CHAPTER XXVI.

DEFINITE INTEGRALS (I.).

988. It has been stated that when $\int \phi(x)\,dx$ can be integrated, and the result of the indefinite integration is $\psi(x)$, then the quantity $\psi(b)-\psi(a)$ is denoted by $\int_a^b \phi(x)\,dx$; and it has been shown that $\psi(b)-\psi(a)$ is the result of obtaining the limit when h is indefinitely small of

$$h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi\{(a+(n-1)h\}],$$

where $b=a+nh$; and the process of obtaining the value of $\int_a^b \phi(x)\,dx$ has been termed a Definite Integration.

We have performed this definite integration in many cases, first of all obtaining the indefinite integral by the rules of the early chapters and so finding $\psi(x)$, and then inserting the values of the limits to obtain the expression $\psi(b)-\psi(a)$; and in doing this our chief attention has been centred upon the discovery of the function $\psi(x)$, whose differential coefficient is $\phi(x)$; *i.e.* upon the reversal of the general problem of differentiation.

It will have been gathered from the last two chapters that the value of the definite integral between certain specific limits can be obtained in many instances by some artifice, even in cases where it is not possible to perform the indefinite integration; *i.e.* that it is possible sometimes to arrive at the value of $\psi(b)-\psi(a)$ without finding the form of $\psi(x)$ at all. Such a case was that of $\int_0^\infty e^{-x^2}\,dx$ discussed in Art. 864, where the indefinite integration of e^{-x^2} could not be expressed in finite terms, but for which the definite integral from 0 to ∞ was discovered to be $\frac{\sqrt{\pi}}{2}$. It is to this class of definite integral in particular that we now turn our attention, and it is to this class—viz. where the integrand does not admit of indefinite integration in finite terms—that the term Definite Integral is by convention mainly confined.

A very large number of such results have been found. A collection of such definite integrals was made by Bierens de Haan, and published under the title *Tables d'Intégrales Définies* (Amsterdam).

989. The artifices employed are numerous and of great variety and ingenuity. It is impossible to give an exhaustive list, but some of the more common devices are as follow:

(*a*) The use of a reduction formula connecting the integral sought with one or more other integrals already found, or more capable of investigation, or with some multiple of itself.

(*b*) The integral $\int_a^d \phi(x)\,dx$ may be regarded as

$$\left(\int_a^b + \int_b^c + \int_c^d\right)\phi(x)\,dx,$$

in which the notation will explain itself. That is, the summation from a to d may be broken up into sections, (a to b), (b to c), etc., and each part may be considered separately.

(*c*) The expansion of the function to be integrated, or of some factor of it in a convergent series, or in partial fractions, with the integration of the several terms and a final summation of the results.

(*d*) Change of the variable with the corresponding change in the limits.

(*e*) Differentiation or integration of a known integral with regard to some constant which it may contain.

(*f*) A factor of the function to be integrated may itself be the result of a known integration between certain constant limits. Upon substituting this integral for the factor a double integral may be formed, and a change in the order of integration or a transformation to a system of new variables may succeed in obtaining the value of the integral under consideration.

(*g*) Investigation of the integral from the original summation definition of an integral.

(*h*) The application of some general theorem such as those already considered in the Eulerian integrals or Dirichlet's integrals, or the theorems of Frullani, Cauchy, Kummer, Poisson or Abel, which will be severally discussed in their proper places.

(*i*) Several of these methods may be combined.

(*j*) The application of Cauchy's theorem in integrating round some closed contour. Contour integration will be reserved for a special chapter.

(*k*) The substitution of a complex quantity for a constant involved in a known integral, and in its result, followed by equating real and unreal parts, frequently suggests new integrals; but the method requires great caution if it is to be regarded as rigidly establishing the values of the resulting definite integrals without further investigation. But it frequently happens that such suggested results can be established by other means.

These are the principal devices used. There are many others applicable to particular forms. A general statement such as the above is necessarily vague on account of its generality. The student should examine the mode of procedure in the numerous cases which we shall have to discuss, and note for himself the method adopted.

990. **Illustrations of Definite Integrals deduced by Change of the Variable.**

1. $I=\int_0^{\frac{\pi}{2}} \log\sin\theta\, d\theta$ [Euler, *Acta Petrop.*, vol. i., p. 2].

Writing $\theta=\frac{\pi}{2}-\phi$, $I=-\int_{\frac{\pi}{2}}^{0} \log\cos\phi\, d\phi=\int_0^{\frac{\pi}{2}} \log\cos\theta\, d\theta.$

Adding, we have

$$2I=\int_0^{\frac{\pi}{2}}(\log\sin\theta+\log\cos\theta)\,d\theta=\int_0^{\frac{\pi}{2}}(\log\sin 2\theta-\log 2)\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}\log\sin 2\theta\,d\theta-\frac{\pi}{2}\log 2,\ \text{and writing } \chi \text{ for } 2\theta,$$

$$\int_0^{\frac{\pi}{2}}\log\sin 2\theta\,d\theta=\frac{1}{2}\int_0^{\pi}\log\sin\chi\,d\chi=\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta=I;$$

$$\therefore\ 2I=I-\frac{\pi}{2}\log 2,\quad \text{giving } I=\frac{\pi}{2}\log\frac{1}{2}.$$

Hence $$\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta=\int_0^{\frac{\pi}{2}}\log\cos\theta\,d\theta=\frac{\pi}{2}\log\frac{1}{2}. \quad \text{.........(1)}$$

It also follows that

$$\int_0^{\frac{\pi}{2}}(\log\sin\theta-\log\cos\theta)\,d\theta=0,\quad \textit{i.e.} \int_0^{\frac{\pi}{2}}\log\tan\theta\,d\theta=0, \ldots(2)$$

and $$\int_0^{\frac{\pi}{2}}\log\sec\theta\,d\theta=\int_0^{\frac{\pi}{2}}\log\operatorname{cosec}\theta\,d\theta=\frac{\pi}{2}\log 2. \quad \text{.........(3)}$$

If we write $\sin\theta=x$ we have another form of the same integral, viz.

$$\int_0^1\frac{\log x}{\sqrt{1-x^2}}\,dx=\frac{\pi}{2}\log\frac{1}{2}; \quad \text{.........................(4)}$$

or again, putting $x=e^{-y}$,

$$\int_0^\infty\frac{y}{\sqrt{e^{2y}-1}}\,dy=\frac{\pi}{2}\log 2 \quad \text{or} \quad \int_0^\infty\frac{xe^{-\frac{x}{2}}}{\sqrt{\sinh x}}\,dx=\frac{\pi}{\sqrt{2}}\log 2; \text{.........(5)}$$

or again, integrating (1) by parts,

$$\Big[\theta\log\sin\theta\Big]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\theta\cot\theta\,d\theta=\frac{\pi}{2}\log\frac{1}{2};$$

$$\therefore\ \int_0^{\frac{\pi}{2}}\theta\cot\theta\,d\theta=\frac{\pi}{2}\log 2; \quad \text{.........................(6)}$$

or integrating again,

$$\left[\frac{\theta^2}{2}\cot\theta\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\frac{\theta^2}{2}\operatorname{cosec}^2\theta\,d\theta=\frac{\pi}{2}\log 2;$$

$$\therefore\ \int_0^{\frac{\pi}{2}}\frac{\theta^2}{\sin^2\theta}\,d\theta=\pi\log 2; \quad \text{.........................(7)}$$

or, which is the same thing, putting $\cot\theta=x$,

$$\int_0^\infty(\cot^{-1}x)^2\,dx=\pi\log 2 \quad \text{.........................(8)}$$

2. $I=\int_0^\pi \frac{\theta \sin\theta\, d\theta}{a+b\cos^2\theta}$, a and b both positive (Poisson, *Journal de l'École Polytechnique*, xvii., p. 624, case where $a=b=1$).

Writing $\pi-\phi$ for θ,

$$I=-\int_\pi^0 \frac{(\pi-\phi)\sin\phi}{a+b\cos^2\phi}d\phi=\pi\int_0^\pi \frac{\sin\phi}{a+b\cos^2\phi}d\phi-I\,;$$

$$\therefore\ 2I=\frac{\pi}{b}\int_0^\pi \frac{\sin\phi\, d\phi}{\frac{a}{b}+\cos^2\phi}=\frac{\pi}{b}\left[-\sqrt{\frac{b}{a}}\tan^{-1}\sqrt{\frac{b}{a}}\cos\phi\right]_0^\pi$$

$$=\frac{\pi}{\sqrt{ab}}\,2\tan^{-1}\sqrt{\frac{b}{a}}\,;\quad \therefore\ I=\frac{\pi}{\sqrt{ab}}\tan^{-1}\sqrt{\frac{b}{a}}.$$

The case $a=b=1$ gives $\int_0^\pi \frac{\theta\sin\theta}{1+\cos^2\theta}d\theta=\pi\tan^{-1}1=\frac{\pi^2}{4}$.

991. **In illustration of the method of expansion** we may, for the same example in the case $a>b$, expand $\left(1+\frac{b}{a}\cos^2\theta\right)^{-1}$. Then

$$I=\frac{1}{a}\int_0^\pi\left[\theta\sin\theta-\frac{b}{a}\theta\sin\theta\cos^2\theta+\frac{b^2}{a^2}\theta\sin\theta\cos^4\theta-\dots\right]d\theta,$$

a convergent expansion if $b<a$.

But

$$\int_0^\pi \theta\sin\theta\cos^{2n}\theta\, d\theta=\left[-\frac{\theta\cos^{2n+1}\theta}{2n+1}\right]_0^\pi+\frac{1}{2n+1}\int_0^\pi \cos^{2n+1}\theta\, d\theta=\frac{\pi}{2n+1}+0\,;$$

$$\therefore\ I=\frac{\pi}{a}\left[\frac{1}{1}-\frac{1}{3}\frac{b}{a}+\frac{1}{5}\frac{b^2}{a^2}-\dots\right]$$

$$=\frac{\pi}{\sqrt{ab}}\tan^{-1}\sqrt{\frac{b}{a}}\ \text{ by Gregory's Series.}$$

If, however, $a<b$ the expansion used would be divergent, and the method would fail.

992. **Illustrations of a Combination of Methods.**

Let $I=\int_0^\pi x\sin^n x\, dx$. Write $x=\pi-y$.

$$I=\int_0^\pi(\pi-y)\sin^n y\, dy=\pi\int_0^\pi \sin^n y\, dy-I\,;$$

$$\therefore\ I=\frac{\pi}{2}\int_0^\pi \sin^n x\, dx=\pi\int_0^{\frac{\pi}{2}}\sin^n x\, dx,$$

and the result can be written down.

This integral is useful, in cases where $F(x)$ is capable of expansion in powers of $\sin x$, for finding $\int_0^\pi x\,F(x)\,dx$.

Ex. 1. $I=\int_0^\pi \frac{x}{\sin x}\log(1+n\sin x)\,dx \quad (n<1)$

$$=\int_0^\pi x\left[n-\frac{n^2}{2}\sin x+\frac{n^3}{3}\sin^2 x-\dots\right]dx$$

$$=\frac{n\pi^2}{2}-\frac{n^2}{2}\pi+\frac{n^3}{3}\pi\frac{1}{2}\frac{\pi}{2}-\frac{n^4}{4}\pi\frac{2}{3}+\frac{n^5}{5}\pi\frac{3}{4}\frac{1}{2}\frac{\pi}{2}-\frac{n^6}{6}\pi\frac{4}{5}\frac{2}{3}+\dots$$

$$=\frac{\pi^2}{2}\left(n+\frac{1}{2}\frac{n^3}{3}+\frac{1\,.\,3}{2\,.\,4}\frac{n^5}{5}+\dots\right)-\pi\left[\frac{n^2}{2}+\frac{2}{3}\frac{n^4}{4}+\frac{2\,.\,4}{3\,.\,5}\frac{n^6}{6}\dots\right]$$

$$=\frac{\pi^2}{2}\sin^{-1}n-\pi\left(\frac{\sin^{-1}n}{2}\right)^2.$$ (See *Diff. Calc.*, p. 90, Ex. 3, Part 3.)

Ex. 2. $I=\int_0^\pi \frac{x\,dx}{1+\cos\alpha\sin x}$

$$=\int_0^\pi x(1-\cos\alpha\sin x+\cos^2\alpha\sin^2 x-\dots)\,dx$$

$$=\pi\left[\frac{\pi}{2}-\cos\alpha+\cos^2\alpha\frac{1}{2}\frac{\pi}{2}-\cos^3\alpha\frac{2}{3}+\cos^4\alpha\frac{3}{4}\frac{1}{2}\frac{\pi}{2}-\cos^5\alpha\frac{4}{5}\frac{2}{3}+\dots\right]$$

$$=-\pi\left[\cos\alpha+\frac{2}{3}\cos^3\alpha+\frac{2\,.\,4}{3\,.\,5}\cos^5\alpha+\dots\right]$$
$$+\frac{\pi^2}{2}\left[1+\frac{1}{2}\cos^2\alpha+\frac{1\,.\,3}{2\,.\,4}\cos^4\alpha+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cos^6\alpha+\dots\right]$$

$$=-\pi\frac{\sin^{-1}\cos\alpha}{\sqrt{1-\cos^2\alpha}}+\frac{\pi^2}{2}(1-\cos^2\alpha)^{-\frac{1}{2}}$$ (See *Diff. Calc.*, Ex. 3, p. 85.)

$$=-\pi\frac{\frac{\pi}{2}-\alpha}{\sin\alpha}+\frac{\pi^2}{2\sin\alpha}=\pi\frac{\alpha}{\sin\alpha}$$ (Wolstenholme.)

This integral might be treated thus:

Write $\pi-x$ for x.

$$I=\int_0^\pi\frac{(\pi-x)\,dx}{1+\cos\alpha\sin x}=\pi\int_0^\pi\frac{dx}{1+\cos\alpha\sin x}-I;$$

$$\therefore\ I=\frac{\pi}{2}\int_0^\pi\frac{dx}{1+\cos\alpha\sin x}=\frac{\pi}{2}\int_0^\pi\frac{\sec^2\frac{x}{2}\,dx}{1+2\cos\alpha\tan\frac{x}{2}+\tan^2\frac{x}{2}}$$

$$=\frac{\pi}{\sin\alpha}\left\{\tan^{-1}\left(\frac{\tan\frac{x}{2}+\cos\alpha}{\sin\alpha}\right)\right\}_0^\pi=\frac{\pi}{\sin\alpha}\left[\frac{\pi}{2}-\tan^{-1}\cot\alpha\right]$$

$$=\frac{\pi}{\sin\alpha}\tan^{-1}(\tan\alpha)=\pi\frac{\alpha}{\sin\alpha}.$$

EXAMPLES.

1. Prove that $\int_{\frac{3}{4}}^{\frac{4}{3}} \sqrt{x^2+1}\, dx = \frac{185}{288} + \frac{1}{2}\log\frac{3}{2}$. [ST. JOHN'S, 1884.]

2. Prove that $\int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta = \frac{1}{\sqrt{2}} + \frac{1}{2}\log(\sqrt{2}+1)$.

[MATH. TRIPOS, 1889.]

3. Prove that

$$\int_0^\infty \phi(x)\, dx = \int_0^1 \left[\phi(x) + \frac{1}{x^2}\phi\left(\frac{1}{x}\right)\right] dx.$$

[ST. JOHN'S, 1882 and 1887.]

4. Show that, n being a positive integer,

$$(n-1)\int \frac{\log x}{(1+x)^n}\, dx = \frac{1}{1+x} + \frac{1}{2(1+x)^2} + \frac{1}{3(1+x)^3} + \dots + \frac{1}{n-2}\frac{1}{(1+x)^{n-2}} + \log\frac{x}{1+x} - \frac{\log x}{(1+x)^{n-1}},$$

and that

(a) $\int_0^\infty \frac{\log x}{(1+x)^n}\, dx = -\frac{1}{n-1}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-2}\right)$.

(b) $\int_0^\infty \frac{\log x}{(1+x)^4}\, dx = -\frac{1}{2}$. [ST. JOHN'S, 1882.]

(c) $\int_0^\infty \frac{1-3x}{(1+x)^5}(\log x)^2\, dx = 1$. [ST. JOHN'S, 1882.]

5. Prove that

$$\int_0^{\frac{\pi}{2}} (\sin\theta - \cos\theta)\log(\sin\theta + \cos\theta)\, d\theta = 0.$$

[ST. JOHN'S, 1884.]

6. Prove that

$$\int_0^\pi \theta^3 \log\sin\theta\, d\theta = \frac{3\pi}{2}\int_0^\pi \theta^2 \log(\sqrt{2}\sin\theta)\, d\theta.$$

[ST. JOHN'S, 1884.]

7. Prove that

(i) $\int_{-\infty}^\infty \frac{dx}{(x^2 \pm ax + a^2)(x^2 \pm bx + b^2)} = \frac{2\pi}{\sqrt{3}}\frac{a+b}{ab(a^2+ab+b^2)}$.

(ii) $\int_{-\infty}^\infty \frac{dx}{(x^2 \pm ax + a^2)(x^2 \mp bx + b^2)} = \frac{2\pi}{\sqrt{3}}\frac{1}{ab(a+b)}$.

[COLLEGES γ, 1891.]

8. Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{\tan x}\,dx = \frac{\pi}{2}\log 2.$ [OXFORD II. P., 1888.]

9. Show that

$$\int_0^{\infty} \left(\frac{\tan^{-1}x}{x}\right)^3 dx = \tfrac{1}{2}\pi(3\log_e 2 - \tfrac{1}{8}\pi^2).$$

[MATH. TRIPOS, 1887.]

10. Show that

$$\int_0^a \sinh px \sin\frac{k\pi x}{a}\,dx = -\frac{ka\pi\sinh(pa)(-1)^k}{p^2a^2 + k^2\pi^2}.$$

[CLARE, CAIUS AND KING'S, 1885.]

11. Prove that

$$(1)\quad \int_0^{\frac{\pi}{4}} \frac{\sin^2 x\,dx}{e^{2mx}(\cos x - m\sin x)^2} = \frac{1}{2m(1+m^2)}\left[\frac{1+m}{1-m}e^{-m\frac{\pi}{2}} - 1\right].$$

$$(2)\quad \int_0^{\frac{\pi}{4}} \frac{\cos^2 x\,dx}{e^{2mx}(\sin x + m\cos x)^2} = \frac{1}{2m(1+m^2)}\left[\frac{1-m}{1+m}e^{-m\frac{\pi}{2}} + 1\right].$$

[ST. JOHN'S, 1886.]

12. Prove that $\displaystyle\int_0^{\pi} \frac{x}{1+\sin^2 x}\,dx = \frac{\pi^2}{2\sqrt{2}}.$ [OXF. II. P., 1885.]

13. Show that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log(\sin x + \cos x)\,dx = -\frac{\pi}{4}\log 2.$$

[COLLEGES, 1886.]

14. Show that $\displaystyle\int_0^{\infty} \frac{dx}{e^x\sqrt{\sinh 2x}} = \frac{\pi}{2\sqrt{2}}.$ [ST. JOHN'S, 1890.]

15. Prove that

$$\int_{1-b}^{b} xf\{x(1-x)\}\,dx = \tfrac{1}{2}\int_{1-b}^{b} f\{x(1-x)\}\,dx.$$

[COLLEGES, 1882.]

16. Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n 2\theta \log\tan\theta\,d\theta = 0$, where n is any positive integer [COLLEGES, 1882.]

17. Prove that

$$\int_b^a \frac{x^{n-1}\{(n-2)x^2 + (n-1)(a+b)x + nab\}}{(x+a)^2(x+b)^2}\,dx = \frac{a^{n-1} - b^{n-1}}{2(a+b)}.$$

[ST. JOHN'S, 1890.]

18. Establish the result

$$\int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\dfrac{\pi}{2}\cos x\right)}{2x - \pi}\,dx = \frac{8}{\pi}.$$

[MATH. TRIPOS, 1882]

19. Prove that

$$\int_0^{\frac{\pi}{2}} \left\{ \left(\frac{3}{x^2} - 1 \right) \sin x - \frac{3}{x} \cos x \right\}^2 dx = \frac{\pi}{4} - \frac{24}{\pi^3}.$$

[Colleges β, 1890.]

20. Prove that $\displaystyle\int_0^{\frac{\pi}{4}} \log (1 + \tan \theta)\, d\theta = \frac{\pi \log 2}{8}$. [Trinity, 1885.]

21. If α be any angle between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, show that

$$\int_0^{\alpha} \log (1 + \tan \alpha \tan x)\, dx = \alpha \log \sec \alpha.$$

[ϵ, 1884.]

22. Prove that, in general,

$$\int_{\beta}^{\alpha} F \left\{ \log_e \frac{x\sqrt{e}}{x+1} \cdot \log_e \frac{x}{(x+1)\sqrt{e}} \right\} \frac{2x+1}{x^2(x+1)^2}\, dx = 0,$$

where

$$\alpha = \frac{1}{\sqrt{e} - 1}, \quad \beta = \frac{\sqrt{e}}{1 - \sqrt{e}},$$

and F is any function. [ϵ, 1881.]

23. Prove that

$$\int_0^{\frac{\pi}{2}} \log (\sin^2\theta + k^2 \cos^2\theta)\, d\theta = \pi \log \frac{1+k}{2} \quad (k \geqslant 0).$$

[Oxf. I. P., 1918.]

24. Prove that

$$\int_{-\infty}^{\infty} f\left(a^2x^2 + \frac{b^2}{x^2}\right) dx = \frac{1}{a} \int_{-\infty}^{\infty} f(x^2 + 2ab)\, dx.$$

993. Integrals of form $\displaystyle\int_0^{\infty} \frac{\sin^m rx}{x^n}\, dx$, $(m \nless n)$, etc.

Consider the integral $I = \displaystyle\int_0^{\infty} \frac{\sin rx}{x}\, dx$, r being a real constant.

If we write $rx = y$, $I = \displaystyle\int_0^{\infty} \frac{\sin y}{y}\, dy = \int_0^{\infty} \frac{\sin x}{x}\, dx$, which is independent of r. But it is obvious upon changing the sign of r in the original integral that the sign of the result must be changed, for all elements of the integrand $\dfrac{\sin rx}{x}$ change sign. Further, when $r = 0$ the value of I is zero. Here then is a curious discontinuity which must be examined.

The integral is of great importance in the theory of definite integrals, and we propose to illustrate by means of it several methods of procedure as mentioned above.

994. METHOD I. **By breaking up the Integration into Sections.**

We have $I \equiv \int_0^\infty \frac{\sin x}{x}\,dx = \left[\left(\int_0^\pi + \int_\pi^{2\pi}\right) + \left(\int_{2\pi}^{3\pi} + \int_{3\pi}^{4\pi}\right) + \dots\right.$

$$\left. + \left(\int_{(2n-2)\pi}^{(2n-1)\pi} + \int_{(2n-1)\pi}^{2n\pi}\right) + \dots\right]\frac{\sin x}{x}\,dx,$$

a notation which will need no explanation.

In these pairs of successive integrals put $x = \pi - y,\ \pi + y$; $3\pi - y,\ 3\pi + y;\ \dots\ (2n-1)\pi - y,\ (2n-1)\pi + y$; etc.

