect to $\mathbb{H}_{o}$.(55) In addition, in the space of functions which are square-integrable over $\mathbb{E}$ with weight $\mathbb{H}_{o}$, the Hermite polynomials are *complete*.(55)

Grad considers only such molecular densities f as can be expanded in a series of Hermite (tensorial) polynomials $B^{\alpha}(\hat{\lambda})$. In order to set down the expansion of the distribution function f around a local Maxwellian f_{o} given by

$$f_{o}(\lambda,x,t) := \rho(x,t)\left[k_{B}T(x,t)\right]^{-3/2} \mathbb{H}_{o}(\hat{\lambda}) \,, \tag{3.4.2a}$$

$$\rho := M^{\rho} \,, \qquad k_{B}T := \frac{1}{3}\, TrM^{2} \Big/ M^{\rho} \,, \tag{3.4.2b}$$

$$\hat{\lambda} := \left(k_{B}T\right)^{-1/2} \bar{\lambda} = \left(k_{B}T\right)^{-1/2} (\lambda - u) \,, \tag{3.4.2c}$$

we let ρ, u, and $(3/2)k_{B}T$ be the mass density, the macroscopic velocity, and the internal energy per unit mass, respectively, that correspond to f.[12] In the result we obtain

$$f(\lambda,x,t) = f_o(\lambda,x,t) \sum_{\alpha=0}^{\infty} \frac{1}{\alpha!} b^{\alpha}(x,t) \cdot B^{\alpha}(\hat{\lambda}) . \tag{3.4.3}$$

If we multiply each side of (3.4.3) by $B^{\alpha}(\hat{\lambda})$, integrate with respect to λ, and make use of the orthogonal properties of $B^{\alpha}(\hat{\lambda})$, then we find that the so-called Hermite expansion coefficients $b^{\alpha} \in \mathbb{E}^{\alpha}_{s}$ of the distribution function are functionally related to f:

$$b^{\alpha} = \frac{1}{\rho} \int_{\mathbb{E}} \mathbb{B}^{\alpha}(\hat{\lambda}) f \, d\lambda . \tag{3.4.4}$$

Both by placing (3.4.1) into (3.4.4) and by appealing to the definitions of ρ, u, and $k_B T$, we arrive at

$$b^{o} = 1, \quad b^{1} = \mathrm{Tr} b^{2} = 0 . \tag{3.4.5}$$

Of course, Grad's expansion is not ideally suited to the irreducible equations of transfer. However, once Eqs. (3.4.1)-(3.4.5) have been set down, we may derive from them, through the elementary operations of multilinear algebra, the alternative representation of the distribution function f that fits in together and with the system (3.3.21), omitting but not disregarding other independent, important approaches fashioned by mathematical apparatus such as spherical tensors and harmonics. [In fact, spherical harmonics have been extremely useful for the calculation of collision integrals for all interaction potentials, not only Maxwellian one; see, e.g., Suchy(70,71) and Weinert.(72,73)] Insofar as we are aware, in the literature on the kinetic theory the nasty problems generated by, and associated with, the *rigorous* transformation of Grad's expansion of the distribution function f into its irreducible analogue

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[12] The quantities k_B and T, which we introduce only formally in this section, can be termed Boltzmann's constant and the absolute temperature, respectively.

have been outflanked rather than logically solved.

We begin by eliminating $B^{\alpha}(\hat{\lambda})$ in (3.4.3) through use of

$$B^{\alpha} = \sum_{\beta=0}^{[\alpha/2]} \ell(\alpha,\beta)k(\alpha-2\beta,\beta)\, B^{\alpha[\beta]} \vee I^{\beta} . \tag{3.4.6}$$

Hence

$$f = f_{\circ} \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{[\alpha/2]} \frac{1}{\alpha!} \ell(\alpha,\beta)k(\alpha-2\beta,\beta)\, b^{\alpha[\beta]} \cdot B^{\alpha[\beta]} . \tag{3.4.7}$$

To say more about f than Eq. (3.4.7) asserts, we must have precise knowledge of $B^{\alpha[\beta]}$. Recalling now(55)

$$B^{\alpha}(\hat{\lambda}) = \sum_{\omega=0}^{[\alpha/2]} (-1)^{\omega}\ell(\alpha,\omega)\Big[\big[\otimes^{\alpha-2\omega}\hat{\lambda}\big] \vee I^{\omega}\Big] \tag{3.4.8}$$

as well as (3.2.9), (3.2.11), and (3.2.19), we obtain for $B^{\alpha[\beta]}$

$$B^{\alpha[\beta]} = \langle \mathrm{Tr}^{\beta} B^{\alpha} \rangle = \sum_{\omega=0}^{\beta} (-1)^{\omega}\ell(\alpha,\omega)|\hat{\lambda}|^{\alpha-2\omega} \times \langle \mathrm{Tr}^{\beta}\Big[\big[\otimes^{\alpha-2\omega} g\big] \vee I^{\omega}\Big]\rangle , \tag{3.4.9}$$

where

$$g := |\hat{\lambda}|^{-1}\hat{\lambda} . \tag{3.4.10}$$

Using for $M^{\alpha-2\omega}$ here the tensor $\otimes^{\alpha-2\omega} g$, and observing that $M^{\alpha-2\omega\,[\gamma]} = Y^{\alpha-2\omega-2\gamma}(g)$, we obtain from the decomposition (3.2.21) in which $\alpha \to \alpha - 2\omega$ the equality

$$\otimes^{\alpha-2\omega} g = \sum_{\gamma=0}^{[(\alpha-2\omega)/2]} \ell(\alpha-2\omega,\gamma)k(\alpha-2\omega-2\gamma,\gamma) \times Y^{\alpha-2\omega-2\gamma}(g) \vee I^{\gamma} . \tag{3.4.11}$$

Clearly, in virtue of (3.4.11), Eq. (3.4.9) may be written as

$$B^{\alpha[\beta]} = \sum_{\omega=0}^{\beta} \sum_{\gamma=0}^{[(\alpha-2\omega)/2]} (-1)^{\omega}\ell(\alpha,\omega)\ell(\alpha-2\omega,\gamma)$$

$$\times k(\alpha-2\omega-2\gamma,\gamma)|\hat{\lambda}|^{\alpha-2\omega} \langle \mathrm{Tr}^{\beta}\left[Y^{\alpha-2\omega-2\gamma}(g) \vee I^{\omega+\gamma}\right]\rangle$$

$$= \sum_{\omega=0}^{\beta} (-1)^{\omega}\ell(\alpha,\omega)\ell(\alpha-2\omega,\beta-\omega)k(\alpha-2\beta,\beta-\omega)$$

$$\times |\hat{\lambda}|^{\alpha-2\omega}\, \mathrm{Tr}^{\beta}\left[Y^{\alpha-2\beta}(g) \vee I^{\beta}\right]. \qquad (3.4.12)$$

Just as before, in (3.4.12) we have made use of (3.2.19). Taking

$$Y^{\alpha-2\beta}(g) \vee I^{\beta} = [\ell(\alpha,\beta)]^{-1}\, Y^{\alpha-2\beta}(g) * I^{\beta} \qquad (3.4.13)$$

and appealing to

$$\mathrm{Tr}^{\beta}\left[Y^{\alpha-2\beta}(g) * I^{\beta}\right] = [k(\alpha-2\beta,\beta)]^{-1}\, Y^{\alpha-2\beta}(g)\ , \qquad (3.4.14)$$

we conclude that

$$B^{\alpha[\beta]} = \sum_{\omega=0}^{\beta} (-1)^{\omega}\ell(\alpha,\omega)\ell(\alpha-2\omega,\beta-\omega)k(\alpha-2\beta,\beta-\omega)$$

$$\times [\ell(\alpha,\beta)k(\alpha-2\beta,\beta)]^{-1}|\hat{\lambda}|^{\alpha-2\omega}\, Y^{\alpha-2\beta}(g)$$

$$= (-1)^{\beta}2^{\beta}\beta!\; Y^{\alpha-2\beta}(\hat{\lambda}) \sum_{\omega=0}^{\beta} (-1)^{\beta+\omega}$$

$$\times \frac{[2(\alpha-2\beta)+2\beta+1]!!}{2^{\omega}\omega!(\beta-\omega)![2(\alpha-2\beta)+2(\beta-\omega)+1]!!}\, z^{\beta-\omega}\ , \qquad (3.4.15)$$

where

$$Y^{\alpha}(\hat{\lambda}) := \langle \otimes^{\alpha}\hat{\lambda} \rangle\ , \qquad z := \frac{1}{2}|\hat{\lambda}|^{2}\ . \qquad (3.4.16)$$

If, as usual, we identify(34,35)

$$\sum_{\omega = 0}^{\beta} (-1)^{\beta + \omega} \frac{(2\alpha + 2\beta + 1)!!}{2^{\omega}\omega!(\beta - \omega)![2\alpha + 2(\beta - \omega) + 1]!!} z^{\beta - \omega} \tag{3.4.17}$$

with the so-called Laguerre polynomials $L_{\beta}^{(\alpha + \frac{1}{2})}(z)$, then we obtain

$$B^{\alpha[\beta]}(\hat{\lambda}) = (-1)^{\beta}2^{\beta}\beta!\, Y^{\alpha - 2\beta}(\hat{\lambda})\, L_{\beta}^{(\alpha - 2\beta + \frac{1}{2})}(z)\ , \tag{3.4.18}$$

and thence by placing (3.4.18) into (3.4.7) we find that the irreducible counterpart of (3.4.3), which fits in *together* and *with* Eqs. (3.3.21), is

$$f = f_{\circ} \sum_{\alpha = 0}^{\infty} \sum_{\beta = 0}^{\infty} (-1)^{\beta}\mathcal{A}(\alpha,\beta)\frac{1}{\alpha!}\, b^{\alpha|\beta}(x,t) \cdot Y^{\alpha}(\hat{\lambda})L_{\beta}^{(\alpha + \frac{1}{2})}(z). \tag{3.4.19}$$

Denoting by $\mathbb{R}^{+}$ the real interval $[0,+\infty)$ and by $\mathbb{K}$ the set of unit vectors, $\mathbb{K} := \{g \in \mathbb{E} : |g| = 1\}$, we see from(34,35)

$$\int_{\mathbb{R}^+} z^{\alpha + \frac{1}{2}} e^{-z} L_{\beta}^{(\alpha + \frac{1}{2})}(z)\, L_{\gamma}^{(\alpha + \frac{1}{2})}(z)\, dz = \frac{\sqrt{\pi}\,(2\alpha + 2\beta + 1)!!}{2^{\alpha + \beta + 1}\beta!}\, \delta_{\beta\gamma} \tag{3.4.20}$$

and(45)

$$\int_{\mathbb{K}} Y^{\alpha}(g) \otimes Y^{\nu}(g)\, dg = \frac{4\pi\alpha!}{(2\alpha + 1)!!}\, \delta_{\alpha\nu}\, \mathbb{E}(\alpha|\alpha) \tag{3.4.21}$$

that the Laguerre-Ikenberry expansion coefficients $b^{\alpha|\beta} \in$ $\in \mathrm{Ker}_{\alpha}\mathrm{Tr}$ of the one-particle density f are the expectations of the corresponding Laguerre-Ikenberry functions $Y^{\alpha}(\hat{\lambda})L_{\beta}^{(\alpha + \frac{1}{2})}(z)$:

$$b^{\alpha|\beta} = (-1)^{\beta}\, \frac{(2\beta)!!}{\rho} \int_{\mathbb{E}} Y^{\alpha}(\hat{\lambda})L_{\beta}^{(\alpha + \frac{1}{2})}(z) f\, d\Lambda\ . \tag{3.4.22}$$

The quantity $\delta_{\alpha\beta}$ appearing in Eqs. (3.4.20) and (3.4.21) denotes the Kronecker delta and $E(\alpha|\alpha)$ is, as we know, the natural projection defined by (3.2.25).

In order to facilitate conversions between $M^{\alpha|\beta}$ and $b^{\alpha|\beta}$, we record here the algebraic equation(34,35) for z^β:

$$z^{\beta} = \frac{1}{2^{\beta}} \sum_{\omega=0}^{\beta} (-1)^{\omega} \begin{bmatrix} \beta \\ \omega \end{bmatrix} (2\omega)!!\,[k(\alpha,\beta)]^{-1} k(\alpha,\omega) L_{\omega}^{(\alpha + \frac{1}{2})}(z). \tag{3.4.23}$$

This equality and the decomposition (3.4.17) for $L_{\beta}^{(\alpha + \frac{1}{2})}(z)$ allow us to write

$$M^{\alpha|\beta} = \rho (k_B T)^{\beta + \alpha/2} \sum_{\omega=0}^{\beta} \begin{bmatrix} \beta \\ \omega \end{bmatrix} [k(\alpha,\beta)]^{-1} k(\alpha,\omega)\; b^{\alpha|\omega}, \tag{3.4.24}$$

$$(-1)^{\beta}\, b^{\alpha|\beta} = \frac{1}{\rho} (k_B T)^{-\alpha/2} \sum_{\omega=0}^{\beta} (-1)^{\omega} \begin{bmatrix} \beta \\ \omega \end{bmatrix} (k_B T)^{-\omega} \times [k(\alpha,\beta)]^{-1} k(\alpha,\omega)\; M^{\alpha|\omega}. \tag{3.4.25}$$

In summary, the expansion (3.4.19) confirms the importance of Laguerre polynomials in the kinetic theory and, at the same time, shows that spherical harmonics may be replaced by those of Ikenberry. According to Coope and Snider,(45) the relation of $Y^{\alpha}(\hat{\lambda}) = |\hat{\lambda}|^{\alpha}\, Y^{\alpha}(g)$ to the properties of three-dimensional space is more apparent than with spherical tensors, where one axis is arbitrarily distinguished,[13] and, furthermore, the tensor equations are far more compact, symmetric, and understandable than the clumsy spherical harmonic results.

Johnston(60) has described a general procedure for expressing f in terms of its tensorial irreducible expansion

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[13] Insofar as the collision process rather than the system of irreducible equations of transfer is concerned, the Coope-Snider statement to demonstrate that one axis has to be arbitrarily distinguished is somewhat dubious, because the axis under consideration points in the direction of the (relative) velocity after a binary encounter; see, e.g., Refs. 70-73.

coefficients $f_\alpha \in \mathrm{Ker}_\alpha \mathrm{Tr}$,

$$f(\lambda,x,t) = \sum_{\alpha=0}^{\infty} f_\alpha\left(|\bar{\lambda}|,x,t\right) \cdot Y^\alpha(g) \,, \tag{3.4.26}$$

and also derived the system of balance equations for them. It is important to observe that Eq. (12) in Ref. 69 and Eq. (3.3.21) in the present work are not completely equivalent. First, from the very beginning Johnston considers only such molecular densities f as may be represented in the *mean* by a series of Ikenberry's coefficients f_α, and second, he makes no use of the expansion

$$f_\alpha = f_\circ \sum_{\beta=0}^{\infty} (-1)^\beta k(\alpha,\beta) \frac{1}{\alpha!} L_\beta^{(\alpha + \frac{1}{2})} b^{\alpha|\beta} \tag{3.4.27}$$

and the transformation rule (3.4.25). Had we chosen the suitable adjustment of Johnston's method, substituting (3.4.27) into Eq. (12) in Ref. 69, we would have been able to see clearly how to obtain the equations of transfer for $b^{\alpha|\beta}$. Of course, because of (3.4.26) and (3.4.27), in the procedure just outlined the one-particle density f must possess the Laguerre-Ikenberry moments $b^{\alpha|\beta}$ of all orders. In contrast, Eq. (3.3.21) is valid, as its derivation in Section 3.3 shows, for a much broader class of functions f, indeed for ones which need possess only a finite number of the direct moments $M^{\alpha|\beta}$.

3.4.B. Irreducible Variant of Grad's Method of Truncation

Before proposing and exploiting the irreducible variant of Grad's truncation scheme, we shall complete our survey of results that conform with those of Section 1.3.C on one-dimensional quasiparticle gases. To this end let us consider the following standard expression for the entropy density h:

$$h = - k_B \int_{\mathbb{E}} f \ln\left(\mathbb{C} f\right) d\lambda \,, \tag{3.4.28}$$

$\mathbb{C}$ being a constant directly associated with that of Planck; the exact form of this constant is of no importance in our further investigations. We can arrive at an approximate formula for an h corresponding to

$$f = f_o(1 + \varepsilon) , \tag{3.4.29a}$$

$$\varepsilon := \sum_{\alpha + 2\beta \geq 2} (-1)^{\beta} k(\alpha,\beta)\frac{1}{\alpha!}\, b^{\alpha|\beta}(x,t) \cdot Y^{\alpha}(\hat{\lambda})L_{\beta}^{(\alpha + \frac{1}{2})}(z), \tag{3.4.29b}$$

which amounts to determining h from $b^{\alpha|\beta}$. In the neighbourhood of the state of local equilibrium, if instead of the logarithm $ln(1 + \varepsilon)$ we use the first two terms in its Taylor expansion $\varepsilon - \frac{1}{2}\varepsilon^2 + \ldots$, we obtain for $h - h_o$

$$h - h_o \cong -\frac{1}{2}\rho k_B \sum_{\alpha + 2\beta \geq 2} \frac{1}{(\alpha + 2\beta)!}\, \ell(\alpha + 2\beta,\beta)k(\alpha,\beta) \times b^{\alpha|\beta} \cdot b^{\alpha|\beta} \leq 0 , \tag{3.4.30}$$

where

$$h_o := - k_B \int_{\mathbb{E}} f_o ln(\mathbb{C}f_o)\, d\lambda = \frac{3}{2}\rho k_B - \rho k_B\, ln\left[\mathbb{C}\rho\left(\frac{1}{2\pi k_B T}\right)^{3/2}\right] , \tag{3.4.31}$$

and the series (3.4.30) converges, because we suppose that $exp\left(\frac{1}{4}\hat{\lambda}^2\right)f$ be square-integrable over $\mathbb{E}$. Interpreting (3.4.30), the choice of the Laguerre-Ikenberry expansion coefficients $b^{\alpha|\beta}$ of f always *diagonalizes* the largest contribution to $h - h_o$ and the right-hand inequality in (3.4.30) suggests that f_o gives h the greatest value it can attain for all molecular densities f corresponding to the same gross condition $\left(b^o = 1, b^1 = Trb^2 = 0\right)$. Obvious, trivial, and well-known as these observations are, they *were* and *will be* of great, almost decisive import in our extensions of the range of validity of Grad's ideas to quasiparticle gaseous systems.(23,25,26)

Given Boltzmann's operator J for any spherically symmetrical model, the balance equation (3.3.21) in which $\partial_t M^{\alpha|\beta}$ occurs contains a contribution from the collision integral

$$P^{\alpha|\beta} = \int_{\mathbb{E}} |\bar{\lambda}|^{2\beta} \, Y^{\alpha}(\bar{\lambda}) J(f) \, d\lambda \ . \tag{3.4.32}$$

We insert the expansion (3.4.19) into (3.4.32) and make use of (3.4.25). The result is an infinite sequence of differential equations in the symmetric traceless moments $M^{\alpha|\beta}$ of f, which in general cannot be solved without some truncation procedure. If we are willing to accept Grad's method, in the system of equations up to $\partial_t M^{\alpha|\beta}$, $\alpha + 2\beta = r$ $(r = 2,3,\dots)$, we may set all Laguerre-Ikenberry coefficients $b^{\alpha|\beta}$ for which $\alpha + 2\beta > r$ equal to zero. Then, due to (3.4.25), we obtain

$$\sum_{\omega=0}^{\beta} (-1)^{\omega} \begin{bmatrix} \beta \\ \omega \end{bmatrix} (k_B T)^{-\omega} \, [\mathcal{A}(\alpha,\beta)]^{-1} \mathcal{A}(\alpha,\omega) \, M^{\alpha|\omega} = 0 \ , \tag{3.4.33a}$$

$$\alpha + 2\beta > r \ . \tag{3.4.33b}$$

On the left differential side, only the final equations for $\partial_t M^{\alpha|\beta}$, $\alpha + 2\beta = r$, themselves are altered; in particular, using (3.4.33) in which $\alpha + 2\beta = r + 1$,

$$M^{\alpha+1|\beta} = \sum_{\omega=0}^{\beta-1} (-1)^{\beta-\omega+1} \begin{bmatrix} \beta \\ \omega \end{bmatrix} (k_B T)^{\beta-\omega} \times [\mathcal{A}(\alpha+1,\beta)]^{-1} \mathcal{A}(\alpha+1,\omega) \, M^{\alpha+1|\omega} \ , \tag{3.4.34a}$$

$$M^{\alpha-1|\beta+1} = \sum_{\omega=0}^{\beta} (-1)^{\beta-\omega} \begin{bmatrix} \beta+1 \\ \omega \end{bmatrix} (k_B T)^{\beta-\omega+1} \times [\mathcal{A}(\alpha-1,\beta+1)]^{-1} \mathcal{A}(\alpha-1,\omega) \, M^{\alpha-1|\omega} \ , \tag{3.4.34b}$$

$$\alpha + 2\beta = r \ . \tag{3.4.34c}$$

On the right collision side, we must replace (3.4.19) by

$$f = f_{\circ} \sum_{\alpha + 2\beta \le r} (-1)^{\beta} A(\alpha,\beta) \frac{1}{\alpha!} b^{\alpha|\beta}(x,t) \cdot Y^{\alpha}(\hat{\lambda}) L_{\beta}^{(\alpha + \frac{1}{2})}(z) \tag{3.4.35}$$

in (3.4.32), except in the case of Maxwellian molecules where the exact expression (3.4.32) for $P^{\alpha|\beta}$, $\alpha + 2\beta \le r$, already has the "proper" form depending on $M^{\nu|\gamma}$, $\nu + 2\gamma \le \alpha + 2\beta \le r$, alone.

The method of Müller and Liu, described in detail in Ref. 15 and based upon the effect of diagonalization of an approximate formula for an h,(22) allows one to establish the precise sense of the statement that the system of differential equations for $M^{\alpha|\beta}$, $\alpha + 2\beta \le r$, just obtained, if supplemented by the principle of conservation of momentum (3.3.7), describes a certain thermomechanical process.

3.5. A Gas of Maxwellian Molecules

3.5.A. The Structure of Collision Integrals

From(55)

$$\otimes^{\alpha} \hat{\lambda} = \sum_{\omega = 0}^{[\alpha/2]} \ell(\alpha,\omega) \left[B^{\alpha - 2\omega}(\hat{\lambda}) \vee I^{\omega} \right] \tag{3.5.1}$$

and the definitions (3.4.2c) and (3.3.6) it follows that

$$P^{\alpha} = \rho (k_B T)^{\alpha/2} \sum_{\omega = 0}^{[\alpha/2]} \ell(\alpha,\omega) \left[Q^{\alpha - 2\omega} \vee I^{\omega} \right] , \tag{3.5.2}$$

where

$$Q^{\alpha} := \frac{1}{\rho} \int_{\mathbb{E}} B^{\alpha}(\hat{\lambda}) J(f) \, d\lambda . \tag{3.5.3}$$

In a gas of Maxwellian molecules, the collision integral Q^{α}, $\alpha \ge 2$, is a linear combination of the Hermite moments of degree α plus a bilinear combination of the Hermite moments of lower degree, the sum of the degrees in each term being α:

$$Q^{\alpha} = Q_{L}^{\alpha} + Q_{N}^{\alpha} , \quad \alpha \geq 2, \tag{3.5.4a}$$

$$Q_{L}^{\alpha} = \rho \sum_{\omega = 0}^{[\alpha/2]} A_{\omega}^{\alpha} \, \hat{b}^{\alpha[\omega]} \vee I^{\omega} , \tag{3.5.4b}$$

$$Q_{N}^{\alpha} = \ldots , \tag{3.5.4c}$$

$$\hat{b}^{\alpha[\omega]} = \mathrm{Tr}^{\omega} \, b^{\alpha} . \tag{3.5.4d}$$

Here the scalar coefficients A_{ω}^{α} are functions of the molecular parameters (constants) alone and are independent of f. In addition, the response of a gas of Maxwellian molecules is exactly the same for P^{α}. That is, if we replace b^{β} by M^{β} in $Q^{\alpha} = \tilde{Q}^{\alpha}(b^{\beta})$, we obtain $P^{\alpha} = \tilde{Q}^{\alpha}(M^{\beta})$. Bearing this(53) in mind, the major breakthrough came from Truesdell and Muncaster,(4) who in their theorem, henceforth referred to as the Truesdell-Muncaster theorem, not only gave the explicit formula for A_{ω}^{α} but also actually succeeded in calculating Q_{N}^{α}. However, since their result is extremely complicated, the missing right-hand-side of (3.5.4c) will not be given here; for more details, see Eqs. (XVI.20) and (XVI.22) on p. 242 in Ref. 4.