Then

$$\int_{(2n-2)\pi}^{(2n-1)\pi} \frac{\sin x}{x}\,dx = -\int_\pi^0 \frac{\sin y}{(2n-1)\pi - y}\,dy = \int_0^\pi \frac{\sin y}{(2n-1)\pi - y}\,dy,$$

and $$\int_{(2n-1)\pi}^{2n\pi} \frac{\sin x}{x}\,dx = -\int_0^\pi \frac{\sin y}{(2n-1)\pi + y}\,dy.$$

Thus, putting $n = 1, 2, 3 \dots$ successively, the integral becomes

$$I = \int_0^\pi \sin y\left[\frac{1}{\pi - y} - \frac{1}{\pi + y} + \frac{1}{3\pi - y} - \frac{1}{3\pi + y} + \dots\right]dy$$

$$= \int_0^\pi \sin y\,\frac{1}{2}\tan\frac{y}{2}\,dy \quad \text{(Hobson, } \textit{Trigonometry}\text{, p. 335.)}$$

$$= \int_0^\pi \sin^2\frac{y}{2}\,dy = \frac{1}{2}\int_0^\pi (1 - \cos y)\,dy = \frac{\pi}{2}.$$

995. If we put $x = -y$ it is clear that

$$\int_0^\infty \frac{\sin x}{x}\,dx = -\int_0^{-\infty} \frac{\sin y}{y}\,dy = \int_{-\infty}^0 \frac{\sin y}{y}\,dy = \int_{-\infty}^0 \frac{\sin x}{x}\,dx.$$

Hence

$$\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \left(\int_{-\infty}^0 + \int_0^\infty\right)\frac{\sin x}{x}\,dx = 2\int_0^\infty \frac{\sin x}{x}\,dx = 2\cdot\frac{\pi}{2} = \pi.$$

996. If r be positive we have, by putting $rx = y$,

$$\int_0^\infty \frac{\sin rx}{x}\,dx = \int_0^\infty \frac{\sin y}{y}\,dy = \frac{\pi}{2}.$$

If r be negative we have, by putting $rx = y$,

$$\int_0^\infty \frac{\sin rx}{x}\,dx = \int_0^{-\infty} \frac{\sin y}{y}\,dy = -\int_{-\infty}^0 \frac{\sin y}{y}\,dy$$

$$= -\int_0^\infty \frac{\sin y}{y}\,dy = -\frac{\pi}{2}.$$

If r be zero the integrand is zero, and

$$\int_0^\infty \frac{\sin rx}{x}\,dx=0.$$

997. If the integrand be regarded as a function of r the discontinuity may be exhibited geometrically by tracing the graph of $y=\int_0^\infty \frac{\sin x\theta}{\theta}\,d\theta$, which will consist of

the straight line $y=-\frac{\pi}{2}$, from $x=-\infty$ to $x=0$;

the point $x=0$, $y=0$, when $x=0$;

the straight line $y=\frac{\pi}{2}$, from $x=0$ to $x=\infty$;

and is shown in Fig. 323.

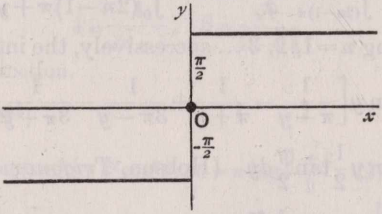

Fig. 323.

998. The graph of the *integrand*, viz. $\frac{\sin x}{x}$, is shown in Fig. 324.

The integral $\int_0^\infty \frac{\sin x}{x}\,dx$ is the difference of the areas between the x-axis and the successive portions of the curve which lie above the x-axis

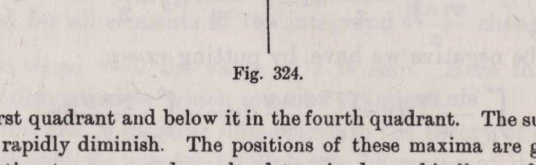

Fig. 324.

in the first quadrant and below it in the fourth quadrant. The successive maxima rapidly diminish. The positions of these maxima are given by the equation $\tan x=x$, and can be determined graphically as the intersections of the graphs of $y=\tan x$ and $y=x$. They occur in each case a little earlier than midway between two successive cuts of the curve $y=\frac{\sin x}{x}$ by the x-axis, but rapidly approximate to the midway as x increases.

999. METHOD II. **A Further Illustration of breaking up the Integration into Sections.**

Since the y-axis is an axis of symmetry for the graph of $\frac{\sin x}{x}$ we may take

$$I\equiv\int_0^\infty \frac{\sin x}{x}\,dx=\frac{1}{2}\int_{-\infty}^\infty \frac{\sin x}{x}\,dx;$$

$$\therefore\ 2I=\left\{\begin{matrix}\int_0^\pi+\int_\pi^{2\pi}+\int_{2\pi}^{3\pi}+\int_{3\pi}^{4\pi}+\dots\\ +\int_{-\pi}^{0}+\int_{-2\pi}^{-\pi}+\int_{-3\pi}^{-2\pi}+\dots\end{matrix}\right\}\left(\frac{\sin x}{x}\,dx\right).$$

In the integrals in the first row put

$$x=y,\quad \pi+y,\quad 2\pi+y,\quad 3\pi+y,\ \text{etc.},$$

and in the second row

$$x=-\pi+y,\quad -2\pi+y,\quad -3\pi+y,\ \text{etc.}$$

Then

$$\begin{aligned}2I&=\int_0^\pi \sin y\left[\frac{1}{y}-\frac{1}{\pi+y}+\frac{1}{2\pi+y}-\frac{1}{3\pi+y}+\dots\right.\\&\qquad\left.-\frac{1}{-\pi+y}+\frac{1}{-2\pi+y}-\frac{1}{-3\pi+y}+\dots\right]dy\\&=\int_0^\pi \sin y\left[\frac{1}{y}-\frac{1}{y+\pi}-\frac{1}{y-\pi}+\frac{1}{y+2\pi}+\frac{1}{y-2\pi}-\dots\right]dy\\&=\int_0^\pi \sin y\,.\,\mathrm{cosec}\,y\,dy=\int_0^\pi 1\,dy=\pi\end{aligned}$$

(Hobson, *Trigonometry*, Art. 295)

giving $I=\frac{\pi}{2}$ as before.

This proof is similar to that of Method I., but makes use of the expression for $\mathrm{cosec}\,y$ in partial fractions instead of that for $\tan\frac{y}{2}$.

1000. METHOD III. **Illustrating Differentiation under an Integration Sign.**

(1) Consider the integral $I=\int_0^\infty e^{-kx}\frac{\sin rx}{x}\,dx$, where r is positive and k any finite positive quantity, which we shall ultimately diminish without limit.

Then so long as k lies between 0 and $+\infty$,

$$\frac{\delta I}{\delta r}=\int_0^\infty e^{-kx}\frac{\sin(r+\delta r)x-\sin rx}{\delta r}\frac{dx}{x}=\int_0^\infty e^{-kx}\cos(r+\theta\,\delta r)x\,dx,\quad(0<\theta<1),$$

$$=\frac{k}{k^2+(r+\theta\,\delta r)^2},\text{ (Art. 96)},$$

and proceeding to the limit when δr is indefinitely small,

$$\frac{dI}{dr}=\frac{k}{k^2+r^2},\quad\text{whence } I\equiv\int_0^\infty e^{-kx}\frac{\sin rx}{x}\,dx=\tan^{-1}\frac{r}{k},$$

no constant being needed since each side vanishes with r.

If in this result we diminish k indefinitely towards zero, the integral tends to the limit $\int_0^\infty \frac{\sin rx}{x}\,dx$, and $\tan^{-1}\frac{r}{k}$ tends to the limit $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as r is positive or negative. But if $r=0$ the integral is obviously zero.

Hence $\int_0^\infty \frac{\sin rx}{x}\,dx=\frac{\pi}{2}$, 0 or $-\frac{\pi}{2}$ according as $r>$, $=$ or <0.

(2) As a further illustration of this method, let

$$I_n\equiv\int_0^{\frac{\pi}{2}}\frac{d\theta}{(a\cos^2\theta+\beta\sin^2\theta)^n},$$

a and β being of the same sign, so that the subject of integration has no infinity between the limits.

Let $\Delta\equiv\frac{\partial}{\partial a}+\frac{\partial}{\partial\beta}$. Then $\Delta I_n=-nI_{n+1}$.

Hence

$$I_{n+1}=\frac{-1}{n}\Delta I_n=\frac{(-1)^2}{n(n-1)}\Delta^2 I_{n-1}=\text{etc.}=\frac{(-1)^n}{n!}\Delta^n I_1.$$

Also $$I_1=\frac{1}{\beta}\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta\,d\theta}{\frac{a}{\beta}+\tan^2\theta}=\frac{1}{\sqrt{a\beta}}\left[\tan^{-1}\left(\sqrt{\frac{\beta}{a}}\tan\theta\right)\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2\sqrt{a\beta}}.$$

Hence

$$I_2=(-1)\frac{\pi}{2}\Delta\frac{1}{\sqrt{a\beta}}=\frac{\pi}{4}\frac{1}{\sqrt{a\beta}}\left(\frac{1}{a}+\frac{1}{\beta}\right).$$

Similarly

$$I_3=\frac{\pi}{16}\frac{1}{\sqrt{a\beta}}\left(\frac{3}{a^2}+\frac{2}{a\beta}+\frac{3}{\beta^2}\right),\text{ and so on.}$$

And since

$$\left(\frac{\partial}{\partial a}\right)^p\left(\frac{\partial}{\partial\beta}\right)^q a^{-\frac{1}{2}}\beta^{-\frac{1}{2}}=\frac{(-1)^{p+q}}{2^{p+q}}(1.3\ldots\overline{2p-1})(1.3\ldots\overline{2q-1})\frac{1}{\sqrt{a\beta}}\frac{1}{a^p\beta^q}$$

$$=\frac{(-1)^{p+q}}{2^{2(p+q)}}\frac{(2p)!\,(2q)!}{(p!)(q!)}\frac{1}{\sqrt{a\beta}}\frac{1}{a^p\beta^q},$$

the general result is

$$I_{n+1}=\frac{\pi}{2}\frac{(-1)^n}{n!}\left(\frac{\partial}{\partial\alpha}+\frac{\partial}{\partial\beta}\right)^n\frac{1}{\sqrt{\alpha\beta}}$$

$$=\frac{\pi}{2}\frac{(-1)^n}{n!}\sum_0^n {}^nC_p\frac{(-1)^n}{2^{2n}}\frac{(2p)!\,(2q)!}{p!\,q!}\frac{1}{\sqrt{\alpha\beta}}\frac{1}{\alpha^p\beta^q},$$

i.e. $$I_{n+1}=\frac{\pi}{2^{2n+1}}\frac{1}{\sqrt{\alpha\beta}}\sum_0^n\frac{(2p)!\,(2q)!}{(p!)^2(q!)^2}\frac{1}{\alpha^p\beta^q},\quad\text{where } p+q=n.$$

Also, since

$$\frac{\partial I_n}{\partial\alpha}=-n\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\,d\theta}{(\alpha\cos^2\theta+\beta\sin^2\theta)^{n+1}},\quad \frac{\partial I_n}{\partial\beta}=-n\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\,d\theta}{(\alpha\cos^2\theta+\beta\sin^2\theta)^{n+1}},$$

all integrals of the forms

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{(\alpha\cos^2\theta+\beta\sin^2\theta)^n},\quad \int_0^{\frac{\pi}{2}}\frac{\cos^2\theta\,d\theta}{(\alpha\cos^2\theta+\beta\sin^2\theta)^n},\quad \int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\,d\theta}{(\alpha\cos^2\theta+\beta\sin^2\theta)^n},$$

can be computed, n being a positive integer and α, β of the same sign.

1001. Since $$\int e^{-ax}\cos bx\,dx=e^{-ax}\frac{b\sin bx-a\cos bx}{a^2+b^2}+\text{const.}$$

and $$\int e^{-ax}\sin bx\,dx=-e^{-ax}\frac{b\cos bx+a\sin bx}{a^2+b^2}+\text{const.},$$

we have
$$\left.\begin{aligned}\int_0^\infty e^{-ax}\cos bx\,dx&=\frac{a}{a^2+b^2}\ \ldots\ldots(1),\\ \int_0^\infty e^{-ax}\sin bx\,dx&=\frac{b}{a^2+b^2}\ \ldots\ldots(2),\end{aligned}\right\}a\text{ being supposed positive.}$$

Integrating the first of these equations with regard to b from 0 to b,

$$\int_0^\infty e^{-ax}\frac{\sin bx}{x}\,dx=\tan^{-1}\frac{b}{a},\ \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and integrating the second from c to b (both positive) and

$$\int_0^\infty e^{-ax}\frac{\cos bx-\cos cx}{x}\,dx=\frac{1}{2}\log\frac{a^2+c^2}{a^2+b^2}.\ \ldots\ldots\ldots\ldots(4)$$

When a diminishes indefinitely the limiting form of (3) is

$$\int_0^\infty\frac{\sin bx}{x}\,dx=\frac{\pi}{2}\ \text{ or }\ \frac{-\pi}{2},\ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

according as b is positive or negative.

If in equation (4) we make a diminish indefinitely,

$$\int_0^\infty\frac{\cos bx-\cos cx}{x}\,dx=\log\frac{c}{b}.\ \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

If we differentiate (1) and (2) $n-1$ times with regard to a,

$$\int_0^\infty x^{n-1}e^{-ax}\cos bx\,dx=(-1)^{n-1}\frac{d^{n-1}}{da^{n-1}}\frac{a}{a^2+b^2}=\frac{(n-1)!}{b^n}\cos n\theta\sin^n\theta,$$

where $\tan\theta=\dfrac{b}{a}$,

and $$\int_0^\infty x^{n-1}e^{-ax}\sin bx\,dx=(-1)^{n-1}\frac{d^{n-1}}{da^{n-1}}\frac{b}{a^2+b^2}=\frac{(n-1)!}{b^n}\sin n\theta\sin^n\theta.$$

Here n is a positive integer and a is positive.

The case when n is not a positive integer is considered later.

1002. METHOD IV. **Deduction of a Definite Integral from the Summation Definition.**

We may employ either of the well-known trigonometrical series

$$\frac{\pi}{2}-\frac{\theta}{2}=\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\dots\ ad\ inf.\quad (\pi>\theta>-\pi),$$

$$\frac{\pi}{4}=\sin\theta+\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta+\dots\ ad\ inf.\quad (\pi>\theta>-\pi),$$

to obtain the value of $\int_0^\infty \frac{\sin x}{x}dx$.

(1) $$\int_0^\infty \frac{\sin x}{x}dx = Lt_{h=0}\, h\left(\frac{\sin h}{h}+\frac{\sin 2h}{2h}+\frac{\sin 3h}{3h}+\dots\right)$$

$$=Lt_{h=0}\left(\frac{\sin h}{1}+\frac{\sin 2h}{2}+\frac{\sin 3h}{3}+\dots\right)$$

$$=Lt_{h=0}\frac{\pi-h}{2}=\frac{\pi}{2}.$$

(2) $$\int_0^\infty \frac{\sin x}{x}dx = Lt_{h=0}\, 2h\left(\frac{\sin h}{h}+\frac{\sin 3h}{3h}+\frac{\sin 5h}{5h}+\dots\right)$$

$$=Lt_{h=2}\, 2\left(\frac{\sin h}{1}+\frac{\sin 3h}{3}+\frac{\sin 5h}{5}+\dots\right)$$

$$=2\times\frac{\pi}{4}=\frac{\pi}{2}.$$

[For the first series see *Diff. Calc.*, p. 108, Ex. 21 (2).

For the second add to the first $\frac{\theta}{2}=\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta-\dots$, or otherwise. (Hobson, *Trigonometry*, p. 288.)

See Bertrand, *Calcul. Diff. et Int.*, vol. i., pages 304, 383.]

1003. METHOD V. **Again illustrating Derivation from the Definition of an Integral as a Summation.**

Consider the series

$$S=\frac{e^{-q\theta}\sin\theta}{1}+\frac{e^{-2q\theta}\sin 2\theta}{2}+\frac{e^{-3q\theta}\sin 3\theta}{3}+\dots\ ad\ inf.$$

Let $$C=\frac{e^{-q\theta}\cos\theta}{1}+\frac{e^{-2q\theta}\cos 2\theta}{2}+\frac{e^{-3q\theta}\cos 3\theta}{3}+\dots.$$

These series are convergent so long as q is positive.

$$C+\iota S=\sum_1^\infty\frac{e^{-nq\theta}e^{n\iota\theta}}{n}=-\log\left(1-e^{-q\theta}e^{\iota\theta}\right)$$

$$=-\log\sqrt{1-2e^{-q\theta}\cos\theta+e^{-2q\theta}}+\iota\tan^{-1}\frac{e^{-q\theta}\sin\theta}{1-e^{-q\theta}\cos\theta};$$

$$\therefore\ S=\tan^{-1}\frac{\sin\theta}{e^{q\theta}-\cos\theta}.$$

In the limit when θ is made indefinitely small,

$$S=\tan^{-1} Lt_{\theta=0}\frac{\cos\theta}{qe^{q\theta}+\sin\theta}=\tan^{-1}\frac{1}{q}=\frac{\pi}{2}-\tan^{-1}q.$$

Now

$$\int_0^\infty \frac{e^{-qx}\sin x}{x}\,dx=Lt_{h=0}\,h\left[\frac{e^{-qh}\sin h}{h}+\frac{e^{-2qh}\sin 2h}{2h}+\frac{e^{-3qh}\sin 3h}{3h}+\dots\right]$$

$$=Lt_{h=0}\left[\frac{e^{-qh}\sin h}{1}+\frac{e^{-2qh}\sin 2h}{2}+\frac{e^{-3qh}\sin 3h}{3}+\dots\right],$$

$$\int_0^\infty \frac{e^{-qx}\sin x}{x}\,dx=\frac{\pi}{2}-\tan^{-1}q.$$

Now let q diminish indefinitely to zero, the limit towards which the result tends without limit is

$$\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

1004. The integral $I=\int_0^\infty \frac{e^{-qx}\sin rx}{x}\,dx=\tan^{-1}\frac{r}{q}$ may be established for the case $q>r$ thus; expanding $\sin rx$, we have

$$I=\int_0^\infty e^{-qx}\left(r-\frac{r^3x^2}{3\,!}+\frac{r^5x^4}{5\,!}-\dots\right)dx.$$

But
$$\int_0^\infty x^n e^{-qx}\,dx=\frac{n\,!}{q^{n+1}};$$

$$\therefore\quad I=\frac{r}{q}-\frac{1}{3}\frac{r^3}{q^3}+\frac{1}{5}\frac{r^5}{q^5}-\dots=\tan^{-1}\frac{r}{q}.$$

This series, however, is divergent if $q<r$. See Art. 1000 (1).

1005. Method VI. **Illustration of Use of Change of Order of Integration.**

Consider the double integral

$$I\equiv\int_0^\infty\int_0^\infty e^{-xy}\sin rx\,dx\,dy.$$

Integrating first with respect to y,

$$I\equiv\int_0^\infty\left[-e^{-xy}\frac{\sin rx}{x}\right]_{y=0}^{y=\infty}dx=\int_0^\infty\frac{\sin rx}{x}\,dx.$$

Changing the order of integration, integrate first with regard to x,

$$I\equiv\int_0^\infty\left[-e^{-xy}\frac{y\sin rx+r\cos rx}{r^2+y^2}\right]_{x=0}^{x=\infty}dy$$

$$=\int_0^\infty\frac{r}{r^2+y^2}\,dy=\left[\tan^{-1}\frac{y}{r}\right]_0^\infty=\frac{\pi}{2}\quad\text{or}\quad-\frac{\pi}{2},$$

according as r is positive or negative;

$$\therefore\quad\int_0^\infty\frac{\sin rx}{x}\,dx=\frac{\pi}{2}\quad\text{or}\quad-\frac{\pi}{2},$$

according as r is positive or negative.

1006. Method VII. The integral may also be established by the method of contour integration. (See Art. 1302.)

1007. The expression for $\cot z$ in partial fractions (Hobson, *Trigonometry*, p. 334) is

$$\cot z = \frac{1}{z} + \frac{1}{z+\pi} + \frac{1}{z-\pi} + \frac{1}{z+2\pi} + \frac{1}{z-2\pi} + \frac{1}{z+3\pi} + \frac{1}{z-3\pi} + \ldots$$

$$= \frac{1}{z} + 2z\sum_1^\infty \frac{1}{z^2 - r^2\pi^2}.$$

If $\phi(z)$ be any periodic function of z with periodicity π, *i.e.* such that $\phi(z) = \phi(z + r\pi)$ for all positive or negative integral values of r, we have

$$\int_{-\infty}^{\infty} \frac{\phi(z)}{z}\,dz = \left\{ \begin{array}{l} \int_0^\pi + \int_\pi^{2\pi} + \int_{2\pi}^{3\pi} + \ldots \\ + \int_{-\pi}^0 + \int_{-2\pi}^{-\pi} + \int_{-3\pi}^{-2\pi} + \ldots \end{array} \right\} \frac{\phi(z)}{z}\,dz.$$

In these integrals, put

$$z = y, \quad \pi + y, \quad 2\pi + y \ldots \text{ in the first row,}$$

and $$-\pi + y, \; -2\pi + y \ldots \text{ in the second row.}$$

$$\int_{r\pi}^{(r+1)\pi} \frac{\phi(z)}{z}\,dz = \int_0^\pi \frac{\phi(r\pi + y)}{r\pi + y}\,dy = \int_0^\pi \frac{\phi(y)}{r\pi + y}\,dy,$$

$$\int_{-r\pi}^{-(r-1)\pi} \frac{\phi(z)}{z}\,dz = \int_0^\pi \frac{\phi(y - r\pi)}{y - r\pi}\,dy = \int_0^\pi \frac{\phi(y)}{y - r\pi}\,dy.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\phi(z)}{z}\,dz = \int_0^\pi \phi(y)\left[\frac{1}{y} + \frac{1}{y+\pi} + \frac{1}{y-\pi} + \frac{1}{y+2\pi} + \frac{1}{y-2\pi} + \ldots\right] dy$$

$$= \int_0^\pi \phi(y)\cot y\,dy,$$

i.e. $$\int_{-\infty}^{\infty} \frac{\phi(x)\,dx}{x} = \int_0^\pi \phi(x)\cot x\,dx, \quad \text{where } \phi(x) = \phi(x + r\pi).$$

Thus, if $\phi(x) = \tan x$, $\displaystyle\int_{-\infty}^{\infty} \frac{\tan x}{x}\,dx = \int_0^\pi \tan x \cot x\,dx = \pi.$

Also $\dfrac{\tan x}{x}$ is not affected by a change of sign of x, and its graph is symmetrical about the y-axis.

Hence $$\int_0^\infty \frac{\tan x}{x}\,dx = \tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\tan x}{x}\,dx = \frac{\pi}{2},$$

and writing rx for x,

$$\int_0^\infty \frac{\tan rx}{x}\,dx = \frac{\pi}{2},\ -\frac{\pi}{2} \text{ or } 0 \text{ as } r \text{ is } +^{\text{ve}},\ -^{\text{ve}} \text{ or zero.}$$

1008. We now proceed to consider some consequences of the result

$$\int_0^\infty \frac{\sin rx}{x}\,dx = \frac{\pi}{2}.$$

By the ordinary method of summation, we have

$$^pC_0 \sin 2px + {}^pC_1 \sin(2p-2)x + \dots + {}^pC_{p-1}\sin 2x = 2^p \cos^p x \sin px;$$

$$\therefore \int_0^\infty \frac{\cos^p x \sin px}{x}\,dx = \frac{1}{2^p}\cdot\frac{\pi}{2}[{}^pC_0 + {}^pC_1 + \dots + {}^pC_{p-1}] = \frac{\pi}{2}\left(1 - \frac{1}{2^p}\right).$$

1009. In the same way

$$^pC_0 \sin 2px - {}^pC_1 \sin(2p-2)x + \dots + (-1)^{p-1}\,{}^pC_{p-1}\sin 2x$$
$$= (-1)^{\frac{p}{2}}\, 2^p \sin^p x \sin px, \quad (p \text{ even})$$
$$\text{or } = (-1)^{\frac{p-1}{2}}\, 2^p \sin^p x \cos px, \quad (p \text{ odd}).$$

Hence $$\int_0^\infty \frac{\sin^{2n} x \sin 2nx\,dx}{x} = \frac{(-1)^n}{2^{2n}}\cdot\frac{\pi}{2}[(1-1)^{2n} - 1] = (-1)^{n+1}\frac{\pi}{2^{2n+1}},$$

and $$\int_0^\infty \frac{\sin^{2n+1} x \cos(2n+1)x\,dx}{x} = \frac{(-1)^n}{2^{2n+1}}\frac{\pi}{2}[(1-1)^{2n+1} + 1] = (-1)^n\frac{\pi}{2^{2n+2}}.$$

1010. Again,

$$\int_0^\infty \frac{\sin^{2n+1} x}{x}\,dx = \frac{(-1)^n}{2^{2n}}\int_0^\infty [\sin(2n+1)x - {}^{2n+1}C_1 \sin(2n-1)x + \dots$$
$$+ (-1)^n\,{}^{2n+1}C_n \sin x]\frac{dx}{x}$$

$$= \frac{(-1)^n}{2^{2n}}\frac{\pi}{2}[1 - {}^{2n+1}C_1 + {}^{2n+1}C_2 - \dots + (-1)^n\,{}^{2n+1}C_n]$$

$$= \frac{1}{2^{2n}}\frac{\pi}{2} \times \text{ coeff. of } z^n \text{ in } (1+z)^{2n+1} \times (1+z)^{-1} = \frac{\pi}{2^{2n+1}}\,{}^{2n}C_n$$

$$= \frac{\pi}{2}\,\frac{1.3.5\dots(2n-1)}{2.4.6\dots 2n}.$$

1011. Let a and b be any two positive quantities ($a > b$).

Then $$\int_0^\infty \frac{\sin(a+b)x}{x}\,dx = \frac{\pi}{2} \quad\text{and}\quad \int_0^\infty \frac{\sin(a-b)x}{x}\,dx = \frac{\pi}{2}.$$

Hence, adding and subtracting,

$$\int_0^\infty \frac{\sin ax \cos bx}{x}\,dx = \frac{\pi}{2} \quad\text{and}\quad \int_0^\infty \frac{\cos ax \sin bx}{x}\,dx = 0.$$

We may then state that

$$\int_0^\infty \frac{\sin px \cos qx}{x}\,dx = \frac{\pi}{2} \text{ or } 0, \text{ according as } p > q \text{ or } < q$$

both being considered positive.