Now, regardless of (3.5.4), the evaluation of $Q^{\alpha|\beta} = \langle \mathrm{Tr}^{\beta} Q^{\alpha + 2\beta} \rangle$, or of $P^{\alpha|\beta}$, was originally proposed by Ikenberry (and Truesdell).(48) One of the most essential aspects of Ikenberry's theorem comes when we recall his result for $Q_{L}^{\alpha|\beta} = \langle \mathrm{Tr}^{\beta} Q_{L}^{\alpha + 2\beta} \rangle$ of the form

$$Q_{L}^{\alpha|\beta} = \rho \, \mathbb{B}_{\beta}^{\alpha} \, b^{\alpha|\beta} , \tag{3.5.5}$$

where $\mathbb{B}_{\beta}^{\alpha}$ stands for the scalar coefficient first determined in Ref. 48. According to Truesdell and Muncaster,(4) the fact that

$Q_L^{\alpha|\beta}$ is a scalar multiple of the symmetric traceless "moment" $b^{\alpha|\beta}$ alone leads to results which would be difficult to extract from Eq. (3.5.4b), although that formula certainly implies them. Since the hope for setting up an undeniable correspondence between both theorems has surprisingly not materialized as yet, here we shall examine this question more closely.

Thus let us consider (3.5.4b) as a starting point. Then, due to the decomposition

$$\hat{b}^{\alpha[\omega]} = \sum_{\gamma=0}^{[(\alpha-2\omega)/2]} \ell(\alpha-2\omega,\gamma)\Bbbk(\alpha-2\omega-2\gamma,\gamma)\, b^{\alpha[\omega+\gamma]} \vee I^{\gamma} \tag{3.5.6}$$

directly resulting from (3.2.21) and (3.2.9), as well as in virtue of the properties (3.2.19) and (3.2.14), we are justified in doing the following sequence of transformations:

$$\begin{aligned} Q_L^{\alpha[\beta]} &= \langle \mathrm{Tr}^{\beta} Q_L^{\alpha}\rangle = \rho \sum_{\omega=0}^{[\alpha/2]} \sum_{\gamma=0}^{[(\alpha-2\omega)/2]} \mathbb{A}_{\omega}^{\alpha}\, \ell(\alpha-2\omega,\gamma) \\ &\qquad \times \Bbbk(\alpha-2\omega-2\gamma,\gamma)\, \langle \mathrm{Tr}^{\beta}\big(b^{\alpha[\omega+\gamma]} \vee I^{\omega+\gamma}\big)\rangle \\ &= \rho \sum_{\omega=0}^{\beta} \mathbb{A}_{\omega}^{\alpha}\, \ell(\alpha-2\omega,\beta-\omega)\Bbbk(\alpha-2\beta,\beta-\omega) \\ &\qquad \times \langle \mathrm{Tr}^{\beta}\big(b^{\alpha[\beta]} \vee I^{\beta}\big)\rangle \\ &= \rho\big[\ell(\alpha,\beta)\Bbbk(\alpha-2\beta,\beta)\big]^{-1} \sum_{\omega=0}^{\beta} \mathbb{A}_{\omega}^{\alpha}\, \ell(\alpha-2\omega,\beta-\omega) \\ &\qquad \times \Bbbk(\alpha-2\beta,\beta-\omega)\, b^{\alpha[\beta]}. \end{aligned} \tag{3.5.7}$$

By making use of $Q_L^{\alpha|\beta} = Q_L^{\alpha+2\beta[\beta]}$, we get a relation for $Q_L^{\alpha|\beta}$ which may easily be identified with Ikenberry's formula (3.5.5), provided that the coefficient $\mathbb{B}_{\beta}^{\alpha}$ is given by

$$\mathbb{B}^{\alpha}_{\beta} = \left[\ell(\alpha + 2\beta,\beta)\Bbbk(\alpha,\beta)\right]^{-1}$$

$$\times \sum_{\omega = 0}^{\beta} \mathbb{A}^{\alpha + 2\beta}_{\omega}\, \ell(\alpha + 2\beta - 2\omega,\beta - \omega)\Bbbk(\alpha,\beta - \omega). \qquad (3.5.8)$$

Hence the derivation of (3.5.5) from (3.5.4b) is complete. Curiously enough, the expression (3.5.8) relating $\mathbb{B}^{\alpha}_{\beta}$ to $\mathbb{A}^{\alpha + 2\beta}_{\omega}$ has never been established before.

Following Ikenberry and Truesdell,(48) we may prove that $\mathbb{B}^{\alpha}_{\beta} < 0$. Finally, making use of (3.5.2), (3.5.5), and (3.4.25), and appealing to the universal method of Section 3.4.A, we are able to arrive at

$$P^{\alpha|\beta} = \rho \sum_{\omega = 0}^{\beta} \sum_{\nu = \omega}^{\beta} (k_B T)^{\beta - \omega} (-1)^{\omega + \nu} \begin{bmatrix} \beta \\ \nu \end{bmatrix} \begin{bmatrix} \nu \\ \omega \end{bmatrix} \mathbb{B}^{\alpha}_{\nu}$$

$$\times \left[\Bbbk(\alpha,\beta)\right]^{-1}\Bbbk(\alpha,\omega)\, M^{\alpha|\omega} + \ldots , \qquad (3.5.9)$$

where the ellipses represents that part of the collision integral $P^{\alpha|\beta}$ which in the linear approximation is considered to be negligible.

3.5.B. Grad's 20-Moment System. Transition to Ordinary Thermodynamics

The basic variables that we use in this section are the following fields:

Name	Symbol
Mass density	$\rho := M^{0\|0} \in \mathbb{R}$
Internal energy per unit mass	$e := \frac{1}{2} (M^{0\|0})^{-1} M^{0\|1} \in \mathbb{R}$

Temperature	$T := \left(3k_B M^{0\|0}\right)^{-1} M^{0\|1} \in \mathbb{R}$
Mean normal pressure	$p := 3^{-1} M^{0\|1} \in \mathbb{R}$
Pressure deviator	$\mathfrak{P} := M^{2\|0} \in \mathrm{Ker}_2\mathrm{Tr}$
Energy flux vector (heat flux)	$q := 2^{-1} M^{1\|1} \in \mathbb{E}$
Traceless part of M^3	$\mathfrak{Q} := M^{3\|0} \in \mathrm{Ker}_3\mathrm{Tr}$

As Truesdell and Muncaster⁽⁴⁾ showed, in the kinetic gas composed of Maxwellian molecules we have

$$\tau := -\frac{1}{3}\,\mathbb{A}^2_0 = \mathbb{A}^2_1 = -\frac{2}{9}\,\mathbb{A}^3_0 = \frac{2}{3}\,\mathbb{A}^3_1 > 0\ . \tag{3.5.10}$$

Choosing r = 3, with the help of (3.3.7), (3.3.21), (3.4.34), (3.5.8)-(3.5.10), and notation just introduced, Grad's 20-moment system of field equations for u and $M^{\alpha|\beta}$, $\alpha + 2\beta \le 3$, may be written down explicitly:

$$\partial_t \rho + \nabla \bullet \rho u = 0, \tag{3.5.11a}$$

$$\partial_t u + u \bullet L + R = K\ , \tag{3.5.11b}$$

$$\partial_t e + u \bullet \nabla e + \frac{1}{\rho}\,\nabla \bullet q + \frac{1}{\rho}\left(p\,I + \mathfrak{P}\right) \bullet L = 0\ , \tag{3.5.11c}$$

$$\partial_t \mathfrak{P} + \nabla \bullet \left(u \otimes \mathfrak{P} + \mathfrak{Q}\right) + \frac{4}{5}\,\nabla \wedge q$$

$$+\ 2L \cap \left(p\,I + \mathfrak{P}\right) = -\,3\rho\tau\ \mathfrak{P}\ , \tag{3.5.11d}$$

$$\partial_t q + \nabla \bullet \left(u \otimes q + \frac{5}{3}\,ep\,I + \frac{7}{3}\,e\mathfrak{P}\right)$$

$$+\left(\mathfrak{Q} + \frac{6}{5}\,q \vee I + q \otimes I\right) \bullet L$$

$$-\left(\frac{5}{2}\,p\,I + \mathfrak{P}\right) \bullet R = -\,2\rho\tau\ q\ , \tag{3.5.11e}$$

$$\partial_t \mathfrak{Q} + \nabla \bullet \left(u \otimes \mathfrak{Q}\right) + 2e\,\nabla \wedge \mathfrak{P} + 2\mathfrak{P} \wedge \nabla e - 3\mathfrak{P} \wedge R$$

$$+\frac{12}{5}\, q \wedge L + 3L \cap \mathfrak{Q} = -\frac{9}{2}\, \rho\tau\, \mathfrak{Q} , \qquad (3.5.11f)$$

where

$$K = K_1 + K_2 , \quad L = \nabla u + L_2 , \qquad (3.5.11g)$$

$$R = \frac{1}{\rho}\, \nabla \cdot \left(p\, I + \mathfrak{P} \right) , \quad p = \frac{2}{3}\, \rho e . \qquad (3.5.11h)$$

Of course, the equations (3.5.11) have been arranged so that the balance laws of ordinary thermodynamics, viz, the conservation laws of mass, momentum, and energy, appear as the first three equations. There is a literature on the 20-moment equations, studied as an end in itself, and some important arguments have been provided by Grad(1,3) in favour of their usefulness in describing such complicated flow problems for which it is no longer permissible to rely on the Navier-Stokes-Fourier theory.

Here we wish to use (3.5.11d) and (3.5.11e) to produce constitutive equations for $\mathfrak{P}$ and q of the type usually adopted in ordinary thermodynamics. This is done by an iterative procedure that is somewhat akin to Maxwellian iteration of Truesdell and Ikenberry; see, e.g., Chapter XXV in Ref. 4. We place the equilibrium values of $\mathfrak{P}$, q, and $\mathfrak{Q}$, denoted by $\mathfrak{P}^{(0)}$, $q^{(0)}$, and $\mathfrak{Q}^{(0)}$, into the l.h.s. of (3.5.11d)-(3.5.11f). In this way Eqs. (3.5.11d)-(3.5.11f) become certain differential equations for the first iterates $\mathfrak{P}^{(1)}$, $q^{(1)}$, and $\mathfrak{Q}^{(1)}$ of $\mathfrak{P}$, q, and $\mathfrak{Q}$. Since the zeroth iterates $\mathfrak{P}^{(0)}$, $q^{(0)}$, and $\mathfrak{Q}^{(0)}$ all vanish, we obtain

$$\mathfrak{P}^{(1)} = -\frac{2}{3}\, k_B T\, \tau^{-1}\, \nabla \wedge u , \qquad (3.5.12a)$$

$$q^{(1)} = -\frac{5}{4}\, k_B^2 T\, \tau^{-1}\, \nabla T , \qquad (3.5.12b)$$

$$\mathfrak{Q}^{(1)} = 0 . \qquad (3.5.12c)$$

The first iterate (3.5.12a) of the pressure deviator $\mathfrak{P}$ obviously represents the classical constitutive relation of Navier-Stokes, in which the viscosity μ_o must be interpreted as

$$\mu_{o} = \frac{2}{3} k_{B}T \tau^{-1} . \qquad (3.5.13a)$$

On the other side, due to the fact that $q^{(4)}$ is proportional to ∇T, just as in Fourier's relation, we may call the coefficient

$$\varkappa_{o} = \frac{5}{4} k_{B}^{2}T \tau^{-1} \qquad (3.5.13b)$$

appearing in Eq. (3.5.12b) the thermal conductivity.

One could proceed in the iteration procedure, but we shall rest content with the first iterates (3.5.12).

3.6. *Final Remarks*

3.6.A. *General Observations*

It may seem odd that, except for Section 3.5.B, we have not yet confronted the routine question, for what reason we should be appreciative of having a completely irreducible tensor description of three-dimensional, classical, moderately rarefied, simple, monatomic gases. Here, quite intentionally, we shall not evaluate our postulational basis by a *mere* listing of the types of predictions of the theory in describing physically meanigful facts. Since the present approach covers very traditional ground, almost all of them have already been convincingly and excellently summarized.(4) Quite apart from the embarrassing observation, whether or not the irreducible tensor description of classical gases can culminate in real physical results, i.e., in numbers, from the very beginning we offer this chapter as a partial and certainly incomplete reply to the following Truesdell-Muncaster statement on p. XIX in their monograph:(4) "..., many of the analysis contain important gaps, ... Our first purpose is to uncover these gaps and to illuminate them as challenges to future research by mathematicians."

Thus, for the time being, Chapter 3 should be considered as an end in itself. However, part of the usefulness of the formalism is that it enables us in the next chapter to extend almost "effortlessly" to quasiparticle gaseous systems a vast quantity of conclusions originally thought limited to a gas of classical molecules.

3.6.B. Comments Regarding Quasiparticle Gases

The essential point we wish to stress here once more is that the severe restrictions on the types of gaseous systems considered in this chapter are *not* basic limitations on the generality of our ideas, but are adopted merely for pedagogical purposes.

Let us suppose that a nonclassical gas composed of phonons, magnons, rotons, etc. is adequately described by the Boltzmann-Peierls equation. Then, by the introduction of the purely mathematical set of functions f satisfying *Condition I* of Section 4.6.A in Chapter 4, a more subtle but very similar theory can be proposed. In consequence of applying *Condition I*,[14] this generalized theory rests primarily on the effect of diagonalization of an approximate formula for an h and provides in essence the *only* real key in establishing a *strict* quasiparticle analogue of the irreducible variant of Grad's expansion of f in terms of Hermite polynomials.

As useful as the characterization of the nonequilibrium occupation probability f of quasiparticle states by its reducible moments has proved to be [in Gurevich's monograph,(2) for instance], it must clearly be conceded that *Condition I* is capable of yielding an irreducible series representation of f which, *in contrast with classical gases*, may later demonstrate to have no reducible counterpart at all. Thus the attempt to invent from the outset in Chapter 4 the irreducible tensor description is *not* only an academic game, intellectually challenging but of no import.

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[14] Those not familiar with the irreducible tensor description on which the theory is based may find sufficient background for *Condition I* in Section 1.3.C of Chapter 1.

Appendix 3.A: Definitions of Π, $\underset{(\beta,\nu)}{\mathrm{Tr}}$, $M^\alpha \cdot M^\beta$, I, and ∇

Let $\mathbb{E}$ be a three-dimensional Euclidean vector space. Choose an orthonormal basis $\{e_1,e_2,e_3\}$ in $\mathbb{E}$ and set

$$e_{r_1\ldots r_\alpha} := e_{r_1} \otimes \ldots \otimes e_{r_\alpha} . \tag{3.A.1}$$

a. The action of the symmetrizer Π on a tensor M^α of degree α ($\alpha \geq 2$) is given by

$$\Pi M^\alpha := \sum_{r_1\ldots r_\alpha = 1}^{3} M^\alpha_{(r_1\ldots r_\alpha)}\, e_{r_1\ldots r_\alpha} , \tag{3.A.2}$$

where the coefficients $M^\alpha_{r_1\ldots r_\alpha}$ are components of M^α with respect to the basis $\{e_1,e_2,e_3\}$ and round brackets enclosing a set of α indices represent symmetrization of these indices, i.e., the sum over $\alpha!$ permutations of the indices, divided by $\alpha!$. Moreover, we define Π to be the identity on $\mathbb{E}$ and $\mathbb{R}$:

$$\Pi M^1 := M^1 , \quad \Pi M^0 := 0 . \tag{3.A.3}$$

Obviously, $\Pi : \mathbb{E}^\alpha \rightarrow \mathbb{E}^\alpha_s$ is a projection operator:

$$\Pi^2 = \Pi . \tag{3.A.4}$$

b. Since the trace operator with respect to the pair (β,ν), denoted by $\underset{(\beta,\nu)}{\mathrm{Tr}}$, determines a linear map:

$$\underset{(\beta,\nu)}{\mathrm{Tr}} : \mathbb{E}^\alpha \rightarrow \mathbb{E}^{\alpha - 2} , \quad \alpha \geq 2, \quad \alpha \geq \beta \geq 1, \quad \alpha \geq \nu \geq 1, \quad \beta \neq \nu ,$$

we only need to consider how it operates on the generator $e_{r_1\ldots r_\alpha}$:

$$\underset{(\beta,\nu)}{\mathrm{Tr}}\, e_{r_1\ldots r_\alpha} := \delta_{r_\beta r_\nu}\, e_{r_1\ldots \hat{r}_\beta \ldots \hat{r}_\nu \ldots r_\alpha} , \tag{3.A.5}$$

where $\delta_{r_\beta r_\nu}$ denotes the Kronecker delta and the hat over r_β and

r_ν tells us that e_{r_β} and e_{r_ν} do not appear in $e_{r_1 \ldots r_\alpha}$.

c. Suppose that $\alpha \geq 1$, $\beta \geq 1$, and $\nu := min(\alpha,\beta)$. Then the bilinear operator

$$\hat{\Gamma} : \mathbb{E}^\alpha \times \mathbb{E}^\beta \to \mathbb{E}^{\alpha + \beta - 2\nu} ,$$

for which we write

$$M^\alpha \cdot M^\beta := \hat{\Gamma}(M^\alpha, M^\beta) , \qquad (3.A.6)$$

is uniquely determined by its action upon the generators $e_{r_1 \ldots r_\alpha}$ and $e_{s_1 \ldots s_\beta}$:

$$e_{r_1 \ldots r_\alpha} \cdot e_{s_1 \ldots s_\beta} := \begin{cases} \delta_{r_1 s_1} \ldots \delta_{r_\beta s_\beta} \, e_{r_{\beta+1} \ldots r_\alpha} & (\alpha > \beta) \\ \delta_{r_1 s_1} \ldots \delta_{r_\beta s_\beta} & (\alpha = \beta) . \\ \delta_{r_1 s_1} \ldots \delta_{r_\alpha s_\alpha} \, e_{s_{\alpha+1} \ldots s_\beta} & (\alpha < \beta) \end{cases} \qquad (3.A.7)$$

In addition, we set

$$M^\alpha \cdot M^0 := M^0 \cdot M^\alpha := M^0 M^\alpha , \quad \alpha \geq 0 . \qquad (3.A.8)$$

It is easily checked that

$$M^\alpha \cdot M^\beta = M^\beta \cdot M^\alpha . \qquad (3.A.9)$$

We call the tensor $M^\alpha \cdot M^\beta$ the inner product of M^α and M^β .

d. The contravariant metric (unit) tensor I of the Euclidean vector space $\mathbb{E}$ is given by

$$I := \sum_{r=1}^{3} e_r \otimes e_r \in \mathbb{E}^2_s . \qquad (3.A.10)$$

e. Let M^α be a differentiable tensor field in $\mathbb{E}$. Then the action of ∇ on a tensor field M^α of degree $\alpha \geq 0$ is defined by

$$\nabla M^{\alpha}|_{x} := \sum_{jr_1\ldots r_\alpha = 1}^{s} \nabla_j M^{\alpha}_{r_1\ldots r_\alpha}\, e_{jr_1\ldots r_\alpha} \in \mathbb{E}^{\alpha+1}, \qquad (3.A.11)$$

where

$$\nabla_j := \frac{\partial}{\partial x_j}, \qquad x = \sum_{j=1}^{s} x_j\, e_j . \qquad (3.A.12)$$

Appendix 3.B: Proof of the Implication $(\beta > 0) \rightarrow (\langle M^{\alpha} * I^{\beta}\rangle := \ell(\alpha + 2\beta,\beta)\,\langle M^{\alpha} \vee I^{\beta}\rangle = 0)$

Let $\left\{\hat{e}_\alpha(r);\ r = 1,\ldots,2\alpha + 1\right\}$ be a basis of $\mathrm{Ker}_\alpha\mathrm{Tr}$. Then the natural projection $E(\alpha|\alpha) \in \mathbb{E}^{2\alpha}$ of $\mathbb{E}^{\alpha}$ onto the irreducible subspace $\mathrm{Ker}_\alpha\mathrm{Tr}$ of symmetric traceless tensors can uniquely be written as a sum

$$E(\alpha|\alpha) = \sum_{r=1}^{2\alpha+1} \hat{e}_\alpha(r) \otimes \hat{e}_\alpha(r) . \qquad (3.B.1)$$

Hence, on recalling (3.2.25), we get

$$\langle \Pi M^{\alpha}\rangle = E(\alpha|\alpha) \cdot M^{\alpha} = \sum_{r=1}^{2\alpha+1} \left[\hat{e}_\alpha(r) \cdot M^{\alpha}\right]\hat{e}_\alpha(r) . \qquad (3.B.2)$$

From (3.B.2) we conclude that, for $\beta > 0$,

$$[\ell(\alpha + 2\beta,\beta)]^{-1}\langle M^{\alpha} * I^{\beta}\rangle = \langle M^{\alpha} \vee I^{\beta}\rangle = \langle \Pi\left(M^{\alpha} \otimes I^{\beta}\right)\rangle$$

$$= \sum_{r=1}^{2\alpha+4\beta+1} \left[\hat{e}_{\alpha+2\beta}(r) \cdot \left[M^{\alpha} \otimes I^{\beta}\right]\right]\hat{e}_{\alpha+2\beta}(r)$$

$$= \sum_{r=1}^{2\alpha+4\beta+1} \left\{\left[\mathrm{Tr}^{\beta}\, \hat{e}_{\alpha+2\beta}(r)\right] \cdot M^{\alpha}\right\} \hat{e}_{\alpha+2\beta}(r) = 0 . \qquad (3.B.3)$$

Thus the implication holds.

Appendix 3.C: Proof of (3.3.9a) *and* (3.3.9b) *for* $\beta = 1$

Recalling the notation of Appendix 3.A and using the Einstein´s summation convention, we obtain

$$\alpha \,\mathrm{Tr}\left(L \cup M^{\alpha}\right) = \alpha \underset{(1,2)}{\mathrm{Tr}} \left(L \cup M^{\alpha}\right)$$

$$= \sum_{r_1 \ldots r_\alpha = 1}^{s} \alpha\, L_{k(r_1} M^{\alpha}_{r_2 \ldots r_\alpha)k} \underset{(1,2)}{\mathrm{Tr}}\, e_{r_1 \ldots r_\alpha}$$

$$= \sum_{r_1 \ldots r_\alpha = 1}^{s} \alpha\, L_{k(r_1} M^{\alpha}_{r_2 \ldots r_\alpha)k}\, \delta_{r_1 r_2}\, e_{r_3 \ldots r_\alpha}$$

$$= \sum_{r_1 \ldots r_\alpha = 1}^{s} (\alpha - 2)\, L_{k(r_3} M^{\alpha}_{r_4 \ldots r_\alpha)k r_1 r_2}\, \delta_{r_1 r_2}\, e_{r_3 \ldots r_\alpha}$$

$$+ \sum_{r_1 \ldots r_\alpha = 1}^{s} 2\, L_{k r_1} M^{\alpha}_{k r_2 r_3 \ldots r_\alpha}\, \delta_{r_1 r_2}\, e_{r_3 \ldots r_\alpha}$$

$$= \sum_{r_3 \ldots r_\alpha = 1}^{s} (\alpha - 2)\, L_{k(r_3} M^{\alpha}_{r_4 \ldots r_\alpha)kpp}\, e_{r_3 \ldots r_\alpha}$$

$$+ \sum_{r_3 \ldots r_\alpha = 1}^{s} 2\, L_{kp} M^{\alpha}_{kp r_3 \ldots r_\alpha}\, e_{r_3 \ldots r_\alpha}$$

$$= (\alpha - 2)\, L \cup \hat{M}^{\alpha[1]} + 2\, L \bullet \hat{M}^{\alpha[0]} \,. \tag{3.C.1}$$

Hence the proof of (3.3.9a) for $\beta = 1$ is complete. Of course, exactly the same view applies to (3.3.9b) for $\beta = 1$.

Appendix 3.D: Proof of the Equality (3.3.17)

Our immediate purpose here is to sketch the proof of Eq. (3.3.17) for the special case in which $\alpha - 2\beta - 4 \geq 0$ and $\beta \geq 1$. It is readily seen then that (3.3.17) holds also for $\alpha - 2\beta \geq 0$.

Before we can proceed, we must begin with the two auxiliary equalities of the form

$$\mathbb{M}^2 \cdot (\mathbb{M}^\alpha * I) = (\mathrm{Tr}\Pi\mathbb{M}^2)\, \mathbb{M}^\alpha + 2\alpha\, (\Pi\mathbb{M}^2) \cup \mathbb{M}^\alpha + (\mathbb{M}^2 \cdot \mathbb{M}^\alpha) * I\ , \tag{3.D.1a}$$

$$\mathbb{M}^2 \cdot (\mathbb{M}^\alpha * I^2) = (\alpha + 1)(\alpha + 2)\, \mathbb{M}^2 \vee \mathbb{M}^\alpha + (\mathrm{Tr}\Pi\mathbb{M}^2)\, \mathbb{M}^\alpha * I$$
$$+ 2\alpha\, [(\Pi\mathbb{M}^2) \cup \mathbb{M}^\alpha] * I + (\mathbb{M}^2 \cdot \mathbb{M}^\alpha) * I^2\ , \tag{3.D.1b}$$

where $\mathbb{M}^2 \in \mathbb{E}^2$ and $\mathbb{M}^\alpha \in \mathbb{E}^\alpha_s$ $(\alpha \geq 2)$.

By making use of the decomposition

$$\hat{\mathbb{M}}^{\alpha[\beta - 1]} = \sum_{\omega = 0}^{[(\alpha - 2\beta + 2)/2]} k(\alpha - 2\beta - 2\omega + 2, \omega) \times \mathbb{M}^{\alpha[\beta + \omega - 1]} * I^\omega \tag{3.D.2}$$

and Eq. (3.2.19), it follows from (3.D.1), (3.2.13), (3.3.20), (3.2.22), and (3.2.23) that

$$\langle L \cdot \hat{\mathbb{M}}^{\alpha[\beta - 1]} \rangle = \sum_{\omega = 0}^{[(\alpha - 2\beta + 2)/2]} k(\alpha - 2\beta - 2\omega + 2, \omega) \times \langle L \cdot \left[\mathbb{M}^{\alpha[\beta + \omega - 1]} * I^\omega\right] \rangle$$

$$= k(\alpha - 2\beta + 2, 0)\, L \cdot \mathbb{M}^{\alpha[\beta - 1]} + k(\alpha - 2\beta, 1)\, \langle L \cdot \left[\mathbb{M}^{\alpha[\beta]} * I\right] \rangle$$

$$+ k(\alpha - 2\beta - 2, 2)\, \langle L \cdot \left[\mathbb{M}^{\alpha[\beta + 1]} * I^2\right] \rangle = L \cdot \mathbb{M}^{\alpha[\beta - 1]}$$

$$+ \frac{1}{2\alpha - 4\beta + 3}\, S\, \mathbb{M}^{\alpha[\beta]} + \frac{2(\alpha - 2\beta)}{2\alpha - 4\beta + 3}\, U \cap \mathbb{M}^{\alpha[\beta]}$$

$$+ \frac{(\alpha - 2\beta)(\alpha - 2\beta - 1)}{(2\alpha - 4\beta - 1)(2\alpha - 4\beta + 1)} L \wedge M^{\alpha[\beta + 1]} . \qquad (3.D.3)$$

Hence the proof of the equality (3.3.17) for the $\alpha - 2\beta - 4 \geq 0$ ($\beta \geq 1$) case is complete.

CHAPTER 4

QUASIPARTICLE GASES: THE IRREDUCIBLE MOMENT REPRESENTATION OF THE DISTRIBUTION FUNCTION

4.1. Introduction

Given a gas consisting of quasiparticles [phonons,(2) magnons,(28) rotons,(27) etc.], the main object of this chapter is both to solve some of the basic tensorial problems generated by the Boltzmann-Peierls equation(2,9,10) and to invent the quasiparticle counterpart to Grad's moment procedure.(3) We first began to study quasiparticle gases, within the framework of the one-dimensional Boltzmann-Peierls equation, in Chapter 1.(23,25,26) The equations of transfer for the moments of the distribution function f and the quasiparticle analogue of Grad's expansion of f in terms of Hermite polynomials were some of the new results we found there. Explicit in our derivation of them was the specific assumption concerning the dispersion relation $\Omega(k)$: $\Omega(k) \sim |k|^r$, $1 < r \leq 2$, k being a wave number. In extending the range of validity of Grad's ideas to the case of three-dimensional quasiparticle gases, here, too, we adopt the method of Chapter 1. However, in contrast with our previous analysis we wish to make use of somewhat deeper properties of the function $\Omega(k)$ than those we have presented so far,(23) and for this reason we shall simply define,(25) and deal from the very outset with, a broad class of *isotropic* dispersion relations.

Here we proceed as follows. In outline, Section 4.2 begins with the introduction of the Boltzmann-Peierls equation, which is a necessary prerequisite for the kinetic theory, and ends with the precise specification of a certain class of isotropic dispersion relations. The model collision operator J is the subject of Section 4.3. In Section 4.4 we define the *symmetric traceless moments* of the distribution function f and derive the exact system of *irreducible* equations of transfer for them.

Yet, in order to render the infinite hierarchy of equations of transfer just mentioned definite, we must necessarily face the difficult problem of evaluating or estimating the collision integrals (which appear on the r.h.s. of those equations) as exhibited series in the symmetric traceless moments of f. Of course, since the collision integrals $P^{\alpha,\beta}$ are functionally related to f, we cannot calculate them without first having to determine the irreducible moment representation of the distribution function, except in the case of a specific choice of the transition rate $\mathcal{R}(k,k_1)$ where the exact expression for $P^{\alpha,\beta}$ already has the "proper" form [cf. Section 4.5]. With these statements in mind, Sections 4.6 and 4.7 may turn out to be the most important. Indeed, in consequence of the introduction of a suitably chosen Hilbert space, these sections provide in essence the only real key for establishing a strict quasiparticle analogue of Grad's expansion of f in terms of Hermite polynomials and Hermite coefficients. Some auxiliary material, being in fact a supplement to Section 4.6, is included as Appendix 4.A. The theoretical method, as developed, is directly applicable to the description of fluctuation phenomena that originate in the molecular activity but manifest themselves in the random variations of the Tchebychef-Ikenberry expansion coefficients $a^{\alpha,\beta}$ of f.(17,65) We discuss this interesting topic in Section 4.8. In addition, in Section 4.8 we try to demonstrate that our framework can be viewed as an important building block not only in constructing the theory which parallels that of Grad for classical gases but also in exploiting the Chapman-Enskog procedure.(36,2)

4.2. The Boltzmann-Peierls Equation and the Definition of a Certain Class of Dispersion Relations

For the purposes of this chapter, the kinetic theory of a three-dimensional gas of quasiparticles should be regarded as a strongly simplified mathematical model in which the nonequilibrium occupation probability f, aside from its obvious dependence on the wave vector $\mathbf{k} \in \mathbb{E}$, is a function of position x and time t satisfying the Boltzmann-Peierls equation of the form(2,9,10)

$$\partial_t f + \nabla_k \Omega \cdot \nabla_x f - \nabla_x \Omega \cdot \nabla_k f = J(f) \, . \tag{4.2.1}$$

Here J stands for the collision operator, whereas $\hbar\,\Omega(\mathbf{k})$ represents the energy of a single quasiparticle in the mode $\mathbf{k}$; the quantity $2\pi\hbar$ denotes, as usual, Planck's constant.

We proceed now to work out the details for three simple and illuminating special classes of isotropic dispersion relations.

Example 1. The isotropic (phonon) model with dispersion. We consider the dispersion relation given by

$$\Omega(\mathbf{k},x) = a\,\mu(y) \;, \quad y := b\,k \;, \quad k := |\mathbf{k}| \;, \tag{4.2.2}$$

where a and b are certain positive constants having the dimensions of frequency and length, respectively. By definition, $\mu(y)$ is a function in $\hat{\mathbb{R}}^+ := (0,\infty)$ with the following properties:[15]

I. $\mu \in C^2(\hat{\mathbb{R}}^+)$, $\mu'(y) > 0$, $\mu''(y) > 0$, $y \in \hat{\mathbb{R}}^+$.

II. $\lim\limits_{y \to 0} \mu(y) = 0$.

III. $z_o := \lim\limits_{y \to 0} \mu'(y) > 0$.

IV. Let $\lambda(z)$, $z \in \mathbb{R}_o^+ := (z_o,\infty)$, be a solution for y of the equation $\mu'(y) = z$.[16] Then as $z \to \infty$ the rate of growth of z is no greater than that of $\mu[\lambda(z)]$.

V. There are positive constants C_1 and C_2 and a number p,

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[15] $C^p(\mathcal{I})$ is a set of real-valued functions on an open interval $\mathcal{I} \subset \mathbb{R} := (-\infty,+\infty)$ that are p times continuously differentiable. The symbols $\mu'(y)$, $\mu''(y)$, and $\lambda'(z)$ denote $d\mu/dy$, $d^2\mu/dy^2$, and $d\lambda/dz$, respectively.

[16] It is not difficult to see that this solution is unique and that $\lambda \in C^1(\mathbb{R}_o^+)$ and $\lambda'(z) > 0$.

$p \ge 0$, such that

$$\lambda'(z)[\lambda(z)]^2 \le \mathbb{C}_1 + \mathbb{C}_2(z - z_o)^p \quad \text{for all} \quad z \in \mathbb{R}^+_o . \tag{4.2.3}$$

In view of Properties I, II, and III we immediately conclude that

$$y < z_o^{-1}\,\mu(y) \quad \text{when} \quad y \in \hat{\mathbb{R}}^+ . \tag{4.2.4}$$

An important example of $\mu(y)$ is provided by the following expression which originates in the theory of phonon excitations in Helium II:(2)

$$\mu(y) = y\left(1 + \theta\, y^m \right) , \tag{4.2.5a}$$

where

$$0 < \theta = \text{const} , \quad 0 < m = \text{const} \le 3 . \tag{4.2.5b}$$

Example 2. The power (magnon) model. We introduce now the dispersion relation of the form

$$\Omega(k,x) = c(x)k^r , \qquad 1 < r \le 2 , \tag{4.2.6}$$

where the dimension-bearing coefficient c in front of k^r is a differentiable function of position x. The usefulness of the $r = 2$ case(23) arises from every attempt to study magnon excitations in a continuum.(28) The "unrealistic" proposition $\Omega \sim k^r$, $1 < r < 2$, still retains the crude qualitative physical properties of acoustic excitations in a continuum that might be important ($\Omega \to 0$ for $k \to 0$ and $\Omega \to \infty$ for $k \to \infty$).

Example 3. The isotropic and dispersionless (phonon) model. Moreover, we shall deal with(25,26)

$$\Omega(k,x) = c(x)k , \tag{4.2.7}$$

where again c is a differentiable function of x. The unquestionable importance of (4.2.7) follows from the fact that the thermal properties of an insulating crystal at low temperature can be discussed assuming a validity of the isotropic and dispersionless approximation to the generally unknown function $\Omega(k,x)$.(2,9,10) *Within the framework of a single acoustic phonon branch and of an isotropic* *long-wavelength approximation*(20) (4.2.7) *to the dispersion relation, the simplest acceptable version of extended irreversible thermodynamics, based upon the 9-moment system of differential equations for the slow and fast gas-state variables, is extensively investigated elsewhere.*(20)

4.3. The Model Collision Operator J

A class of collision operators which we shall find to include, in a sense, a counterpart of results that follow from the kinetic theory of a gas of Maxwellian molecules(4) may be defined by the integral equality

$$J(f) = -\frac{1}{\tau}\left(f - f_{o} \right)$$

$$+ \int_{\mathbb{E}} \left[\mathcal{R}(k_{1},k)f_{1}(1 + f) - \mathcal{R}(k,k_{1})f(1 + f_{1}) \right] dk_{1}, \qquad (4.3.1a)$$

in which

$$f_{1} := f(k_{1},x,t) \ , \qquad (4.3.1b)$$

$$\mathcal{R}(k,k_{1}) := G(\kappa)\ H(g \cdot g_{1})\ \delta(\kappa - \kappa_{1}) \ , \qquad (4.3.1c)$$

$$\kappa_{1} := |k_{1}| \ , \quad g := \kappa^{-1}k \ , \quad g_{1} := \kappa_{1}^{-1}k_{1} \ . \qquad (4.3.1d)$$

Here $G(\kappa)$ and $H(\zeta)$, $|\zeta| \le 1$, are certain positive functions of their arguments, and $\delta(\kappa - \kappa_{1})$ is Dirac's symbol. The quantity τ can be interpreted as the mean (k-independent) relaxation time; in contrast with $G(\kappa)$ and $H(\zeta)$ the precise specification of τ is of no greater import in our further investigations. Just as in Chapter 1, the f_{o} denotes the Bose-Einstein distribution function that corresponds to f:

$$f_{o} := \left(e^{\omega} - 1\right)^{-1} \ , \qquad \omega := \frac{\hbar\ \Omega}{k_{B}T} \ , \qquad (4.3.2a)$$

$$\int_{\mathbb{E}} \Omega\left(f - f_{o}\right) dk = 0 \ . \qquad (4.3.2b)$$

Insofar as the physical content of (4.3.1) is concerned, the integral on the r.h.s. of (4.3.1a), which here appears as a mere definition, was systematically exploited by Gurevich⁽²⁾ in his studies on the elastic scattering of phonons by different defects (imperfections) of a dielectric crystal. Of course, for such a scattering process the energy of a single phonon remains unchanged, i.e., the transition rate $\mathcal{R}(k,k_1)$ vanishes unless $\Omega((k,x) = \Omega(k_1,x)$, and the simplest possible assumption that

$$J(f) = \int_{\mathbb{E}} [\, \mathcal{R}(k_1,k)f_1(1 + f) - \mathcal{R}(k,k_1)f(1 + f_1)\,]\, dk_1 \qquad (4.3.3)$$

evidently prevents the quasiparticle system from reaching local or absolute equilibrium. Thus if we wish to model the real collision operator adequately, we should slightly modify the r.h.s. of (4.3.3) by adding a contribution of Callaway's type as well.

Elementary inspection shows that the proposition (4.3.1) for J(f) reproduces correctly the three basic features of the Boltzmann-Peierls equation:

(i) A one-particle density is unaffected by collisions if and only if it is a Bose-Einstein distribution function:

$$J(f) = 0 \quad \Leftrightarrow \quad f = f_o \,. \qquad (4.3.4a)$$

(ii) The fundamental conservation equation of quasiparticle hydrodynamics, which expresses the time variation of the local density of energy as the divergence of the corresponding flow, is fulfilled because of the property

$$\int_{\mathbb{E}} \Omega J(f)\, dk = 0 \,. \qquad (4.3.4b)$$

(iii) The Boltzmann-Peierls equation satisfies an H-theorem:

$$\int_{\mathbb{E}} J(f) \ln\left[\frac{1 + f}{f}\right] dk \geq 0 \,. \qquad (4.3.4c)$$

The ansatz (4.3.1c) concerns all positive functions $H(\xi)$, $|\xi| \leq 1$, such as to make the integral expression in (4.3.1a) exist, not merely those $H(\xi)$ that can be expanded in a series of the Legendre polynomials⁽³⁴,³⁵⁾ $P_\nu(\xi)$, $\nu = 0,1,\ldots$, as follows:

$$H(\xi) = \sum_{\nu=0}^{\infty} d_\nu P_\nu(\xi) \,. \qquad (4.3.5)$$

Let $\mathcal{N}$ be any function of $\mathscr{k}$. Then the expansion (4.3.5) serves as basis from which to evaluate the collision integral of the form

$$\wp[Y^{\alpha}(g)\mathcal{N}(\mathscr{k})] := \int_{\mathbb{E}} Y^{\alpha}(g)\mathcal{N}(\mathscr{k})J(f)\, d\mathscr{k} , \qquad (4.3.6)$$

$Y^{\alpha}(g)$, $\alpha = 0,1,\dots$, being Ikenberry's harmonics.[4,24] As a step toward calculating $\wp[Y^{\alpha}(g)\mathcal{N}(\mathscr{k})]$ from (4.3.1), (4.3.5), and (4.3.6), we recall the remarkable formula of Coope and Snider:[45]

$$P_{\nu}(g \cdot g_1) = \frac{(2\nu)!}{2^{\nu}(\nu!)^2}\, Y^{\nu}(g) \cdot Y^{\nu}(g_1) . \qquad (4.3.7)$$

If we also adopt (3.4.21), then we obtain

$$\wp[Y^{\alpha}(g)\mathcal{N}(\mathscr{k})] = -\frac{1}{\tau}\int_{\mathbb{E}} Y^{\alpha}(g)\mathcal{N}(\mathscr{k})(f - f_o)\, d\mathscr{k}$$

$$+ 4\pi c_{\alpha}\int_{\mathbb{E}} Y^{\alpha}(g)\mathscr{k}^2\mathcal{N}(\mathscr{k})G(\mathscr{k})(f - f_o)\, d\mathscr{k} , \qquad (4.3.8a)$$

where

$$c_{\alpha} := \frac{1}{2\alpha + 1}\, d_{\alpha} - d_o \quad \left(\Rightarrow c_o = 0 \right) . \qquad (4.3.8b)$$

4.4. Irreducible Equations of Transfer

4.4.A. The Isotropic (Phonon) Model with Dispersion

After substituting for Ω from (4.2.2), the Boltzmann-Peierls equation (4.2.1) becomes

$$\partial_t f + a\, \delta\, \mu'(y)\, g \cdot \nabla f = J(f) , \qquad \nabla := \nabla_x . \qquad (4.4.1)$$

We take as the relations determining the symmetric traceless moments $N^{\alpha,\beta} \in \mathrm{Ker}_{\alpha}\mathrm{Tr}$, $\alpha,\beta = 0,1,\dots$, of f those which are obtained if $Y^{\alpha}(g)\mu(y)f$ is multiplied by

$$S_{\beta} := \begin{cases} [\, \mu'(y)\,]^{\nu} & \text{for } \beta = 0,2,\dots, \\ \dfrac{y}{\mu(y)}\, [\, \mu'(y)\,]^{\nu} & \text{for } \beta = 1,3,\dots, \end{cases} \qquad (4.4.2)$$

ν being the greatest integer $\le \beta/2$,[17] and integrated over all values of **k**:

$$N^{\alpha,\beta} := \int_{\mathbb{E}} Y^{\alpha}(\mathbf{g})\mu(y)S_{\beta}f \, d\mathbf{k} \, . \tag{4.4.3}$$

Looking at the definitions of the fields of physical interest, we see that all of them are proportional to certain moments of f, regarded as a function of **k**:

Name	Symbol
Energy per unit volume	$\varepsilon := \hbar a(8\pi^3)^{-1}N^{0,0}$
Quasimomentum per unit volume	$\mathbf{Q} := \hbar(8\pi^3\delta)^{-1}\mathbf{N}^{1,1}$
Heat flux	$\mathbf{q} := \hbar a^2\delta(8\pi^3)^{-1}\mathbf{N}^{1,2}$

With the aid of (4.4.1)-(4.4.3), the explicit calculation of the irreducible equations of transfer for $N^{\alpha,\beta}$ presents no difficulties in principle. Techniques for treating this problem we developed in Section 3.3 of Chapter 3. Although individual methods differ in detail, depending on whether (3.3.1) or (4.4.1) is under consideration, all are based on exactly the same structural ingredients; in the result we obtain

$$\partial_t N^{\alpha,\beta} + a\,\delta\,\nabla \cdot N^{\alpha + 1,\beta + 2} + \frac{\alpha}{2\alpha + 1}\,a\,\delta\,\nabla \wedge N^{\alpha - 1,\beta + 2} = P^{\alpha,\beta} \, , \tag{4.4.4a}$$

$$\alpha,\beta = 0,1,\ldots \, , \quad N^{-1,\beta} := 0 \, , \tag{4.4.4b}$$

where

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[17] Whether or not a set of functions $\left\{S_{\beta};\ \beta = 0,1,\ldots\right\}$ consists of linearly independent elements, cannot be deduced without first having to determine $\mu(y)$. As an example, for a class of dispersion relations defined by (4.2.5), we see at a glance that $m\,S_{2\beta + 1} + S_{2\beta + 3} = (m + 1)\,S_{2\beta}$, $\beta = 0,1,\ldots$. Hence only the subset $\left\{S_0, S_1, S_{2\beta};\ \beta = 1,2,\ldots\right\}$ of the set $\left\{S_{\beta};\ \beta = 0,1,\ldots\right\}$ is linearly independent.

$$P^{\alpha,\beta} := \int_{\mathbb{E}} Y^{\alpha}(g)\mu(y)S_{\beta}J(f)\, dk\ . \tag{4.4.5}$$

By choosing $\mathcal{N}(k)$ to be $\mu(y)S_{\beta}$ in Eq. (4.3.8a), we arrive, in virtue of $P^{\alpha,\beta} = \mathcal{P}[Y^{\alpha}(g)\mu(y)S_{\beta}]$, at

$$P^{\alpha,\beta} = -\frac{1}{\tau}\left(N^{\alpha,\beta} - N^{\alpha,\beta}_{eq} \right)$$

$$+\ 4\pi c_{\alpha} \int_{\mathbb{E}} Y^{\alpha}(g)k^{2}G(k)\mu(y)S_{\beta}(f - f_{o})\, dk\ , \tag{4.4.6}$$

where

$$N^{\alpha,\beta}_{eq} := \int_{\mathbb{E}} Y^{\alpha}(g)\mu(y)S_{\beta}f_{o}\, dk\ . \tag{4.4.7}$$

It is easily verified that $N^{\alpha,\beta}_{eq} = 0$ unless $\alpha = 0$.

4.4.B. The Power (Magnon) Model

If we substitute (4.2.6) into (4.2.1), then we obtain

$$\partial_{t}f + rcv(k)\, g \cdot \nabla_{x}f - k^{r}\, W \cdot \nabla_{k}f = J(f)\ , \tag{4.4.8a}$$

where

$$v(k) := k^{r-1}\ , \qquad W := \nabla_{x}c\ . \tag{4.4.8b}$$

The symmetric traceless moments of f are defined as follows:

$$N^{\alpha,\beta} := \int_{\mathbb{E}} Y^{\alpha}(g)k^{r}[v(k)]^{\beta - 1}f\, dk\ , \tag{4.4.9a}$$

$$\alpha,\beta = 0,1,\dots\ . \tag{4.4.9b}$$

Multiplying the Boltzmann-Peierls equation (4.4.8a) by $Y^{\alpha}(g)k^{r}[v(k)]^{\beta - 1}$ and integrating the result over $\mathbb{E}$ yield

$$\partial_{t}N^{\alpha,\beta} + rc\, \nabla \cdot N^{\alpha+1,\beta+1} + \frac{\alpha}{2\alpha+1}\, rc\, \nabla \wedge N^{\alpha-1,\beta+1}$$

$$+ [2r + (\beta - 1)(r - 1) - \alpha]\, W \cdot N^{\alpha+1,\beta+1}\ +$$

$$\frac{\alpha}{2\alpha+1}\,[2r + (\beta - 1)(r - 1) + \alpha + 1]\, W \wedge N^{\alpha-1,\beta+1} = P^{\alpha,\beta}, \tag{4.4.10a}$$

$$\alpha,\beta = 0,1,\dots , \qquad N^{-1,\beta} := 0 , \tag{4.4.10b}$$

where

$$P^{\alpha,\beta} := \int_{\mathbb{E}} Y^{\alpha}(g)k^{r}[v(k)]^{\beta - 1}J(f)\, d\mathbf{k}$$

$$= -\frac{1}{\tau}\left(N^{\alpha,\beta} - N^{\alpha,\beta}_{eq} \right)$$

$$+ 4\pi c_{\alpha}\int_{\mathbb{E}} Y^{\alpha}(g)k^{r + 2}G(k)[v(k)]^{\beta - 1}(f - f_{o})\, d\mathbf{k} . \tag{4.4.11}$$

Here the $N^{\alpha,\beta}_{eq}$ is given by

$$N^{\alpha,\beta}_{eq} := \int_{\mathbb{E}} Y^{\alpha}(g)k^{r}[v(k)]^{\beta - 1}f_{o}\, d\mathbf{k} . \tag{4.4.12}$$

As to the physical interpretation of some of the irreducible moments $N^{\alpha,\beta}$ of f, we have

$$\varepsilon = \frac{\hbar c}{8\pi^{3}} N^{0,1} , \qquad Q = \frac{\hbar}{8\pi^{3}} N^{1,0} , \qquad q = \frac{r\hbar c^{2}}{8\pi^{3}} N^{1,2} . \tag{4.4.13}$$

4.4.C. The Isotropic and Dispersionless (Phonon) Model

Using (4.2.1) and (4.2.7), it is a straightforward matter to show that

$$\partial_{t}f + cg \cdot \nabla_{x}f - k\, W \cdot \nabla_{k}f = J(f) . \tag{4.4.14}$$

Within the framework set up here, the symmetric traceless moments of f are given by

$$N^{\alpha,\beta} := \int_{\mathbb{E}} Y^{\alpha}(g)k^{\beta + 1}f\, d\mathbf{k} , \tag{4.4.15a}$$

$$\alpha,\beta = 0,1,\dots . \tag{4.4.15b}$$

It follows directly from (4.4.14) and (4.4.15) that

$$\partial_{t}N^{\alpha,\beta} + c\, \nabla \cdot N^{\alpha + 1,\beta} + \frac{\alpha}{2\alpha + 1} c\, \nabla \wedge N^{\alpha - 1,\beta}$$

$$+ (\beta - \alpha + 2)\, W \bullet N^{\alpha + 1,\beta} + \frac{\alpha}{2\alpha + 1} (\alpha + \beta + 3)\, W \wedge N^{\alpha - 1,\beta}$$

$$= P^{\alpha,\beta} , \qquad (4.4.16a)$$

$$\alpha,\beta = 0,1,\dots , \qquad N^{-1,\beta} := 0 , \qquad (4.4.16b)$$

$P^{\alpha,\beta}$ being the collision integral:

$$P^{\alpha,\beta} := \int_{\mathbb{E}} Y^{\alpha}(g)\mathscr{k}^{\beta + 1} J(f)\, dk = - \frac{1}{\tau}\left(N^{\alpha,\beta} - N^{\alpha,\beta}_{eq} \right)$$

$$+ 4\pi c_{\alpha} \int_{\mathbb{E}} Y^{\alpha}(g)\mathscr{k}^{\beta + 3} G(\mathscr{k})(f - f_{o})\, dk , \qquad (4.4.17a)$$

$$N^{\alpha,\beta}_{eq} := \int_{\mathbb{E}} Y^{\alpha}(g)\mathscr{k}^{\beta + 1} f_{o}\, dk . \qquad (4.4.17b)$$

Regardless of the r.h.s. of (4.4.16a), the differential equations just obtained are *unsymmetrically* coupled (uncoupled) in that a single branch $\mathcal{M}_{\gamma} := \left\{ N^{0,\gamma}, N^{1,\gamma}, \dots \right\}$ of moments with $\gamma \neq \beta$ is not allowed to affect the space-time behaviour of $\mathcal{M}_{\beta}$, while $N^{\alpha + 1,\beta}$ influences only $N^{\alpha - 1,\beta}$ and $N^{\alpha,\beta}$, none of the other variables.