If $p=q$,

$$\int_0^\infty \frac{\sin px \cos qx}{x}\,dx = \frac{1}{2}\int_0^\infty \frac{\sin 2px}{x}\,dx = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

1012. Graphical Illustrations.

Consider the graph of $y=\int_0^\infty \frac{\sin x\theta \cos\theta}{\theta}\,d\theta.$

We may write this as $y=\frac{1}{2}\int_0^\infty \frac{\sin(x+1)\theta+\sin(x-1)\theta}{\theta}\,d\theta.$

If $x>1$, $\quad y=\frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{\pi}{2}.$

If $x=1$, $\quad y=\frac{1}{2}\left(\frac{\pi}{2}+0\right)=\frac{\pi}{4}.$

If $x<1$ and >-1, $\quad y=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{2}\right)=0.$

If $x=-1$, $\quad y=\frac{1}{2}\left(0-\frac{\pi}{2}\right)=-\frac{\pi}{4}.$

If $x<-1$, $\quad y=\frac{1}{2}\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)=-\frac{\pi}{2}.$

Hence the graph is discontinuous and as shown in Fig. 325.

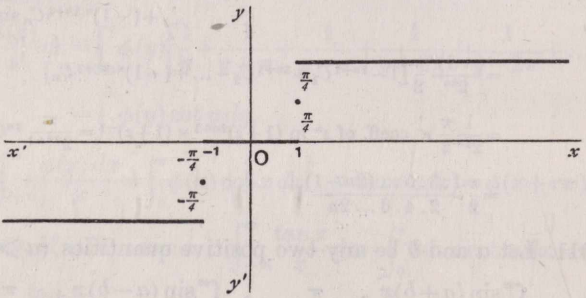

Fig. 325.

1013. Graph of $y=\int_0^\infty \frac{\sin\theta\cos x\theta}{\theta}\,d\theta$

$$=\frac{1}{2}\int_0^\infty \frac{\sin(1+x)\theta+\sin(1-x)\theta}{\theta}\,d\theta.$$

Here, if $x>1$, $y=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{2}\right)=0$;

$x=1$, $y=\frac{1}{2}\left(\frac{\pi}{2}+0\right)=\frac{\pi}{4}$;

$-1<x<1$, $y=\frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{\pi}{2}$;

$x=-1$, $y=\frac{1}{2}\left(0+\frac{\pi}{2}\right)=\frac{\pi}{4}$;

$x<-1$, $y=\frac{1}{2}\left(-\frac{\pi}{2}+\frac{\pi}{2}\right)=0$;

and the graph is as shown in Fig. 326.

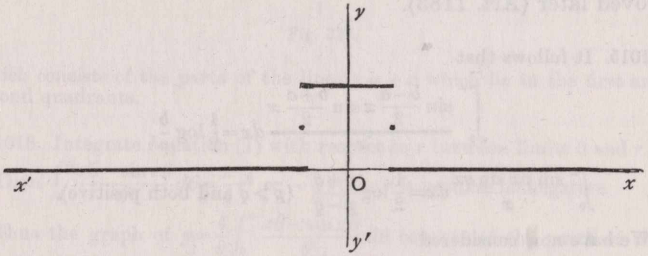

Fig. 326.

being again discontinuous at $x=1$ and $x=-1$.

1014. Consider the integral

$$\int_0^h \frac{\cos z-1}{z}\,dz,$$

and put $z=ax$ and $z=bx$ therein alternately.

Then $$\int_0^{\frac{h}{a}} \frac{\cos ax-1}{x}\,dx=\int_0^{\frac{h}{b}} \frac{\cos bx-1}{x}\,dx,$$

i.e. $$\int_0^{\frac{h}{a}} \frac{\cos ax-\cos bx}{x}\,dx-\int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\cos bx}{x}\,dx=\int_{\frac{h}{b}}^{\frac{h}{a}} \frac{1}{x}\,dx=\log\frac{b}{a}.$$

Now $$\int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\cos bx}{x}\,dx=\left[\frac{\sin bx}{b}\cdot\frac{1}{x}\right]_{\frac{h}{a}}^{\frac{h}{b}}+\frac{1}{b}\int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\sin bx}{x^2}\,dx,$$

and when h is increased indefinitely, becomes $\frac{1}{b}\int_{\frac{h}{a}}^{\frac{h}{b}} \frac{\sin bx}{x^2}\,dx$, and this must lie in numerical magnitude intermediate between the results obtained by replacing $\sin bx$ by -1 and by $+1$ respectively, *i.e.* between $\pm\frac{1}{b}\left[-\frac{1}{x}\right]_{\frac{h}{a}}^{\frac{h}{b}}$ or $\pm\frac{a \sim b}{bh}$, *i.e.* ± 0.

Therefore the second integral, for the infinite interval between $\frac{h}{a}$ and $\frac{h}{b}$ vanishes, and we have

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\frac{b}{a}.$$

This is a special case of a theorem due to Frullani to be proved later (Art. 1183).

1015. It follows that

$$\int_0^\infty \frac{\sin\frac{b-a}{2}x\sin\frac{b+a}{2}x}{x}\,dx = \frac{1}{2}\log\frac{b}{a},$$

i.e. $$\int_0^\infty \frac{\sin px\sin qx}{x}\,dx = \frac{1}{2}\log\frac{p+q}{p-q} \quad (p>q \text{ and both positive}).$$

We have now considered

$$\int_0^\infty \frac{\sin px\sin qx}{x}\,dx = \frac{1}{2}\log\frac{p+q}{p-q},$$

and $$\int_0^\infty \frac{\sin px\cos qx}{x}\,dx = \frac{\pi}{2} \text{ or } 0, \text{ as } p> \text{ or } <q \quad \text{(Art. 1011).}$$

Also $$\int_0^\infty \frac{\cos px\cos qx}{x}\,dx \text{ is infinite} \quad \text{(Art. 348).}$$

1016. Taking $y=\int_0^\infty \frac{\sin r\theta}{\theta}\,d\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$,

as r is positive or negative, or 0 if $r=0$, integrate with regard to r from $r=0$ to $r=r$,

$$y=\int_0^\infty \frac{1-\cos r\theta}{\theta^2}\,d\theta = \frac{\pi r}{2} \text{ or } -\frac{\pi r}{2}, \quad \text{............(1)}$$

as r is positive or negative, or 0 if $r=0$; *i.e.* putting $2r$ for r,

$$y=\int_0^\infty \frac{\sin^2 r\theta}{\theta^2}\,d\theta = \frac{\pi r}{2} \text{ or } -\frac{\pi r}{2}, \quad \text{...............(2)}$$

as r is positive or negative, or 0 if r be zero.

1017. To illustrate this geometrically, consider the graph of

$$y=\frac{2}{\pi}\int_0^\infty \frac{\sin^2 x\theta}{\theta^2}\,d\theta,$$

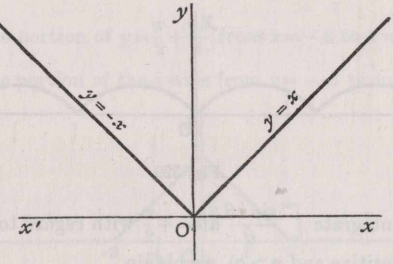

Fig. 327.

which consists of the parts of the lines $y=\pm x$ which lie in the first and second quadrants.

1018. Integrate equation (1) with respect to r between limits 0 and r. Then $\int_0^\infty \frac{r\theta-\sin r\theta}{\theta^3}\,d\theta=\frac{\pi}{4}r^2$ or $-\frac{\pi}{4}r^2$, as r is positive or negative.

Thus the graph of $y=\frac{4}{\pi}\int_0^\infty \frac{x\theta-\sin x\theta}{\theta^3}\,d\theta$ consists of the parts of the two parabolas $y=x^2$ and $y=-x^2$, as x is positive or negative, which lie in the first and third quadrants.

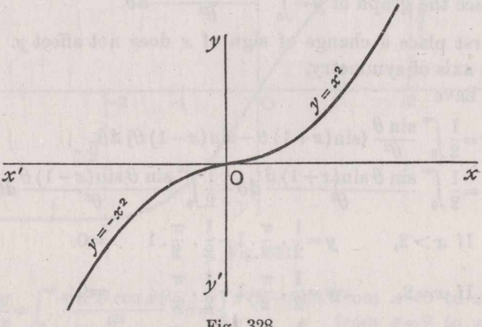

Fig. 328.

Similarly we might proceed to further integrations.

1019. Graph of $\qquad y=\frac{2a}{\pi}\int_0^\infty \frac{\sin^2\left(\theta\sin\frac{x}{a}\right)}{\theta^2}\,d\theta.$

Since a change of sign of x evidently does not affect the value of the integral, the y-axis is an axis of symmetry.

Also

$$y=a\sin\frac{x}{a} \text{ if } \sin\frac{x}{a} \text{ be positive and } y=-a\sin\frac{x}{a} \text{ if } \sin\frac{x}{a} \text{ be negative.}$$

Hence the graph is that shown in Fig. 329.

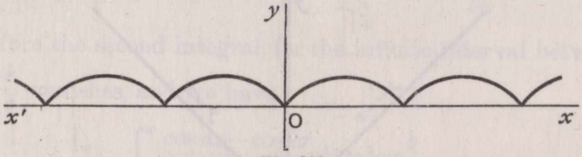

Fig. 329.

1020. If we integrate $\int_0^\infty \frac{\sin r\theta}{\theta}\,d\theta=+\frac{\pi}{2}$ with regard to r between limits q and p (both positive and $p>q$), we obtain

$$\int_0^\infty \frac{\cos q\theta-\cos p\theta}{\theta^2}\,d\theta=\frac{\pi}{2}(p-q),$$

i.e.
$$\int_0^\infty \frac{\sin\frac{p+q}{2}\theta\sin\frac{p-q}{2}\theta}{\theta^2}\,d\theta=\frac{\pi}{4}(p-q),$$

or putting $p+q=2a$, $p-q=2b$,

$$\int_0^\infty \frac{\sin a\theta\sin b\theta}{\theta^2}\,d\theta=\frac{\pi}{2}b,$$

where b is the smaller of the two quantities a and b.

1021. Trace the graph of $y=\int_0^\infty \frac{\sin^2\theta\cos x\theta}{\theta^2}\,d\theta.$

In the first place a change of sign of x does not affect y. Hence the y-axis is an axis of symmetry.

Also we have

$$y=\frac{1}{2}\int_0^\infty \frac{\sin\theta}{\theta^2}\{\sin(x+1)\theta-\sin(x-1)\theta\}\,d\theta$$
$$=\frac{1}{2}\int_0^\infty \frac{\sin\theta\sin(x+1)\theta}{\theta^2}\,d\theta-\frac{1}{2}\int_0^\infty \frac{\sin\theta\sin(x-1)\theta}{\theta^2}\,d\theta.$$

If $x>2$, $\quad y=\frac{1}{2}\cdot\frac{\pi}{2}.1-\frac{1}{2}\cdot\frac{\pi}{2}.1 \quad =0.$

If $x=2$, $\quad y=\frac{1}{2}\cdot\frac{\pi}{2}.1-\frac{1}{2}\cdot\frac{\pi}{2} \quad =0.$

If $2>x>1$, $\quad y=\frac{1}{2}\cdot\frac{\pi}{2}.1-\frac{1}{2}\cdot\frac{\pi}{2}(x-1)=\frac{\pi}{4}(2-x).$

If $x=1$, $\quad y=\frac{1}{2}\cdot\frac{\pi}{2}.1-0 \quad =\frac{\pi}{4}.$

If $1>x>0$, $\quad y=\frac{1}{2}\cdot\frac{\pi}{2}.1+\frac{1}{2}\cdot\frac{\pi}{2}(1-x)=\frac{\pi}{4}(2-x).$

If $x=0$, $\quad y=\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{2}\cdot\frac{\pi}{2} \quad =\frac{\pi}{2}.$

The graph therefore consists of:

(*a*) the portion of the x-axis from $x=2$ to $x=\infty$,

(*b*) the portion of the line $y=\frac{\pi}{2}-\frac{\pi x}{4}$ from $x=0$ to $x=2$,

(*c*) the portion of $y=\frac{\pi}{2}+\frac{\pi x}{4}$ from $x=-2$ to $x=0$,

(*d*) the portion of the x-axis from $x=-\infty$ to $x=-2$.

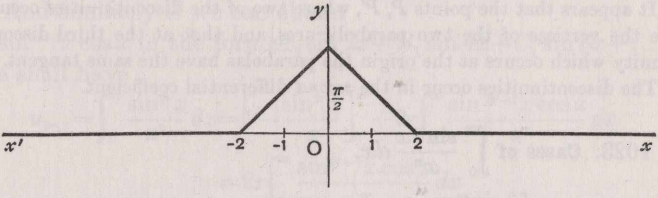

Fig. 330.

And the discontinuous nature is shown in the illustration (Fig. 330).

1022. Trace the graph of $y=\int_0^\infty \frac{\sin^2\theta \sin x\theta}{\theta^3}\,d\theta$. (Math. Tripos, 1895.)

We note in the first place that a change of sign of x gives a change of sign of y. That is, the origin is a centre of symmetry.

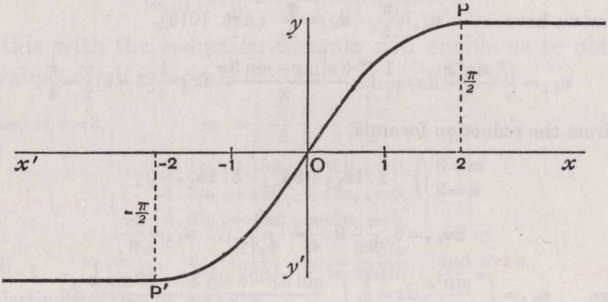

Fig. 331.

Also $\frac{dy}{dx}=\int_0^\infty \frac{\sin^2\theta\cos x\theta}{\theta^2}\,d\theta=\begin{cases}\pi(2-x)/4 & \text{from } x=0 \text{ to } x=2,\\ 0 & \text{from } x=2 \text{ to } x=\infty;\end{cases}$

$$\therefore\ y=\begin{cases}A+\pi(4x-x^2)/8 & \text{from } x=0 \text{ to } x=2,\\ B & \text{from } x=2 \text{ to } x=\infty,\end{cases}$$

where A and B are constants.

Moreover, the difference of adjacent ordinates at $x-\epsilon$, $x+\epsilon$, being to the first order $2\epsilon\int_0^\infty \frac{\sin^2\theta\cos x\theta}{\theta^2}\,d\theta$, ultimately vanishes with ϵ, and therefore there is no abrupt change of *ordinate* at any point on the graph.

Again, $y=0$ if $x=0$; $\therefore\ A=0$;

and at $x=2$, $A+\pi(4.2-2^2)/8=B$; $\therefore\ B=\frac{\pi}{2}$.

Therefore the graph in the first quadrant consists of a portion of the parabola $y=\pi(4x-x^2)/8$ from $x=0$ to $x=2$, the vertex being at $(2, \pi/2)$, and a line, $2y=\pi$, parallel to the x-axis from $x=2$ to $x=\infty$.

And remembering that there is symmetry with regard to the origin, the graph is as shown in Fig. 331.

It appears that the points P, P', where two of the discontinuities occur, are the vertices of the two parabolic arcs, and that at the third discontinuity which occurs at the origin the parabolas have the same tangent.

The discontinuities occur in the *second* differential coefficient.

1023. Cases of $\int_0^\infty \frac{\sin^m x}{x^n}\,dx$.

Let $u_{m,n}=\int_0^\infty \frac{\sin^m x}{x^n}\,dx$, where m is not less than n, and m, n are either both odd or both even positive integers >2. We have proved in Art. 265 a reduction formula connecting $u_{m,n}$, $u_{m,n-2}$ and $u_{m-2,n-2}$, viz.

$$(n-1)(n-2)u_{m,n}+m^2u_{m,n-2}-m(m-1)u_{m-2,n-2}=0.$$

Now we have $u_{1,1}=\frac{\pi}{2}$, $u_{2,2}=\frac{\pi}{2}$ (Art. 1016),

and $$u_{3,1}=\int_0^\infty \frac{\sin^3 x}{x}\,dx=\frac{1}{4}\int_0^\infty \frac{3\sin x-\sin 3x}{x}\,dx=\frac{1}{4}[3-1]\frac{\pi}{2}=\frac{\pi}{4};$$

and from the reduction formula,

$$\left.\begin{matrix}m=3\\ n=3\end{matrix}\right\}\quad 2.1u_{3,3}+9u_{3,1}-3.2u_{1,1}=0;$$

$$\therefore\ 2u_{3,3}=6.\frac{\pi}{2}-9.\frac{\pi}{4}=\frac{3\pi}{4};\quad \therefore\ u_{3,3}=\frac{3\pi}{8}.$$

Also $$u_{5,1}=\int_0^\infty \frac{\sin^5 x}{x}\,dx=\frac{1}{2^4}\int_0^\infty \frac{\sin 5x-5\sin 3x+10\sin x}{x}\,dx$$

$$=\frac{1}{2^4}(1-5+10)\frac{\pi}{2}=\frac{3\pi}{16}.$$

Then the reduction formula,

$$\left.\begin{matrix}m=5\\ n=3\end{matrix}\right\}\text{ gives }2.1u_{5,3}+25u_{5,1}-5.4u_{3,1}=0,$$

and $$\left.\begin{matrix}m=5\\ n=5\end{matrix}\right\}\text{ gives }4.3u_{5,5}+25u_{5,3}-5.4u_{3,3}=0;$$

whence $$u_{5,3}=\frac{5\pi}{32},\quad u_{5,5}=\frac{115}{384}\pi,\text{ etc.}$$

1024. In order to generalise these results it will be plain that it is necessary to express $\sin^{2r+1}x$ in the form

$$A\sin x+B\sin 3x+C\sin 5x+\dots,$$

and then we shall have

$$u_{2r+1,1}=\int_0^\infty \frac{\sin^{2r+1}x}{x}\,dx=\frac{\pi}{2}(A+B+C+\dots)$$

(see Art. 1010).

And similarly if we can obtain

$\sin^{2r-1}x\cos x$ in the form $A_1\sin 2x+B_1\sin 4x+C_1\sin 6x+\dots$, we shall have

$$u_{2r,2}=\int_0^\infty \frac{\sin^{2r}x}{x^2}\,dx=\left[-\frac{\sin^{2r}x}{x}\right]_0^\infty+2r\int_0^\infty \frac{\sin^{2r-1}x\cos x}{x}\,dx$$

$$=2r\int_0^\infty \frac{\sin^{2r-1}x\cos x}{x}\,dx$$

$$=2r(A_1+B_1+C_1+\dots)\frac{\pi}{2},$$

and the sums $A+B+C+\dots$, and $A_1+B_1+C_1+\dots$ are easy to find. (Art. 1026.)

1025. It has been shown in Art. 1010 that

$$u_{2r+1,1}=\frac{\pi}{2}\,\frac{1\,.\,3\,.\,5\dots(2r-1)}{2\,.\,4\,.\,6\dots 2r},$$

and this with the reduction formula will enable us to obtain the values of all integrals of form $u_{2n+1,2p+1}$ $(n \nless p)$.

Thus, if $r=3$, $\qquad u_{7,1}=\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{\pi}{2}=\frac{5\pi}{2^5}$,

and
$$\left.\begin{aligned}2\,.\,1u_{7,3}+49u_{7,1}-42u_{5,1}&=0,\\4\,.\,3u_{7,5}+49u_{7,3}-42u_{5,3}&=0,\\6\,.\,5u_{7,7}+49u_{7,5}-42u_{5,5}&=0,\end{aligned}\right\}$$

giving $\quad u_{7,3}=\frac{7\pi}{64},\quad u_{7,5}=\frac{77\pi}{768},\quad u_{7,7}=\frac{5887\pi}{23040}$, and so on.

Collecting the results, we have

$$u_{1,1}=\frac{\pi}{2},$$

$$u_{3,1}=\frac{\pi}{4},\quad u_{3,3}=\frac{3\pi}{8},$$

$$u_{5,1}=\frac{3\pi}{16},\quad u_{5,3}=\frac{5\pi}{32},\quad u_{5,5}=\frac{115\pi}{384},$$

$$u_{7,1}=\frac{5\pi}{32},\quad u_{7,3}=\frac{7\pi}{64},\quad u_{7,5}=\frac{77\pi}{768},\quad u_{7,7}=\frac{5887\pi}{23040},$$

etc.

$u_{2r+1,1}=\frac{1\,.\,3\,.\,5\dots(2r-1)}{2\,.\,4\,.\,6\dots 2r}\cdot\frac{\pi}{2}$, the same result as $\int_0^{\frac{\pi}{2}}\sin^{2r}\theta\,d\theta$.

1026. Again by differentiating the formula

$$2^{2r}(-1)^r \sin^{2r} x = 2 \sum_{s=0}^{s=r-1} (-1)^s\, {}^{2r}C_s \cos(2r-2s)x + (-1)^r\, {}^{2r}C_r,$$

we obtain

$$2r \sin^{2r-1} x \cos x = \frac{(-1)^{r-1}}{2^{2r-1}} \sum_{s=0}^{s=r-1} (-1)^s (2r-2s)\, {}^{2r}C_s \sin(2r-2s)x,$$

and the sum of the coefficients required (Art. 1024) is

$$\frac{(-1)^{r-1}}{2^{2r}\,.\,r}\{2r\,{}^{2r}C_0 - (2r-2)\,{}^{2r}C_1 + (2r-4)\,{}^{2r}C_2 - \ldots + (-1)^{r-1} 2\,{}^{2r}C_{r-1}\}$$

$$= \frac{1}{2^{2r-1}\,.\,r}\{{}^{2r}C_{r-1} - 2\,{}^{2r}C_{r-2} + 3\,{}^{2r}C_{r-3} - \ldots + (-1)^{r-1} r\,{}^{2r}C_0\}$$

$$= \frac{1}{2^{2r-1}\,.\,r} \times \text{coef. of } z^{r-1} \text{ in } (1+z)^{2r} \times (1+z)^{-2}$$

$$= \frac{1}{2^{2r-1}\,.\,r} \times \text{coef. of } z^{r-1} \text{ in } (1+z)^{2r-2} = \frac{1}{2^{2r-1}\,.\,r}\,\frac{(2r-2)!}{\{(r-1)!\}^2}.$$

Hence $\quad u_{2r,2} \equiv \int_0^\infty \frac{\sin^{2r} x}{x^2}\,dx = \frac{1\,.\,3\,.\,5\,\ldots\,(2r-3)}{2\,.\,4\,.\,6\,\ldots\,(2r-2)}\,\frac{\pi}{2}$ if $r \nless 2$,

and $=\frac{\pi}{2}$ if $r=1$, and $=\frac{\pi}{4}$ if $r=2$.