Chief among $N^{\alpha,\beta}$ are the following fields:

$$\varepsilon = \frac{\hbar\, c}{8\pi^{3}} N^{0,0} , \qquad Q = \frac{\hbar}{8\pi^{3}} N^{1,0} , \qquad q = \frac{\hbar c^{2}}{8\pi^{3}} N^{1,0} . \qquad (4.4.18)$$

4.5. Explicit Collision Integrals for a Special Choice of the Transition Rate $\mathcal{R}(k,k_1)$: $\Omega(k,x) = a\, \mu(y)$

In Section 4.3 we defined and explained the model collision operator J in general terms. One possible way of making further progress is to descend to special cases of that operator. When the transition rate $\mathcal{R}(k,k_1)$ in (4.3.1c) is specialized by letting $G(\mathscr{k})$ have the form

$$G(\mathscr{k}) = \mathscr{k}^{-2} \sum_{\nu = 0}^{p} \mathbb{B}_{\nu} \left[\mu'(y) \right]^{\nu} , \qquad p = 0,1,\dots , \qquad (4.5.1)$$

$\mathbb{B}_\nu$, $0 \le \nu \le p$, being certain constants, then placing (4.5.1) into (4.4.6) and using the definition (4.4.3) we obtain

$$P^{\alpha,\beta} = -\frac{1}{\tau}\left(N^{\alpha,\beta} - N^{\alpha,\beta}_{eq} \right)$$

$$+ 4\pi c_\alpha \sum_{\nu=0}^{p} \mathbb{B}_\nu \left(N^{\alpha,\beta+2\nu} - N^{\alpha,\beta+2\nu}_{eq} \right). \tag{4.5.2}$$

Thus we see now that the ansatz (4.5.1) delivers the collision integral (4.4.5) as an *exhibited* linear function of finitely many moments of f, and the predictions of the kinetic theory of quasiparticle gases for (4.3.1), (4.3.5), and (4.5.1) have in a sense the same quality as those for a classical gas composed of Maxwellian molecules.⁽⁴⁾ [However, as we have demonstrated in Section 3.5.A of Chapter 3, in a gas of Maxwellian molecules the r.h.s. of (3.3.21) does not introduce, *in contrast with that to* (4.4.4a), the so-called forward coupling of the equations of moments.] Of course, our calculations are rather simple. But whether there be another quasiparticle model such as to satisfy the extra requirement of evaluating $P^{\alpha,\beta}$ in terms of $N^{\alpha,\beta}$ alone without first having to determine the irreducible moment representation of f, is not presently known!

The functions $G(\mathscr{k})$ as defined by (4.5.1) are of physical interest whenever $p = 0,1,\ldots$ and $\mathbb{B}_\nu$, $0 \le \nu \le p$, are such that $G(\mathscr{k}) > 0$ for any $\mathscr{k} \in \hat{\mathbb{R}}^+$, and also each acceptable $G(\mathscr{k})$ cannot increase without bound with decreasing $\mathscr{k}$. Thus, considering the proposition (4.2.5) as useful illustration, we easily arrive, denoting by s the greatest integer $\le 2/m$, at the following restrictions on the admissible values of p and $\mathbb{B}_\nu$, $0 \le \nu \le p$:

(i) $p > s$;

(ii) $\mathbb{B}_\nu$, $s < \nu \le p$, are positive;

(iii) $\mathbb{B}_\nu$, $0 \le \nu \le s$, are calculated from $\mathbb{B}_\nu$, $s < \nu \le p$:

$$\sum_{\nu=\gamma}^{s} \begin{bmatrix} \nu \\ \gamma \end{bmatrix} \mathbb{B}_\nu = - \sum_{\nu=1+s}^{p} \begin{bmatrix} \nu \\ \gamma \end{bmatrix} \mathbb{B}_\nu , \tag{4.5.3a}$$

$$0 \leq \gamma \leq s \, . \qquad (4.5.3b)$$

After perusing the foregoing calculations for $\Omega(k,x) = a\ \mu(y)$, the reader will conclude, almost effortlessly, that there are reasons in support of the universality of the method when other dispersion relations, namely, (4.2.6) and (4.2.7), are taken. This is then a good point to leave simplifying assumptions, such as, for instance, (4.5.1), and move to the general problem of finding the irreducible moment representation of f.

4.6. A Quasiparticle Analogue of Grad's Moment Method for $\Omega(k,x) = a\ \mu(y)$

4.6.A. The Tchebychef-Ikenberry Expansion of the Distribution Function

The choice (4.5.1) of $G(\boldsymbol{k})$ ensures the satisfaction of (4.5.2): it also implies that we can outflank the original Boltzmann-Peierls equation itself by working directly with the equations of transfer for the expectations $N^{\alpha,\beta}$ of $Y^{\alpha}(g)\mu(y)S_{\beta}$. As, however, the function $G(\boldsymbol{k})$ other than (4.5.1) causes difficulty in the rigorous determination of $P^{\alpha,\beta}$ from finitely many moments of f, the general problem of formally evaluating $P^{\alpha,\beta}$ as *infinite* series in the moments may be solved in a manner analogous to that employed for a one-dimensional quasiparticle gas; for more details, see Section 1.3.A in Chapter 1. Equation (4.4.6) indicates in turn that in order to calculate $P^{\alpha,\beta}$, it is necessary first to express f in terms of $N^{\alpha,\beta}$ and therefore to propose the irreducible moment representation of f. The only object of this section is both to invent such a representation and to investigate its elementary properties.

a. To this end, let us introduce the following definitions:

$$\mathbb{H}(z;\Theta) := \frac{\lambda'\lambda^{2}\omega^{2}e^{\omega}}{\left(e^{\omega}-1\right)^{2}} \, , \quad z \in \mathbb{R}_{0}^{+} \, , \qquad (4.6.1a)$$

$$\{\varphi_1|\varphi_2\}_1 := \int_{\mathbb{R}_o^+} \varphi_1(z)\varphi_2(z)\, \mathbb{W}(z;\Theta)\, dz \,, \tag{4.6.1b}$$

$$||\varphi||_1 := \left(\{\varphi|\varphi\}_1\right)^{1/2} , \tag{4.6.1c}$$

$$\mathbb{L}^2\left(\mathbb{R}_o^+; \mathbb{W}(z;\Theta)dz \right) := \left\{\varphi : ||\varphi||_1 < \infty\right\} , \tag{4.6.1d}$$

in which

$$\Theta := \frac{\hbar\, a}{k_B T} \,, \quad \omega = \Theta\, \mu[\lambda(z)] \,. \tag{4.6.1e}$$

It is clear from (4.6.1d) that $\mathbb{L}^2\left(\mathbb{R}_o^+; \mathbb{W}(z;\Theta)dz \right)$ is a real Hilbert space with the scalar product $\{\ |\ \}_1$ given by (4.6.1b). Referring back to Section 4.2, we can utilize Properties I,...,V of $\mu(y)$ so as to demonstrate that, for certain functions $\mathbb{M}$ and $\mathbb{C}$ of Θ,

$$\mathbb{W}(z;\Theta) \le \mathbb{M}\, e^{-\mathbb{C}z} \,, \quad z \in \mathbb{R}_o^+ \,, \tag{4.6.2}$$

and to prove, by appealing to the *lemma* of Dijkstra and van Leeuwen as formulated in Ref. 33 on p. 468, that the collection of polynomials defined on $\mathbb{R}_o^+$ is a dense subset of $\mathbb{L}^2\left(\mathbb{R}_o^+; \mathbb{W}(z;\Theta)dz \right)$. Now, writing S_β, $\beta = 0,1,\ldots,$ from (4.4.2), in the form

$$S_{2\nu}(z) = z^\nu \qquad \text{when} \quad \beta = 0,2,\ldots, \tag{4.6.3a}$$

$$S_{2\nu+1}(z) = \frac{\lambda(z)}{\mu[\lambda(z)]}\, z^\nu \qquad \text{when} \quad \beta = 1,3,\ldots, \tag{4.6.3b}$$

and applying the inequality $\lambda(z) < z_o^{-1}\mu[\lambda(z)]$ which parallels that in (4.2.4), we see at a glance not only that $S_\beta \in \mathbb{L}^2\left(\mathbb{R}_o^+; \mathbb{W}(z;\Theta)dz \right)$ but also that the functions S_β, $\beta = 0,1,\ldots,$ are dense in our Hilbert space. Let $\left\{S_\beta;\ \beta = 0,1,\ldots\right\}$ be a linearly independent set. Then, as we know from the monographs of Szegö(34) and Sansone,(35) it is possible to construct, via a standard procedure called orthogonalization, an *orthonormal* set of Tchebychef functions $A_\beta(z;\Theta)$, $\beta = 0,1,\ldots,$ such that

$$A_\beta(z;\Theta) = \sum_{\nu=0}^{\beta} c_{\beta\nu}(\Theta)\, S_\nu(z) \,, \tag{4.6.4a}$$

$$S_\beta(z) = \sum_{\nu=0}^{\beta} d_{\beta\nu}(\Theta)\, A_\beta(z;\Theta) \,, \tag{4.6.4b}$$

$$c_{\beta\beta} > 0 \,, \quad d_{\beta\beta} = 1/c_{\beta\beta} \,, \tag{4.6.4c}$$

$c_{\beta\nu}$ and $d_{\beta\nu}$ being certain uniquely determined functions of the parameter Θ.[18] The Tchebychef functions $A_\beta(z;\Theta)$, $\beta = 0,1,\ldots,$ are complete, i.e., they form a basis in $\mathbb{L}^2(\,\mathbb{R}^+_0;\ \mathbb{M}(z;\Theta)dz\,)$.

b. Of course, a sequence of Ikenberry's harmonics $Y^\alpha(g)$, $\alpha = 0,1,\ldots,$ can be treated as a complete orthogonal[(45)] [but, according to Eq. (3.4.21), not normalized] set of "angular" functions of the "point" g of the unit sphere $\mathbb{K}$ for immediate use in expanding an arbitrary element of the real Hilbert space $\mathbb{L}^2(\,\mathbb{K};\ dg\,)$ generated by the scalar product

$$\{\varphi_1|\varphi_2\}_2 := \int_{\mathbb{K}} \varphi_1(g)\varphi_2(g)\, dg \,. \tag{4.6.5}$$

c. Let $\mathbb{R}^+_0 \times \mathbb{K}$ be a set of all ordered pairs (z,g), $z \in \mathbb{R}^+_0$, $g \in \mathbb{K}$, and consider the induced [by $\mathbb{L}^2(\,\mathbb{R}^+_0;\ \mathbb{M}(z;\Theta)dz\,)$ and $\mathbb{L}^2(\,\mathbb{K};\ dg\,)$] Hilbert space $\mathbb{L}^2(\,\mathbb{R}^+_0 \times \mathbb{K};\ \mathbb{M}(z;\Theta)dzdg)$ with the scalar product $\{\ |\ \}_0$ given by

$$\{\varphi_1|\varphi_2\}_0 := \int_{\mathbb{R}^+_0}\int_{\mathbb{K}} \varphi_1(z,g)\varphi_2(z,g)\, \mathbb{M}(z;\Theta)\, dzdg \,. \tag{4.6.6}$$

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[18] If some of the functions S_β are linearly dependent, then we must reject all unnecessary elements, leaving the procedure of orthogonalization otherwise unchanged. As an example, in case of (4.2.5) the set $\left\{S_\beta;\ \beta = 0,1,\ldots\right\}$ should be replaced by its subset $\left\{S_0, S_1, S_{2\beta};\ \beta = 1,2,\ldots\right\}$.

Inspection shows that the system $\left\{ Y^{\alpha}(g)A_{\beta}(z;\Theta);\ \alpha,\beta = 0,1,\ldots \right\}$ is a basis in $\mathbb{L}^2(\mathbb{R}^+_o \times \mathbb{K};\ \mathbb{H}(z;\Theta)dzdg)$.

Remembering that $y = \delta|\mathbf{k}|$, to each wave vector $\mathbf{k} \in \mathbb{E} \setminus \{0\}$ corresponds a uniquely determined element $(z,g) \in \mathbb{R}^+_o \times \mathbb{K}$ such that $z = \mu'(y)$ and $g = |\mathbf{k}|^{-1}\mathbf{k}$. Since $\mathbf{k} = \delta^{-1}\lambda(z)g$, the opposite statement is also true. Whenever there is no possibility of confusion, it may be convenient not to distinguish between $\mathbb{E} \setminus \{0\}$ and $\mathbb{R}^+_o \times \mathbb{K}$, but to regard them as the same set.

Condition I. Let φ be a certain function defined on $\mathbb{R}^+_o \times \mathbb{K}$ and suppose that

$$\hat{\varphi}(z,g) := \frac{(e^{\omega} - 1)}{\omega\, e^{\omega}}\ \frac{[\varphi(z,g) - f_o(\omega)]}{f_o(\omega)}\ , \tag{4.6.7}$$

f_o being equal to $(e^{\omega} - 1)^{-1}$. Then φ satisfies *Condition I* if $\hat{\varphi} \in \mathbb{L}^2(\mathbb{R}^+_o \times \mathbb{K};\ \mathbb{H}(z;\Theta)dzdg)$.

Now, if the distribution function satisfies *Condition I*, then f has a unique expansion

$$f = f_o\left[1 + \frac{\omega\, e^{\omega}}{e^{\omega} - 1} \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} a^{\alpha,\beta}(x,t) \cdot Y^{\alpha}(g)A_{\beta}(z;\Theta) \right] , \tag{4.6.8}$$

and this expansion converges in the *mean* to the function f. The Tchebychef-Ikenberry coefficients $a^{\alpha,\beta} \in \mathrm{Ker}_{\alpha}\mathrm{Tr}$ have a simple representation in terms of $f - f_o$. Because of the orthogonality properties of $A_{\beta}(z;\Theta)$ and $Y^{\alpha}(g)$, we find that

$$a^{\alpha,\beta} = \frac{(2\alpha + 1)!!}{4\pi\alpha!}\, \delta^3 \int_{\mathbb{E}} \omega\, Y^{\alpha}(g)A_{\beta}(z;\Theta)(f - f_o)\, d\mathbf{k}\ , \tag{4.6.9}$$

and if we assume that f_o is a Bose-Einstein distribution function which corresponds to f, we get

$$a^{0,0} = 0\ . \tag{4.6.10}$$

For every fixed α, $\alpha = 0,1,\ldots$, each member of one of the two sets $\left\{N^{\alpha,\beta} - N^{\alpha,\beta}_{eq};\ \beta = 0,\ldots,\gamma\right\}$ and $\left\{a^{\alpha,\beta};\ \beta = 0,\ldots,\gamma\right\}$ is a linear combination of the members of the other. Indeed, using (4.4.3), (4.4.7), (4.6.9), (4.6.4), we arrive at

$$a^{\alpha,\beta} = \frac{(2\alpha + 1)!!}{4\pi\alpha!}\, \Theta\, \delta^3 \sum_{\nu = 0}^{\beta} c_{\beta\nu} \left(N^{\alpha,\nu} - N^{\alpha,\nu}_{eq} \right) , \qquad (4.6.11a)$$

$$N^{\alpha,\beta} - N^{\alpha,\beta}_{eq} = \frac{4\pi\alpha!}{(2\alpha + 1)!!} \left(\Theta\, \delta^3\right)^{-1} \sum_{\nu = 0}^{\beta} d_{\beta\nu}\, a^{\alpha,\nu} . \qquad (4.6.11b)$$

In this sense the $a^{\alpha,\beta}$ are simply an alternative set of moments of f, just as were the Tchebychef coefficients b_n , $n = 0,1,\ldots$, introduced in Section 1.3.A of Chapter 1.

One final point to note is as follows. Our interest in studying the irreducible moment representation of f rests altogether on the fact that, in contrast with the irreducible expansion of f for classical gases which results from (3.4.3), there is *no* reducible counterpart of (4.6.8) at all. Although, instead of (4.4.3) and (4.4.4a), we could equally well have defined in Section 4.4.A the reducible moments of f:

$$\hat{N}^{\alpha,\beta} := \int_{\mathbb{E}} \left(\otimes^{\alpha} g\right) \mu(y) S_{\beta} f \, dk , \qquad (4.6.12)$$

and also derived there the balance equations for them, the benefit which arises if we deal with (4.4.3) rather than (4.6.12) is manifestly strengthened by the *nonexistence* of the reducible variant of (4.6.11) as well. [It is well known that a complete set of orthonormal polynomials in three variables x_1, x_2, and x_3 can be obtained by using products of such polynomials in a single variable. Such a procedure lacks symmetry, and there is sometimes an advantage to be gained by expressing the polynomials in tensor invariant (reducible or irreducible) notation. Thus, insofar as classical gases are concerned, Grad(55) defined the Hermite function $B^{\alpha}(x)$, $x \in \mathbb{E}$, which is a reducible symmetric tensor of order α, whose components are polynomials of degree α in the coordinates x_1, x_2, and x_3 of x. Due to the unique fact that $exp(|x|^2) = exp(x_1^2 + x_2^2 + x_3^2) = exp(x_1^2)\ exp(x_2^2)\ exp(x_3^2)$, the Hermite polynomials $B^\alpha(x) \in \mathbb{E}_s^\alpha$ are orthogonal with respect to the weight $(2\pi)^{-3/2}\ exp(-\frac{1}{2}|x|^2)$. Since the same reasoning cannot be proposed for other weight functions, our description of quasiparticle gases is inherently in tensor irreducible notation.]

The procedure of this section can be repeated, essentially word for word with only slight technical changes in the method, to take into account (4.2.6) as well as (4.2.7). Some discussion of these cases is offered in Appendix 4.A [cf. also Refs. 25 and 26].

4.6.B. The Scheme of Constructing Tchebychef Functions

In connection with the expansion (4.6.8) we introduced the set of real-valued continuous functions $A_\beta(z;\Theta)$ on $\mathbb{R}_o^+$ which are orthonormal with respect to weight $\mathbb{M}(z;\Theta)$: $\{A_\beta | A_\nu\}_1 = \delta_{\beta\nu}$, $\delta_{\beta\nu}$ being Kronecker's delta and $\{\ |\ \}_1$ being a scalar product defined by (4.6.1b). The importance of Tchebychef functions $A_\beta(z;\Theta)$ is obvious, for it is only they that permit us even to frame such questions as whether Grad's ideas(3,4) are of interest in the kinetic theory of quasiparticle gases. Of course, it is one thing to state that there are certain special Tchebychef functions of $z \in \mathbb{R}_o^+$ and Θ, but it is quite another to determine them. Thus we wish to conclude Section 4.6.A with a brief mention of the method of constructing $A_\beta(z;\Theta)$.

We begin by defining the class of auxiliary quantities to be considered:

$$k_\beta := \left\{ zA_\beta - 2 \,|\, A_\beta - 2 \right\}_1 , \tag{4.6.13a}$$

$$\ell_\beta := \left\{ z^2A_\beta - 2 \,|\, A_\beta - 2 \right\}_1 , \tag{4.6.13b}$$

$$n_\beta := \left\{zA_{\beta - 2}|A_{\beta - 1}\right\}_1 . \tag{4.6.13c}$$

We are now ready to establish the useful

Theorem. The following relation holds for any five consecutive orthonormal Tchebychef functions:

$$A_\beta(z;\Theta) = (e_\beta z - f_\beta)A_{\beta - 2}(z;\Theta) - g_\beta A_{\beta - 1}(z;\Theta) - h_\beta A_{\beta - 3}(z;\Theta)$$

$$- i_\beta A_{\beta - 4}(z;\Theta) , \quad \beta = 4,5,\ldots . \tag{4.6.14a}$$

Here e_β, f_β, g_β, h_β, and i_β are functions of Θ alone; $e_\beta > 0$, $f_\beta > 0$, and $i_\beta > 0$. If the highest coefficient $c_{\beta\beta}(\Theta)$ of $A_\beta(z;\Theta)$ is denoted by τ_β in Eqs. (4.6.4), then we have

$$e_\beta = \tau_\beta(\tau_{\beta - 2})^{-1} , \quad i_\beta = e_\beta(e_{\beta - 2})^{-1} , \tag{4.6.14b}$$

$$g_\beta = e_\beta n_\beta , \quad h_\beta = e_\beta n_{\beta - 1} , \tag{4.6.14c}$$

$$f_\beta = e_\beta k_\beta , \tag{4.6.14d}$$

$$\tau_\beta = \tau^2_{\beta - 2}\left[(\ell_\beta - k^2_\beta - n^2_\beta - n^2_{\beta - 1})\tau^2_{\beta - 2} - \tau^2_{\beta - 4}\right]^{-1/2} . \tag{4.6.14e}$$

For the proof, we first observe that $zS_{\beta - 2} = S_\beta$ and then determine e_β so that $A_\beta - e_\beta zA_{\beta - 2}$ is a linear combination $\delta_0 A_0 + \ldots + \delta_{\beta - 1}A_{\beta - 1}$. Because of the orthogonality properties of Tchebychef functions it is readily seen that $\delta_\nu = 0$ if $\nu < \beta - 4$. Therefore (4.6.14a) follows. The first part of (4.6.14b) is a consequence of (4.6.14a); the second part follows from

$$\int_{\mathbb{R}^+_0} A_{\beta - 4}(z;\Theta)A_\beta(z;\Theta)\mathbb{W}(z;\Theta)\, dz = 0 = - i_\beta$$

$$+ e_\beta \int_{\mathbb{R}^+_0} zA_{\beta - 4}(z;\Theta)A_{\beta - 2}(z;\Theta)\mathbb{W}(z;\Theta)\, dz ,$$

since the integral of the right-hand member is equal to

$$\tau_{\beta - 4}\left\{S_{\beta - 2} | A_{\beta - 2}\right\}_1 = (e_{\beta - 2})^{-1} .$$

The proof of (4.6.14c)-(4.6.14e) may be obtained by combining (4.6.14a) with $\{A_\beta | A_\nu\}_1 = \delta_{\beta\nu}$ where $\nu = \beta - 4, \ldots, \beta$. ■

The recurrence formulas (4.6.14), which tell us that A_β is *uniquely* determined from $A_{\beta - 4}$, $A_{\beta - 3}$, $A_{\beta - 2}$, and $A_{\beta - 1}$, are valid also for $\beta = 2,3$ if we write $A_{-2} = A_{-1} = \tau_{-2} = \tau_{-1} = 0$.

In summary, Eqs. (4.6.14) deliver through routine algebra as many of the orthonormal Tchebychef functions A_β as be desired, provided only that A_0 and A_1 shall have been calculated first. The theorem we present here has not been published before,(25) so far as we know. We were led to it in an attempt to generalize some results of Szegö on Tchebychef polynomials [cf. Ref. 34, p. 42].

4.6.C. Formal Evaluation of Collision Integrals

So as to be able to arrive at the collision integrals $P^{\alpha,\beta}$ on the r.h.s. of (4.4.4a) that depend upon $N^{\alpha,\beta} - N^{\alpha,\beta}_{eq}$, we follow, step by step, Grad's line of thought.(1,3) Thus we formally substitute the representation (4.6.8) of f into (4.4.6) and then, applying $\text{ƙ}^2 \text{đ}k = \delta^{-3}\lambda'\lambda^4 dz dg$, (3.4.21), (4.3.8b), (4.3.2a), (4.6.1e), (4.6.1a), and (4.6.11a), expand the result:

$$P^{\alpha,\beta} = -\frac{1}{\tau}\left(N^{\alpha,\beta} - N^{\alpha,\beta}_{eq} \right)$$

$$+ 4\pi c_\alpha \int_{\mathbb{E}} \text{đ}k \; Y^\alpha(g)\,\text{ƙ}^2 G(\text{ƙ})\mu(y) S_\beta$$

$$\times f_o\left[1 + \frac{\omega e^\omega}{e^\omega - 1} \sum_{\nu = 0}^{\infty} \sum_{\gamma = 0}^{\infty} a^{\nu,\gamma} \cdot Y^\nu(g) A_\gamma(z;\Theta)\right]$$

$$= -\frac{1}{\tau}\left(N^{\alpha,\beta} - N^{\alpha,\beta}_{eq} \right)$$

$$\underline{+\ 4\pi c_{\alpha}\delta^{-2} \sum_{\nu=0}^{\infty} \sum_{\gamma=\nu}^{\infty} p_{\beta\gamma} c_{\gamma\nu}\left(N^{\alpha,\nu} - N^{\alpha,\nu}_{eq} \right)} \ , \qquad (4.6.15a)$$

in which

$$p_{\beta\gamma} := \int_{\mathbb{R}^+_0} \left(\lambda(z)\right)^2 G\left(\delta^{-1}\lambda(z)\right) S_{\beta}(z) A_{\gamma}(z;\Theta)\mathbb{M}(z;\Theta)\, dz \ . \qquad (4.6.15b)$$

The effect of using (4.6.15) to determine the r.h.s. of (4.4.4a) is not only the system of equations of transfer for the expectations $N^{\alpha,\beta}$ of the Tchebychef-Ikenberry functions $Y^{\alpha}(g)\mu(y)S_{\beta}$ but also the precise algorithm (4.6.15b) for expressing the collision elements $p_{\beta\gamma}$ in (4.6.15a) in terms of certain functions of Θ alone. Of course, just as in Grad's approach, our results are only formal since the underlined expression in (4.6.15a) involves the product of two series, neither of which need converge pointwise. If we set G given by (4.5.1) in (4.6.15b), it follows from

$$\left(\lambda(z)\right)^2 G\left(\delta^{-1}\lambda(z)\right) S_{\beta}(z) = \delta^2 \sum_{\nu=0}^{p} \mathbb{B}_{\nu}\, S_{\beta+2\nu}(z) \ , \qquad (4.6.16)$$

(4.6.4), and the orthogonality properties of $A_{\beta}(z;\Theta)$ that $p_{\beta\gamma}$ vanishes if $\gamma > \beta + 2p$ and that the series

$$\sum_{\gamma=\nu}^{\infty} p_{\beta\gamma} c_{\gamma\nu} = \sum_{\gamma=\nu}^{\beta+2p} p_{\beta\gamma} c_{\gamma\nu} \qquad (4.6.17)$$

is equal to 0 unless $\nu = \beta + 2s$ where $s = 0,\ldots,p$. For functions $G(k)$ other than (4.5.1) all the collision elements are usually nonzero, but presumably in many cases of interest the dominant ones appear to be those that are already nonzero if $G(k)$ is to be approximated fairly closely by the sum

$$\sum_{\nu=0}^{p} \mathbb{B}_{\nu}\, [\mu'(y)]^{\nu}$$

in which the choice of $p = 0,1,\ldots$ and $\mathbb{B}_{\nu}$, $0 \leq \nu \leq p$, depends on the shape of $G(\mathbb{A})$. By this means it is easy to think of at least one reasonable way of simplifying the underlined expression in (4.6.15a):

$$4\pi c_{\alpha}\delta^{-2} \sum_{\nu=0}^{\infty} \sum_{\gamma=\nu}^{\infty} p_{\beta\gamma} c_{\gamma\nu} \left(N^{\alpha,\nu} - N^{\alpha,\nu}_{eq} \right)$$

$$\rightarrow 4\pi c_{\alpha}\delta^{-2} \sum_{\nu=\beta}^{\beta+2p} \sum_{\gamma=\nu}^{\beta+2p} p_{\beta\gamma} c_{\gamma\nu} \left(N^{\alpha,\nu} - N^{\alpha,\nu}_{eq} \right) . \qquad (4.6.18)$$

Now, if we replace $P^{\alpha,\beta}$ by the last two terms on the r.h.s. of (4.6.15a) in Eq. (4.4.4a) and then adopt (4.6.18), we see that the ansatz (4.6.18) causes the so-called forward coupling of the equations of moments, as first indicated in Section 4.5 in a slightly different context.

On the understanding that either $G(\mathbb{A})$ is given by (4.5.1) or the transformation (4.6.18) may occur approximately (in the sense not yet made precise) when a more general function $G(\mathbb{A})$ is used, the method of solving the infinite system of equations of transfer for $N^{\alpha,\beta}$ will be discussed elsewhere.

4.7. The Approximate Formulae for the Entropy Density h *and its Flux* Φ*:* $\Omega(k,x) = a\,\mu(y)$

The tradition of the kinetic theory of quasiparticle gases chooses the following definitions of the entropy density h and the entropy flux Φ:

$$h := k_B (8\pi^3)^{-1} \int_{\mathbb{E}} \mathbb{F}\, dk \ , \qquad (4.7.1a)$$

$$\Phi := k_B (8\pi^3)^{-1} a\, \delta \int_{\mathbb{E}} g\, \mu'(y) \mathbb{F}\, dk \ , \qquad (4.7.1b)$$

where

$$\mathbb{F} := (1 + f)\ln(1 + f) - f\ln f \,. \tag{4.7.1c}$$

Of course, when we wish to compare propositions of the kinetic theory with phenomenological statements about entropy and its flux, we must multiply (4.4.1) by a function $(k_B/8\pi^3)[\ln(1 + f) - \ln f]$ and integrate the result over $\mathbb{E}$. Then, using (4.3.4c), we obtain the entropy principle of the form

$$\partial_t h + \nabla \cdot \Phi \geq 0 \,. \tag{4.7.2}$$

The crucial role of the Bose-Einstein distribution function f_o, especially in relation to the concept of local equilibrium, provides a persuasive reason for estimating the deviation of the entropy density h from its equilibrium counterpart h_o for f_o:

$$h_o := k_B(8\pi^3)^{-1}\int_{\mathbb{E}} \mathbb{F}_o \, dk \,. \tag{4.7.3a}$$

Here we have set

$$\mathbb{F}_o := (1 + f_o)\ln(1 + f_o) - f_o\ln f_o \,. \tag{4.7.3b}$$

If f satisfies *Condition I* of Section 4.6.A, i.e., if

$$\hat{f}(z,g) = \frac{(e^{\omega} - 1)}{\omega\, e^{\omega}} \; \frac{[f(z,g) - f_o(\omega)]}{f_o(\omega)}$$

belongs to the space of functions which are square-integrable over $\mathbb{R}^+_o \times \mathbb{K}$ with weight $\mathbb{W}(z;\Theta)$, then we are justified in using for $f_o^{-1}(f - f_o)$ here the function ε_o as defined by

$$\varepsilon_o := \frac{\omega\, e^{\omega}}{e^{\omega} - 1}\,\theta \,, \tag{4.7.4a}$$

$$\theta := \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} a^{\alpha,\beta}(x,t) \cdot Y^{\alpha}(g)A_{\beta}(z;\Theta) \,. \tag{4.7.4b}$$

In order to arrive at an approximate formula for an h corresponding to $f = f_o(1 + \varepsilon_o)$ which amounts to expressing $h - h_o$ in terms of the Tchebychef-Ikenberry coefficients $a^{\alpha,\beta}$ of f, we replace the logarithm $\ln(1 + X)$ $(|X| < 1)$ by the first two terms in its Taylor expansion $X - \frac{1}{2}X^2 + \ldots$, and so we obtain

$$\mathbb{F} = [1 + f_o(1 + \varepsilon_o)]\ln[(1 + f_o)(1 + e^{-\omega}\varepsilon_o)]$$

$$- f_o(1 + \varepsilon_o)\ln[f_o(1 + \varepsilon_o)] = \mathbb{F}_o + f_o\omega\,\varepsilon_o$$

$$+ (1 + f_o)(1 + e^{-\omega}\varepsilon_o)\ln(1 + e^{-\omega}\varepsilon_o) - f_o(1 + \varepsilon_o)\ln(1 + \varepsilon_o)$$

$$= \mathbb{F}_o + f_o\omega\,\varepsilon_o - \frac{1}{2}\,e^{-\omega}\varepsilon_o^2 + \ldots$$

$$\cong \mathbb{F}_o + (\lambda'\lambda^2)^{-1}\mathbb{H}(z;\Theta)\theta - \frac{1}{2}\,(\lambda'\lambda^2)^{-1}\mathbb{H}(z;\Theta)\theta^2\,. \qquad (4.7.5)$$

Substituting (4.7.5) into (4.7.1a) and referring to the orthogonality properties of $A_\beta(z;\Theta)$ and $Y^\alpha(g)$, we easily show that

$$h - h_o \cong -\frac{k_B}{4\pi^2\delta^3}\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\frac{\alpha!}{(2\alpha+1)!!}\,a^{\alpha,\beta}\cdot a^{\alpha,\beta}\,. \qquad (4.7.6)$$

In deriving (4.7.6) from (4.7.1a) and (4.7.5) we have used (4.6.10). Routine calculation shows that

$$\left\{\hat{f}\,|\,\hat{f}\right\}_o = 4\pi\sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\frac{\alpha!}{(2\alpha+1)!!}\,a^{\alpha,\beta}\cdot a^{\alpha,\beta}\,, \qquad (4.7.7)$$

where $\{\ |\ \}_o$ is a scalar product given by (4.6.6). Therefore if we assume that f satisfies *Condition I*, the series on the r.h.s. of (4.7.6) converges. Clearly, our study of the dependence of $h - h_o$ upon $a^{\alpha,\beta}$ applies only to small values of $f_o^{-1}(f - f_o)$, they being the only ones for which a plausible argument, however limited, in favour of the practical usefulness of the specific expansion (4.6.8) could be provided [cf. also Section 4.8]. But we have explained already these problems and of general interest in this regard is our discussion in Section 1.3.C of Chapter 1.

We are now ready to calculate Φ. By use of (4.7.1b), (4.7.5), (4.7.4b), $\mu'(y) = z$, $dk = \delta^{-3}\lambda'\lambda^2 dz dg$, (4.6.14), $\{A_\beta | A_\nu\}_1 = \delta_{\beta\nu}$, and (3.4.21) it is a simple but very tedious matter to do so. The result is

$$\Phi - \frac{1}{T} q \cong - \frac{k_B a}{2\pi^2 \delta^2} \sum_{\alpha=1}^{\infty} \sum_{\beta=0}^{\infty} \frac{\alpha!}{(2\alpha+1)!!} a^{\alpha,\beta} \circ c^{\alpha-1,\beta}, \tag{4.7.8a}$$

where

$$c^{\alpha,\beta} := \tau_\beta(\tau_{\beta+2})^{-1} a^{\alpha,\beta+2} + \tau_{\beta-2}(\tau_\beta)^{-1} a^{\alpha,\beta-2}$$

$$+ n_{\beta+2} a^{\alpha,\beta+1} + n_{\beta+1} a^{\alpha,\beta-1} + k_{\beta+2} a^{\alpha,\beta}, \tag{4.7.8b}$$

$$c^{\alpha,\beta} \in \mathrm{Ker}_\alpha \mathrm{Tr}, \quad a^{\alpha,-2} = a^{\alpha,-1} = 0 . \tag{4.7.8c}$$

In going from (4.7.1b) to (4.7.8) we have utilized

$$\frac{k_B a}{6\pi^2 \delta^2} \sum_{\beta=0}^{2} \left\{ z | A_\beta \right\}_1 a^{1,\beta} = \frac{1}{T} q , \tag{4.7.9}$$

q being the heat flux and T being the kinetic theory temperature.

As a step toward obtaining an estimate of the r.h.s. of (4.7.8a), let us observe that by (4.6.13) and (4.6.14e) we arrive at

$$k_\beta \le \sqrt{\ell_\beta} , \tag{4.7.10a}$$

$$|n_\beta| \le \min(\sqrt{\ell_\beta}, \sqrt{\ell_{\beta+1}}) , \tag{4.7.10b}$$

$$\tau_{\beta-2}(\tau_\beta)^{-1} \le \min(\sqrt{\ell_\beta}, \sqrt{\ell_{\beta+2}}) . \tag{4.7.10c}$$

Then, combining (4.7.10) with

$$\frac{\alpha!}{(2\alpha+1)!!} < \frac{1}{2} \frac{(\alpha-1)!}{(2\alpha-1)!!} , \quad \alpha = 1,2,\ldots \tag{4.7.11a}$$

and

$$|a^{\alpha,\beta} \bullet a^{\alpha - 1,\nu}| \le 3\, |a^{\alpha,\beta}|\, |a^{\alpha - 1,\nu}|$$

$$\le \frac{3}{2} \left(a^{\alpha,\beta} \bullet a^{\alpha,\beta} + a^{\alpha - 1,\nu} \bullet a^{\alpha - 1,\nu} \right) , \qquad (4.7.11b)$$

we may easily find that

$$\left| \sum_{\alpha = 1}^{\infty} \sum_{\beta = 0}^{\infty} \frac{\alpha!}{(2\alpha + 1)!!} a^{\alpha,\beta} \bullet c^{\alpha - 1,\beta} \right|$$

$$< \mathfrak{X} := 10 \sum_{\alpha = 0}^{\infty} \sum_{\beta = 0}^{\infty} \frac{\alpha!}{(2\alpha + 1)!!} \varkappa_{\beta}\, a^{\alpha,\beta} \bullet a^{\alpha,\beta} , \qquad (4.7.12a)$$

where

$$\varkappa_{\beta} := \max\left(1, \sqrt{l_{\beta + 2}} \right) . \qquad (4.7.12b)$$

It follows from the definition of $\mathfrak{X}$ and the inequality (4.7.12a) that if $\mathfrak{X} < \infty$, then f satisfies *Condition I* and the series on the r.h.s. of (4.7.8a) converges in the sense of the norm in $\mathbb{E}$; in this context, see Section 3.2.A in Chapter 3.

4.8. Final Remarks

In closing, it is pertinent to summarize the main feature of the analysis.

a. The derivation of a set of equations of transfer and of the irreducible moment representation of f is only the first step in constructing a kinetic theory of flow phenomena for quasiparticle gases. There still remains the problem of obtaining a solution (or solutions) to Eqs. (4.4.4a) and the subsequent utilization of these solutions so as to be able to demonstrate that the sum of the series (4.6.8) of which the Tchebychef-Ikenberry coefficients $a^{\alpha,\beta}$ are gained by (4.6.11a) is a solution of the original Boltzmann-Peierls equation (4.4.1). Insofar as the model proposition (4.3.1) and the simplifying assumptions (4.3.5) and (4.5.1) are concerned, research in this direction can be outlined and is under way.

b. Of particular, even decisive importance in our analysis is the method of defining the weight (4.6.1a) and of introducing the Tchebychef functions $A_\beta(z;\theta)$ that culminates in, and leads to, the approximate formula (4.7.6) for $h - h_o$ with no cross terms coupling different Tchebychef-Ikenberry coefficients. Of course, by a mere substituting (4.6.8) into (4.7.1a) we cannot *diagonalize* the entropy density exactly, but provided that $f_o^{-1}(f - f_o) \ll 1$ it is worthwhile applying the transformation (4.6.11) to uncouple the largest contribution to $h - h_o$ —— that of great interest in considering the axioms of extended irreversible thermodynamics.(65) Thus the unique advantage of having (4.7.6) is certain to be elaborate.(26)

c. Notice that Eq. (4.7.6) contains complete information about the so-called *second moments* of the equilibrium fluctuations of the Tchebychef-Ikenberry coefficients $a^{\alpha,\beta}$ of f;(47) the fluctuations are supposed to occur around $a^{\alpha,\beta} = 0$ provided the $a^{o,o}$ is being kept equal to zero. Since $a^{\alpha,\beta} = 0$ gives $h - h_o$ the greatest value it can attain for one-particle densities f corresponding to the same gross condition $a^{o,o} = 0$, we may follow in a sense Einstein and Smoluchowski and assume that the (relative) probability density of fluctuations $\mathbb{P}$ is given by

$$\mathbb{P} = exp\left[k_B^{-1} V(h - h_o)\right] ,$$

V being the volume of the gaseous system. In this way the approximate expression (4.7.6) for $h - h_o$ defines the simplest acceptable potential for the random variations of $a^{\alpha,\beta}$, or of $N^{\alpha,\beta}$, near the *absolute* equilibrium state. Of course, this becomes a statement with meaning only if we first tell what is meant by probability. There are many ways of assigning mathematically a probability, i.e., a normalized measure in the space of sequences $\left\{a^{\alpha,\beta}\right\}$ such that

$$\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{\alpha!}{(2\alpha+1)!!} a^{\alpha,\beta} \cdot a^{\alpha,\beta} < \infty ,$$

but if expectations according to this probability are to correspond to the interesting ideas of Jou,(75) we demand a physical explanation of it. Since $\mathbb{P}$ is, roughly speaking, a joint probability density of infinitely many random variables, these problems deserve some more attention in future investigations and will be considered elsewhere.

d. We should note one more thing. Our decision to study the Ikenberry-Tchebychef expansion (4.6.8) is considerably strengthened by the fact that the abstract space $\mathbb{L}^2\big(\mathbb{R}_o^+ \times \mathbb{K};\ \mathbb{H}(z;\theta)dzdg\big)$ invented in Section 4.6.A and the Hilbert space $\mathbb{L}^2\big(\mathbb{E};\ f_o(1+f_o)dk\big)$ in which the linearized Boltzmann-Peierls operator $\mathbb{C}(f_o,\cdot)$ is symmetric and nonpositive are *canonically isomorphic*. Now, let $\mathbb{L}^2_\perp$ be a set consisting of either

(1) functions $\psi \in \mathbb{L}^2\big(\mathbb{E};\ f_o(1+f_o)dk\big)$ such that

$\int_{\mathbb{E}} \Omega\, \psi\, f_o(1+f_o)\, dk = 0$, or

(2) functions $\psi \in \mathbb{L}^2\big(\mathbb{E};\ f_o(1+f_o)dk\big)$ such that

$\int_{\mathbb{E}} \Omega\, \psi\, f_o(1+f_o)\, dk = 0, \qquad \int_{\mathbb{E}} k\, \psi\, f_o(1+f_o)\, dk = 0$,

and denote by $\mathbb{C}_L$ the restriction of $\mathbb{C}(f_o,\cdot)$ to $\mathbb{L}^2_\perp$. Then, recalling both the Chapman-Enskog method(36) and Gurevich's monograph,(2) we know that the problem of calculating a variety of transport coefficients is simply one of solving a Fredholm equation for ψ of the form

$\mathbb{C}_L \psi = \mathbb{D}$,

$\mathbb{D}$ being given in terms of k, Ω, $\nabla_k \Omega$, T, and the first space-derivatives of the conserved moments of f. That this equation has a solution for many common quasiparticle models, is ensured by the Riesz-Schauder theory of completely continuous operators, as presented, for example, in Chapter X of Yosida's book.(56) What is more important for the present purpose, however, is the observation as follows. The specific and well-founded assumption of the expressibility of $\Omega^{-1}\psi$ by the sum of a convergent series of Tchebychef functions $A_\beta(z;\Theta)$ allows one to estimate the expansion coefficients, which are quantities independent of k, through a sequence of approximations usually made when dealing with the integral equation of Fredholm's type [see, e.g., Ref. 36, pp. 124-129]. Clearly this is a formulation of the basic step in the argument —— that the Tchebychef basis $\{A_\beta;\ \beta = 0,1,\dots\}$ is very useful in exploiting the Chapman-Enskog procedure as well.

Appendix 4.A: The Tchebychef-Ikenberry Expansion for the Dispersion Relations (4.2.6) and (4.2.7)

4.A.a. The Power (Magnon) Model: $\Omega(k,x) = c(x)\mathcal{k}^r$, $1 < r \le 2$

We begin with the definitions:

$$\iota := \frac{r}{r-1}\,, \qquad d := \frac{r+4}{r}\,, \qquad \Theta := \frac{\hbar c}{k_B T}\,, \tag{4.A.1}$$

$$z := \Theta^{\iota^{-1}} v(\mathcal{k}) \qquad \left[\Rightarrow \omega = \frac{\hbar\Omega}{k_B T} = z^{\iota}\right], \tag{4.A.2}$$

$$\mathbb{M}_r(z) := \frac{\omega^{d}\, e^{\omega}}{\left(e^{\omega}-1\right)^2} \le \mathbb{M}\, e^{-\mathbb{C}z}\,. \tag{4.A.3}$$

If in addition we replace $\mathbb{L}^2(\mathbb{R}^+_o \times \mathbb{K};\ \mathbb{M}(z;\Theta)dzdg)$ by $\mathbb{L}^2(\mathbb{R}^+ \times \mathbb{K};\ \mathbb{M}_r(z)dzdg)$ in Section 4.6.A, then by a mere application of the transformations

$$\left\{S_\beta(z);\ \beta = 0,1,\dots\right\} \Rightarrow \left\{1,\ z^{-1},\ z^{\beta};\ \beta = 1,2,\dots\right\}, \tag{4.A.4}$$

$$A_\beta(z;\Theta) \Rightarrow A_\beta(z)\,, \tag{4.A.5}$$

$$\left\{c_{\beta\nu}(\Theta),\ d_{\beta\nu}(\Theta)\right\} \Rightarrow \left\{c_{\beta\nu},\ d_{\beta\nu}\right\}, \tag{4.A.6}$$

$$\delta \Rightarrow (r-1)^{1/2}\Theta^{1/r}\,, \tag{4.A.7}$$

we conclude that (4.6.4) and (4.6.8)-(4.6.10) are still valid.

4.A.b. The Isotropic and Dispersionless (Phonon) Model: $\Omega(k,x) = c(x)\mathscr{k}$

In this case we have

$$z := \theta\,\mathscr{k} \ , \qquad \theta := \frac{\hbar\, c}{k_B T} \ , \tag{4.A.8}$$

$$\mathbb{W}(z) := \frac{z^4\, e^z}{(e^z - 1)^2} \leq \mathbb{M}\, e^{-\mathbb{C} z} \ , \tag{4.A.9}$$

and

$$\mathbb{L}^2\big(\, \mathbb{R}_o^+ \times \mathbb{K};\ \mathbb{W}(z;\theta)dzdg\,\big) \to \mathbb{L}^2\big(\, \mathbb{R}^+ \times \mathbb{K};\ \mathbb{W}(z)dzdg\,\big) \ , \tag{4.A.10}$$

$$\Big\{S_\beta(z);\ \beta = 0,1,\ldots\Big\} \to \Big\{z^\beta;\ \beta = 0,1,\ldots\Big\} \ , \tag{4.A.11}$$

$$A_\beta(z;\theta) \to A_\beta(z) \ , \tag{4.A.12}$$

$$\Big\{c_{\beta\nu}(\theta),\ d_{\beta\nu}(\theta)\Big\} \to \Big\{c_{\beta\nu},\ d_{\beta\nu}\Big\} \ , \tag{4.A.13}$$

$$\delta \to \theta \ . \tag{4.A.14}$$

PART C

DENSE GASES AND FLUIDS: MODEL SYSTEMS

CHAPTER 5

EXTENDED THERMODYNAMICS OF A MODEL HARD-SPHERE FLUID: THE REVISED ENSKOG EQUATION

The joint labours of specialists in mechanics and thermodynamics have suggested certain purely phenomenological hypotheses regarding the structure of extended irreversible thermodynamics of gases and fluids: the microscopic counterparts to these propositions are, however, known for very few kinds of kinetic models of continuum systems (Boltzmann's equation). While Boltzmann's H-theorem lies at the very basis of the kinetic origin of extended irreversible thermodynamics of dilute gases — in particular, in justifying the local constitutive assumption underlying the Liu-Müller theory — no such interrelation between macroscopic and microscopic approaches has been proved as yet for the revised Enskog equation with the appropriate H-theorem; see, however, Ref. 20. [The revised Enskog equation, although restricted to the model hard-sphere fluid, has proved successful in giving a microscopic foundation to the local balance equation for the entropy density h with the non-negative entropy production σ_h for arbitrarily strong spatial inhomogeneities.] Accordingly, the purpose of this chapter is to construct extended irreversible thermodynamics of dense gases and fluids within the framework of the revised Enskog equation and to investigate the nature of the resulting discrepancies with the existing phenomenological theories.