1027. Thus

$$u_{2,2} = \frac{\pi}{2}; \quad u_{4,2} = \frac{\pi}{4}; \quad u_{6,2} = \frac{1\,.\,3}{2\,.\,4}\,.\,\frac{\pi}{2} = \frac{3\pi}{16}; \quad u_{8,2} = \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\,.\,\frac{\pi}{2} = \frac{5\pi}{32}; \text{ etc.,}$$

the first of these having been found before.

And now the reduction formula can be used,

$$(n-1)(n-2)\,u_{m,n} + m^2 u_{m,n-2} - m(m-1)\,u_{m-2,n-2} = 0 \quad (m \nless n),$$

$$\left.\begin{matrix} m=4 \\ n=4 \end{matrix}\right\}\ 3\,.\,2u_{4,4} + 16u_{4,2} - 4\,.\,3u_{2,2} = 0;$$

$$\left.\begin{matrix} m=6 \\ n=4 \end{matrix}\right\}\ 3\,.\,2u_{6,4} + 36u_{6,2} - 6\,.\,5u_{4,2} = 0;$$

$$\left.\begin{matrix} m=6 \\ n=6 \end{matrix}\right\}\ 5\,.\,4u_{6,6} + 36u_{6,4} - 6\,.\,5u_{4,4} = 0;$$

etc.,

giving $\quad u_{4,4} = \frac{\pi}{3}, \quad u_{6,4} = \frac{\pi}{8}, \quad u_{6,6} = \frac{11\pi}{40}$, etc.,

and collecting the results,

$$u_{2,2} = \frac{\pi}{2},$$

$$u_{4,2} = \frac{\pi}{4}, \qquad u_{4,4} = \frac{\pi}{3},$$

$$u_{6,2}=\frac{3\pi}{16},\quad u_{6,4}=\frac{\pi}{8},\quad u_{6,6}=\frac{11\pi}{40},$$

$$u_{8,2}=\frac{5\pi}{32},\text{ etc.};$$

and generally,

$$u_{2r,2}=\frac{1.3.5\dots(2r-3)}{2.4.6\dots(2r-2)}\cdot\frac{\pi}{2}\quad(r \nless 2),\text{ and therefore}=\int_0^{\frac{\pi}{2}}\sin^{2r-2}\theta\,d\theta.$$

1028. A result due to Wolstenholme follows at once, viz.

$$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}F(\sin^2 x)dx=\int_0^{\pi}F(\sin^2 x)\,dx,$$

provided $F(z)$ be any function of z which can be expanded in a convergent series of positive integral powers of z. For let

$$F(z)\equiv A_0+A_1 z+A_2 z^2+\dots.$$

Then

$$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}F(\sin^2 x)dx=2\int_0^{\infty}\frac{\sin^2 x}{x^2}(A_0+A_1\sin^2 x+A_2\sin^4 x+\dots)dx$$

$$=2(A_0 u_{2,2}+A_1 u_{4,2}+A_2 u_{6,2}+\dots)$$

$$=2\int_0^{\frac{\pi}{2}}(A_0+A_1\sin^2 x+A_2\sin^4 x+\dots)\,dx$$

$$=2\int_0^{\frac{\pi}{2}}F(\sin^2 x)\,dx=\int_0^{\pi}F(\sin^2 x)\,dx.$$

1029. It is also plain that if $F(\sin\theta, \cos\theta)$ can be expressed in the form $\quad A\sin p\theta+B\sin q\theta+C\sin r\theta+\dots,$

where $p, q, r \dots$ are all positive; then

$$\int_0^{\infty}\frac{F(\sin\theta,\cos\theta)}{\theta}d\theta=(A+B+C+\dots)\frac{\pi}{2},$$

or if $F(\sin\theta, \cos\theta)$ can be expressed as

$$A\cos p\theta+B\cos q\theta+C\cos r\theta+\dots,$$

where p, q, r are all positive, and if $A+B+C+\dots=0$, then

$$\int_0^{\infty}\frac{F(\sin\theta,\cos\theta)}{\theta^2}d\theta=\int_0^{\infty}\frac{A\cos p\theta+B\cos q\theta+\dots}{\theta^2}d\theta$$

$$=\int_0^{\infty}\frac{A(\cos p\theta-1)+B(\cos q\theta-1)+\dots}{\theta^2}d\theta$$

$$=-\frac{\pi}{2}(Ap+Bq+Cr+\dots),$$

and evidently other propositions of similar kind may be enunciated.

1030. Ex. 1. Since $u_{2r+1,1}=\dfrac{1.3.5\ldots(2r-1)}{2.4.6\ldots 2r}\cdot\dfrac{\pi}{2}$, we have

$$\int_0^\infty \log\frac{1+n\sin ax}{1-n\sin ax}\frac{dx}{x}\quad (n<1,\ a>0)$$

$$=2\int_0^\infty\left(\frac{n}{1}\sin ax+\frac{n^3}{3}\sin^3 ax+\frac{n^5}{5}\sin^5 ax+\ldots\right)\frac{dx}{x}$$

$$=2\left[\frac{n}{1}\cdot\frac{\pi}{2}+\frac{n^3}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{n^5}{5}\cdot\frac{1.3}{2.4}\cdot\frac{\pi}{2}+\ldots\right]=\pi\sin^{-1}n\ \textit{(Diff. Calc., p. 85)};$$

$$\therefore \int_0^\infty \tanh^{-1}(n\sin ax)\frac{dx}{x}=\frac{\pi}{2}\sin^{-1}n;$$

and if $n=\frac{1}{2}$, $$\int_0^\infty \tanh^{-1}(\tfrac{1}{2}\sin ax)\frac{dx}{x}=\frac{\pi^2}{12}.$$

Ex. 2. Since $$u_{2r,2}=\frac{1.3.5\ldots(2r-3)}{2.4.6\ldots(2r-2)}\cdot\frac{\pi}{2}\quad (r>2),$$

$$\int_0^\infty \log\frac{1+n\sin^2 ax}{1-n\sin^2 ax}\frac{dx}{x^2}\quad (n<1,\ a>0)$$

$$=2\int_0^\infty\left(\frac{n}{1}\sin^2 ax+\frac{n^3}{3}\sin^6 ax+\frac{n^5}{5}\sin^{10}ax+\ldots\right)\frac{dx}{x^2}$$

$$=2a\frac{\pi}{2}\left(n+\frac{n^3}{3}\,\frac{1.3}{2.4}+\frac{n^5}{5}\,\frac{1.3.5.7}{2.4.6.8}+\ldots\right);$$

$$\therefore \int_0^\infty \tanh^{-1}(n\sin^2 ax)\frac{dx}{x^2}=\frac{\pi a}{2}\{\sqrt{1+n}-\sqrt{1-n}\}.$$

Ex. 3. $$I\equiv\int_{-\infty}^\infty \frac{\sin^2 x}{x^2}\tanh^{-1}\left(\cos\frac{a}{2}\sin^2 x\right)dx.$$

By Wolstenholme's principle given above, this integral

$$=2\int_0^{\frac{\pi}{2}}\tanh^{-1}\left(\cos\frac{a}{2}\sin^2 x\right)dx$$

$$=2\int_0^{\frac{\pi}{2}}\left[\cos\frac{a}{2}\sin^2 x+\frac{1}{3}\cos^3\frac{a}{2}\sin^6 x+\frac{1}{5}\cos^5\frac{a}{2}\sin^{10}x+\ldots\right]dx$$

$$=2\frac{\pi}{2}\left[\cos\frac{a}{2}\,\frac{1}{2}+\frac{1}{3}\cos^3\frac{a}{2}\,\frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}+\frac{1}{5}\cos^5\frac{a}{2}\,\frac{9}{10}\,\frac{7}{8}\,\frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}+\ldots\right].$$

Now $$\frac{(1-z)^{-\frac{1}{2}}-(1+z)^{-\frac{1}{2}}}{2z}=\frac{1}{2}+\frac{1.3.5}{2.4.6}z^2+\frac{1.3.5.7.9}{2.4.6.8.10}z^4+\ldots;$$

$$\therefore\ \frac{1}{2}\int_0^z\left[\frac{1}{(1-z)^{\frac{1}{2}}}-\frac{1}{(1+z)^{\frac{1}{2}}}\right]\frac{dz}{z}=\frac{1}{2}z+\frac{1.3.5}{2.4.6}\frac{z^3}{3}+\frac{1.3.5.7.9}{2.4.6.8.10}\frac{z^5}{5}+\ldots;$$

and writing $z=\cos 2\theta$, this integral

$$=-\frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\theta}\left(\frac{1}{\sin\theta}-\frac{1}{\cos\theta}\right)\frac{4\sin\theta\cos\theta}{\cos 2\theta}d\theta$$

$$=\sqrt{2}\int_\theta^{\frac{\pi}{4}}\frac{d\theta}{\cos\theta+\sin\theta}=\log\cot\left(\frac{\theta}{2}+\frac{\pi}{8}\right).$$

Hence putting $4\theta=a$, $I=\pi\log\cot\dfrac{\pi+a}{8}$.

1031. If p and q be positive integers and $p \not< q$, the integral $I=\int_0^\infty \frac{\sin^p x}{x^q}\,dx$ may be investigated by a method *which does not entail the successive calculation of previous results of the same form* leading up to this integral, as was done in Art. 1023.

Since
$$\int_0^\infty z^{q-1}e^{-xz}\,dz=\frac{\Gamma(q)}{x^q},$$
we have
$$\int_0^\infty \frac{\sin^p x}{x^q}\,dx=\frac{1}{(q-1)!}\int_0^\infty\int_0^\infty z^{q-1}e^{-xz}\sin^p x\,dz\,dx.$$

Now, p being taken greater than unity, and a positive
$$\int_0^\infty e^{-ax}\sin^p x\,dx=\frac{p(p-1)}{p^2+a^2}\int_0^\infty e^{-ax}\sin^{p-2}x\,dx \quad \text{(Art. 104)}$$
$$=\frac{p!}{(a^2+p^2)\{a^2+(p-2)^2\}\dots(a^2+2^2)}\frac{1}{a} \text{ if } p \text{ be even}$$
or
$$=\frac{p!}{(a^2+p^2)\{a^2+(p-2)^2\}\dots(a^2+1^2)} \text{ if } p \text{ be odd.}$$

Hence
$$\int_0^\infty \frac{\sin^p x}{x^q}\,dx=\frac{p!}{(q-1)!}\int_0^\infty \frac{z^{q-1}\,dz}{z(z^2+2^2)(z^2+4^2)\dots(z^2+p^2)} \text{ if } p \text{ be even}$$
and
$$=\frac{p!}{(q-1)!}\int_0^\infty \frac{z^{q-1}\,dz}{(z^2+1^2)(z^2+3^2)(z^2+5^2)\dots(z^2+p^2)} \text{ if } p \text{ be odd.}$$

The integrand can then be put into partial fractions of the form:

p even, $\quad \sum_1^{\frac{p}{2}}\frac{A_{2k}}{z^2+(2k)^2}$ (q even) $\quad$ or $\quad \sum_1^{\frac{p}{2}}\frac{A_{2k}z}{z^2+(2k)^2}$ (q odd);

p odd, $\quad \sum_1^{\frac{p+1}{2}}\frac{B_{2k-1}}{z^2+(2k-1)^2}$ (q odd) $\quad$ or $\quad \sum_1^{\frac{p+1}{2}}\frac{B_{2k-1}z}{z^2+(2k-1)^2}$ (q even);

and their coefficients have been found in Arts. 162 to 165.

In the two cases $\left.\begin{array}{l}p \text{ even,}\\ q \text{ even,}\end{array}\right\}$ $\left.\begin{array}{l}p \text{ odd,}\\ q \text{ odd,}\end{array}\right\}$ the integrals are of the inverse tangent species, viz.
$$\int_0^\infty \frac{dz}{z^2+n^2}=\left[\frac{1}{n}\tan^{-1}\frac{z}{n}\right]_0^\infty=\frac{1}{n}\frac{\pi}{2};$$
but in the remaining cases the integrals are logarithmic.

1032. Particular cases are simple.

Thus $$\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\,!}{2\,!}\int_0^\infty \frac{z^2\,dz}{(z^2+1^2)(z^2+3^2)} = 3\int_0^\infty \left[-\frac{1}{8}\frac{1}{z^2+1^2}+\frac{9}{8}\frac{1}{z^2+3^2}\right]dz$$
$$= \frac{3}{8}\left[\frac{9}{3}\tan^{-1}\frac{z}{3}-\tan^{-1}z\right]_0^\infty = \frac{3}{8}\cdot\left(\frac{9}{3}-1\right)\frac{\pi}{2} = \frac{3\pi}{8},$$
$$\int_0^\infty \frac{\sin^3 x}{x^2}\,dx = \frac{3\,!}{1\,!}\int_0^\infty \frac{z\,dz}{(z^2+1^2)(z^2+3^2)} = 6\int_0^\infty \left(\frac{1}{8}\frac{z}{z^2+1^2}-\frac{1}{8}\frac{z}{z^2+3^2}\right)dz$$
$$= \frac{3}{4}\cdot\frac{1}{2}\left[\log\frac{z^2+1^2}{z^2+3^2}\right]_0^\infty = \frac{3}{8}\log 3^2 = \frac{3}{4}\log 3.$$

1033. The general result is not difficult to obtain; the integrations have already been performed in Arts. 162, etc.

$$\int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{p\,!}{(q-1)!}\int_0^\infty \frac{z^{q-2}\,dz}{(z^2+2^2)(z^2+4^2)\dots(z^2+p^2)}\ \left(\begin{matrix}p \text{ even,}\\ q \text{ even}\end{matrix}\right\} \text{ and } p \not< q\Big);$$

and by writing $\left.\begin{matrix} q-2 \text{ for } 2q \\ \text{and } p \text{ for } 2n\end{matrix}\right\}$ in result (A) of Art. 162,

$$\int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{(-1)^{\frac{p+q}{2}}}{(q-1)!}\frac{\pi}{2^p}\left[{}^pC_0 p^{q-1} - {}^pC_1(p-2)^{q-1} + \dots + (-1)^{\frac{p}{2}-1}\,{}^pC_{\frac{p}{2}-1}\,2^{q-1}\right]. \dots \text{(A)}$$

And if $\left.\begin{matrix} p \text{ be odd} \\ \text{and } q \text{ be odd}\end{matrix}\right\}$ and $p \not< q$,

$$\int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{p\,!}{(q-1)!}\int_0^\infty \frac{z^{q-1}\,dz}{(z^2+1^2)(z^2+3^2)\dots(z^2+p^2)};$$

and writing $\left.\begin{matrix} q-1 \text{ for } 2q \\ \text{and } p \text{ for } 2n-1\end{matrix}\right\}$ in result (C) of Art. 164, the integral

$$= \frac{(-1)^{\frac{q-p}{2}}}{(q-1\,!}\frac{\pi}{2^p}\left[{}^pC_0 p^{q-1} - {}^pC_1(p-2)^{q-1} + {}^pC_2(p-4)^{q-1} - \dots + (-1)^{\frac{p-1}{2}}\,{}^pC_{\frac{p-1}{2}}1^{q-1}\right]. \text{ (B)}$$

If $\left.\begin{matrix} p \text{ be even} \\ \text{and } q \text{ be odd}\end{matrix}\right\}$ and $p \not< q$,

$$\int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{p\,!}{(q-1)!}\int_0^\infty \frac{z^{q-3}z\,dz}{(z^2+2^2)(z^2+4^2)\dots(z^2+p^2)};$$

and writing $\left.\begin{matrix} q-3 \text{ for } 2q \\ \text{and } p \text{ for } 2n\end{matrix}\right\}$ in result (B) of Art. 163, the indefinite integral is

$$\frac{1}{(q-1)!}\frac{(-1)^{\frac{p+q-1}{2}}}{2^p}\Big[{}^pC_0 p^{q-1}\log(z^2+p^2) - {}^pC_1(p-2)^{q-1}\log\{z^2+(p-2)^2\} + \dots$$
$$+ (-1)^{\frac{p}{2}-1}\,{}^pC_{\frac{p}{2}-1}\,2^{q-1}\log(z^2+2^2)\Big].$$

Now in the expansion of $(e^x - e^{-x})^p \equiv (2x+\dots)^p = 2^p x^p + \dots$ there are no terms of lower degree than x^p. Hence, if q be $\not> p$, the coefficient of x^{q-1} is zero; *i.e.* the coefficient of x^{q-1} in

$${}^pC_0 e^{px} - {}^pC_1 e^{(p-2)x} + {}^pC_2 e^{(p-4)x} - \dots + (-1)^{\frac{p}{2}-1}\,{}^pC_{\frac{p}{2}-1}\,e^{2x} + (-1)^{\frac{p}{2}}\,{}^pC_{\frac{p}{2}}$$
$$+ (-1)^{\frac{p}{2}+1}\,{}^pC_{\frac{p}{2}+1}\,e^{-2x} + \dots + {}^pC_p e^{-px}$$

is zero; and p being even and q odd,

$$ {}^pC_0 p^{q-1} - {}^pC_1(p-2)^{q-1} + {}^pC_2(p-4)^{q-1} - \ldots + (-1)^{\frac{p}{2}-1}\,{}^pC_{\frac{p}{2}-1}\,2^{q-1} $$

vanishes identically. Hence, multiplying this expression by $\log z^2$, and subtracting it from the portion of the indefinite integral in square brackets, we have

$$ {}^pC_0 p^{q-1}\log\left(1+\frac{p^2}{z^2}\right) - {}^pC_1(p-2)^{q-1}\log\left\{1+\frac{(p-2)^2}{z^2}\right\} + \ldots $$
$$ + (-1)^{\frac{p}{2}-1}\,{}^pC_{\frac{p}{2}-1}\,2^{q-1}\log\left(1+\frac{2^2}{z^2}\right), $$

which vanishes when z is infinitely large.

Hence

$$ \int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{(-1)^{\frac{p+q+1}{2}}}{(q-1)!}\,\frac{1}{2^{p-1}}\Big[{}^pC_0 p^{q-1}\log p - {}^pC_1(p-2)^{q-1}\log(p-2) $$
$$ + {}^pC_2(p-4)^{q-1}\log(p-4) - \ldots + (-1)^{\frac{p}{2}-1}\,{}^pC_{\frac{p}{2}-1}\,2^{q-1}\log 2\Big]. \quad \text{....(C)} $$

Finally, if p be odd and q be even $\Big\}$ and $p \not< q$,

$$ \int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{p!}{(q-1)!}\int_0^\infty \frac{z^{q-2}z\,dz}{(z^2+1^2)(z^2+3^2)(z^2+5^2)\ldots(z^2+p^2)}; $$

and writing $q-1$ for $2q+1$ and p for $2n-1$ $\Big\}$ in result (D) of Art. 165, the indefinite integral is

$$ \frac{1}{(q-1)!}\,\frac{(-1)^{\frac{q-p-1}{2}}}{2^p}\Big[{}^pC_0 p^{q-1}\log(z^2+p^2) - {}^pC_1(p-2)^{q-1}\log\{z^2+(p-2)^2\} + \ldots $$
$$ + (-1)^{\frac{p-1}{2}}\,{}^pC_{\frac{p-1}{2}}\,1^{q-1}\log(z^2+1^2)\Big]; $$

and in this case (p odd, q even) we have, in the same way as before,

$$ {}^pC_0 p^{q-1} - {}^pC_1(p-2)^{q-1} + {}^pC_2(p-4)^{q-1} - \ldots + (-1)^{\frac{p-1}{2}}\,{}^pC_{\frac{p-1}{2}}\,1^{q-1} \equiv 0, $$

an identity. Multiplying by $\log z^2$ and subtracting from the portion of the indefinite integral in square brackets, we get

$$ {}^pC_0 p^{q-1}\log\left(1+\frac{p^2}{z^2}\right) - {}^pC_1(p-2)^{q-1}\log\left\{1+\frac{(p-2)^2}{z^2}\right\} + \ldots $$
$$ + (-1)^{\frac{p-1}{2}}\,{}^pC_{\frac{p-1}{2}}\,1^{q-1}\log\left(1+\frac{1}{z^2}\right), $$

which vanishes when z is infinitely large.

Hence we get

$$ \int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{(-1)^{\frac{q-p+1}{2}}}{(q-1)!}\,\frac{1}{2^{p-1}}\Big[{}^pC_0 p^{q-1}\log p - {}^pC_1(p-2)^{q-1}\log(p-2) + \ldots $$
$$ + (-1)^{\frac{p-1}{2}}\,{}^pC_{\frac{p-1}{2}}\,1^{q-1}\log 1\Big], \quad \text{...........(D)} $$

the last term vanishing.

Hence, **summing up**, the four results may be written as

$$\int_0^\infty \frac{\sin^p x}{x^q}\,dx = \frac{(-1)^{\frac{p-q}{2}}}{(q-1)!}\,\frac{\pi}{2^p}\Big[p^{q-1} - p(p-2)^{q-1} + \frac{p(p-1)}{1\,.\,2}(p-4)^{q-1} - \dots\Big]$$

to $\frac{p}{2}$ or $\frac{p+1}{2}$ terms, if $p-q$ be even, or as

$$= \frac{(-1)^{\frac{p-q-1}{2}}}{(q-1)!}\,\frac{1}{2^{p-1}}\Big[p^{q-1}\log p - p(p-2)^{q-1}\log(p-2) + \frac{p(p-1)}{1\,.\,2}(p-4)^{q-1}\log(p-4) - \dots\Big]$$

to $\frac{p}{2}$ or $\frac{p+1}{2}$ terms, if $p-q$ be odd ; p being $\nless q$.

This generalisation is due to the late Prof. Wolstenholme.

It will be noticed that more is effected by the treatment of $\int_0^\infty \frac{\sin^p x}{x^q}\,dx$ in this article than in Art. 1023, as the limitation p, q, *both* even or *both* odd, is now avoided.

1034. Thus, for instance,

$$\int_0^\infty \frac{\sin^6 x}{x^4}\,dx = \frac{(-1)}{3!}\,\frac{\pi}{2^6}[6^3 - 6\,.\,4^3 + 15\,.\,2^3] = -\frac{\pi}{3\,.\,2^7}(-48) = \frac{\pi}{2^3} = \frac{\pi}{8},$$

$$\int_0^\infty \frac{\sin^5 x}{x^4}\,dx = \frac{1}{3!}\,\frac{1}{2^4}\{5^3\log 5 - 5\,.\,3^3\log 3\}.$$

EXAMPLES.

1. Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{4}{3}\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = \frac{3}{2}\int_0^\infty \left(\frac{\sin x}{x}\right)^4 dx.$$

[MATH. TRIPOS, 1884.]

2. Prove that (1) $\displaystyle\int_0^\infty \frac{\sin^{2n+1}x}{x}\,dx = \frac{1\,.\,3\,\dots\,(2n-1)}{2\,.\,4\,\dots\,2n}\,.\,\frac{\pi}{2}$;

(2) $\displaystyle\int_0^\infty \frac{\sin^{2n+1}x}{x^3}\,dx = \frac{1\,.\,3\,\dots\,(2n-3)(2n+1)}{2\,.\,4\,\dots\,2n}\,.\,\frac{\pi}{4}$.

[TRINITY, 1889.]

3. Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{\tan x}{x}\,dx = \frac{\pi}{2}$. [MATH. TRIPOS, 1887.]

4. Find the value of $\displaystyle\int_0^1 \left(\sin x + \sin\frac{1}{x}\right)\frac{dx}{x}$. [COLLEGES β, 1888.]

5. Trace the locus $\displaystyle y = \int_0^\infty \frac{\cos\theta\,\sin^3\theta x}{\theta}\,d\theta$.

6. Describe the discontinuous surface $\frac{\pi z}{2}=\int_0^\infty \frac{\sin x\theta \cos y\theta}{\theta}\,d\theta.$

[TRINITY, 1888.]

7. Show that

$$\int_0^\infty \{\phi 0 - x^2\phi 1 + x^4\phi 2 - \text{etc.}\}\,dx = \tfrac{1}{2}\pi\phi(-\tfrac{1}{2}),$$

and apply this theorem to find $\int_0^\infty \frac{\sin ax}{x}\,dx.$ [GLAISHER.]