5.1. Notation

Throughout this chapter we use direct notation for vectors and tensors as introduced in Section 3.2 and Appendix 3.A. In addition, for the present purpose further useful tensorial abbreviations are of interest. We define them in the following way:

a. Let $M^\alpha \in \mathbb{E}^\alpha$ and $M^\beta \in \mathbb{E}^\beta$ be two arbitrary tensors and choose an orthonormal basis $\{e_1, e_2, e_3\}$ in $\mathbb{E}$. Then

$$M^\alpha \circledast M^\beta := \sum_{r=1}^{3} e_r \otimes \left[(e_r \cdot M^\alpha) \vee M^\beta \right] , \qquad (5.1.1a)$$

$$M^\alpha \blacksquare M^\beta := \sum_{r=1}^{3} \sum_{s=1}^{3} e_r \otimes e_s \otimes \left[(e_r \cdot M^\alpha) \cdot (e_s \cdot M^\beta) \right] , \qquad (5.1.1b)$$

$$M^\alpha \square M^\beta := \sum_{r_1 \ldots r_{\alpha-\beta}=1}^{3} e_{r_1 \ldots r_{\alpha-\beta}} \otimes \left[(e_{r_1 \ldots r_{\alpha-\beta}} \cdot M^\alpha) \cdot M^\beta \right] , \quad \alpha \geq \beta , \qquad (5.1.1c)$$

$$M^\alpha \square M^\beta := \sum_{r_1 \ldots r_{\beta-\alpha}=1}^{3} e_{r_1 \ldots r_{\beta-\alpha}} \otimes \left[M^\alpha \cdot (e_{r_1 \ldots r_{\beta-\alpha}} \cdot M^\beta) \right] , \quad \beta > \alpha . \qquad (5.1.1d)$$

b. Let $M^\alpha : \mathbb{E}^\beta \Rightarrow \mathbb{E}^\alpha$ be a differentiable tensor function and choose an orthonormal basis in $\mathbb{E}$. Then a mapping $\varphi_{\alpha\beta} : \mathbb{E}^\beta \Rightarrow \mathbb{E}^{\alpha+\beta}$, for which we write

$$\varphi_{\alpha\beta}(M^\beta) := \frac{\partial M^\alpha}{\partial M^\beta} , \qquad (5.1.2a)$$

is defined by

$$\varphi_{\alpha\beta}(M^\beta) := \sum_{r_1 \ldots s_\beta = 1}^{3} \frac{\partial M^\alpha_{r_1 \ldots r_\alpha}}{\partial M^\beta_{s_1 \ldots s_\beta}} e_{r_2 \ldots r_\alpha r_1 s_1 \ldots s_\beta} , \qquad (5.1.2b)$$

where the coefficients $M^\alpha_{r_1 \ldots r_\alpha}$ and $M^\beta_{s_1 \ldots s_\beta}$ are components of

M^{α} and M^{β}, respectively.

It is easy to show that the above definitions are independent of the choice of an orthonormal basis in $\mathbb{E}$.

5.2. Balance Equations of Extended Thermodynamics of Dense Gases and Fluids and their Microscopic Background

5.2.A. The Revised Enskog Equation

Let us consider the simplest case of a system of N identical rigid elastic spheres with diameter r_o and molecular mass μ which are contained in a macroscopic volume V. In the dense gas (fluid) of hard spheres the calculation of the macroscopic observables requires only precise knowledge of the pair correlation function $\chi_2(x,y|\rho(t))$, which has the same functional dependence[1] on the mass density $\rho(x,t)$ as in the case of an inhomogeneous equilibrium gas (fluid) of hard spheres,(57) and of the one-point distribution function $f(x,\lambda,t)$. According to Boltzmann, f is defined in such a way that

$$n\, f(x,\lambda,t)dx d\lambda \qquad \left[\, n := \frac{N}{V} \,\right]$$

expresses the number of particles within the element of space volume dx whose velocity vectors λ have their endpoints within an element $d\lambda$ of the velocity-space for these particles.

The exact kinetic equation for a dense gas (fluid) of hard spheres can be displayed only in the most formal way(58) at the present time. Consequently, an intuitive analysis is of interest. Ignoring the non-Markovian effects which follow from the velocity correlations between the particles, the derivation of the revised Enskog equation by Van Beijeren and Ernst(59) builds on the pioneering work of Enskog.(36) The revised Enskog equation, governing the time evolution of f, has the form

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[1] The quantity $A(x,y,t)$, functionally depending on $\rho(x,t)$ for all x, is often written $A(x,y|\rho(t))$.

$$\partial_t f + \lambda \cdot \nabla_x f = J(f) \ , \tag{5.2.1a}$$

where the collision operator J is defined by

$$J(f) := nr_o^2 \iiint dy\,d\omega\,d\nu \ [f_2(x,\lambda^*,y,\omega^*,t)\delta(y - x - r_o\ell)$$

$$- f_2(x,\lambda,y,\omega,t)\delta(y - x + r_o\ell)], \tag{5.2.1b}$$

$$f_2(x,\lambda,y,\omega,t) := \chi_2(x,y|\rho(t)) \ f(x,\lambda,t)f(y,\omega,t), \tag{5.2.1c}$$

$$d\nu := d\ell \ (c \cdot \ell)H(c \cdot \ell), \quad c := \omega - \lambda \ , \tag{5.2.1d}$$

$$\lambda^* := \lambda + (c \cdot \ell)\ell \ , \quad \omega^* := \omega - (c \cdot \ell)\ell \ . \tag{5.2.1e}$$

Here ℓ, δ, and H are the unit vector, the Dirac symbol, and the Heaviside function, respectively.

In the course of time $f(x,\lambda,t)$ changes owing to free streaming and collisions. For hard spheres the duration of binary encounters is zero. Thus, when two particles of equal mass collide, their velocities (λ,ω) take instantaneously post-collisional values λ^* and ω^*. The step function H and the distributions δ in Eqs. (5.2.1d) and (5.2.1b) impose the conditions: $c \cdot \ell > 0$ and $|y - x| = r_o$, respectively, which must be satisfied at the moment of impact. The density of pairs of particles in collisional configurations is described by the two-particle distribution f_2. Although restricted to the model hard-sphere fluid, the revised Enskog equation has proved successful in that the transport coefficients computed from it agreed quite well with experimental data over a wide range of densities. Two effects, not present in the Boltzmann description, have been taken into account: first, the covolume effect leading to exact equilibrium correlations between the colliding spheres, and second, the mechanism of collisional transfer of momentum and energy, influencing significantly transport phenomena.(11-14,50)

5.2.B. Grad's Arguments Regarding the Dilute-Gas Limit

Following Grad,(3) consider a fixed domain as a container for the fluid (gas) and a fixed distribution function f defined for x in the given domain. In the dilute-gas limit we let the total number of particles, N, increase without bound. The mass of each particle, μ, is allowed to approach zero in such a way that μn = const. At the same time, the molecular diameter r_o decreases with increasing N according to the rule nr_o^2 = const. Obviously, the pair correlation function χ_2 approaches 1 and the revised Enskog equation is replaced by the Boltzmann equation for elastic spheres in this limit.

5.2.C. Derivation of the Moment Equations within the Framework of the Revised Enskog Equation

Let $\mathscr{N}(\lambda)$ be any function of the molecular velocity λ. Multiplication of the collision operator J(f) by $\mathscr{N}(\lambda)$ in the revised Enskog equation and integration throughout the velocity-space as well as the subsequent use of the Irving-Kirkwood formula,(60) i.e., of the integral equality[20]

$$\int dy\ A(x,y) = -\frac{1}{2}\nabla\cdot\left[\int dz\ z\otimes\int_0^1 d\xi\ A\big(x+\xi z, x+(\xi-1)z\big)\right] \tag{5.2.2}$$

for the quantity A(x,y) that is antisymmetric on interchange of x and y, yield

$$\int d\lambda\ J(f)\mathscr{N}(\lambda) = P[\mathscr{N}\ ] - \nabla\cdot N[\mathscr{N}\ ]\ , \tag{5.2.3a}$$

where

$$P[\mathscr{N}\ ] := \frac{1}{4}\, nr_o^2 \iiint d\lambda d\omega dl\ [\mathscr{N}(\lambda^*) + \mathscr{N}(\omega^*) - \mathscr{N}(\lambda) - \mathscr{N}(\omega)] \times [F_2(\lambda,\omega|0) + F_2(\lambda,\omega|1)]\ , \tag{5.2.3b}$$

$$N[\mathscr{N}\ ] := \frac{1}{4}\, nr_o^3 \int_0^1\iiint d\xi d\lambda d\omega dl\ l\otimes[\mathscr{N}(\lambda^*) - \mathscr{N}(\omega^*) - \mathscr{N}(\lambda) + \mathscr{N}(\omega)]F_2(\lambda,\omega|\xi)\ , \tag{5.2.3c}$$

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[20] For more details, see the derivation of Eq. (5.13) in Ref. 61.

$$F_2(\lambda,\omega|\zeta) := f_2\big(x + \zeta r_o l,\lambda,x + (\zeta - 1)r_o l,\omega,t\big) \; . \qquad (5.2.3d)$$

The significance of $P[\mathcal{N}\,]$ and $N[\mathcal{N}\,]$ is readily seen; $P[\mathcal{N}\,]$ measures the rate of change of $\int d\Lambda\; \mathcal{N}(\lambda)f(x,\lambda,t)$ owing to encounters and $N[\mathcal{N}\,]$ describes the collisional transfer of the molecular property $\mathcal{N}(\lambda)$ across an element of area. Since for the molecular property $\mathcal{N}(\lambda)$ conserved after encounter a collision does not alter the total amount of $\mathcal{N}(\lambda)$ possessed by the molecules $\big(P[\mathcal{N}\,] = 0\big)$, but produces only a transfer of $\mathcal{N}(\lambda)$ across an element of area, we call $P[\mathcal{N}\,]$ and $N[\mathcal{N}\,]$ the $\mathcal{N}$-collision production (integral) and the $\mathcal{N}$-flux, respectively.

The method we use is related to the explicit calculation of the moment equations from the revised Enskog equation. We define, therefore, the *relative flux moment* $N^\alpha \in \mathbb{E}^\alpha$ *of degree* α and the collision integral $P^\alpha \in \mathbb{E}^\alpha_s$ of degree α as

$$N^\alpha := M^\alpha + \mu n\, N[\,\otimes^{\alpha - 1}\bar{\lambda}\,] \; , \qquad \alpha = 1,2,\dots \; , \qquad (5.2.4a)$$

$$P^\alpha := \mu n\, P[\,\otimes^{\alpha}\bar{\lambda}\,] \; , \qquad \alpha = 0,1,\dots \; , \qquad (5.2.4b)$$

$$\bar{\lambda} := \lambda - u \; , \qquad (5.2.4c)$$

u being the macroscopic velocity field of the gas (fluid):

$$u := \int d\Lambda\; \lambda f \left[\int d\Lambda\; f\right]^{-1} . \qquad (5.2.4d)$$

The relative flux moment N^α consists of two structurally different parts, one of which, denoted by M^α and called the *relative kinetic moment of degree* α, arises from the thermal motion of the molecules:

$$M^\alpha := \mu n \int d\Lambda\; \big(\otimes^{\alpha}\bar{\lambda}\big)f \; , \qquad \alpha = 0,1,\dots \; , \qquad (5.2.4e)$$

and the second, denoted by $\mu n\, N\big[\otimes^{\alpha - 1}\bar{\lambda}\,\big]$, from the intermolecular hard-sphere forces. Given the definitions (5.2.4), with the help of (5.2.3) and the revised Enskog equation (5.2.1) we arrive at the following system of balance equations for M^α:

$$\partial_t M^\alpha + \nabla \cdot \big(u \otimes M^\alpha + N^{\alpha + 1}\big) + \alpha\big(L \cup N^\alpha - R \vee M^{\alpha - 1}\big) = P^\alpha \; ,$$

$$\alpha = 0,1,\dots \; , \qquad (5.2.5a)$$

$$L := \nabla u\ , \quad R := (M^0)^{-1}\nabla \cdot N^2\ , \tag{5.2.5b}$$

$$M^{-1} := 0\ , \quad N^0 := 0\ , \quad M^1 = N^1 = 0\ . \tag{5.2.5c}$$

It is also traditional in continuum mechanics to supplement (5.2.5a) by the equation of change of u:

$$\partial_t u + u \cdot L + R = 0\ . \tag{5.2.5d}$$

By using (5.2.3b) and (5.2.3c) we are led to some interesting conclusions about the general properties of the tensors M^α, N^α, and P^α of degree α. In the frame of the purely macroscopic considerations there is room for freedom of interpretation. First, let us observe that the hard-sphere forces, which are taken into account, imply the natural statement that the relative kinetic moment M^α and the relative flux moment N^α are not the same, as in the case of a rarefied gas. Thus, insofar as dense gases are concerned, we may say that the introduction of the equality $M^\alpha = N^\alpha$ is no longer a well-founded assumption. Secondly, we should note that the symmetries in all indices can be attributed only to the tensors M^α and P^α:

$$\Pi M^\alpha = M^\alpha, \qquad \Pi P^\alpha = P^\alpha\ . \tag{5.2.6a}$$

Indeed, since the internal structure of N^α is considerably affected by the hard-sphere forces, the relative flux moments N^α are not, generally speaking, symmetric tensors but possess the far weaker properties of the type(19,20)

$$\Pi(e_r \cdot N^\alpha) = e_r \cdot N^\alpha\ , \tag{5.2.6b}$$

$\{e_1, e_2, e_3\}$ being an orthonormal basis in $\mathbb{E}$.

Chief among fields which can be constructed from M^α and N^α are the following quantities:

Name	Symbol
Mass density	$\rho := M^0 \in \mathbb{R}$
Internal energy per unit mass	$e := (2M^0)^{-1}\mathrm{Tr}M^2 \in \mathbb{R}$

Temperature	$T := \mu\left(3M^{0}\right)^{-1}\mathrm{Tr}M^{2} \in \mathbb{R}$
Pressure tensor	$\mathfrak{M} := N^{2} \in \mathbb{E}_{s}^{2}$
Mean normal pressure	$p := \frac{1}{3}\,\mathrm{Tr}N^{2} \in \mathbb{R}$
Pressure deviator	$\mathfrak{P} := \langle N^{2}\rangle \in \mathrm{Ker}_{2}\mathrm{Tr}$
Energy flux vector (heat flux)	$q := \frac{1}{2}\,\underset{(2,3)}{\mathrm{Tr}}\;N^{3} \in \mathbb{E}$

5.2.D. The H-Theorem for the Revised Enskog Equation

Among the objectives of any kinetic theory, one stands out as being of immediate importance —— that is a proof that the local entropy inequality

$$\partial_{t}h + \nabla \cdot (hu + \Phi) = \sigma_{h} \geq 0 \tag{5.2.7}$$

(where h, Φ, and σ_h are the entropy density, the entropy flux, and the entropy production, respectively) underlying all the work in rational thermodynamics(16,40) follows from the kinetic equation for f.

As far as the revised Enskog equation is concerned, such an amount of literature has grown up around the H-theorem, starting with the pioneering work by Résibois,(11) that we cannot hope to treat every detail of this problem. Instead we recall only the remarkable result of Mareschal et al.:(14)

$$h = \ldots - \mu n \int d\lambda\; f(x,\lambda,t)\,\ln\left[\mathbb{C}f(x,\lambda,t)\right] , \tag{5.2.8a}$$

$$\Phi = \ldots - \mu n \int d\lambda\; \vec{\lambda}\; f(x,\lambda,t)\,\ln\left[\mathbb{C}f(x,\lambda,t)\right]$$

$$- \frac{1}{4}\,\mu n^{2} r_{o}^{3} \int_{0}^{1}\!\iiint d\zeta\, d\lambda\, d\omega\, d\nu\;\; l\; F_{2}(\lambda,\omega|\zeta)\,\ln\left[\frac{F_{2}(\lambda^{*},\omega|\zeta)}{F_{2}(\lambda,\omega^{*}|\zeta)}\right] , \tag{5.2.8b}$$

$$\sigma_{h} = \frac{1}{2}\,\mu n^{2} r_{o}^{2} \iiint d\lambda\, d\omega\, d\nu\; \Big\{F_{2}(\omega^{*},\lambda^{*}|1) - F_{2}(\omega,\lambda|1)$$

$$- F_2(\omega,\lambda|1)ln\left[\frac{F_2(\omega^*,\lambda^*|1)}{F_2(\omega,\lambda|1)}\right]\Bigg\} . \qquad (5.2.8c)$$

The ellipses occurring in either (5.2.8a) or (5.2.8b) represents the "hydrodynamic" part of h and Φ functionally depending on $\rho(\cdot,t)$ and $u(\cdot,t)$, and $\mathbb{C}$ is a certain constant.

The inequality

$$X - Y - Yln(X/Y) \geq 0, \quad X > 0,\ Y > 0 \qquad (5.2.9)$$

implies the important result

$$\sigma_h \geq 0 . \qquad (5.2.10)$$

5.3. The Second Axiom of Thermodynamics in the Form of Müller-Liu and its Consequences

Of course, the H-theorem, bringing out the kinetic origin of (5.2.7), is the real key of this chapter, but the argument is easier to follow if we state here the local entropy inequality separately from its microscopic interpretation. The kinetic nature of (5.2.7) will be reconsidered in Section 5.4.

5.3.A. A Break of the Equipresence of Constitutive Variables in Extended Thermodynamics of Fluids

The underlying idea of linear irreversible thermodynamics is to take the physically relevant fields, viz., the slow conserved densities ρ, u, and e, as independent fluid-state variables and supplement the balance equations for them by the constitutive relations of the Navier-Stokes-Fourier type. The differential equations of transfer arising from such an approach are parabolic and exhibit infinite speed of propagation of signals. In order to avoid the well-known limitations of ordinary theory, in extended irreversible thermodynamics of degree r, r ≥ 2, one assumes that the space of state variables consists of the union of two subsets, the subset formed by the conserved local densities $\{\rho, u, e\}$, and the subset of nonconserved moments $\left\{\langle M^2\rangle, M^3, \ldots, M^r\right\}$.

Of course, there is an aspect of the equations of transfer of degree r that leaves something to be desired as they stand —— in neither case can the above system of balance equations serve as field equations for u and $A^r_{-1} := \left\{M^0, M^2, \ldots, M^r\right\}$, because additional quantities have appeared, namely, the relative flux moments $\left\{N^2, \ldots, N^{r+1}\right\}$ and the collision integrals $\left\{P^2, \ldots, P^r\right\}$. Consequently, these quantities should be related to u and A^r_{-1} in a materially dependent manner. Within the framework of the local constitutive assumption, we postulate that the excessive moments N^α do not depend upon the time-space derivatives of u and M^α, i.e., that

$$N^\alpha = N^\alpha(A^r_{-1}), \qquad \alpha = 2, \ldots, r+1 . \tag{5.3.1a}$$

Since, regardless of a debate over the validity of the principle of material frame-indifference,(62) we require constitutive equations to be Galilean invariant, the velocity vector u does not appear on the r.h.s. of (5.3.1a).

In extended irreversible thermodynamics of degree r, inserting (5.3.1a) into (5.2.5a) yields expressions which are bilinear sums of

$$\nabla A^r_{-1} := \left\{\nabla M^0, \nabla M^2, \ldots, \nabla M^r\right\}$$

and certain tensorial, local, and generally nonlinear functions of A^r_{-1}. Once this fact is seen clearly, there is no reason whatsoever to eliminate, through the local constitutive assumption attached to P^α, the space derivatives $\nabla u = L$ and ∇M^α on the r.h.s. of Eq. (5.2.5a). It is, then, no surprise that we assign the same, roughly speaking, constitutive structure to the collision integral P^α as that resulting from substituting (5.3.1a) into (5.2.5a). Thus we assume that[21]

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[21] An anti-discriminatory assumption.

$$P^{\alpha} = \wp^{\alpha}(\mathbb{A}^{r}_{-1}) + \mathbb{P}^{\alpha + 2}(\mathbb{A}^{r}_{-1}) \circ L$$

$$+ \sum_{\beta = 0}^{r} \mathbb{P}_{\beta}^{\alpha + \beta + 1}(\mathbb{A}^{r}_{-1}) \circ \nabla M^{\beta} , \quad \alpha = 2,\dots,r , \qquad (5.3.1b)$$

$$\mathbb{P}_{1}^{\alpha + 2} := 0 , \qquad (5.3.1c)$$

where the constitutive quantities $\wp^{\alpha}$, $\mathbb{P}^{\alpha + 2}$, and $\mathbb{P}_{\beta}^{\alpha + \beta + 1}$, locally depending upon the fluid-state variables $\mathbb{A}^{r}_{-1}$, are certain tensors of degrees α, $\alpha + 2$, and $\alpha + \beta + 1$, respectively.

5.3.B. Formal Thermodynamic Restrictions upon Basic and Auxiliary Constitutive Functions

Qualitatively speaking a comparison of (5.2.7) with (5.2.5a) immediately reveals that h and Φ play the same role in (5.2.7) as macroscopic moments M^{α} and N^{α} in their balance equations, namely, they act there as the density and the flux, respectively. Since in extended irreversible thermodynamics of degree r h and Φ are the auxiliary constitutive quantities, the entropy density h and the entropy flux Φ should be derivable from the fluid-state variables u and $\mathbb{A}^{r}_{-1}$ in a materially dependent manner. According to this remark, the local constitutive assumption attached to h and Φ suggests a straightforward way of handling such a problem:

$$h = h(\mathbb{A}^{r}_{-1}), \qquad \Phi = \Phi(\mathbb{A}^{r}_{-1}) . \qquad (5.3.2a)$$

Each admissible thermodynamic process, i.e., "each" solution of the field equations of degree r, must be "dissipative". In other words, the basic constitutive relations (5.3.1) and the auxiliary equations (5.3.2a) are required to be such as to ensure the satisfaction of the entropy inequality in every thermodynamic process [*compatible with the constitutive assumptions* (!!); in this context, see the Banach-Piekarski discussion at the end of Section III B in Ref. 26]. This requirement severely restricts the generality of Eqs. (5.3.1) and (5.3.2a). Since the equations of transfer of degree r form constraints for the fields u and $\mathbb{A}_{-1}^{r}$ that must satisfy (5.2.7), one can get rid of those constraints by the use of Lagrange multipliers Δ^{α}, $\alpha = 0,2,\dots,r$; they are determined from the fluid-state variables as well.[22] However, in contrast with the previous fields which split into different classes, say, densities, fluxes, and productions, there is no *a priori* condition for the presence or absence of $\nabla\mathbb{A}_{-1}^{r}$ in Δ^{α}, so that at the outset we propose that

$$\Delta^{\alpha} = \Delta^{\alpha}(\mathbb{A}_{-1}^{r}, \nabla\mathbb{A}_{-1}^{r}) . \tag{5.3.2b}$$

Liu(37) has shown that the new [with respect to (5.2.7)] inequality holds for all fields:

$$\partial_t h + \nabla \cdot (hu + \Phi) - \sum_{\alpha=0}^{r} \Delta^{\alpha} \cdot [\partial_t M^{\alpha} + \nabla \cdot (u \otimes M^{\alpha} + N^{\alpha+1}) + \alpha(L \cup N^{\alpha} - R \vee M^{\alpha-1}) - P^{\alpha}] \geq 0 , \quad \Delta^{1} := 0 . \tag{5.3.3}$$

Substituting (5.3.1) and (5.3.2) into (5.3.3) and carrying out the differentiations, we find, after a tedious tensorial calculus, that

$$[\mathcal{A}_{u}^{2}] \cdot \nabla u + \sum_{\alpha=0}^{r} \Big\{ [\Pi\mathcal{A}_{h}^{\alpha}] \cdot \partial_t M^{\alpha} + \Big[u \otimes \Pi\mathcal{A}_{h}^{\alpha} + \sum_{k=1}^{s} e_k \otimes \Pi(e_k \cdot \mathcal{A}_{\Phi}^{\alpha+1})\Big] \cdot \nabla M^{\alpha} + \Delta^{\alpha} \cdot \wp^{\alpha} \Big\} \geq 0 , \tag{5.3.4a}$$

where

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[22] In order to avoid burdensome notation, and because it makes no changes in Eqs. (5.3.6), the Lagrange multiplier associated with Eq. (5.2.5d) does not appear in our approach.

$$\mathcal{M}_u^2 := h\,I - \sum_{\beta=0}^{r}\left[(M^\beta \cdot \Delta^\beta)I + \beta(N^\beta \blacksquare \Delta^\beta) - \Delta^\beta \circ \mathbb{P}^{\beta+2}\right], \tag{5.3.4b}$$

$$\mathcal{M}_h^\alpha := \frac{\partial h}{\partial M^\alpha} - \Delta^\alpha, \tag{5.3.4c}$$

$$\mathcal{M}_\Phi^{\alpha+1} := \frac{\partial \Phi}{\partial M^\alpha} - \sum_{\beta=0}^{r}\Big[\Delta^\beta \circ \left[\frac{\partial N^{\beta+1}}{\partial M^\alpha} - \mathbb{P}_\alpha^{\beta+\alpha+1}\right]$$

$$- \beta(M^0)^{-1}\delta_{\alpha 2}\, I \otimes (\Delta^\beta \cdot M^{\beta-1})\Big]. \tag{5.3.4d}$$

Since $\partial_t M^\alpha$ can be chosen *arbitrarily* and *independently* of any other term in (5.3.4a), it follows that $\Pi\mathcal{M}_h^\alpha = 0$. This result asserts in turn that $\nabla\mathbb{A}_{-1}^r$ drops out of the r.h.s. of (5.3.2b):

$$\Delta^\alpha = \Delta^\alpha(\mathbb{A}_{-1}^r). \tag{5.3.5}$$

Having this in mind, exactly the same view applies to ∇u and ∇M^α and we obtain that the quantities in square brackets $[\cdots]$ occurring on the l.h.s. of (5.3.4a) must vanish lest the inequality (5.3.4a) be violated by certain choices of the time-space derivatives $\partial_t M^\alpha$, ∇u, and ∇M^α:

$$\mathcal{M}_u^2 = 0, \tag{5.3.6a}$$

$$\Pi\mathcal{M}_h^\alpha = 0, \quad \alpha = 0,2,\ldots,r, \tag{5.3.6b}$$

$$\Pi(e_k \cdot \mathcal{M}_\Phi^{\alpha+1}) = 0, \quad \alpha = 0,2,\ldots,r, \quad k = 1,2,3. \tag{5.3.6c}$$

The remaining (residual) inequality is

$$\sum_{\alpha=0}^{r} \Delta^\alpha \circ \mathcal{P}^\alpha \geq 0, \tag{5.3.7a}$$

where

$$\mathcal{P}^0 = 0, \quad \mathcal{P}^1 = 0, \quad \mathrm{Tr}\mathcal{P}^2 = 0. \tag{5.3.7b}$$

Although the thermodynamic restrictions just obtained have the effect of interrelating the basic and auxiliary constitutive representations, i.e., the unknown functions of $\mathbb{A}_{-1}^r$

appearing in (5.3.1), (5.3.2a), and (5.3.5), they fail —— once and for all$^{(20)}$ —— both to determine them explicitly and to privilege any particular macroscopic theory of hard-sphere fluids, especially that we look for; rather, Eqs. (5.3.6) are typical of all of them.$^{(20)}$ Certainly, more specific results follow, as they should, when rarefied (hard-sphere) gases are laid down [the dilute-gas limit$^{(3,20)}$], or, what is equally important, when we put forward additional kinetic arguments [cf. Section 5.4].