8. Discuss the locus

$$y=\int_0^\infty \sin\frac{(2x-n+1)\theta}{2}\sin\frac{n\theta}{2}\operatorname{cosec}\frac{\theta}{2}\frac{d\theta}{\theta}$$

where n is a positive integer.

9. If $0<\alpha<\pi$, prove that

(i) $\int_0^\infty \log\frac{1+\sin\alpha\sin x}{1-\sin\alpha\sin x}\frac{dx}{x}=\pi\alpha;$

(ii) $\int_0^\infty \log\frac{1+\sin^2\alpha\sin^2 x}{1-\sin^2\alpha\sin^2 x}\frac{dx}{x^2}=\pi(\sqrt{1+\sin^2\alpha}-\cos\alpha).$

10. Prove that $\int_{-\infty}^\infty \frac{\sin^2 x\cos^2 x}{x^2(1+\sin^2 x)}\,dx=\pi(\sqrt{2}-1).$

1035. Let $I_1=\int e^{-ax}\cos bx\,dx,\quad I_2=\int e^{-ax}\sin bx\,dx,\quad (a+^{\text{ve}}).$

Then $I_1=e^{-ax}\frac{-a\cos bx+b\sin bx}{a^2+b^2}$, and $\Big[I_1\Big]_0^\infty=\frac{a}{a^2+b^2}$,

$I_2=e^{-ax}\frac{-a\sin bx-b\cos bx}{a^2+b^2}$, and $\Big[I_2\Big]_0^\infty=\frac{b}{a^2+b^2}$.

Integrating each with regard to a, from $a=p$ to $a=q$,

$$\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\cos bx\,dx=\frac{1}{2}\log\frac{p^2+b^2}{q^2+b^2}. \quad\ldots\ldots\ldots\ldots(1)$$

$$\int_0^\infty \frac{e^{-qx}-e^{-px}}{x}\sin bx\,dx=\tan^{-1}\frac{p}{b}-\tan^{-1}\frac{q}{b}. \quad\ldots\ldots\ldots(2)$$

The case $\left.\begin{array}{l}p=\infty,\\ q=0\end{array}\right\}$ in (2) gives

$$\int_0^\infty \frac{\sin bx}{x}\,dx=\pm\frac{\pi}{2} \text{ as } b \text{ is } +^{\text{ve}} \text{ or } -^{\text{ve}}.$$

1036. Again starting with the same integrals, integrate with regard to b; then

$$\int_0^\infty e^{-ax}\frac{\sin px-\sin qx}{x}\,dx=\tan^{-1}\frac{p}{a}-\tan^{-1}\frac{q}{a}, \quad\ldots\ldots\ldots(3)$$

$$\int_0^\infty e^{-ax}\frac{\cos px-\cos qx}{x}\,dx=\frac{1}{2}\log\frac{a^2+q^2}{a^2+p^2}. \quad\ldots\ldots\ldots\ldots(4)$$

Then

$$\int_0^\infty e^{-ax}\frac{\sin px}{x}\,dx=\tan^{-1}\frac{p}{a};\quad \int_0^\infty e^{-ax}\frac{\text{vers}\,px}{x}\,dx=\frac{1}{2}\log\left(1+\frac{p^2}{a^2}\right).$$

1037. **Consider the Integral** $I=\int_0^\infty e^{-x^2}\cos ax\,dx$.

(Laplace, *Mémoires de l'Institut*, 1809, p. 367.)

Differentiating with regard to a,

$$\frac{dI}{da}=-\int_0^\infty e^{-x^2}x\sin ax\,dx=\left[\frac{e^{-x^2}}{2}\sin ax\right]_0^\infty-\frac{a}{2}\int_0^\infty e^{-x^2}\cos ax\,dx$$

$$=-\frac{a}{2}I;$$

$\therefore\ I=Ae^{-\frac{a^2}{4}}$ where A is independent of a. Putting $a=0$,

$$I_{a=0}=\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2};\ \therefore\ A=\frac{\sqrt{\pi}}{2}.\ \text{ Hence } I=\frac{\sqrt{\pi}}{2}e^{-\frac{a^2}{4}}.$$

The proof is that of Legendre (*Exercices*, p. 362).

1038. Laplace established the result by aid of the integral

$$\int_0^\infty x^{2n}e^{-x^2}dx=\frac{1}{2}\Gamma\left(\frac{2n+1}{2}\right),$$

viz.
$$I=\int_0^\infty e^{-x^2}\left(1-\frac{a^2x^2}{2\,!}+\frac{a^4x^4}{4\,!}-\ldots\right)dx$$

$$=\frac{\sqrt{\pi}}{2}\left(1-\frac{a^2}{2\,!}\frac{1}{2}+\frac{a^4}{4\,!}\cdot\frac{1\,.\,3}{2\,.\,2}-\ldots\right)$$

$$=\frac{\sqrt{\pi}}{2}\left(1-\frac{a^2}{4}+\frac{1}{1\,.\,2}\frac{a^4}{4^2}-\frac{1}{1\,.\,2\,.\,3}\frac{a^6}{4^3}+\ldots\right)=\frac{\sqrt{\pi}}{2}e^{-\frac{a^2}{4}}.$$

1039. Differentiating I n times with respect to a (*D.C.*, Art. 106),

$$\int_0^\infty e^{-x^2}x^n\cos\left(ax+\frac{n\pi}{2}\right)dx=\frac{\sqrt{\pi}}{2}\frac{d^n}{da^n}\left(e^{-\frac{a^2}{4}}\right)$$

$$=(-1)^n\frac{\sqrt{\pi}}{2}e^{-\frac{a^2}{4}}\left\{\frac{(2a)^n}{4^n}-\frac{n(n-1)}{1\,!}\frac{(2a)^{n-2}}{4^{n-1}}+\frac{n(n-1)(n-2)(n-3)}{2\,!}\frac{(2a)^{n-4}}{4^{n-2}}-\ldots\right\}$$

$$=(-1)^n\frac{\sqrt{\pi}}{2^{n+1}}e^{-\frac{a^2}{4}}\left\{a^n-\frac{n(n-1)}{1\,!}a^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\,!}a^{n-4}-\ldots\right\}.$$

1040. Integrating I with regard to a, from 0 to a,

$$\int_0^\infty e^{-x^2}\frac{\sin ax}{x}dx=\frac{\sqrt{\pi}}{2}\int_0^a e^{-\frac{a^2}{4}}da=\frac{\sqrt{\pi}}{2}\int_0^a\left(1-\frac{a^2}{4}+\frac{1}{2!}\frac{a^4}{4^2}-\dots\right)da$$

$$=\frac{\sqrt{\pi}}{2}\left\{a-\frac{a^3}{4.3}+\frac{1}{2!}\frac{a^5}{4^2.5}-\frac{1}{3!}\frac{a^7}{4^3.7}+\dots\right),$$

a rapidly converging series for small values of a, but not capable of summation by means of the known algebraic or trigonometric functions.

1041. Laplace's integral $I=\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}}$ * follows immediately from the form of Art. 1037 by writing therein $\frac{2b}{a}$ for a and $x=ay$. It should be noted that the *process of differentiation in Art.* 1037 *is legitimate though the upper limit is infinite.* (See remarks in Art. 356.)

For, taking the present form, the integrand $e^{-a^2x^2}\cos 2bx$ remains finite for all values of x. Change b to $b+\delta b$. Then

$$I+\delta I=\int_0^\infty e^{-a^2x^2}\cos 2(b+\delta b)x\,dx.$$

Hence

$$\frac{\delta I}{\delta b}=\int_0^\infty e^{-a^2x^2}\frac{\cos 2(b+\delta b)x-\cos 2bx}{\delta b}dx=\int_0^\infty e^{-a^2x^2}\{-2x\sin 2bx+\epsilon\}dx,$$

where ϵ is a finite quantity which vanishes in the limit when δb is made infinitesimally small,

i.e.
$$\frac{\delta I}{\delta b}=-2\int_0^\infty xe^{-a^2x^2}\sin 2bx\,dx+\int_0^\infty \epsilon\,.\,e^{-a^2x^2}dx.$$

If ϵ_1 be the greatest numerical value of ϵ in the range of values of x from 0 to ∞, the second term is numerically $<\epsilon_1\int_0^\infty e^{-a^2x^2}dx$, *i.e.* $<\epsilon_1\frac{\sqrt{\pi}}{2a}$, and therefore vanishes in the limit when ϵ is infinitesimally small.

The process of differentiation is therefore justifiable.

Proceeding as before, $\quad \frac{dI}{db}=-\frac{2b}{a^2}I,\quad I=Ae^{-\frac{b^2}{a^2}};$

and putting $b=0$, $\quad I=\int_0^\infty e^{-a^2x^2}dx=\frac{\sqrt{\pi}}{2a};\quad \therefore\ A=\frac{\sqrt{\pi}}{2a},$

and
$$I=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{a^2}}.$$

* *Mémoires de l'Institut*, 1810, p. 290.

EXAMPLES.

1. Show that

$$\int_0^\infty xe^{-x^2}\sin ax\,dx = \frac{\sqrt{\pi}}{4}e^{-\frac{a^2}{4}}a,$$

$$\int_0^\infty x^2e^{-x^2}\cos ax\,dx = \frac{\sqrt{\pi}}{4}e^{-\frac{a^2}{4}}\left(1-\frac{a^2}{2}\right),$$

$$\int_0^\infty x^3e^{-x^2}\sin ax\,dx = \frac{\sqrt{\pi}}{8}e^{-\frac{a^2}{4}}\left(3a-\frac{a^3}{2}\right),$$

$$\int_0^\infty x^4e^{-x^2}\cos ax\,dx = \frac{\sqrt{\pi}}{8}e^{-\frac{a^2}{4}}\left(3-3a^2+\frac{a^4}{4}\right),$$

$$\int_0^\infty x^5e^{-x^2}\sin ax\,dx = \frac{\sqrt{\pi}}{16}e^{-\frac{a^2}{4}}\left(15a-5a^3+\frac{a^5}{4}\right),$$

and show that we can calculate

$$\int_0^\infty [\phi(x^2)\cos ax + \psi(x^2)x\sin ax]e^{-x^2}\,dx$$

when $\phi(x^2)$ and $\psi(x^2)$ are rational integral functions of x^2.

[LEGENDRE, *Exercices*, p. 363.]

2. Show that if $I=\int_0^\infty e^{-x^2}\sin ax\,dx$, then

$$I=\frac{1}{2}e^{-\frac{a^2}{4}}\int_0^a e^{\frac{a^2}{4}}\,da = \frac{1}{2}\left(a-\frac{a^3}{2.3}+\frac{a^5}{3.4.5}-\frac{a^7}{4.5.6.7}+\dots\right).$$

[LEGENDRE, *ibid.*]

3. If $I=\frac{1}{2}e^{-\frac{a^2}{4}}\int_0^a e^{\frac{a^2}{4}}\,da$, prove that

$$\int_0^\infty e^{-x^2}x\cos ax\,dx = \frac{1}{2}-\frac{1}{2}aI,$$

$$\int_0^\infty e^{-x^2}x^2\sin ax\,dx = \frac{1}{4}a+\frac{1}{2}I\left(1-\frac{a^2}{2}\right),$$

$$\int_0^\infty e^{-x^2}x^3\cos ax\,dx = \frac{1}{2}-\frac{a^2}{8}-\frac{1}{4}I\left(3a-\frac{a^3}{2}\right),$$

$$\int_0^\infty e^{-x^2}x^4\sin ax\,dx = \frac{5}{8}a-\frac{a^3}{16}+\frac{1}{4}I\left(3-3a^2+\frac{a^4}{4}\right),$$

etc. [LEGENDRE, *ibid.*]

4. Show that

(i) $$\int_0^\infty e^{-x^2}\left(\tfrac{1}{2}a\sin ax + x\cos ax\right)dx = \tfrac{1}{2};$$

(ii) $$\int_0^\infty e^{-x^2}\left(1-\tfrac{1}{2}a^2-2x^2\right)\sin ax\,dx = -\tfrac{1}{2}a.$$

[LEGENDRE, *ibid.*]

1042. The Integral $I\equiv\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}=\frac{\pi}{2}\,\frac{1}{ab(a+b)}$ **is useful in a certain class of Definite Integrals,** (a and b both $+^{\text{ve}}$).

Since $\frac{1}{(a^2+x^2)(b^2+x^2)}=\frac{1}{b^2-a^2}\left(\frac{1}{a^2+x^2}-\frac{1}{b^2+x^2}\right)$, we have

$$I\equiv\frac{1}{b^2-a^2}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}-\frac{1}{b}\tan^{-1}\frac{x}{b}\right]_0^\infty=\frac{1}{b^2-a^2}\left(\frac{1}{a}-\frac{1}{b}\right)\frac{\pi}{2}=\frac{\pi}{2}\,\frac{1}{ab(a+b)}.$$

Thus, if
$$u=\int_0^\infty \frac{\tan^{-1}\frac{x}{a}}{x(b^2+x^2)}dx,\quad (a,\ b\text{ both }+^{\text{ve}}),$$

$$\frac{du}{da}=-\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}=-\frac{\pi}{2}\,\frac{1}{ab(a+b)}=\frac{\pi}{2b^2}\left(\frac{1}{a+b}-\frac{1}{a}\right);$$

$$\therefore\ u=\frac{\pi}{2b^2}\log\frac{a+b}{a}+A,$$

where A is independent of a. But when $a=\infty$, $u=0$; $\therefore\ A=0$;

$$\therefore\ \int_0^\infty \frac{\tan^{-1}\frac{x}{a}}{x(b^2+x^2)}dx=\frac{\pi}{2b^2}\log\left(1+\frac{b}{a}\right). \quad\text{.................(1)}$$

Putting $x=b\tan\theta$, we have $\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}\left(\frac{b}{a}\tan\theta\right)}{\tan\theta}d\theta=\frac{\pi}{2}\log\left(1+\frac{b}{a}\right)$,

or writing c for $\frac{b}{a}$, $\int_0^{\frac{\pi}{2}}\cot\theta\tan^{-1}(c\tan\theta)\,d\theta=\frac{\pi}{2}\log(1+c)$. ....(2)

The particular case $c=1$ gives $\int_0^{\frac{\pi}{2}}\theta\cot\theta\,d\theta=\frac{\pi}{2}\log 2$. ....(3)

Integrating by parts, $[\theta\log\sin\theta]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta=\frac{\pi}{2}\log 2$,

or
$$\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta=\frac{\pi}{2}\log\tfrac{1}{2}, \quad\text{......(4)}$$
as in Art. 990.

1043. The Integral $I=\int_0^b \frac{\tan^{-1}\frac{x}{a}}{x\sqrt{b^2-x^2}}dx$, ($b \not> a$), is of similar form, but best evaluated by expansion. Put $x=b\sin\theta$.

$$I=\int_0^{\frac{\pi}{2}}\frac{\tan^{-1}\left(\frac{b}{a}\sin\theta\right)}{b\sin\theta}d\theta=\frac{1}{a}\int_0^{\frac{\pi}{2}}\left(1-\frac{b^2}{a^2}\,\frac{\sin^2\theta}{3}+\frac{b^4}{a^4}\,\frac{\sin^4\theta}{5}-\dots\right)d\theta$$

$$=\frac{\pi}{2b}\left(\frac{b}{a}-\frac{1}{2}\cdot\frac{1}{3}\,\frac{b^3}{a^3}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5}\,\frac{b^5}{a^5}-\dots\right)=\frac{\pi}{2b}\sinh^{-1}\left(\frac{b}{a}\right),$$

i.e.
$$\int_0^{\frac{\pi}{2}}\operatorname{cosec}\theta\tan^{-1}(c\sin\theta)\,d\theta=\frac{\pi}{2}\sinh^{-1}c=\frac{\pi}{2}\log\left(c+\sqrt{1+c^2}\right),$$

or, for the case $c=1$,

$$\int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(\sin\theta)}{\sin\theta}\,d\theta = \frac{\pi}{2}\log(1+\sqrt{2}).$$

1044. Let $I \equiv \int_0^\infty \frac{\log(1+a^2x^2)}{b^2+x^2}\,dx$. Then $\frac{dI}{da} = \int_0^\infty \frac{2ax^2}{(1+a^2x^2)(b^2+x^2)}\,dx$

$$=\frac{2}{1-a^2b^2}\int_0^\infty\left(\frac{1}{a}\cdot\frac{1}{\frac{1}{a^2}+x^2} - ab^2\frac{1}{b^2+x^2}\right)dx = \frac{2}{1-a^2b^2}\,\frac{\pi}{2}[1-ab] = \frac{\pi}{1+ab},$$

(a, b each being taken $+^{\text{ve}}$).

Hence $I = \frac{\pi}{b}\log(1+ab) + A$, where A is independent of a. Also $I=0$ if $a=0$; $\therefore\ A=0$;

$$\therefore \int_0^\infty \frac{\log(1+a^2x^2)}{b^2+x^2}\,dx = \frac{\pi}{b}\log(1+ab).$$

It follows that $\int_0^\infty \frac{\log(c^2+x^2)}{b^2+x^2}\,dx = \int_0^\infty \frac{\log c^2}{b^2+x^2}\,dx + \int_0^\infty \frac{\log\left(1+\frac{x^2}{c^2}\right)}{b^2+x^2}\,dx$

$$=\frac{\log c^2}{b}\cdot\frac{\pi}{2} + \frac{\pi}{b}\log\left(1+\frac{b}{c}\right) = \frac{\pi}{b}\log(c+b),$$

(b, c each $+^{\text{ve}}$).

And writing $x = b\tan\theta$,

$\int_0^{\frac{\pi}{2}} \log(c^2+b^2\tan^2\theta)\,d\theta = \pi\log(b+c)$; and adding $\int_0^{\frac{\pi}{2}} \log\cos^2\theta\,d\theta = \pi\log\frac{1}{2}$,

$$\int_0^{\frac{\pi}{2}} \log(b^2\sin^2\theta + c^2\cos^2\theta)\,d\theta = \pi\log\frac{b+c}{2}, \quad (b,\ c\ +^{\text{ve}}).$$

1045. Again, taking the expression for $\frac{\tan x}{x}$ in partial fractions (logarithmic differential of $\cos x$ expressed in factors), viz.

$$\frac{\tan x}{x} = \sum_1^\infty \frac{2\cdot 2^2}{(2r-1)^2\pi^2 - 2^2x^2},$$

put $x = \pi kz\iota$; then

$$\frac{\pi\tanh\pi kz}{kz} = \sum_1^\infty \frac{2\cdot 2^2}{(2r-1)^2 + 2^2k^2z^2},$$

and $$\frac{\pi}{k}\int_0^\infty \frac{\tanh\pi kz}{(a^2+z^2)}\frac{dz}{z} = \sum_1^\infty \int_0^\infty \frac{2\,dz}{k^2(a^2+z^2)\left\{\left(\frac{2r-1}{2k}\right)^2+z^2\right\}}$$

$$=\sum_1^\infty \frac{2}{k^2}\,\frac{\pi}{2a\cdot\frac{2r-1}{2k}\left(a+\frac{2r-1}{2k}\right)},$$

and $$\int_0^\infty \frac{\tanh\pi kz}{(a^2+z^2)}\frac{dz}{z} = \frac{4k}{a}\sum_1^\infty \frac{1}{(2r-1)(2ka+2r-1)}.$$

Thus, in the case $a=k=1$,

$$\int_0^\infty \frac{\tanh \pi z}{(1+z^2)}\frac{dz}{z}=4\left[\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\dots \textit{ad. inf.}\right]$$

$$=2\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\dots\right]=2,$$

or taking $a=1$ and k any positive integer,

$$\int_0^\infty \frac{\tanh k\pi z}{(1+z^2)}\frac{dz}{z}=4k\left[\frac{1}{1.(2k+1)}+\frac{1}{3(2k+3)}+\frac{1}{5(2k+5)}+\dots\right]$$

$$=2\left[\left(\frac{1}{1}-\frac{1}{2k+1}\right)+\left(\frac{1}{3}-\frac{1}{2k+3}\right)+\dots\left(\frac{1}{2k+1}-\frac{1}{4k+1}\right)+\dots\right]$$

$$=2\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2k-1}\right),$$

and if a, k be any two positive integers, the series will terminate as in the last case.

$$\int_0^\infty \frac{\tanh k\pi z}{(a^2+z^2)}\frac{dz}{z}=\frac{4k}{a}\frac{1}{2ka}\Sigma\left(\frac{1}{2r-1}-\frac{1}{2ka+2r-1}\right)$$

$$=\frac{2}{a^2}\left[\left(\frac{1}{1}-\frac{1}{2ka+1}\right)+\left(\frac{1}{3}-\frac{1}{2ka+3}\right)+\dots+\left(\frac{1}{2ka+1}-\frac{1}{4ka+1}\right)+\dots\right]$$

$$=\frac{2}{a^2}\left[\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2ka-1}\right].$$

If $k=\frac{1}{2}$ and a an even number, the series will also terminate.

Thus $$\int_0^\infty \frac{\tanh\frac{\pi z}{2}}{(a^2+z^2)}\frac{dz}{z}=\frac{2}{a^2}\Sigma\left(\frac{1}{2r-1}-\frac{1}{a+2r-1}\right).$$

If $a=2n$, this becomes

$$\int_0^\infty \frac{\tanh\frac{\pi z}{2}}{\{(2n)^2+z^2\}}\frac{dz}{z}=\frac{2}{4n^2}\left[\left(\frac{1}{1}-\frac{1}{2n+1}\right)+\left(\frac{1}{3}-\frac{1}{2n+3}\right)+\dots\right]$$

$$=\frac{1}{2n^2}\left(\frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{2n-1}\right).$$

But if a be odd, $=2n+1$, the series does not terminate.

$$\int_0^\infty \frac{\tanh\frac{\pi z}{2}}{\{(2n+1)^2+z^2\}}\frac{dz}{z}=\frac{2}{(2n+1)^2}\left\{\left(\frac{1}{1}-\frac{1}{2n+2}\right)+\left(\frac{1}{3}-\frac{1}{2n+4}\right)+\dots\right\}$$

$$=\frac{2}{(2n+1)^2}\left[\log 2+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2n}\right]$$

$$=\frac{1}{(2n+1)^2}\left[\log 4+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right].$$

Similarly if $2k$ be any odd number $=2p+1$, *i.e.* $k=\frac{2p+1}{2}$,

$$\int_0^\infty \frac{\tanh\frac{2p+1}{2}\pi z}{(a^2+z^2)}\frac{dz}{z}=\frac{2}{a^2}\Sigma\left(\frac{1}{2r-1}-\frac{1}{(2p+1)a+2r-1}\right),$$

and this will terminate, or will not terminate, according as a is even or odd.

If a be even, $=2n$, the result is

$$=\frac{1}{2n^2}\left[\left(\frac{1}{1}-\frac{1}{2n(2p+1)+1}\right)+\left(\frac{1}{3}-\frac{1}{2n(2p+1)+3}\right)+\dots\right]$$

$$=\frac{1}{2n^2}\left\{\frac{1}{1}+\frac{1}{3}+\dots+\frac{1}{2n(2p+1)-1}\right\}.$$

If a be odd, $=2n+1$, the result is

$$=\frac{2}{(2n+1)^2}\left[\log 2+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots+\frac{1}{(2n+1)(2p+1)-1}\right].$$

1046. Let $$I\equiv\int_0^\infty e^{-c^2\left(x^2+\frac{a^2}{x^2}\right)}\,dx. \quad (a+^{\text{ve}}).$$

[Laplace, *Mém. de l'Inst.*, 1820, for the case $c=1$.]

The integrand is finite for the whole range of integration. Change a to $a+\delta a$.

Then $$I+\delta I=\int_0^\infty e^{-c^2\left\{x^2+\frac{(a+\delta a)^2}{x^2}\right\}}\,dx.$$

Hence $$\frac{\delta I}{\delta a}=\int_0^\infty e^{-c^2x^2}\left\{e^{-\frac{a^2c^2}{x^2}}\left(-\frac{2c^2a}{x^2}\right)+\epsilon\right\}dx,$$

where ϵ becomes infinitesimally small and ultimately vanishes when δa is indefinitely diminished.

$$\therefore\ \frac{\delta I}{\delta a}=-2c^2a\int_0^\infty\frac{1}{x^2}e^{-c^2\left(x^2+\frac{a^2}{x^2}\right)}\,dx+\int_0^\infty\epsilon\,.\,e^{-c^2x^2}\,dx.$$

Let ϵ_1 be the greatest numerical value of ϵ in the range of x.

Then the second term is $<\epsilon_1\int_0^\infty e^{-c^2x^2}\,dx$; *i.e.* $<\epsilon_1\,.\,\frac{\sqrt{\pi}}{2c}$, and ultimately vanishes with δa.

Hence the process of differentiation with regard to a under the integration sign with an infinite limit is justifiable.

In the first put $x=a/y$.

Then

$$\frac{dI}{da}=2c^2\int_\infty^0 e^{-c^2\left(\frac{a^2}{y^2}+y^2\right)}\,dy=-2c^2I;\quad\therefore\ I=Ae^{-2c^2a},$$

where A is independent of a.

But when $a=0$,

$$I_{a=0}=\int_0^\infty e^{-c^2x^2}\,dx=\frac{\sqrt{\pi}}{2c};\quad\therefore\ A=\frac{\sqrt{\pi}}{2c},\ c\text{ being supposed }+^{\text{ve}}.$$

Hence

$$I \equiv \int_0^\infty e^{-c^2\left(x^2+\frac{a^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2c}e^{-2c^2a}\left(\text{or } -\frac{\sqrt{\pi}}{2c}e^{-2c^2a} \text{ if } c \text{ be } -^{\text{ve}}\right).$$

Laplace's form, viz. the case $c=1$, gives

$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}dx = \frac{\sqrt{\pi}}{2}e^{-2a}, \quad (a+^{\text{ve}}).$$

If we replace a^2 by b^2a^2 and c^2 by $\dfrac{k}{a^2}$, we have the form

$$\int_0^\infty e^{-k\left(\frac{x^2}{a^2}+\frac{b^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}\frac{a}{\sqrt{k}}e^{-2k\frac{b}{a}}, \quad \text{.................(1)}$$

where a, b, k are positive.

This result may be written

$$\int_0^\infty e^{-k\left(\frac{x}{a}-\frac{b}{x}\right)^2}dx = \frac{a}{2}\sqrt{\frac{\pi}{k}}. \quad \text{.....................(2)}$$

1047. Cor. 1. If $k=1$ and $a=b$, we have

$$I_1 \equiv \int_0^\infty e^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}ae^{-2}. \quad \text{.....................(3)}$$

Cor. 2. If we differentiate I_1 with respect to a, we have

$$\frac{dI_1}{da} \equiv \int_0^\infty \left(\frac{2x^2}{a^3}-\frac{2a}{x^2}\right)e^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \frac{\sqrt{\pi}}{2}e^{-2},$$

i.e.

$$\int_0^\infty \left(\frac{x^2}{a^2}-\frac{a^2}{x^2}\right)e^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \tfrac{1}{4}\sqrt{\pi}ae^{-2}. \quad \text{..................(4)}$$

Differentiating (1) with regard to k, and then putting $k=1$ and $a=b$,

$$\int_0^\infty \left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)e^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \tfrac{5}{4}\sqrt{\pi}ae^{-2}. \quad \text{..............(5)}$$

(4) and (5) give

$$\int_0^\infty \frac{x^2}{a^2}e^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \tfrac{3}{4}\sqrt{\pi}ae^{-2}, \ldots(6) \qquad \int_0^\infty \frac{a^2}{x^2}e^{-\left(\frac{x^2}{a^2}+\frac{a^2}{x^2}\right)}dx = \tfrac{1}{2}\sqrt{\pi}ae^{-2}, \ldots(7)$$

Cor. 3. We also have

$$\int_0^\infty e^{-k\frac{x^2}{a^2}}\left(e^{-k\frac{b_1^2}{x^2}}-e^{-k\frac{b_2^2}{x^2}}\right)dx = \frac{\sqrt{\pi}a}{2\sqrt{k}}\left(e^{-2k\frac{b_1}{a}}-e^{-2k\frac{b_2}{a}}\right),$$

and making a indefinitely large,

$$\int_0^\infty \left(e^{-k\frac{b_1^2}{x^2}}-e^{-k\frac{b_2^2}{x^2}}\right)dx = \frac{\sqrt{\pi}}{2\sqrt{k}}.\,2k(b_2-b_1) = \sqrt{\pi k}(b_2-b_1). \quad \text{.......(8)}$$

1048. Let $I=\int_0^\infty \frac{\cos rx}{a^2+x^2}\,dx$ (a positive).

We have $\int_0^\infty 2ze^{-(a^2+x^2)z^2}\,dz=\frac{1}{a^2+x^2}$.

Then

$$I=\int_0^\infty\int_0^\infty \cos rx\, 2ze^{-(a^2+x^2)z^2}\,dx\,dz$$

$$=\int_0^\infty 2ze^{-a^2z^2}\left(\int_0^\infty e^{-x^2z^2}\cos rx\,dx\right)dz$$

$$=\int_0^\infty 2ze^{-a^2z^2}\left(\frac{\sqrt{\pi}}{2z}e^{-\frac{r^2}{4z^2}}\right)dz=\sqrt{\pi}\int_0^\infty e^{-\left(a^2z^2+\frac{r^2}{4z^2}\right)}dz$$

$$=\sqrt{\pi}\,.\,\frac{\sqrt{\pi}}{2a}e^{-ar} \text{ or } \sqrt{\pi}\frac{\sqrt{\pi}}{2a}e^{+ar}, \text{ as } r \text{ is positive or negative.}$$

$$\therefore\ I=\int_0^\infty \frac{\cos rx}{a^2+x^2}dx=\frac{\pi}{2a}e^{-ar} \text{ or } \frac{\pi}{2a}e^{ar}, \text{ as } r \text{ is positive or negative.}$$

This integral is more commonly written as

$$\int_0^\infty \frac{\cos rx}{1+x^2}\,dx=\frac{\pi}{2}e^{-r} \text{ or } \frac{\pi}{2}e^{r}, \text{ as } r \text{ is positive or negative.}$$

This result is due to Laplace (*Bulletin de la Soc. Phil.* 1811).

1049. Both results may be expressed in one as

$$\int_0^\infty \frac{\cos rx}{1+x^2}\,dx=\frac{\pi}{2}\left\{\frac{e^r}{1+0^{-r}}+\frac{e^{-r}}{1+0^r}\right\},$$

for 0^r is zero or infinite according as r is positive or negative.

This form was given in Crelle's *Journal*, vol. x., and is due to Libri. (See Gregory's *Examples*, p. 486.)

1050. Differentiating with regard to r, we obtain the integral ($a+^{\text{ve}}$)

$$\int_0^\infty \frac{x\sin rx}{a^2+x^2}dx=\frac{\pi}{2}e^{-ar} \text{ or } -\frac{\pi}{2}e^{ar}, \text{ as } r \text{ is positive or negative.}$$

This integral vanishes if $r=0$.

The differentiation under the integral sign may be shown to be justifiable, although the upper limit is infinite, in the same manner as in previous cases.

1051. If we integrate with respect to r between limits r_1 and r_2 (both positive),

$$\int_0^\infty \frac{\sin r_2x-\sin r_1x}{x(a^2+x^2)}\,dx=\frac{\pi}{2a^2}(e^{-ar_1}-e^{-ar_2}).$$

If $r_1=0$, we have

$$\int_0^\infty \frac{\sin rx}{x(a^2+x^2)}\,dx=\frac{\pi}{2a^2}(1-e^{-ar}),$$

a result given by Laplace (*Mémoires de l'Académie*, 1782).

If we write $x=\tan\theta$ in the integral

$$\int_0^\infty \frac{\cos rx}{1+x^2}\,dx=\frac{\pi}{2}e^{-r} \text{ or } \frac{\pi}{2}e^{r},$$

we have
$$\int_0^{\frac{\pi}{2}} \cos(r\tan\theta)\,d\theta=\frac{\pi}{2}e^{-r} \text{ or } \frac{\pi}{2}e^{r},$$

according as r is positive or negative.

1052. **Graphical Illustrations.**

Graph of
$$y=\frac{2}{\pi}\int_0^\infty \frac{\cos x\theta}{1+\theta^2}\,d\theta.$$

We have $y=e^{-x}$ or $y=e^{x}$, according as x is positive or negative, the y-axis being an axis of symmetry.

The logarithmic curve is traced in *Diff. Calc.*, Art. 442.

The graph now required consists of the two portions of the above curves which run asymptotically to the x-axis from their point of intersection upon the y-axis (Fig. 332).

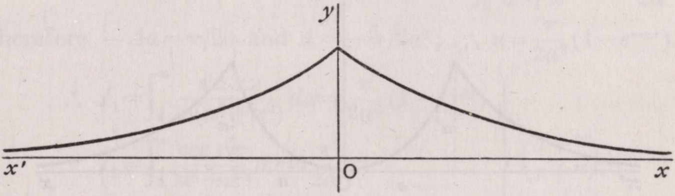

Fig. 332.

1053. Graph of
$$y=\frac{2}{\pi}\int_0^\infty \frac{\cos x\theta\cos a\theta}{1+\theta^2}\,d\theta.$$

The y-axis is again an axis of symmetry,

$$y=\frac{1}{\pi}\int_0^\infty \frac{\cos(x+a)\theta}{1+\theta^2}\,d\theta+\frac{1}{\pi}\int_0^\infty \frac{\cos(x-a)\theta}{1+\theta^2}\,d\theta.$$

If a be regarded as a positive constant and $x>a$, we have

$$y=\frac{1}{\pi}\left[\frac{\pi}{2}e^{-(x+a)}+\frac{\pi}{2}e^{-(x-a)}\right]=\cosh a\,.\,e^{-x}.$$

If $a>x>0$, we have

$$y=\frac{1}{\pi}\left[\frac{\pi}{2}e^{-(x+a)}+\frac{\pi}{2}e^{(x-a)}\right]=\cosh x\,.\,e^{-a}.$$

The graph therefore consists of a portion of a catenary from $x=0$ to $x=a$

and a portion of the logarithmic curve from $x=a$ to $x=\infty$, with the image with regard to the y-axis of these portions (Fig. 333).

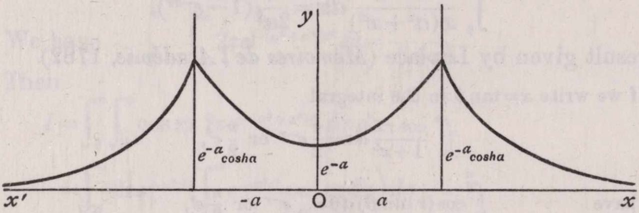

Fig. 333.

1054. Graph of $$\frac{\pi y}{2a}=\int_0^\infty \frac{\cos\left(\theta\log\dfrac{a^2}{x^2}\right)}{1+\theta^2}\,d\theta.$$

Here, if $x<a$, $$\frac{\pi y}{2a}=\frac{\pi}{2}e^{-\log\frac{a^2}{x^2}}=\frac{\pi}{2}e^{\log\frac{x^2}{a^2}}=\frac{\pi}{2}\frac{x^2}{a^2},$$

i.e. $x^2=ay$, a parabola ;

if $x>a$, $$\frac{\pi y}{2a}=\frac{\pi}{2}e^{\log\frac{a^2}{x^2}}=\frac{\pi}{2}\frac{a^2}{x^2},$$
$$x^2y=a^3,$$

and the y-axis is obviously an axis of symmetry (Fig. 334).

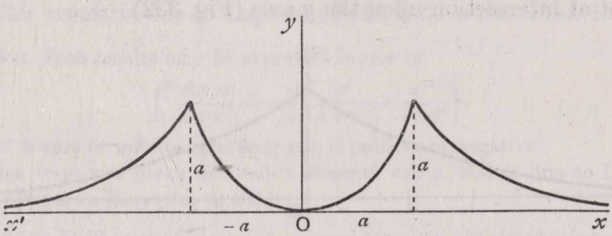

Fig. 334.

1055. Graph of $$\frac{\pi y}{2a}=\int_0^\infty \frac{\cos\left(\theta\log\sin^2\dfrac{x}{a}\right)}{1+\theta^2}\,d\theta.$$

$\log\sin^2\dfrac{x}{a}$ is negative. Hence

$$\frac{\pi y}{2a}=\frac{\pi}{2}e^{\log\sin^2\frac{x}{a}}=\frac{\pi}{2}\sin^2\frac{x}{a} \quad\text{and}\quad y=a\sin^2\frac{x}{a} \text{ (Fig. 335).}$$

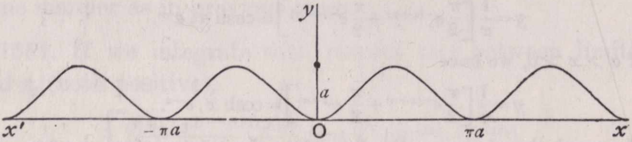

Fig. 335.

1056. Another mode of discussing the integrals of Arts. 1048 to 1051 is as follows:

Let $$u=\int_0^\infty \frac{\sin rx}{x(a^2+x^2)}\,dx, \quad (a \text{ positive}).$$

Then $$\frac{du}{dr}=\int_0^\infty \frac{\cos rx}{a^2+x^2}\,dx, \quad \frac{d^2u}{dr^2}=-\int_0^\infty \frac{x\sin rx}{a^2+x^2}\,dx;$$

$$\therefore \frac{d^2u}{dr^2}-a^2u=-\int_0^\infty\left(x+\frac{a^2}{x}\right)\frac{\sin rx}{a^2+x^2}\,dx=-\int_0^\infty \frac{\sin rx}{x}\,dx$$

$$=-\pi/2,\ 0 \text{ or } +\pi/2, \text{ as } r \text{ is } +^{\text{ve}}, \text{ zero or } -^{\text{ve}};$$

$\therefore u=\pi/2a^2+Ae^{-ar}+Be^{ar}$ for any positive value of r

(*I.C. for Beginners*, p. 250),

where A and B are constants as regards r.

But u is finite when r is infinite; $\therefore B=0$. Also there is obviously no discontinuity in the value of $\frac{du}{dr}$, which is also finite for all values of r, as r diminishes through the value zero and becomes negative; for a small negative value of r gives the same value to $\int_0^\infty \frac{\cos rx}{a^2+x^2}\,dx$ as an equal small positive value, and when r is zero the value is $\int_0^\infty \frac{dx}{a^2+x^2}$, *i.e.* $\frac{\pi}{2a}$.

Therefore $-Aa=\pi/2a$ and $A=-\pi/2a^2$; $\therefore u=\frac{\pi}{2a^2}(1-e^{-ar})$.

$$\left.\begin{aligned} \therefore I_1&=\int_0^\infty \frac{\sin rx}{x(a^2+x^2)}\,dx=\frac{\pi}{2a^2}(1-e^{-ar})\\ I_2&=\int_0^\infty \frac{\cos rx}{a^2+x^2}\,dx=\frac{\pi}{2a}e^{-ar}\\ I_3&=\int_0^\infty \frac{x\sin rx}{a^2+x^2}\,dx=\frac{\pi}{2}e^{-ar}\end{aligned}\right\}\begin{pmatrix}a+^{\text{ve}}\\ r+^{\text{ve}}\end{pmatrix}.$$

The collected results are for the various signs of a and r:

	$a+$ $r+$	$a+$ $r-$	$a-$ $r+$	$a-$ $r-$
I_1	$\frac{\pi}{2a^2}(1-e^{-ar})$	$-\frac{\pi}{2a^2}(1-e^{ar})$	$\frac{\pi}{2a^2}(1-e^{ar})$	$-\frac{\pi}{2a^2}(1-e^{-ar})$
I_2	$\frac{\pi}{2a}e^{-ar}$	$\frac{\pi}{2a}e^{ar}$	$-\frac{\pi}{2a}e^{ar}$	$-\frac{\pi}{2a}e^{-ar}$
I_3	$\frac{\pi}{2}e^{-ar}$	$-\frac{\pi}{2}e^{ar}$	$\frac{\pi}{2}e^{ar}$	$-\frac{\pi}{2}e^{-ar}$

1057. A Reduction Formula.

Let $I_n=\int_0^\infty \frac{\cos rx}{(a^2+x^2)^n}dx.$ Then $I_1=\frac{\pi}{2}a^{-1}e^{-ra}$,

and $$\frac{dI_n}{da}=-2na\int_0^\infty \frac{\cos rx}{(a^2+x^2)^{n+1}}dx=-2naI_{n+1}.$$

Therefore the successive integrals for the cases $n=2$, $n=3$, etc., may be calculated by the rule $I_{n+1}=-\frac{1}{2na}\frac{dI_n}{da}$.

In each case $\frac{\pi}{2}e^{-ra}$ will appear as a factor. Let $I_n=\frac{\pi}{2}A_n e^{-ra}$

Then $\frac{dI_n}{da}=\frac{\pi}{2}\left(\frac{dA_n}{da}-rA_n\right)e^{-ra}$ and $I_{n+1}=\frac{\pi}{2}A_{n+1}e^{-ra}$.

Hence the form of A_n may be calculated by successive applications of the formula

$$A_{n+1}=\frac{1}{2n}\left[r\frac{A_n}{a}-\frac{1}{a}\frac{dA_n}{da}\right], \text{ where } A_1=a^{-1}.$$

Thus $$A_2=\frac{1}{2}\frac{1}{1\,!}[ra^{-2}+a^{-3}],$$

$$A_3=\frac{1}{2^2}\frac{1}{2\,!}[r^2a^{-3}+3ra^{-4}+3a^{-5}],$$

$$A_4=\frac{1}{2^3}\frac{1}{3\,!}[r^3a^{-4}+6r^2a^{-5}+15ra^{-6}+15a^{-7}], \text{ and so on.}$$

So that if

$$A_n=\frac{1}{2^{n-1}}\frac{1}{(n-1)\,!}[K_1r^{n-1}a^{-n}+K_2r^{n-2}a^{-(n+1)}+K_3r^{n-3}a^{-(n+2)}+\dots \text{ to } n \text{ terms}],$$

$$A_{n+1}=\frac{1}{2^n}\frac{1}{n\,!}[K_1r^na^{-(n+1)}+K_2r^{n-1}a^{-(n+2)}+K_3r^{n-2}a^{-(n+3)}+\dots$$
$$+nK_1r^{n-1}a^{-(n+2)}+(n+1)K_2r^{n-2}a^{-(n+3)}+\dots],$$

and the coefficients in A_{n+1} are

$K_1\ (=1)$, K_2+nK_1, $K_3+(n+1)K_2$, $K_4+(n+2)K_3$, etc. ..., $(2n-1)K_n$,

and the law of formation of the successive sets of coefficients is easy.

It may be shown by induction that the general formula is

$$A_n=\frac{1}{2^{n-1}(n-1)\,!}\Big[r^{n-1}a^{-n}+\frac{n(n-1)}{2}r^{n-2}a^{-(n+1)}$$
$$+\frac{(n+1)n(n-1)(n-2)}{2\,.\,4}r^{n-3}a^{-(n+2)}$$
$$+\frac{(n+2)(n+1)n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,6}r^{n-4}a^{-(n+3)}+\dots\Big].$$

Thus $\int_0^\infty \frac{\cos rx}{(a^2+x^2)^n}dx = \frac{\pi}{2^n}\frac{e^{-ar}}{(n-1)!}\Big[r^{n-1}a^{-n}+\frac{n(n-1)}{2}r^{n-2}a^{-(n+1)}$

$$+\frac{(n+1)n(n-1)(n-2)}{2\,.\,4}r^{n-3}a^{-(n+2)}+\dots \text{ to } n \text{ terms}\Big].$$

In the same way $\int_0^\infty \frac{x\sin rx}{(a^2+x^2)^n}dx = \frac{\pi}{4}\frac{e^{-ra}}{(n-1)}rA_{n-1}$,

or we may deduce the result from the former by differentiation with regard to r.

1058. Consider the Integral $I=\int_0^\infty \frac{\sin rx\,dx}{x(x^4+2a^2x^2\cos 2\alpha+a^4)}$.

We have

$$\frac{dI}{dr}=\int_0^\infty \frac{\cos rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4};\quad \frac{d^2I}{dr^2}=\int_0^\infty \frac{-x\sin rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4};$$

$$\frac{d^3I}{dr^3}=\int_0^\infty \frac{-x^2\cos rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4};\quad \frac{d^4I}{dr^4}=\int_0^\infty \frac{x^3\sin rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4}.$$

Hence, when the first of these integrals has been found, the other four of this particular class follow by differentiation. Adding the fifth to $(-2a^2\cos 2\alpha)$ times the third and a^4 times the first, we have

$$\frac{d^4I}{dr^4}-2a^2\cos 2\alpha\frac{d^2I}{dr^2}+a^4I=\int_0^\infty \frac{\sin rx}{x}dx=\frac{\pi}{2},\ 0\ \text{ or }\ -\frac{\pi}{2},$$

according as r is positive, zero or negative. We shall assume r positive, for the case r negative will be at once deducible from our result by changing the sign of r. We also take a positive and α an acute angle.

The differential equation is of the ordinary class with linear coefficients (*I.C. for Beginners*, pages 244 to 263). It may be written

$$[\{D^2-a^2\cos 2\alpha\}^2+a^4\sin^2 2\alpha]\,I=\frac{\pi}{2},$$

and the general solution is

$$I=\frac{\pi}{2a^4}+e^{-ar\cos\alpha}\{A_1\cos(ar\sin\alpha)+A_2\sin(ar\sin\alpha)\}$$
$$+e^{ar\cos\alpha}\{A_3\cos(ar\sin\alpha)+A_4\sin(ar\sin\alpha)\}.$$

Since an infinite value of r does not make I infinite, the last two terms must vanish, *i.e.* $A_3=A_4=0$. And when r is diminished indefinitely to zero, I should vanish. Therefore we have $A_1=-\frac{\pi}{2a^4}$.