5.4. Explicit Constitutive Relations in Linear Extended Thermodynamics of Degree 3

5.4.A. Structure of Certain Functionals of the Distribution Function in the Case of Weak Spatial Inhomogeneity

The pair correlation function $\chi_2\big(x + \zeta r_o l, x + (\zeta - 1)r_o l \,|\, \rho(t)\big)$, occurring in (5.2.3b), (5.2.3c), (5.2.8b), and (5.2.8c) through $F_2(\lambda,\omega|\zeta)$ [in this context, see the definitions (5.2.3d) and (5.2.1c)], can be calculated to various degrees of accuracy; within the framework of the local constitutive assumption attached to $N[\mathcal{N}\,]$ and Φ, χ_2 is exactly the Enskog local equilibrium factor χ that does not involve the spatial gradient of the mass density, higher powers of this gradient, as well as the space derivatives of higher order than the first:

$$\chi[\,\rho(x,t)] := \chi_2\big(x' + \zeta r_o l, x' + (\zeta - 1)r_o l \,|\, \rho(x,t)\big)\ . \qquad (5.4.1)$$

Of course, since $\rho(\bullet,t)$ is now replaced by the spatially constant function $\rho(x,t)$ in $\chi_2\big(x' + \zeta r_o l, x' + (\zeta - 1)r_o l \,|\, \rho(t)\big)$ [x stated once and for all], χ_2 depends upon the distance between $x' + \zeta r_o l$ and $x' + (\zeta - 1)r_o l$, and, consequently, ζ and x' do not occur on the l.h.s. of (5.4.1).

Aside from the transition rule (5.4.1) itself, which is valid for $N[\mathcal{N}\,]$ and Φ, an appropriate treatment of the collision integral (5.2.3b) differs from the previous one

$$F_2(\lambda,\omega|\zeta) \rightarrow \chi[\,\rho(x,t)]\ f(x,\lambda,t)f(x,\omega,t) \qquad (5.4.2a)$$

with regard to (5.2.3c) and (5.2.8b) in that it assumes from the outset that the following transformations are applicable:

$$F_2(\lambda,\omega|0) \Rightarrow \chi[\,\rho(x - \tfrac{1}{2}\, r_o \ell, t)]\ f(x,\lambda,t) f(x - r_o \ell, \omega, t), \quad (5.4.2b)$$

$$F_2(\lambda,\omega|1) \Rightarrow \chi[\,\rho(x + \tfrac{1}{2}\, r_o \ell, t)]\ f(x + r_o \ell, \lambda, t) f(x,\omega,t), \quad (5.4.2c)$$

where the functions on the r.h.s. of (5.4.2b) and (5.4.2c) should be expanded around $\chi[\,\rho(x,t)]\ f(x,\lambda,t)f(x,\omega,t)$; as χ and f typically vary only very slowly with x in regions of molecular dimensions, we confine ourselves to the zeroth and first order terms in these series expansions. The salient aspect of the above two transformations, justifying an unusual occurrence in (5.4.2b) and (5.4.2c) of χ rather than χ_2, is the important result of Van Beijeren and Ernst(30) that the original Enskog equation for a single system of hard spheres is accurate up to $\mathcal{O}(\nabla)$ terms inclusive.

Taking (5.4.2) in the microscopic expressions for $P[\mathcal{N}\,]$, $N[\mathcal{N}\,]$, and $\mathcal{S}$, we find that (5.2.3b), (5.2.3c), and (5.2.8b) are then replaced, respectively, by

$$P[\mathcal{N}\,] = \frac{1}{2}\, n r_o^2 \chi \iint d\lambda\, d\omega\ f(\lambda) f(\omega)\ \Phi_1[\mathcal{N}\,](\lambda,\omega)$$

$$+ \frac{1}{2}\, n r_o^3 \chi \iint d\lambda\, d\omega \Big[f(\omega)\nabla f(\lambda) - f(\lambda)\nabla f(\omega)\Big] \cdot \Phi_2[\mathcal{N}\,](\lambda,\omega), \quad (5.4.3a)$$

$$N[\mathcal{N}\,] = \frac{1}{2}\, n r_o^3 \chi \iint d\lambda\, d\omega\ f(\lambda) f(\omega)\ \Phi_2[\mathcal{N}\,](\lambda,\omega)\ , \quad (5.4.3b)$$

and

$$\mathcal{S} = -\,\mu n \int d\lambda\ \bar{\lambda}\ f(\lambda)\, \ln[\mathbb{C} f(\lambda)]$$

$$- \frac{1}{2}\, \mu n^2 r_o^3 \chi \iint d\lambda\, d\omega\ f(\lambda) f(\omega)\ \Phi_2\big[\ln(\mathbb{C} f)\big](\lambda,\omega)\ , \quad (5.4.3c)$$

$\Phi_1[\mathcal{N}\,]$ and $\Phi_2[\mathcal{N}\,]$ being functions of λ and ω, generated by $\mathcal{N}(\lambda)$ and defined as follows:

$$\Phi_1[\mathcal{N}\,](\lambda,\omega) := \int d\nu\ [\mathcal{N}(\lambda^*) - \mathcal{N}(\lambda)]\ , \quad (5.4.3d)$$

$$\Phi_2[\mathcal{N}\,](\lambda,\omega) := \int d\nu\ \ell \otimes [\mathcal{N}(\lambda^*) - \mathcal{N}(\lambda)]\ . \quad (5.4.3e)$$

In so doing, wherever χ and f occur, they must be evaluated at the same space-time point (x,t).

Now, let us observe that if $\mathscr{N}(\lambda)$ is a polynomial of degree $\alpha - 1$ in the components of λ, then $\Phi_2[\mathscr{N}](\lambda,\omega)$ is a homogeneous polynomial of degree α in the components of λ and ω. Thus, given (5.4.3b), the value of $N^\alpha = M^\alpha + \mu n\, N[\circledast^{\alpha - 1}\bar{\lambda}]$ is a linear combination of the moments of order α plus a bilinear combination of the moments of the lower order, the sum of the orders in each term being α. The coefficients in the bilinear combination[23] are functions of r_o, μ, and χ alone and in this sense are independent of f. Also, denoting by $\mathscr{P}[\mathscr{N}]$ the underlined expression on the r.h.s. of (5.4.3a), the elementary reasoning, which relies upon the property of $\Phi_2[\mathscr{N}]$ just mentioned, shows that we can evaluate the value of

$$\mu n\, P[\circledast^\alpha\bar{\lambda}] - \mu n\, \mathscr{P}[\circledast^\alpha\bar{\lambda}] = P^\alpha - \mu n\, \mathscr{P}[\circledast^\alpha\bar{\lambda}] \qquad (5.4.4)$$

directly in terms of $\mathbb{A}^{\alpha + 1}_{-1}$, $\nabla\mathbb{A}^{\alpha + 1}_{-1}$, and[24] L without first having to determine f, as in the case of N^α where we arrive at

$$N^\alpha = N^\alpha(\mathbb{A}^\alpha_{-1}) . \qquad (5.4.5)$$

As a direct consequence of (5.4.5), in extended irreversible thermodynamics of degree r one obtains for the hard-sphere fluid the following important result:

$$\frac{\partial N^\alpha}{\partial M^\beta} = 0 , \quad r \geq \beta > \alpha , \quad \alpha = 2,3,\ldots,r - 1 , \qquad (5.4.6)$$

the only one that under the single assumption (5.4.2a) is simple enough in mathematical structure to yield the kind of

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[23] We are able, suitably modifying and "complicating" the Truesdell-Ikenberry-Muncaster method [cf., e.g., Chapter XVI in Ref. 4], to put out enormous drudgery to calculate such a bilinear combination, but we have not thought the matter of sufficient importance to carry it through here.

[24] Consistent with our previous notation, the symbols $\mathbb{A}^\alpha_\beta$ and $\nabla\mathbb{A}^\alpha_\beta$ stand for $\left\{M^{\beta + 1},\ldots,M^\alpha\right\}$ and $\left\{\nabla M^{\beta + 1},\ldots,\nabla M^\alpha\right\}$, respectively.

conclusions we call "theorems valid for each value of r". In order to make further progress, however, we must restrict the generality of our approach and appeal, among other things, not only to Grad's moment representation of f but also to his truncation procedure, insofar as linear local constitutive theory regarding N^{α}, h, and Φ is concerned.

5.4.B. Grad's 20-Moment Approximation to f and his Method of Calculating Some of the Constitutive Functions

5.4.B.a. Preliminaries

If the one-point distribution function f can be expanded about a locally Maxwellian molecular density f_o:

$$f_o := \frac{\rho}{\mu n}\left[\frac{3}{4\pi e}\right]^{3/2} exp\left(-\frac{1}{2}\hat{\lambda}^2\right) , \qquad (5.4.7a)$$

$$\hat{\lambda} := \left[\frac{3}{2e}\right]^{1/2}\bar{\lambda} , \qquad (5.4.7b)$$

in a series of the Hermite tensor polynomials $B^{\alpha}(\hat{\lambda}) \in \mathbb{E}^{\alpha}_{s}$ as

$$f = f_o\left[1 + \sum_{\alpha=2}^{\infty} \frac{1}{\alpha!} b^{\alpha} \cdot B^{\alpha}\right] , \qquad (5.4.7c)$$

and if (5.4.7c) can be valid in the sense of mean-square convergence, then $f(x,\lambda,t)$ is determined solely in terms of its Hermite tensor coefficients $b^{\alpha}(x,t) \in \mathbb{E}^{\alpha}_{s}$ that are obtained from f by using the orthogonality properties of $B^{\alpha}(\hat{\lambda})$:(55)

$$b^{\alpha} = \mu n \rho^{-1} \int d\lambda\, B^{\alpha}(\hat{\lambda}) f . \qquad (5.4.7d)$$

In the neighbourhood of the state of local equilibrium, a valid approximation to h and Φ may be obtained if one replaces the functions f and $ln(\mathbb{C}f)$, respectively, by (5.4.7c) and

$$ln(\mathbb{C}f_o) + \sum_{\alpha=2}^{\infty} \frac{1}{\alpha!} b^{\alpha} \cdot B^{\alpha}$$

$$-\frac{1}{2}\sum_{\alpha=2}^{\infty}\sum_{\beta=2}^{\infty}\frac{1}{\alpha!\beta!}\left(b^{\alpha}\circledast b^{\beta}\right)\cdot\left(B^{\alpha}\circledast B^{\beta}\right)$$

in Eqs. (5.2.8a) and (5.4.3c) and then drops from the result all terms of higher than second order in the coefficients b^{α}, $\alpha = 2,3,\dots$. With these statements in mind, we arrive at

$$h = h_{o} - \frac{1}{2}\rho\sum_{\alpha=2}^{\infty}\frac{1}{\alpha!}b^{\alpha}\cdot b^{\alpha}\ , \tag{5.4.8a}$$

$$\Phi = \rho\left(\frac{2e}{3}\right)^{1/2}\left[\frac{1}{2}\left(1+\frac{2}{5}\rho\hat{\chi}\right)\mathrm{Tr}b^{3} - \sum_{\alpha=2}^{\infty}\frac{1}{\alpha!}b^{\alpha}\square c^{\alpha+1}\right], \tag{5.4.8b}$$

where

$$\hat{\chi} := \pi r_{o}^{3}\mu^{-1}\chi\ , \tag{5.4.8c}$$

$$h_{o} := \dots + \frac{3}{2}\rho - \rho\,\ln\left[\mathbb{C}\rho(\mu n)^{-1}\left(\frac{3}{4\pi e}\right)^{3/2}\right], \tag{5.4.8d}$$

$$c^{\alpha} := b^{\alpha} + \mu n\,\rho^{-1}\left(\frac{3}{2e}\right)^{1/2}N_{L}[B^{\alpha-1}(\hat{\lambda})]\ , \tag{5.4.8e}$$

and $N_{L}[B^{\alpha-1}(\hat{\lambda})]$ represents this part of $N[B^{\alpha-1}(\hat{\lambda})]$ which arises if we simply cast away from $N[B^{\alpha-1}(\hat{\lambda})]$ all products $b^{\alpha}\circledast b^{\beta}$. Eq. (5.4.8d) is very similar to the definition (3.4.31) in Chapter 3, the only essential difference in form being that in Eq. (3.4.31) an h_{o} contains no interparticle contribution, which here is briefly represented by the ellipses. It is also not difficult to analyse the behaviour of $c^{\alpha}\in\mathbb{E}^{\alpha}$ as $f = f_{o}$. Directly from (5.4.3b), (5.4.3e), (5.2.1d), (5.2.1e), (5.4.7a), and the well-known orthogonality properties of $B^{\alpha}(\hat{\lambda})$ we see that $N[B^{\alpha-1}(\hat{\lambda})] = 0$, and hence that $c^{\alpha} = 0$, when $f = f_{o}$ and $\alpha = 3,4,\dots$. Of course, since

$$M^{\alpha} = \rho\left[\frac{2e}{3}\right]^{\alpha/2} \sum_{\beta = 0}^{[\alpha/2]} \ell(\alpha,\beta) \left[b^{\alpha - 2\beta} \vee I^{\beta}\right] , \qquad (5.4.9a)$$

$$b^{\alpha} = \frac{1}{\rho}\left[\frac{3}{2e}\right]^{\alpha/2} \sum_{\beta = 0}^{[\alpha/2]} (-1)^{\beta}\, \ell(\alpha,\beta)\left[\frac{2e}{3}\right]^{\beta} \left[M^{\alpha - 2\beta} \vee I^{\beta}\right] , \qquad (5.4.9b)$$

and

$$N[B^{\alpha}(\hat{\lambda})] = \left[\frac{3}{2e}\right]^{\alpha/2} \sum_{\beta = 0}^{[\alpha/2]} (-1)^{\beta}\, \ell(\alpha,\beta)\left[\frac{2e}{3}\right]^{\beta} \times \left[N[\circledast^{\alpha - 2\beta}\bar{\lambda}] \circledast I^{\beta}\right] , \qquad (5.4.9c)$$

both $N[B^{\alpha - 1}(\hat{\lambda})]$ and $N_L[B^{\alpha - 1}(\hat{\lambda})]$ are insensitive to the choice of the values of $\mathbb{B}^{\infty}_{\alpha}$ but[25] depend only upon ρ, e, and $\mathbb{B}^{\alpha}_{1}$.

As we know, in linear extended irreversible thermodynamics of degree r one sets all higher order Hermite coefficients $\mathbb{B}^{\infty}_{r}$ equal to zero (*Grad's ansatz*), at least insofar as (5.4.8a) and (5.4.8b) are concerned. If we introduce a considerable simplification of the integral (5.4.3b) for $\mathcal{N}(\lambda) = \mu n(\circledast^{\alpha - 1}\bar{\lambda})$, i.e., if we make use of (5.4.9a) and neglect in $\mu n\, N[\circledast^{\alpha - 1}\bar{\lambda}]$ all terms which are quadratic in the fields b^{α}, $\alpha = 2,3,\ldots$, then exactly the same view based upon $b^{r+1} = 0$ applies to the constitutive flux moment N^{r+1}. Since, however, the main ingredients that are needed in such a method of estimating N^{r+1}, as well as $N^{2},\ldots,N^{r}$,[26] are particularly tedious and difficult when r is large, in Section 5.4.B.b we shall only provide some

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[25] The symbol $\mathbb{B}^{\alpha}_{\beta}$ stands for $\left\{b^{\beta+1},\ldots,b^{\alpha}\right\}$.

[26] At this point we must still remember that N^{α}, $\alpha = 2,\ldots,r$, are expressible in terms of ρ, e, and $\mathbb{B}^{\alpha}_{1}$, or of $\mathbb{A}^{\alpha}_{-1}$, without any knowledge of the expansion (5.4.7c) and Grad's simplifying ansatz.

specimens of the calculation according to linear extended irreversible thermodynamics of degree 3.

Now, by means of (5.4.8a), Grad's ansatz, (5.4.9b), (5.3.4c), and (5.3.6b) we can write the Lagrange multipliers Δ^{α} in the explicit form as follows:

$$\Delta^{\alpha} = \Delta_{o}^{\alpha} - \frac{1}{\alpha!}\left(\frac{3}{2e}\right)^{\alpha/2} \sum_{\beta = 0}^{[(r - \alpha)/2]} (-1)^{\beta} \frac{1}{2^{\beta}\beta!} \mathrm{Tr}^{\beta} b^{\alpha + 2\beta} ,$$

$$\alpha = 0,2,\ldots,r , \qquad (5.4.10a)$$

where

$$\Delta_{o}^{0} = \ldots - 1 - \ln\left[\mathbb{C}\rho(\mu n)^{-1}\left(\frac{3}{4\pi e}\right)^{3/2}\right] . \qquad (5.4.10b)$$

$$\Delta_{o}^{2} = \frac{3}{4e} I , \qquad \Delta_{o}^{\alpha} = 0 , \qquad \alpha = 3,4,\ldots,r . \qquad (5.4.10c)$$

5.4.B.b. Estimate of N^{α}, h, Φ, *and* Δ^{α} *for* r = 3: *The Linear Constitutive Approximation*

Upon substituting for b^{α}, $\alpha > r = 3$, from Grad's ansatz ($b^{\alpha} = 0$, $\alpha > r = 3$) in the Hermite expansion (5.4.7c) and upon using Eq. (5.4.9b) with $\alpha = 2,3$ as well as $B^{2}(\hat{\lambda}) = \bullet^{2}\hat{\lambda} - I$, $B^{3}(\hat{\lambda}) = \bullet^{3}\hat{\lambda} - 3\hat{\lambda} \vee I$, and the definition (5.4.7b) of $\hat{\lambda}$, we obtain the so-called 20-moment approximation $\bar{f}$ to f in the explicit form

$$\bar{f} := f_{o}\left[1 + \frac{9}{8\rho e^{2}} \langle M^{2}\rangle \cdot (\bullet^{2}\bar{\lambda}) \right.$$

$$\left. + \frac{9}{16\rho e^{3}} M^{3} \cdot (\bullet^{3}\bar{\lambda} - 2e\, \bar{\lambda} \vee I) \right] . \qquad (5.4.11)$$

On the understanding that $\mathcal{N}(\lambda) = \mu n(\bullet^{\alpha - 1}\bar{\lambda})$, $\alpha = 2,\ldots,4$, we now return to the precise calculation of $N[\mathcal{N}]$, or of N^{α}, $2 \le \alpha \le 4$, in the linear 20-moment approximation. To this end, we substitute $f = \bar{f}$ in Eq.(5.4.3b) and consequently arrive, with the neglected terms involving all possible products of the "nonequilibrium densities" $\langle M^2\rangle$ and M^3, at the following constitutive representations for

The Relative Flux Moments:

$$N^2 = \frac{4}{15}\rho^2 e\hat{\chi}\ I + \left[\ 1 + \frac{4}{15}\rho\hat{\chi}\ \right] M^2 \ , \qquad (5.4.12a)$$

$$N^3 = \frac{2}{15}\rho\hat{\chi}\ I \bullet \mathrm{Tr}M^3 + \left[\ 1 + \frac{4}{15}\rho\hat{\chi}\ \right] M^3 \ , \qquad (5.4.12b)$$

$$N^4 = 2e\left[\ 1 + \frac{18}{35}\rho\hat{\chi}\ \right] I \bullet M^2 + 2e\left[\ 1 + \frac{26}{105}\rho\hat{\chi}\ \right] M^2 \bullet I$$

$$- \frac{4}{3}\rho e^2\left[\ 1 + \frac{2}{21}\rho\hat{\chi}\ \right] I \vee I \ . \qquad (5.4.12c)$$

The procedure for obtaining the analogue of (5.4.12a)-(5.4.12c) for h, $\mathbf{s}$, and Δ^{α}, $\alpha = 0,2,3$, differs in only minor ways from that just described with respect to the relative flux moments. The details will, therefore, be omitted [cf., however, Section 5.4.B.a]. In the result we have

The Entropy Density:

$$h = \ldots + \frac{3}{2}\rho - \rho\ \ln\left[\mathbb{C}\rho(\mu n)^{-1}\left(\frac{3}{4\pi e}\right)^{3/2}\right]$$

$$- \frac{9}{16\rho e^2}\left(\langle M^2\rangle \cdot \langle M^2\rangle + \frac{1}{2e} M^3 \cdot M^3\right) \ ; \qquad (5.4.12d)$$

The Entropy Flux:

$$\mathbf{s} = \frac{3}{4e}\left(\ 1 + \frac{2}{5}\rho\hat{\chi}\ \right) \mathrm{Tr}M^3 - \frac{9}{8\rho e^2}\left[\left(\ 1 + \frac{82}{105}\rho\hat{\chi}\ \right) \langle M^2\rangle \cdot M^3\right.$$

$$\left. + \frac{4}{35}\rho\hat{\chi}\ \langle M^2\rangle \cdot \mathrm{Tr}M^3\right] \ ; \qquad (5.4.12e)$$

The Lagrange Multipliers:

$$\Delta^0 = \ldots - 1 - \ln\left[\mathbb{C}\rho(\mu n)^{-1}\left(\frac{3}{4\pi e}\right)^{3/2}\right] \ , \qquad (5.4.12f)$$

$$\Delta^2 = \frac{3}{4e} I - \frac{9}{8\rho e^2} \langle M^2 \rangle \ , \qquad (5.4.12g)$$

$$\Delta^3 = - \frac{9}{16\rho e^3} M^3 \ . \qquad (5.4.12h)$$

Having obtained the above formulas, it is important to stress once more that the constitutive representations (5.4.12) are ones according to linear extended irreversible thermodynamics of degree 3.