To determine the remaining constant A_2, we may differentiate with regard to r; we obtain

$$\frac{dI}{dr}=-a\cos\alpha e^{-ar\cos\alpha}\{A_1\cos(ar\sin\alpha)+A_2\sin(ar\sin\alpha)\}$$
$$-a\sin\alpha e^{-ar\cos\alpha}\{A_1\sin(ar\sin\alpha)-A_2\cos(ar\sin\alpha)\},$$

and when r is diminished indefinitely to zero this becomes in the limit

$$\frac{dI}{dr}=-a\cos\alpha\,.\,A_1+a\sin\alpha\,.\,A_2.$$

But when r is diminished indefinitely to zero, we ultimately have

$$\frac{dI}{dr}=\int_0^\infty \frac{dx}{x^4+2a^2x^2\cos 2\alpha+a^4}=\frac{\pi}{4a^3\cos\alpha} \quad \text{(see p. 159, Vol. I.)};$$

$$\therefore\ a\sin\alpha\,.\,A_2-a\cos\alpha\,.\,A_1=\frac{\pi}{4a^3\cos\alpha},$$

i.e.

$$a\sin\alpha\,.\,A_2=-\frac{\pi}{2a^3}\cos\alpha+\frac{\pi}{4a^3\cos\alpha}=-\frac{\pi}{4a^3}\frac{\cos 2\alpha}{\cos\alpha},$$

and

$$A_2=-\frac{\pi}{2a^4}\cot 2\alpha.$$

Hence

$$I=\frac{\pi}{2a^4}[1-e^{-ar\cos\alpha}\{\cos(ar\sin\alpha)+\cot 2\alpha\sin(ar\sin\alpha)\}]$$

$$=\frac{\pi}{2a^4}\left\{1-e^{-ar\cos\alpha}\frac{\sin(ar\sin\alpha+2\alpha)}{\sin 2\alpha}\right\},$$

i.e. we have for values of $r>0$

$$\int_0^\infty \frac{\sin rx\,dx}{x(x^4+2a^2x^2\cos 2\alpha+a^4)}=\frac{\pi}{2a^4}\left\{1-e^{-ar\cos\alpha}\frac{\sin(ar\sin\alpha+2\alpha)}{\sin 2\alpha}\right\},$$

$$\int_0^\infty \frac{\cos rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4}=\frac{\pi}{2a^3}e^{-ar\cos\alpha}\frac{\sin(\alpha+ar\sin\alpha)}{\sin 2\alpha},$$

$$\int_0^\infty \frac{x\sin rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4}=\frac{\pi}{2a^2}e^{-ar\cos\alpha}\frac{\sin(ar\sin\alpha)}{\sin 2\alpha},$$

$$\int_0^\infty \frac{x^2\cos rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4}=\frac{\pi}{2a}e^{-ar\cos\alpha}\frac{\sin(\alpha-ar\sin\alpha)}{\sin 2\alpha},$$

$$\int_0^\infty \frac{x^3\sin rx\,dx}{x^4+2a^2x^2\cos 2\alpha+a^4}=\frac{\pi}{2}e^{-ar\cos\alpha}\frac{\sin(2\alpha-ar\sin\alpha)}{\sin 2\alpha}.$$

1059. Taking for instance the case when $\alpha=\frac{\pi}{4}$, $a=c\sqrt{2}$, so that $a\sin\alpha=c$,

$$\int_0^\infty \frac{\sin rx\,dx}{x(x^4+4c^4)}=\frac{\pi}{8c^4}\left\{1-e^{-rc}\sin\left(rc+\frac{\pi}{2}\right)\right\}=\frac{\pi}{8c^4}(1-e^{-rc}\cos rc),$$

$$\int_0^\infty \frac{\cos rx\,dx}{x^4+4c^4}=-\frac{\pi}{4c^3\sqrt{2}}e^{-rc}\sin\left(rc+\frac{5\pi}{4}\right)=\frac{\pi}{8c^3}e^{-rc}(\sin rc+\cos rc),$$

$$\int_0^\infty \frac{x\sin rx\,dx}{x^4+4c^4}=\frac{\pi}{4c^2}e^{-rc}\sin\left(rc+\frac{8\pi}{4}\right)=\frac{\pi}{4c^2}e^{-rc}\sin rc.$$

$$\int_0^\infty \frac{x^2\cos rx\,dx}{x^4+4c^4}=\frac{\pi}{2c\sqrt{2}}e^{-rc}\sin\left(rc+\frac{11\pi}{4}\right)=\frac{\pi}{4c}e^{-rc}(\cos rc-\sin rc),$$

$$\int_0^\infty \frac{x^3\sin rx\,dx}{x^4+4c^4}=-\frac{\pi}{2}e^{-rc}\sin\left(rc+\frac{14\pi}{4}\right)=\frac{\pi}{2}e^{-rc}\cos rc.$$

1060. **Consider** $I=\displaystyle\int_0^\infty \frac{\sin rx}{x(x^6+a^6)}dx,\quad \left.\begin{array}{l} r \text{ positive,} \\ a \text{ positive.}\end{array}\right\}$

We have

$$\frac{dI}{dr}=\int_0^\infty \frac{\cos rx}{x^6+a^6}dx;\qquad \frac{d^2I}{dr^2}=-\int_0^\infty \frac{x\sin rx}{x^6+a^6}dx;$$

$$\frac{d^3I}{dr^3}=-\int_0^\infty \frac{x^2\cos rx}{x^6+a^6}dx;\qquad \frac{d^4I}{dr^4}=\int_0^\infty \frac{x^3\sin rx}{x^6+a^6}dx;$$

$$\frac{d^5I}{dr^5}=\int_0^\infty \frac{x^4\cos rx}{x^6+a^6}\,dx\,;\qquad \frac{d^6I}{dr^6}=-\int_0^\infty \frac{x^5\sin rx}{x^6+a^6}\,dx\,;$$

$$\therefore\ \frac{d^6I}{dr^6}-a^6I=-\int_0^\infty\left(x^5+\frac{a^6}{x}\right)\frac{\sin rx}{x^6+a^6}\,dx=-\int_0^\infty\frac{\sin rx}{x}\,dx=-\frac{\pi}{2}.$$

Solving this equation,

$$I=\frac{\pi}{2a^6}+A_1e^{-ar}+A_2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+A_3\right)+B_1e^{ar}+B_2e^{\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+B_3\right).$$

Now, since the integral obviously remains finite when r becomes infinite, the terms with positive indices in their exponential factors must disappear.

Hence $B_1=0$ and $B_2=0$, and the form of the integral reduces to

$$I=\frac{\pi}{2a^6}+A_1e^{-ar}+A_2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+A_3\right).$$

Now I, $\dfrac{d^2I}{dr^2}$, $\dfrac{d^4I}{dr^4}$ ultimately vanish with r.

These considerations will determine A_1, A_2, A_3.

Now $\dfrac{d^nI}{dr^n}=A_1(-a)^ne^{-ar}+A_2a^ne^{-\frac{ar}{2}}\cos\left(\dfrac{ar\sqrt{3}}{2}+A_3+n\dfrac{2\pi}{3}\right)$;

we therefore have

$$\left.\begin{aligned}0&=\frac{\pi}{2a^6}+A_1\quad+A_2\cos A_3,\\ 0&=\qquad+A_1a^2+A_2a^2\cos\left(A_3+\frac{4\pi}{3}\right),\\ 0&=\qquad+A_1a^4+A_2a^4\cos\left(A_3+\frac{8\pi}{3}\right),\end{aligned}\right\}\ \text{whence } A_3=0,\ A_2=2A_1=-\frac{\pi}{3a^6}.$$

Hence, for values of $r>0$,

$$\int_0^\infty\frac{\sin rx}{x(x^6+a^6)}\,dx=\frac{\pi}{6a^6}\left[3-e^{-ar}-2e^{-\frac{ar}{2}}\cos\frac{ar\sqrt{3}}{2}\right],$$

$$\int_0^\infty\frac{\cos rx}{x^6+a^6}\,dx=\frac{\pi}{6a^5}\left[\ e^{-ar}-2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+\frac{2\pi}{3}\right)\right],$$

$$\int_0^\infty\frac{x\sin rx}{x^6+a^6}\,dx=\frac{\pi}{6a^4}\left[\ e^{-ar}+2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+\frac{4\pi}{3}\right)\right],$$

$$\int_0^\infty\frac{x^2\cos rx}{x^6+a^6}\,dx=\frac{\pi}{6a^3}\left[-e^{-ar}+2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+\frac{6\pi}{3}\right)\right],$$

$$\int_0^\infty\frac{x^3\sin rx}{x^6+a^6}\,dx=\frac{\pi}{6a^2}\left[-e^{-ar}-2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+\frac{8\pi}{3}\right)\right],$$

$$\int_0^\infty\frac{x^4\cos rx}{x^6+a^6}\,dx=\frac{\pi}{6a}\left[\ e^{-ar}-2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+\frac{10\pi}{3}\right)\right],$$

$$\int_0^\infty\frac{x^5\sin rx}{x^6+a^6}\,dx=\frac{\pi}{6}\left[\ e^{-ar}+2e^{-\frac{ar}{2}}\cos\left(\frac{ar\sqrt{3}}{2}+\frac{12\pi}{3}\right)\right],$$

some of which admit of a little simplification, but are left in their present form as exhibiting the general law followed by the several members of the group.

1061. The same process may evidently be extended to any integral of the class
$$\int_0^\infty \frac{\sin rx\,dx}{x(x^{4n}+2a^{2n}x^{2n}\cos n\alpha+a^{4n})},$$
and its family of $2n$ other integrals may be obtained by differentiating $2n$ times with regard to r. But we exhibit another method of procedure in Art. 1067, which avoids the labour of determination of the various constants.

1062. We have seen that
$$\int_0^\infty \frac{\cos rx\,dx}{x^2+a^2}=\frac{\pi}{2a}e^{-ar} \quad\text{or}\quad \frac{\pi}{2a}e^{ar},$$
according as r is positive or negative, a being supposed positive.

If a be negative, since the integrand is unaltered, the result will be $-\frac{\pi}{2a}e^{ar}$ or $-\frac{\pi}{2a}e^{-ar}$, according as r is positive or negative (see Art. 1056). The result must be positive in either case, and the index of the exponential must be negative, for the integral does not become infinite when r becomes infinite.

The four results are therefore
$$\frac{\pi}{2a}e^{-ar},\ \binom{a+^{\text{ve}}}{r+^{\text{ve}}};\qquad \frac{\pi}{2a}e^{ar},\ \binom{a+^{\text{ve}}}{r-^{\text{ve}}};$$
$$-\frac{\pi}{2a}e^{ar},\ \binom{a-^{\text{ve}}}{r+^{\text{ve}}};\qquad -\frac{\pi}{2a}e^{-ar},\ \binom{a-^{\text{ve}}}{r-^{\text{ve}}}.$$

Taking the case a and r both positive, it is clear that the integrand is not affected by a change of sign of x.

Hence
$$\int_{-\infty}^0 \frac{\cos rx}{x^2+a^2}\,dx=\int_0^\infty \frac{\cos rx}{x^2+a^2}\,dx, \quad\text{and}\quad \int_{-\infty}^\infty \frac{\cos rx}{x^2+a^2}\,dx=\frac{\pi}{a}e^{-ar}, \ldots(1)$$
with the modifications above specified, if a or r or both of them be negative.

Again,
$$\int_{-\infty}^\infty \frac{\sin rx}{x^2+a^2}\,dx=0, \ldots\ldots\ldots\ldots\ldots\ldots(2)$$
for elements of the summation represented by the integral, for which the values of x are equal but of opposite sign, cancel each other.

1063. These facts enable us to calculate

$$I=\int_{-\infty}^{\infty}\frac{\cos rx}{(x-b)^2+a^2}\,dx.$$

For, putting $x=b+z$, $I=\int_{-\infty}^{\infty}\frac{\cos rb\cos rz-\sin rb\sin rz}{z^2+a^2}\,dz$

$$=\cos rb\int_{-\infty}^{\infty}\frac{\cos rz}{z^2+a^2}\,dz-\sin rb\int_{-\infty}^{\infty}\frac{\sin rz}{z^2+a^2}\,dz,$$

i.e. $$\int_{-\infty}^{\infty}\frac{\cos rx}{(x-b)^2+a^2}\,dx=\frac{\pi}{a}e^{-ar}\cos br\ \begin{pmatrix}r>0\\a>0\end{pmatrix}. \quad \text{.......(3)}$$

It will be observed that this is independent of the sign of b, but subject to the same modifications as before with regard to the signs of a and r.

Differentiating (3) with regard to r,

$$\int_{-\infty}^{\infty}\frac{x\sin rx}{(x-b)^2+a^2}\,dx=\frac{\pi}{a}e^{-ar}(a\cos br+b\sin br)\,; \quad \text{......(4)}$$

and integrating (3) with regard to r from $r=0$ to $r=r$,

$$\int_{-\infty}^{\infty}\frac{\sin rx\,dx}{x\{(x-b)^2+a^2\}}=\frac{\pi}{a(a^2+b^2)}\{a-e^{-ar}(a\cos br-b\sin br)\}, \quad \text{...(5)}$$

where each formula is subject to the same modifications as before with regard to the signs of a and r if they be not both $+^{\text{ve}}$.

Putting $b=p\cos\alpha$, $a=p\sin\alpha$, $\alpha<\pi$, p positive, we have the integrals

$$\int_{-\infty}^{\infty}\frac{\sin rx\,dx}{x(x^2-2px\cos\alpha+p^2)}=\frac{\pi}{p^2}+\frac{\pi}{p^2\sin\alpha}e^{-pr\sin\alpha}\sin(pr\cos\alpha-\alpha),$$

$$\int_{-\infty}^{\infty}\frac{\cos rx\,dx}{x^2-2px\cos\alpha+p^2}=\frac{\pi}{p\sin\alpha}e^{-pr\sin\alpha}\cos(pr\cos\alpha),$$

$$\int_{-\infty}^{\infty}\frac{x\sin rx\,dx}{x^2-2px\cos\alpha+p^2}=\frac{\pi}{\sin\alpha}e^{-pr\sin\alpha}\sin(pr\cos\alpha+\alpha),$$

which again can be readily modified as before for the cases in which any of the constants involved have negative values.

1064. Again, differentiating $\int_{-\infty}^{\infty}\frac{\sin rx}{x^2+a^2}\,dx=0$ with regard to r, we have

$$\int_{-\infty}^{\infty}\frac{x\cos rx}{x^2+a^2}\,dx=0\,;$$

and from this we may obtain the value of the integral

$$I_1\equiv\int_{-\infty}^{\infty}\frac{x\cos rx}{(x-b)^2+a^2}\,dx.$$

Putting $x=b+z$, $I_1=\int_{-\infty}^{\infty}\frac{(b+z)\cos r(b+z)}{z^2+a^2}dz$

$$=\int_{-\infty}^{\infty}\frac{b\cos br\cos rz+z\cos br\cos rz-b\sin br\sin rz-z\sin br\sin rz}{z^2+a^2}dz$$

$$=b\cos br\int_{-\infty}^{\infty}\frac{\cos rz}{z^2+a^2}dz-\sin br\int_{-\infty}^{\infty}\frac{z\sin rz}{z^2+a^2}dz,$$

since the other two integrals vanish,

$$=b\cos br\frac{\pi}{a}e^{-ar}-\sin br\,\pi e^{-ar};$$

$$\therefore \int_{-\infty}^{\infty}\frac{x\cos rx\,dx}{(x-b)^2+a^2}=\frac{\pi}{a}e^{-ar}(b\cos br-a\sin br),$$

and

$$\int_{-\infty}^{\infty}\frac{x\cos rx\,dx}{x^2-2px\cos\alpha+p^2}=\frac{\pi}{\sin\alpha}e^{-pr\sin\alpha}\cos(pr\cos\alpha+\alpha),$$

where $b=p\cos\alpha$, $a=p\sin\alpha$, and it is understood that a is positive, p positive, $\sin\alpha$ positive; and the formula can be readily modified as before to meet other cases, and other integrals may be deduced by integration with regard to r.

1065. **The integral** $I=\int_{-\infty}^{\infty}\frac{\sin rx}{(x-b)^2+a^2}dx$ may also be obtained in the same way. Put $x=b+z$.

$$I=\int_{-\infty}^{\infty}\frac{\sin br\cos rz+\cos br\sin rz}{z^2+a^2}dz$$

$$=\sin br\int_{-\infty}^{\infty}\frac{\cos rz}{z^2+a^2}dz+\cos br\int_{-\infty}^{\infty}\frac{\sin rz}{z^2+a^2}dz=\frac{\pi}{a}e^{-ar}\sin br,$$

for the second integral vanishes.

Since
$$\int_{-\infty}^{\infty}\frac{\cos rx}{(x-b)^2+a^2}dx=\frac{\pi}{a}e^{-ar}\cos br;$$

$$\int_{-\infty}^{\infty}\frac{\sin rx}{(x-b)^2+a^2}dx=\frac{\pi}{a}e^{-ar}\sin br;$$

$$\int_{-\infty}^{\infty}\frac{x\cos rx}{(x-b)^2+a^2}dx=\frac{\pi}{a}e^{-ar}(b\cos br-a\sin br);$$

$$\int_{-\infty}^{\infty}\frac{x\sin rx}{(x-b)^2+a^2}dx=\frac{\pi}{a}e^{-ar}(a\cos br+b\sin br),$$

it follows that by differentiating $n-1$ times with respect to a^2, we can obtain the following integrals:

$$\int_{-\infty}^{\infty}\frac{\cos rx}{\{(x-b)^2+a^2\}^n}dx=P\cos br;\quad \int_{-\infty}^{\infty}\frac{\sin rx}{\{(x-b)^2+a^2\}^n}dx=P\sin br;$$

$$\int_{-\infty}^{\infty}\frac{x\cos rx}{\{(x-b)^2+a^2\}^n}dx=Pb\cos br-Q\sin br;$$

$$\int_{-\infty}^{\infty}\frac{x\sin rx}{\{(x-b)^2+a^2\}^n}dx=Q\cos br+Pb\sin br,$$

where

$$P \equiv \frac{(-1)^{n-1}\pi}{(n-1)!}\left(\frac{d}{2a\,da}\right)^{n-1}\left(\frac{e^{-ar}}{a}\right),\quad Q \equiv \frac{(-1)^{n-1}\pi}{(n-1)!}\left(\frac{d}{2a\,da}\right)^{n-1}(e^{-ar}).$$

1066. It follows that if $f(x)$ and $\phi(x)$ be rational integral algebraic functions of x, of which the degree of $f(x)$ in x is lower than that of $\phi(x)$, and if the roots of $\phi(x)=0$ be all unreal, then since $\dfrac{f(x)}{\phi(x)}$ may be expressed as the sum of a set of partial fractions of the types

$$\frac{Ax+B}{(x-b)^2+a^2},\quad \frac{A'x+B'}{\{(x-b')^2+a'^2\}^n},$$

the latter only occurring in the case of $\phi(x)$ having repeated imaginary roots, we can obtain the value of any definite integral of either of the forms

$$\int_{-\infty}^{\infty}\frac{f(x)}{\phi(x)}\sin rx\,dx \quad\text{or}\quad \int_{-\infty}^{\infty}\frac{f(x)}{\phi(x)}\cos rx\,dx.$$

Ex. 1. $\displaystyle\int_{-\infty}^{\infty}\frac{\cos rx\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}=\Sigma\frac{1}{(a^2-b^2)(a^2-c^2)}\int_{-\infty}^{\infty}\frac{\cos rx}{x^2+a^2}dx=\text{etc.}$

Ex. 2. $\displaystyle\int_{-\infty}^{\infty}\frac{\sin rx\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}=0.$

1067. **Integrals of the class** $\displaystyle\int_0^{\infty}\frac{\cos rx}{x^{2n}+a^{2n}}dx$ **may also be conveniently treated as follows,** without the formation of a differential equation as used in Art. 1060.

Putting $\dfrac{1}{x^{2n}+a^{2n}}$ into partial fractions, we have

$$\frac{1}{x^{2n}+a^{2n}}=\frac{1}{na^{2n-1}}\sum_{\lambda=0}^{\lambda=n-1}\frac{a-x\cos\alpha_\lambda}{(x-a\cos\alpha_\lambda)^2+a^2\sin^2\alpha_\lambda},$$

where $\alpha_\lambda=\dfrac{2\lambda+1}{2n}\pi$, and α_λ is less than π for the whole range of values of λ from 0 to $n-1$, and $\sin\alpha_\lambda$ is therefore positive.

$$\int_{-\infty}^{\infty}\frac{\cos rx}{x^{2n}+a^{2n}}dx=\frac{1}{na^{2n-1}}\sum_0^{n-1}\int_{-\infty}^{\infty}\frac{(a-x\cos\alpha_\lambda)\cos rx\,dx}{(x-a\cos\alpha_\lambda)^2+a^2\sin^2\alpha_\lambda}$$

$$=\frac{1}{na^{2n-1}}\sum_0^{n-1}\frac{\pi}{\sin\alpha_\lambda}e^{-ar\sin\alpha_\lambda}\{\cos(ar\cos\alpha_\lambda)-\cos\alpha_\lambda\cos(ar\cos\alpha_\lambda+\alpha_\lambda)\}$$

$$=\frac{\pi}{na^{2n-1}}\sum_0^{n-1}e^{-ar\sin\alpha_\lambda}\sin(ar\cos\alpha_\lambda+\alpha_\lambda);$$

and since the integrand $\dfrac{\cos rx}{x^{2n}+a^{2n}}$ is not affected by a change of sign of x, we have

$$\int_0^\infty \frac{\cos rx}{x^{2n}+a^{2n}}\,dx=\frac{1}{2}\int_{-\infty}^\infty \frac{\cos rx}{x^{2n}+a^{2n}}\,dx.$$

Therefore $I\equiv\displaystyle\int_0^\infty \frac{\cos rx}{x^{2n}+a^{2n}}\,dx$

$$=\frac{\pi}{2na^{2n-1}}\sum_0^{n-1} e^{-ar\sin\frac{2\lambda+1}{2n}\pi}\sin\left(ar\cos\frac{2\lambda+1}{2n}\pi+\frac{2\lambda+1}{2n}\pi\right).$$

The other members of the family of integrals obtainable from this are $\displaystyle\int_0^\infty \frac{\sin rx}{x(x^{2n}+a^{2n})}\,dx$ by integration with regard to r, from $r=0$ to $r=r$, and

$$\int_0^\infty \frac{x\sin rx}{x^{2n}+a^{2n}}\,dx,\ \int_0^\infty \frac{x^2\cos rx}{x^{2n}+a^{2n}}\,dx,\ \int_0^\infty \frac{x^3\sin rx}{x^{2n}+a^{2n}}\,dx,\ \ldots \int_0^\infty \frac{x^{2n-1}\sin rx}{x^{2n}+a^{2n}}\,dx,$$

the latter system by differentiation with regard to r.

Since

$$\frac{d}{dr}e^{-ar\sin\alpha}\sin(ar\cos\alpha+\alpha)=ae^{-ar\sin\alpha}\sin\left(ar\cos\alpha+2\alpha+\frac{\pi}{2}\right),$$

we have

$$\int_0^\infty \frac{x^k\cos\left(rx+\frac{k\pi}{2}\right)}{x^{2n}+a^{2n}}\,dx=\frac{d^kI}{dr^k}$$

$$=\frac{\pi}{2na^{2n-1}}a^k\sum_0^{n-1}e^{-ar\sin\frac{2\lambda+1}{2n}\pi}\sin\left[ar\cos\frac{2\lambda+1}{2n}\pi+(k+1)\frac{2\lambda+1}{2n}\pi+\frac{k\pi}{2}\right],$$

where $k \not> 2n-1$, which gives all the integrals from

$$\int_0^\infty \frac{x\sin rx}{x^{2n}+a^{2n}}\,dx \ldots \text{ to } \int \frac{x^{2n-1}\sin rx}{x^{2n}+a^{2n}}\,dx.$$

The integral $\displaystyle\int_0^\infty \frac{\sin rx}{x(x^{2n}+x^{2n})}\,dx$ is of the form

$$A+\frac{\pi}{2na^{2n-1}}a^{-1}\sum_0^{n-1}e^{-ar\sin\frac{2\lambda+1}{2n}\pi}\sin\left(ar\cos\frac{2\lambda+1}{2n}\pi-\frac{\pi}{2}\right),$$

where A is a quantity, independent of r, to be found.

And since the integral vanishes with r,

$$0=A+\frac{\pi}{2na^{2n}}\sum_0^{n-1}\sin\left(-\frac{\pi}{2}\right)=A-\frac{\pi}{2a^{2n}};\quad\therefore\ A=\frac{\pi}{2a^{2n}};$$

$$\therefore\ \int_0^\infty \frac{\sin rx}{x(x^{2n}+a^{2n})}\,dx=\frac{\pi}{2a^{2n}}-\frac{\pi}{2na^{2n}}\sum_0^{n-1}e^{-ar\sin\frac{2\lambda+1}{2n}\pi}\cos\left(ar\cos\frac{2\lambda+1}{2n}\pi\right).$$

1068. Those interested in the history of the subject may refer to an article by Poisson in the *Jour. de l'École Polyt.*, xvi. p. 225, where the integral of $\int_0^\infty \frac{\cos rx}{1+x^{2n}}dx$ is discussed, and to articles by Catalan in the *Journal de Mathématiques*, vol. v. p. 110,* for integrals of form $\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^n}$.

1069. In the same way we may evaluate the integral

$$\int_0^\infty \frac{\cos rx\,dx}{x^{4n}-2a^{2n}x^{2n}\cos 2n\alpha+a^{4n}} \quad \begin{pmatrix} a>0 \\ \alpha<\pi \end{pmatrix}$$

with its attendant family of integrals derivable by differentiation and integration with regard to r.

For

$$\frac{1}{x^{4n}-2a^{2n}x^{2n}\cos 2n\alpha+a^{4n}}=\frac{1}{2n\sin 2n\alpha}\frac{1}{a^{4n-1}}\sum\frac{a\sin 2n\chi-x\sin(2n-1)\chi}{(x-a\cos\chi)^2+a^2\sin^2\chi},$$

where $\chi=\alpha+\frac{\lambda\pi}{n}$, the summation being for $2n$ consecutive integral values of λ.