5.4.C. Evaluation of Collision Integrals for r = 3: *The Linear Constitutive Approximation*

While Grad's ansatz appears to be right for the system in the dilute-gas limit:

$$f(x,\lambda,t) \to f_o(x,\lambda,t)\left[1 + S(\bar{\lambda};\mathbb{A}^r_{-1})\right] \ , \qquad (5.4.13a)$$

$S(\bar{\lambda};\mathbb{A}^r_{-1})$ being a scalar function defined by

$$S(\bar{\lambda};\mathbb{A}^r_{-1}) := \sum_{\alpha = 2}^{r} \frac{1}{\alpha!} b^\alpha\left(\mathbb{A}^\alpha_{-1} \right) \cdot B^\alpha(\hat{\lambda}) \ , \qquad (5.4.13b)$$

and although it allows the satisfactory estimate of N^α, h, Φ, and Δ^α within the framework of the linear local constitutive assumption —— irrespective of whether or not the rarefied gas is taken into account ——, the proposition (5.4.13) seems to be of no interest in evaluating the collision integrals P^α for Enskog's hard-sphere fluid. As the simplest fluid situation to consider, let f be determined, in *linear* extended irreversible thermodynamics of degree r, by $\mathbb{A}^r_{-1}$, L, and $\nabla\mathbb{A}^r_{-1}$ as follows:

$$f(x,\lambda,t) \to f_o(x,\lambda,t)\Big[1 + S(\bar{\lambda};\mathbb{A}^r_{-1}) + S^2_u(\bar{\lambda};\rho,e) \cdot L$$
$$+ \sum_{\alpha = 0}^{r} S^{\alpha + 1}_M(\bar{\lambda};\rho,e) \cdot \nabla M^\alpha \Big] \ , \qquad (5.4.14a)$$

$$S_{M}^{2} := 0 \,, \tag{5.4.14b}$$

where[27] with respect to $S_{u}^{2} \in \mathbb{E}^{2}$ and $S_{M}^{\alpha} \in \mathbb{E}^{\alpha}$ we impose the natural orthogonality properties

$$\int d\lambda \; f_{o} \, \mathcal{N} \otimes S_{u}^{2} = 0 \,, \tag{5.4.15a}$$

$$\int d\lambda \; f_{o} \, \mathcal{N} \otimes S_{M}^{\alpha} = 0 \,, \quad \alpha = 1,3,\dots,r+1 \tag{5.4.15b}$$

for

$$\mathcal{N}(\lambda) = 1,\ \bar{\lambda},\ \otimes^{2}\bar{\lambda},\ \otimes^{3}\bar{\lambda} \,. \tag{5.4.15c}$$

Otherwise no more precise knowledge of S_{u}^{2} and S_{M}^{α} is available at the present time.

We now wish to calculate, directly or indirectly, the constitutive functions which appear on the r.h.s. of (5.3.1b):

$$\mathbb{P}^{\alpha+2}(\mathbb{A}_{-1}^{r}) \rightarrow \mathbb{P}^{\alpha+2}(\rho,e) \,, \tag{5.4.16a}$$

$$\mathbb{P}_{\beta}^{\alpha+\beta+1}(\mathbb{A}_{-1}^{r}) \rightarrow \mathbb{P}_{\beta}^{\alpha+\beta+1}(\rho,e) \,. \tag{5.4.16b}$$

Assuming that $\mathcal{N}(\lambda) = \mu n(\otimes^{\alpha}\bar{\lambda})$, $\alpha = 2,\dots,r$, and combining (5.4.3a) and (5.4.14a), Eq. (5.4.3a) indicates that P^{α}, $\alpha = 2,\dots,r$, is in general made up of three structurally different parts. First, there is the ordinary production of M^{α} resulting from the instantaneous point encounters. Indeed, the underlined integral in (5.4.3a), whose calculation rests essentially on (5.4.13), reproduces correctly $\mathcal{P}^{\alpha}(\mathbb{A}_{-1}^{r})$ in (5.3.1b) as a linear function of the Hermite tensor coefficients $b^{\alpha}(\mathbb{A}_{-1}^{\alpha})$. As an example, a straightforward but somewhat tedious calculus shows that if r = 3, then

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[27] Generally speaking, the tensor functions S_{u}^{2} and $S_{M}^{\alpha+1}$ depend on position x and time t through $\bar{\lambda} = \lambda - u$ and $\mathbb{A}_{-1}^{r}$. In the linear approximation, however, $\langle M^{2}\rangle$ and $\mathbb{A}_{2}^{r}$ drop out of S_{u}^{2} and $S_{M}^{\alpha+1}$.

$$\mathcal{P}^2 = -\frac{16}{5}\,\hat{\chi}\,r_o^{-1}\rho\left[\frac{2e}{3\pi}\right]^{1/2}\left(M^2 - \frac{2}{3}\,\rho e\,I\right)\,, \qquad (5.4.17a)$$

$$\mathcal{P}^3 = -\frac{24}{5}\,\hat{\chi}\,r_o^{-1}\rho\left[\frac{2e}{3\pi}\right]^{1/2}\left(M^3 - \frac{1}{3}\,I \vee \mathrm{Tr}M^3\right)\,. \qquad (5.4.17b)$$

Next, to the second *nonlocal* term on the r.h.s. of (5.4.3a), which can be evaluated with the aid of (5.4.13) as well, must be added the separate contribution of exactly the same order arising from inserting the gradient part of (5.4.14a) into the underlined integral. Having, however, no complete knowledge of S_u^2 and $S_M^{\alpha+1}$, except for (5.4.15), this direct procedure does not determine, in contrast with $\mathcal{P}^{\alpha}(A_{-1}^{r})$, the constitutive functions $\mathbb{P}^{\alpha+2}(\rho,e)$ and $\mathbb{P}_{\beta}^{\alpha+\beta+1}(\rho,e)$ —— see (5.3.1) and (5.4.16) —— and an alternative approach concerning them can potentially become a subject of intense interest.

To illustrate our indirect method of dealing with the collision integrals P^{α}, $\alpha = 2,\dots,r$, let us set once more $r = 3$. Then, according to the general principles of continuum mechanics, in linear extended irreversible thermodynamics of degree 3 the constitutive representations for P^2 and P^3 are assumed to be of the general form

$$\begin{aligned} P^2 = \mathcal{P}^2 &+ \frac{8}{15}\,d_o\,\rho^2 e\hat{\chi}\,\nabla \wedge u \\ &+ \frac{2}{105}\,\rho\hat{\chi}\left(d_1\,\nabla \wedge \mathrm{Tr}M^3 + 8d_2\,\langle\nabla \cdot M^3\rangle\right)\,, \qquad (5.4.18a)\end{aligned}$$

$$\begin{aligned} P^3 = \mathcal{P}^3 &+ \frac{4}{5}\,\rho e\hat{\chi}\left(d_3\,\nabla \vee M^2 + d_4\,I \vee \nabla\mathrm{Tr}M^2\right. \\ &\left. - \frac{1}{3}\,d_5\,I \vee \nabla \cdot M^2 - \frac{4}{9}\,d_6 e\,I \vee \nabla\rho\right)\,, \qquad (5.4.18b)\end{aligned}$$

the coefficients $d_o,\dots,d_6$ being certain dimensionless numbers. Were the distribution function f independent of L and ∇M^{α}, $\alpha = 0,2,3$:

$$S_u^2 = 0, \qquad S_M^{\alpha} = 0, \qquad \alpha = 1,3,4,\dots, \qquad (5.4.19)$$

as typically in the case of rarefied gases, the values of d_0, ..., d_6 in (5.4.18) would be evaluated with the help of the microscopic definition of $P[\mathcal{N}]$, i.e., by mere substitution of (5.4.13a) into (5.4.3a). Thus, ignoring the axiom of entropy growth altogether, we could arrive at the following result:

$$d_n = 1\,, \qquad n = 0,\dots,3,5,6\,, \qquad d_4 = 0\,. \tag{5.4.20}$$

In fact, using (5.4.12) and (5.4.18), making the change of state variables

$$\left\{u, M^0, M^2, M^3\right\} \rightarrow \left\{u, \rho, e, \langle M^2\rangle, M^3\right\} \tag{5.4.21}$$

in (5.3.6), and finally exploiting [in the sense of the first-order perturbation theory(20,26)[28]] the Liu-Müller restrictions upon the basic and auxiliary constitutive representations, we obtain for d_0, ..., d_6, instead of (5.4.20),

$$d_0 = d_1 = d_6 = 1, \quad d_2 = -\frac{27}{8}\,, \tag{5.4.22a}$$

$$d_3 = -\frac{2}{3}\,,\; d_4 = \frac{1}{3}\,,\; d_5 = -1\,. \tag{5.4.22b}$$

In view of the foregoimg, we conclude that this work is in favour of the validity of the statement that in general

$$P^\alpha - \wp^\alpha \neq 0; \qquad P^\alpha - \wp^\alpha = 0 \tag{5.4.23}$$

only in Grad's dilute-gas limit [$r_0^{-1}\hat{\chi}$ = const in (5.4.17) and $\hat{\chi} \rightarrow 0$ in (5.4.18); cf. also Section 5.2.B]. Thus our extended theory does not agree in all respects with the thermodynamic description of dense gases and fluids by Liu(63) and Kremer,(64) which bases from the outset upon the local constitutive assumption applied to P^α ($P^\alpha = \wp^\alpha$).

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[28] For more details of general import, see our remarks at the end of Section 1.5.B in Chapter 1. Here let us only observe that $\langle M^2\rangle \sim b^2$ and $M^3 \sim b^3$.

5.5. Transition to Ordinary Thermodynamics

5.5.A. The 20-Moment System of Field Equations

Appeal to (5.2.5), $M^0 = \rho$, $P^0 = 0$, and $P^1 = 0$ shows that

$$\partial_t \rho + \nabla \cdot \rho u = 0 \ , \tag{5.5.1a}$$

$$\partial_t u + u \cdot L + R = 0 \ , \tag{5.5.1b}$$

$$\partial_t M^2 + \nabla \cdot (u \otimes M^2 + N^3) + 2\, L \cup N^2 - \Delta\mathcal{P}^2 = \mathcal{P}^2 \ , \tag{5.5.1c}$$

$$\partial_t M^3 + \nabla \cdot (u \otimes M^3 + N^4) + 3(L \cup N^3 - R \vee M^2) - \Delta\mathcal{P}^3 = \mathcal{P}^3 \ , \tag{5.5.1d}$$

where

$$\Delta\mathcal{P}^2 := P^2 - \mathcal{P}^2 \ , \quad \Delta\mathcal{P}^3 := P^3 - \mathcal{P}^3 \ , \tag{5.5.1e}$$

and these, together with (5.2.5b), (5.4.12a)-(5.4.12c), $e = \frac{1}{2}\rho^{-1}\mathrm{Tr}M^2$, (5.4.17), (5.4.18), and (5.4.22), constitute the 20-moment system of field equations for ρ, u, M^2, and M^3, each of which is of first order in both t and x. On the other side, the Navier-Stokes-Fourier equations as well as all other equations arising from normal solutions (with the single exception of the Euler equations) are of first order in t and of second (third, etc.) order in x and, in contrast with our system of differential equations, exhibit infinite speed of propagation of signals.

5.5.B. The Navier-Stokes-Fourier Relations

In Section 3.5.B of Chapter 3 we have studied a formal iterative procedure, somewhat akin to the Maxwellian iteration of Ikenberry and Truesdell,(4) designed to approximate $\langle M^2 \rangle$ and $\frac{1}{2}\mathrm{Tr}M^3$ by functions of the derivatives of the conserved (principal) moments —— derivatives not only with respect to place but also with respect to time. Our present purpose is to examine some of the implications of the iterative procedure applied to the differential system (5.5.1) of field equations.

To do so, suppose that N^3_T is a "traceless" part of N^3 given by

$$N^3_T := \sum_{k=1}^{3} e_k \otimes \langle e_k \cdot N^3 \rangle \, . \tag{5.5.2}$$

Then it follows from (5.5.1c) that

$$\partial_t \langle M^2 \rangle + \nabla \cdot \left(u \otimes \langle M^2 \rangle + N^3_T \right) + 2\, L \cap N^2 - \Delta \wp^2 = \wp^2 \, . \tag{5.5.3}$$

Now, setting $\langle M^2 \rangle$ and M^3 equal to zero in (5.5.3) and (5.5.1d) wherever they do not appear on the r.h.s. of those equations, it is a routine matter to find, by use of (5.4.12a)-(5.4.12c), (5.4.17), (5.4.18), (5.4.22), and the definition of T introduced at the end of Section 5.2.C, that

$$\langle M^2 \rangle = - 2\chi^{-1} \left(1 + \frac{4}{15} \rho \hat{\chi} \right) \mu_B \, \nabla \wedge u \, , \tag{5.5.4a}$$

$$\frac{1}{2} \operatorname{Tr} M^3 = - \chi^{-1} \left(1 + \frac{2}{5} \rho \hat{\chi} \right) \varkappa_B \, \nabla \frac{T}{\mu} \, , \tag{5.5.4b}$$

$$\langle M^3 \rangle = 0 \, , \tag{5.5.4c}$$

where

$$\mu_B = \frac{5}{16} \, \mu r_o^{-2} \left[\frac{T}{\mu \pi} \right]^{1/2} , \tag{5.5.4d}$$

$$\varkappa_B = \frac{75}{64} \, \mu r_o^{-2} \left[\frac{T}{\mu \pi} \right]^{1/2} . \tag{5.5.4e}$$

As we know, the values of μ_B and $\varkappa_B$ given here are what Chapman and Cowling(30) call the first (very good) approximations to their true values. [The coefficients μ_B and $\varkappa_B$ can be interpreted as the shear viscosity and thermal conductivity of the moderately rarefied hard-sphere gas, i.e., of the model gas at normal densities.] At the same time, by comparing (5.5.4a) and (5.5.4b) with the so-called second approximations to the values of the expectations $\langle M^2 \rangle$ and $\frac{1}{2} \operatorname{Tr} M^3$ of $\langle \otimes^2 \bar{\lambda} \rangle$ and $\frac{1}{2} |\bar{\lambda}|^2 \bar{\lambda}$ we obtain a direct check on the iterative procedure, involving no adjustment of the functions of the mass density ρ in front of

$\mu_s \nabla \wedge u$ and $\varkappa_s \nabla (T/\mu)$. The agreement is striking, insofar as the explicit calculations on p. 280 in Ref. 36 are concerned.

The quantities $M^2 = \rho (T/\mu) I + \langle M^2 \rangle$ and $\frac{1}{2}\,\mathrm{Tr} M^3$ give the parts of the pressure tensor $\mathfrak{M}$ and the heat flux q that arise from the transport of momentum and energy by the motion of molecules from point to point. To these must be added the contributions arising from the transport of momentum and energy by molecular collisions. These will now be evaluated, correct to the second approximation.

Consider first the molecular expression for $\mathfrak{M} := N^2$. After using $\mathfrak{M} = M^2 + \mu n\, N[\bar{\lambda}\ ]$ and replacing $\int_0^1 d\zeta\; F_2(\lambda,\omega|\zeta)$ by

$$\chi[\,\rho(x,t)\,]\; f(x,\lambda,t) f(x,\omega,t)$$
$$+ \frac{1}{2}\, r_o \chi[\,\rho(x,t)\,]\; [\; f_o(x,\omega,t) \nabla f_o(x,\lambda,t)$$
$$- f_o(x,\lambda,t) \nabla f_o(x,\omega,t)\;] \cdot l$$

in Eq. (5.2.3c), we arrive at

$$\mathfrak{M} = \rho\, \frac{T}{\mu}\, \Big(1 + \frac{2}{3}\, \rho \hat{\chi} \Big)\, I + \Big(1 + \frac{4}{15}\, \rho \hat{\chi} \Big)\, \langle M^2 \rangle$$
$$- \frac{2}{3}\, n_o \Big[\, \big(\nabla \cdot u \big)\, I + \frac{6}{5}\, \nabla \wedge u \,\Big]\, , \qquad (5.5.5a)$$

where

$$n_o := \frac{2}{3}\, \rho^2 \hat{\chi}\; r_o \Big[\frac{T}{\mu\, \pi} \Big]^{1/2} . \qquad (5.5.5b)$$

Of course, exactly the same view applies to $q = \frac{1}{2}\,\mathrm{Tr} M^3 + \frac{1}{2}\,\mu n\, N\big[\, |\bar{\lambda}|^2 \big]$ and we obtain

$$q = \Big(1 + \frac{2}{5}\, \rho \hat{\chi} \Big)\, \frac{1}{2}\, \mathrm{Tr} M^3 - n_o\, \nabla\, \frac{T}{\mu}\, . \qquad (5.5.5c)$$

Now, if we insert in Eqs. (5.5.5a) and (5.5.5c) the values of $\langle M^2 \rangle$ and $\frac{1}{2}\,\mathrm{Tr} M^3$ found in (5.5.4a) and (5.5.4b), *which result in our extended theory without classical arguments of the Chapman-Enskog method*,(36) then these equations become

$$\mathfrak{M} = (p_E - \eta_E \nabla \cdot u) I - 2 \mu_E \nabla \wedge u , \tag{5.5.6a}$$

$$q = - \varkappa_E \nabla \frac{T}{\mu} , \tag{5.5.6b}$$

where

$$p_E := \rho \frac{T}{\mu} (1 + \frac{2}{3} \rho \hat{\chi}) \tag{5.5.6c}$$

denotes the equilibrium part of the mean normal pressure $p = p_E - \eta_E \nabla \cdot u$ and the quantities

$$\eta_E := \frac{2}{3} n_o , \tag{5.5.6d}$$

$$\mu_E := \chi^{-1} (1 + \frac{4}{15} \rho \hat{\chi})^2 \mu_B + \frac{2}{5} n_o , \tag{5.5.6e}$$

and

$$\varkappa_E := \chi^{-1} (1 + \frac{2}{5} \rho \hat{\chi})^2 \varkappa_B + n_o \tag{5.5.6f}$$

may be interpreted as the coefficients, respectively, of bulk viscosity, shear viscosity, and thermal conductivity.

Had we chosen the other, at first sight more attractive alternative, offering the physically relevant fields $\{ \rho, u, e, \mathfrak{P}, q, \ldots \}$ rather than $\{ \rho, u, e, \langle M^2 \rangle, \frac{1}{2} \mathrm{Tr} M^3, \ldots \}$ as independent fluid-state variables in extended irreversible thermodynamics of degree 3, we would not have been able to see clearly how to derive the appropriate formulae for μ_E and $\varkappa_E$. In other words, leaving the theoretical method otherwise unchanged but neglecting only, just as in Refs. 63 and 64, the ∇-part of the r.h.s. of (5.3.1b) and subsequently replacing $\langle M^2 \rangle$ by the pressure deviator $\mathfrak{P} := \langle N^2 \rangle$ and $\frac{1}{2} \mathrm{Tr} M^3$ by the heat flux q in the set of independent fluid-state variables, the iterative procedure yields immediately the following *incorrect* expressions for the coefficients of shear viscosity and thermal conductivity:

$$\mu_E = \chi^{-1} (1 + \frac{2}{3} \rho \hat{\chi}) (1 + \frac{4}{15} \rho \hat{\chi}) \mu_B , \tag{5.5.7a}$$

$$\varkappa_E = \chi^{-1}\left(1 + \frac{2}{3}\,\rho\hat{\chi} \right)\left(1 + \frac{2}{5}\,\rho\hat{\chi} \right) \varkappa_B \,. \qquad (5.5.7b)$$

To an observer looking at the interesting, purely macroscopic description of fluids disclosed in other approaches,(63,64,76) this problem is in many respects obscured by the very fact that, in spite of structurally and qualitatively acceptable results —— the Navier-Stokes relations, Fourier law, etc.—— no microscopic (quantitative) interpretation is available. In summary, some (*not all*) of the axioms in the pioneering works by Liu(63) and Kremer-Müller,(64,76) besides being doubtful on account of the generally unjustifiable simplification $P^\alpha - \mathcal{P}^\alpha = 0$, do not recognize an important role of both the definite choice of the set of independent fluid-state variables and also the two different steps leading to μ_E and λ_E —— one way or another, the iterative procedure itself does not determine the shear viscosity and thermal conductivity of the model hard-sphere fluid.

5.6. *Final Remarks*

Although we have set down in Section 5.4.C an *intuitive* method of replacing the one-particle density f by a function which depends on finitely many moments only, the ansatz (5.4.14a) is not one of the successive approximations to the exact (unknown) solutions of the revised Enskog equation. Rather, our framework should be regarded as a procedure for *formally approximating the kinetic theory by a sequence of theories of continuum mechanics*, just as in the case of the *far less intricate* Grad's scheme of dealing with Boltzmann's equation. In other words, instead of directly treating the revised Enskog equation, the infinite hierarchy of the moment equations is studied with the help of a structurally complicated, indirect truncation scheme. This comes to taking the first r equations of transfer together with some constitutive relations of interest for weak spatial inhomogeneity, so that a closed set of field equations is gained. The solutions of this system are subject to the entropy principle in the form of Liu-Müller. Thus such a system of "thermomechanical" equations constitutes extended irreversible thermodynamics of degree r. Then with some right one conjectures that as r → ∞ at least the lower moments up to a certain r′ are 'correct'(in the sense not yet made precise). The ansatz (5.4.14a) is the simplest compatible with the axiom of entropy growth, practically manageable generalization of that of Grad concerning rarefied gases. It is hoped that in this form already for "small" r a better estimate of the lower moments is gained than in Grad's approach structured around (5.4.13).

Irrespective of these possibilities we may also ask for improvements or replacements of the Navier-Stokes-Fourier theory in circumstances outside its range of validity and look to the 20-moment system of field equations. If we are willing to accept (5.5.1), then we can use it not only to determine *correctly* $\langle M^2 \rangle$ and $\frac{1}{2}\,\mathrm{Tr}M^3$ in terms of conserved (principal) moments but also to suggest the way in which the pressure deviator and the heat flux decay in case of f independent of place. This is important because the essential phenomena of relaxation are presently beyond our graphs if approached through the Boltzmann kinetic theory itself, i.e., without any reference to the truncation procedure.

EPILOGUE

This work has been devoted to the detailed development of Grad's theory of quasiparticle and/or classical gases (fluids). Throughout, we have emphasized one particular approach to this problem, extensively describing the relationship between the moment procedure(s) and so-called extended irreversible thermodynamics. In closing, it is pertinent to summarize the main feature of the analysis.

(a) The description of quasiparticle gases via an infinite hierarchy of the moment equations requires, as a first step, resolution of the difficulty afforded by the fact that in general we cannot evaluate the collision integrals directly in terms of expectations without first having to determine f. At this point we wish to stress that in the treatment of the collision integrals appearing on the r.h.s. of the equations of transfer, the old Grad's ideas regarding classical gases have served as useful guides to new methods [cf. Chapters 1 and 4]; conversely, the new theories have helped to clarify the earlier points of view [cf. Chapters 2 and 3]. In particular, the universal method of obtaining the irreducible moment representation (4.6.8) of the distribution function f, somewhat similar to, and more subtle than, the scheme outlined in Section 3.4 of Chapter 3, rests altogether upon the following modification of Grad's ascertainments on pp. 267 and 284 in Ref. 3: if we are interested in diagonalizing the largest contribution to the deviation of the entropy density h from its equilibrium counterpart h_o, then (by the choice of a local or absolute Bose-Einstein distribution function f_o) we have simply no alternative to the use of the weight (4.6.1a) and of the one-dimensional Tchebychef functions $A_\beta(z;\Theta)$ orthonormal with respect to it. It is not a matter for surprise, therefore, that the Tchebychef-Ikenberry representation (4.6.8) can be viewed as a quasiparticle analogue of the irreducible variant of Grad's expansion of f in terms of Hermite polynomials.

(b) Although we have shown that there is conceptual consis-

tency at all points between Grad's representation of f and that of Banach-Piekarski for quasiparticle gases,(23,25,26) and it is known that these representations lead to the evaluation of collision integrals as infinite series in the moments, the theory immediately faces the problem of truncating the coupled hierarchy of differential equations of transfer. Since it is rather common that the axiom of entropy growth is relegated to the status of a forgotten or even disregarded object in the kinetic theory of gases and fluids, we have tried to prove that the different truncation procedures of Grad's type yield the closed sets of mesoscopic equations of transfer which are in keeping with the Liu-Müller *perturbational formulation* of linear and/or "slightly nonlinear" extended irreversible thermodynamics. Thus it appears not unlikely that our analysis can serve the purpose of verifying some of the statements that have been used in the "older" work(41,67) to cast *unfounded doubt*(26,68) upon the compatibility of Grad's moment truncation procedure with the axiom of entropy growth.

(c) The utilization of extended irreversible thermodynamics, alone with Grad's framework, aims both at outflanking the kinetic theory itself and at replacing it by a theory which is much simpler. The precise status of this replacement is still obscure [cf., e.g., Sections 1.4.A, 1.4.B.b, and 5.6], but the main object is to obtain the phenomenological (pseudo-macroscopic, mesoscopic, etc.) equations which generalize the usual ones in cases where it is no longer permissible to rely on normal solutions. Also, our combination of kinetics and thermodynamics might be described as interpolation between the two extreme specifications of a state: via one-particle distributions in the kinetic theory on the one hand and via conserved, principal moments in ordinary thermodynamics on the other. That the moment methods of Grad's type give rise to results identical with those by the first-order Chapman-Enskog theory, at least in the lowest approximation in the Laguerre (Sonine) polynomial expansion and for classical, moderately rarefied gases, is now a well-established, almost trivial fact. But the essential, even decisive aspect of the present approach

is that exactly the same observation applies to the model hard-sphere fluid [cf. Section 5.5.B] and, *what is much more important*, to a variety of quasiparticle gaseous systems; of course in the latter case the set of Laguerre (Sonine) polynomials must be replaced by that of Tchebychef functions.

If the theory of quasiparticle and/or classical gases (fluids) presented herein serves as a stimulus for further research, the efforts put forth in the preparation of the text will have been rewarded.

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