And it is to be noted that χ is greater than 0 and less than π (and therefore $\sin\chi$ positive) for values of λ such that $\lambda\frac{\pi}{n}>-\alpha$ and $<\pi-\alpha$ respectively,

i.e. $\qquad \lambda>-\frac{n\alpha}{\pi}$ and $\lambda<n-\frac{n\alpha}{\pi}$,

i.e. for $\lambda=-k,\ -k+1,\ldots n-k-1$, where k is the greatest integer in $\frac{n\alpha}{\pi}$; and that $\sin\chi$ is negative for values of λ from $\lambda=n-k$ up to $\lambda=2n-k-1$.

Now

$$\int_{-\infty}^{\infty}\frac{\cos rx\,dx}{(x-a\cos\chi)^2+a^2\sin^2\chi}=\frac{\pi}{a\sin\chi}e^{-ar\sin\chi}\cos(ar\cos\chi) \quad \text{if } \sin\chi \text{ be } +^{\text{ve}}$$

and

$$=-\frac{\pi}{a\sin\chi}e^{ar\sin\chi}\cos(ar\cos\chi) \quad \text{if } \sin\chi \text{ be } -^{\text{ve}},$$

and

$$\int_{-\infty}^{\infty}\frac{x\cos rx\,dx}{(x-a\cos\chi)^2+a^2\sin^2\chi}=\frac{\pi}{\sin\chi}e^{-ar\sin\chi}\cos(ar\cos\chi+\chi) \quad \text{if } \sin\chi \text{ be } +^{\text{ve}}$$

and

$$=-\frac{\pi}{\sin\chi}e^{ar\sin\chi}\cos(ar\cos\chi-\chi) \quad \text{if } \sin\chi \text{ be } -^{\text{ve}}.$$

* Gregory, *Examples*, p. 486.

Hence $2n \sin 2n\alpha \dfrac{a^{4n-1}}{\pi} \int_{-\infty}^{\infty} \dfrac{\cos rx\, dx}{x^{4n} - 2a^{2n}x^{2n}\cos 2n\alpha + a^{4n}}$

$$= \sum_{-k}^{n-k-1} \frac{e^{-ar\sin\chi}}{\sin\chi}[\sin 2n\chi \cos(ar\cos\chi) - \sin(2n-1)\chi\cos(ar\cos\chi+\chi)]$$

$$- \sum_{n-k}^{2n-k-1} \frac{e^{ar\sin\chi}}{\sin\chi}[\sin 2n\chi\cos(ar\cos\chi) - \sin(2n-1)\chi\cos(ar\cos\chi-\chi)]$$

$$= \sum_{-k}^{n-k-1} e^{-ar\sin\chi}\cos\{ar\cos\chi-(2n-1)\chi\} - \sum_{n-k}^{2n-k-1} e^{ar\sin\chi}\cos\{ar\cos\chi+(2n-1)\chi\}$$

where k is the greatest integer in $\dfrac{n\alpha}{\pi}$ and $\chi = \alpha + \dfrac{\lambda\pi}{n}$.

Also, since the integrand is not affected by a change in the sign of x,

$$\int_0^{\infty} \frac{\cos rx\, dx}{x^{4n} - 2a^{2n}x^{2n}\cos 2n\alpha + a^{4n}} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos rx\, dx}{x^{4n} - 2a^{2n}x^{2n}\cos 2n\alpha + a^{4n}}.$$

The attendant family of integrals formed by differentiating $4n-1$ times with regard to r can now be written down, and are of type

$$4n \sin 2n\alpha \cdot \frac{a^{4n-p-1}}{\pi}\int_0^{\infty} \frac{x^p \cos\left(rx + p\frac{\pi}{2}\right) dx}{x^{4n} - 2a^{2n}x^{2n}\cos 2n\alpha + a^{4n}}$$

$$= \sum_{-k}^{n-k-1} e^{-ar\sin\chi}\cos\left\{ar\cos\chi - (2n-1)\chi + p\left(\frac{\pi}{2}+\chi\right)\right\}$$

$$- \sum_{n-k}^{2n-k-1} e^{ar\sin\chi}\cos\left\{ar\cos\chi + (2n-1)\chi + p\left(\frac{\pi}{2}-\chi\right)\right\},$$

and the integration with regard to r from 0 to r furnishes the remaining member of the family, viz.

$$4n \sin 2n\alpha \cdot \frac{a^{4n}}{\pi}\int_0^{\infty} \frac{\sin rx\, dx}{x(x^{4n} - 2a^{2n}x^{2n}\cos 2n\alpha + a^{4n})} - 2n\sin 2n\alpha$$

$$= \sum_{-k}^{n-k-1} e^{-ar\sin\chi}\cos\left\{ar\cos\chi - (2n-1)\chi - \left(\frac{\pi}{2}+\chi\right)\right\}$$

$$- \sum_{n-k}^{2n-k-1} e^{ar\sin\chi}\cos\left\{ar\cos\chi + (2n-1)\chi - \left(\frac{\pi}{2}-\chi\right)\right\}$$

$$= \sum_{-k}^{n-k-1} e^{-ar\sin\chi}\sin(ar\cos\chi - 2n\alpha) - \sum_{n-k}^{2n-k-1} e^{ar\sin\chi}\sin(ar\cos\chi + 2n\alpha),$$

k and χ being as defined before.

1070. It will be noted further that the integral

$$\int_0^{\infty} \frac{\cos rx\, dx}{x^{4n} + 2a^{2n}x^{2n}\cos 2n\beta + a^{4n}}$$

and its accompanying family of integrals can be deduced from the above family by writing $\alpha = \dfrac{\pi}{2n} - \beta$.

PROBLEMS.

1. Prove that

$$\left[\int_\infty^x e^{-ax}\cos bx\,dx\right]^2+\left[\int_\infty^x e^{-ax}\sin bx\,dx\right]^2=e^{-2ax}/(a^2+b^2).$$

[γ, 1893.]

2. If $u_n=\int_0^\infty x^n e^{-ax^2}dx$, show that $u_n=\dfrac{n-1}{2a}u_{n-2}$.

Hence calculate u_n where n is any positive integer. [Trinity, 1881.]

3. Show that $\displaystyle\int_0^\infty \frac{e^{-x}\sin^4 x}{x}dx=\frac{1}{16}\log\frac{625}{17}$. [$\beta$, 1891.]

4. Evaluate $\displaystyle\int_{-\infty}^\infty e^{-(ax^2+bx+c)}dx$. [Colleges, 1879.]

5. Deduce from the integral $\displaystyle\int_0^\infty \frac{\cos rx}{1+x^2}dx$ the result

$$\int_0^\infty \frac{\sin rx}{x(n^2+x^2)}dx=\frac{\pi}{2n^2}(1-e^{-nr});\quad \begin{pmatrix} n+{}^{\text{ve}} \\ r+{}^{\text{ve}}\end{pmatrix}.$$

6. Find the value of $\displaystyle\int_0^\infty\left(\frac{1}{e^{mx}+e^{-mx}}\right)^n dx$, where n is a positive integer. [Math. Trip., Pt. I., 1890.]

7. Show that $\displaystyle\int_0^\infty \frac{\sin qx\sinh qx}{(\cosh qx+\cos qx)^2}dx=\frac{1}{2q}$. [$\beta$, 1891.]

8. Show that $\displaystyle\int_0^\infty \frac{\sinh px\sin qx}{(\cosh px+\cos qx)^2}dx=\frac{q}{p^2+q^2}$.

9. Show that, if p be a positive quantity,

$$\int_0^\infty \frac{\sinh px}{x}\left(\frac{1}{\cosh px+\cos ax}-\frac{1}{\cosh px+\cos bx}\right)dx=\frac{1}{2}\log\frac{p^2+a^2}{p^2+b^2}.$$

[Math. Tripos, 1890.]

10. Prove that

$$(a)\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{x\,dx}{\sin x\cos x}=\frac{\pi}{4}\log 3;\quad (b)\int_{\frac{\pi}{4}-\alpha}^{\frac{\pi}{4}+\alpha}\frac{x\,dx}{\sin x\cos x}=\pi\tanh^{-1}(\tan\alpha).$$

11. Prove that $\displaystyle\int_0^{\frac{\pi}{2}}(\cos^{\frac{1}{n}}\theta+\sin^{\frac{1}{n}}\theta)^{-2n}d\theta=\frac{(n-1)!}{n(n+1)(n+2)\dots(2n-1)}$,
where n is a positive integer. [Math. Tripos, 1889.]

12. Prove that $\displaystyle\int_{-\infty}^\infty e^{-x^2}\cos 2nx\,dx=\sqrt{\pi}\,e^{-n^2}$. [$\epsilon$, 1883.]

13. Prove that $\displaystyle\int_0^\infty \frac{\sin rx}{x}\,\frac{\cosh x\theta}{\cosh x\frac{\pi}{2}}dx=\cot^{-1}\left(\frac{\cos\theta}{\sinh r}\right)$. [$\alpha$, 1885.]

14. Prove that $\int_0^\infty \frac{\cos bx}{x} \tanh \frac{\pi x}{2} dx = \log\left(\coth \frac{b}{2}\right)$. [Colleges, 1879.]

15. Prove that $\int_0^u \frac{du}{(e\cosh u - 1)^n} = \frac{1}{(e^2-1)^{n-\frac{1}{2}}} \int_0^\theta (e\cos\theta + 1)^{n-1} d\theta$,

if $(e\cos\theta + 1)(e\cosh u - 1) = e^2 - 1$. [Math. Tripos, 1885.]

16. Prove that $\int_0^\infty \frac{\cos 4kx \tanh x}{x} dx = \log_e \coth k\pi$.

[Math. Tripos, 1889.]

17. Prove that, if α lies between $-\pi/4$ and $\pi/4$,

$$\int_0^\pi \frac{d\theta}{1 - 2\sin 2\alpha \cos\theta + \cos^2\theta} = \frac{\pi \cos\alpha}{\sqrt{2}\cos^3 2\alpha}.$$

[Math. Tripos, 1885.]

18. Prove that $\int_0^1 \frac{dx}{(1-x^{2n})^{\frac{n}{2}}} = \frac{\pi}{2n \sin\frac{\pi}{2n}}$. [$\beta$, 1888.]

19. Prove that $4\int_0^1 \frac{dx}{(1-x^4)^{\frac{1}{2}}} = 2\int_0^1 \frac{dx}{(1-x^2)^{\frac{3}{4}}} = \frac{\{\Gamma(\frac{1}{4})\}^2}{(2\pi)^{\frac{1}{2}}}$.

[Trinity, 1889.]

20. Evaluate

$$(a)\ e^{-x^4}\int_0^x x^2 e^{x^4} dx\,; \quad (b)\ e^{-x^4}\int_0^x x^3 e^{x^4} dx\,;$$

$$(c)\ e^{-x^4}\int_0^x x^4 e^{x^4} dx\,; \quad (d)\ xe^{-x^2}\int_0^x e^{x^2} dx,$$

where in each case x becomes infinite.

21. Prove that $\int_0^\infty e^{-x} \frac{\sin tx}{\sinh x} dx = \frac{\pi}{2}\coth t\frac{\pi}{2} - \frac{1}{t}$.

22. Show that $\int_0^\infty \frac{\cos x}{1+x^2} dx = \int_0^\infty \frac{\cos x}{(1+x^2)^2} dx = \frac{8}{7}\int_0^\infty \frac{\cos x}{(1+x^2)^3} dx$.

[Math. Tripos, 1876.]

23. Evaluate $\int_{-\infty}^\infty \frac{\cos mx}{1+x+x^2} dx$. [Math. Tripos, 1892.]

24. Prove that, if m be positive,

$$\int_0^\infty \frac{\cos mx}{1+x^2+x^4} dx = \frac{\pi}{\sqrt{3}} e^{-\frac{1}{2}m\sqrt{3}} \sin\left(\tfrac{1}{2}m + \tfrac{1}{6}\pi\right).$$

[Math. Tripos, 1892.]

25. Show that (i) $\int_0^\infty \frac{\cos ax}{1+x^4} dx = \frac{\pi}{2\sqrt{2}} e^{-\frac{a}{\sqrt{2}}}\left\{\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}}\right\}$.

[Laplace, *Mém. de l'Inst.*, 1810.]

(ii) $\int_0^\infty \frac{\cos x\, dx}{x^4 + 4a^4} = \frac{\pi e^{-a}}{8a^3}(\cos a + \sin a)$.

[Math. Tripos, Pt. I., 1914.]

26. Show that $\int_0^\infty \frac{x\sin 2ax}{x^4+1}\,dx = \frac{\pi}{2}e^{-a\sqrt{2}}\sin a\sqrt{2}.$ [St. John's, 1883.]

27. Show that $\int_0^\infty \frac{x\sin bx}{(1+x^2)(a^2+x^2)}\,dx = \frac{\pi}{2}\frac{e^{-b}-e^{-ab}}{a^2-1};\quad \begin{pmatrix} a+^{\text{ve}} \\ b+^{\text{ve}} \end{pmatrix}.$

28. Prove that

$$\int_0^\infty \frac{x^{m-1}dx}{(1+2x\cos\alpha+x^2)^m(1+x^n)} = \int_0^1 \frac{x^{m-1}dx}{(1+2x\cos\alpha+x^2)^m}$$

$$= \frac{1}{2^m\sin^{2m-1}\alpha}\int_0^\alpha (\cos x-\cos\alpha)^{m-1}dx$$

$$= \frac{1}{2^m(m-1)!}\left\{\frac{1}{\sin\alpha}\frac{d}{d\alpha}\right\}^{m-1}\left(\frac{\alpha}{\sin\alpha}\right).$$

[Wolstenholme, *Educ. Times.*]

29. Prove that $\int_0^\infty \frac{\sin\pi x}{x(1-x^2)}\,dx = \pi.$ [Math. Trip., Pt. II., 1919.]

30. Prove that

$$\int_a^\infty e^{-nx-x^2}dx = \frac{e^{-na-a^2}}{2a+n}\left\{1-\frac{2}{(2a+n)^2}+\frac{12}{(2a+n)^4}\right\} \text{ approximately.}$$

[γ, 1891.]

31. Prove that $\int_0^\infty e^{-x^2\cos\theta}\sin(x^2\sin\theta)\,dx = \frac{\sqrt{\pi}}{2}\sin\frac{\theta}{2}.$ [Coll., 1892.]

32. From the integral $\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}dx = \frac{1}{2}\sqrt{\pi}\,e^{-2a}$, show that

$$\int_0^\infty\int_0^\pi e^{-r-\frac{a^2\operatorname{cosec}^2\theta}{r}}\,dr\,d\theta = \pi e^{-2a}.$$

[Trinity, 1886.]

33. Express the sum of the series $1+x^{\sqrt{1}}+x^{\sqrt{2}}+x^{\sqrt{3}}+\dots$ *ad inf.* by means of a definite integral, x being a real quantity less than unity. [Trinity, 1895.]

34. Prove the formula

$$\int_{-\infty}^\infty e^{-a^2x^2}\frac{\sin(2n+1)bx}{\sin bx}\,dx = \frac{\sqrt{\pi}}{a}\left[1+2\sum_{r=1}^{r=n} e^{-\frac{r^2b^2}{a^2}}\right].$$

[St. John's, 1881.]

35. Prove that $\int_\infty^b\int_\infty^a \frac{dx\,dy}{xy(x+y)} = \log\left\{\frac{(a+b)^{\frac{1}{a}+\frac{1}{b}}}{a^{\frac{1}{b}}b^{\frac{1}{a}}}\right\}.$ [Trinity, 1886.]

36. Show that $\int_0^\infty \tan^{-1}\frac{x}{a}\tan^{-1}\frac{x}{b}\frac{dx}{x^2} = \frac{\pi}{2}\log\frac{(a+b)^{\frac{1}{a}+\frac{1}{b}}}{a^{\frac{1}{b}}b^{\frac{1}{a}}}.$

[Bertrand, *Calc. Int.*, p. 200.]

37. Show that $\int_0^\infty \phi\left(\frac{x}{a}\right)\phi\left(\frac{x}{b}\right)dx = \log\left[(a+b)^{a+b}a^{-a}b^{-b}\right]$,

where $$\phi(x) = \int_x^\infty \frac{e^{-u}\,du}{u}.$$ [MATH. TRIP., 1882.]

38. Prove that $$\int_{-\infty}^\infty e^{-\frac{1}{2}u^2+ux\sqrt{2\lambda}}\,du = \sqrt{2\pi}\,e^{\lambda x^2},$$

and deduce $$x^n e^{\lambda x^2} = \frac{(-1)^n}{\sqrt{2\pi}(2\lambda)^{\frac{n}{2}}}\int_{-\infty}^\infty e^{ux\sqrt{2\lambda}}\frac{d^n(e^{-\frac{1}{2}u^2})}{du^n}\,du.$$

[ST. JOHN'S, 1882.]

39. Having given that

$$\int_0^\infty e^{-x^2-\frac{a^2}{x^2}}\,dx = \frac{\sqrt{\pi}}{2}e^{-2a},$$

prove that $$\int_0^\infty x^2 e^{-x^2-\frac{1}{x^2}}\,dx = \frac{3\sqrt{\pi}}{4e^2}.$$ [COLLEGES, 1882.]

40. Having given that $\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$, deduce the value of $\int_0^\infty e^{-ax^2}\cos bx\,dx$. [COLLEGES, 1879.]

41. Prove that $\int_0^\infty e^{-x^2}\cos ax\{(a^2-6)x-4x^3\}\,dx = 1$.

42. Find the value of $$\int_0^\infty e^{-\frac{1}{2}x^2}\cos x\,dx,$$

and prove that $$\int_0^\infty e^{-\frac{1}{2}x^2}\sin x\,dx = \frac{1}{\sqrt{e}}\int_0^1 e^{\frac{1}{2}y^2}\,dy.$$ [ST. JOHN'S, 1886.]

43. Prove that $\int_0^\infty \frac{\sin 2nx}{\sin x}\frac{dx}{1+x^2} = \frac{\pi}{4e^n}\frac{\sinh n}{\sinh 1}$, n being a positive integer.

44. Starting with $\int_0^1 x^p\,dx = \frac{1}{p+1}$, deduce $\int_0^1 \frac{x^p - x^q}{\log x}\,dx = \log\frac{p+1}{q+1}$.

Putting $p = a\sqrt{-1}$ and $q = b\sqrt{-1}$, deduce the values of the integrals

$$\int_0^\infty e^{-t}\frac{\cos bt - \cos at}{t}\,dt \quad \text{and} \quad \int_0^\infty e^{-t}\frac{\sin bt - \sin at}{t}\,dt,$$

and verify your results by a rigorous independent method.

Show that $$\int_0^1 \frac{\sin(p\log x)}{\log x}\,dx = \tan^{-1}p.$$

45. Prove that

$$\int_0^1 \log\cos\left(\frac{\pi}{2}\sqrt{1-x^2}\right)dx = \log\frac{\pi}{4} - 2\left\{1 - \frac{1}{3}\frac{a_1}{2^3} + \frac{1}{5}\frac{a_2}{2^5} - \dots\right\},$$

where $$a_n = \sum_{r=1}^{r=\infty}\left\{\frac{1}{r(r+1)}\right\}^n.$$ [ST. JOHN'S, 1885.]

46. Prove that $\int_0^\infty \left(e^{-\frac{p^2}{x^2}} - e^{-\frac{q^2}{x^2}}\right) dx = \sqrt{\pi}(q-p)$. [MATH. TRIPOS.]

47. Prove that

$$\int_0^{\frac{\pi}{2}} \phi(\sin 2x)\cos x\,dx = \int_0^{\frac{\pi}{2}} \phi(\cos^2 x)\cos x\,dx.$$

[BESGE, *Liouville's Journal*, xviii.]

48. Deduce from Laplace's Integral

$$\int_0^\infty dx\,e^{-\left(x^2+\frac{a^2}{x^2}\right)} = \frac{\sqrt{\pi}}{2}e^{-2a},$$

the results *

$$\int_0^\infty \cos\left(x^2+\frac{a^2}{x^2}\right)dx = \frac{\sqrt{\pi}}{2}\cos\left(\frac{\pi}{4}+2a\right),$$

$$\int_0^\infty \sin\left(x^2+\frac{a^2}{x^2}\right)dx = \frac{\sqrt{\pi}}{2}\sin\left(\frac{\pi}{4}+2a\right),$$

$$\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)\cos\theta}\cos\left\{\left(x^2+\frac{a^2}{x^2}\right)\sin\theta\right\}dx = \frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\cos\left(2a\sin\theta+\frac{\theta}{2}\right)$$

$$\int_0^\infty e^{-\left(x^2+\frac{a^2}{x^2}\right)\cos\theta}\sin\left\{\left(x^2+\frac{a^2}{x^2}\right)\sin\theta\right\}dx = \frac{\sqrt{\pi}}{2}e^{-2a\cos\theta}\sin\left(2a\sin\theta+\frac{\theta}{2}\right).$$

[CAUCHY, *Mém. des Sav. Ét.*]

49. From Laplace's Integral

$$\int_0^\infty e^{-a^2x^2}\cos 2rx\,dx = \frac{\sqrt{\pi}}{2a}e^{-\frac{r^2}{a^2}},$$

deduce *

$$\int_0^\infty \cos a^2x^2\cos 2rx\,dx = \frac{\sqrt{\pi}}{2a}\cos\left(\frac{\pi}{4}-\frac{r^2}{a^2}\right),$$

$$\int_0^\infty \sin a^2x^2\cos 2rx\,dx = \frac{\sqrt{\pi}}{2a}\sin\left(\frac{\pi}{4}-\frac{r^2}{a^2}\right).$$

[FOURIER, *T. de la Chal.*]

50. Prove that if $f^{(r)}(z) \equiv \left(\frac{d}{dz}\right)^r f(z)$, and all the differential coefficients up to the $(r-1)^{\text{th}}$ inclusive remain continuous from $z=-1$ to $z=1$, then will

$$\int_0^\pi f^{(r)}(\cos x)\sin^{2r}x\,dx = 1\,.\,3\,.\,5\,\ldots\,(2r-1)\int_0^\pi f(\cos x)\cos rx\,dx.$$

[JACOBI, *Crelle's J.*, xv.; GREGORY, *Examples*, p. 501.]

* See remarks on the use of imaginaries (Arts. 1189 to 1201).

51. Prove that

$$a\int_0^{-a}(t^2-a^2)^m\cos tx\,dt = 2^m m!\left(\frac{1}{x}\frac{d}{dx}\right)^{m+1}\cos ax,$$

m being a positive integer. [CULLEN, *Educ. Times*, 14808.]

52. Prove that

$$\int_0^\infty \frac{\sin\left(\frac{r\pi}{2}+ax\right)}{x^{n-r}}dx = \frac{(n-1)(n-2)\ldots(n-r)}{\Gamma(n)}\frac{\pi a^{n-r-1}}{2\sin\frac{n\pi}{2}},$$

r being an integer and $1>n>0$. [U. C. GHOSH, *Educ. Times*, 14954.]

53. Show that if

$$A=\int_0^\infty e^{-ax^2}\cos bx^2\,dx,\quad B=\int_0^\infty e^{-ax^2}\sin bx^2\,dx\quad (a>0),$$

then A^2+B^2 and $2AB$ can be expressed in terms of elementary functions. [MATH. TRIPOS, PT. II., 1914.]

54. Show that $\int_0^\infty \left(\frac{\tan^{-1}x}{x}\right)^3 dx = \frac{1}{2}\pi\left(3\log_e 2-\frac{1}{8}\pi^2\right)$.

[MATH. TRIPOS, PT. I., 1887.]

55. If $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^n\frac{x^{2n}}{(2n)!}X$

and $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+(-1)^n\frac{x^{2n-1}}{(2n-1)!}X'$,

prove that $\int_0^\infty \frac{X}{x}dx = \frac{\pi}{2} = \int_0^\infty \frac{X'}{x}dx.$ [MATH. TRIPOS, 1875.]

56. If a and y be positive, prove that the value of

$$\int_0^\infty \frac{\sin(yx)\cos(ax)}{x}dx$$

is $\frac{1}{2}\pi$ or 0 according as y is greater or less than a.

By multiplying by $e^{-by}\cos cy$ and integrating with respect to y from a to ∞, or otherwise, prove that

$$\int_0^\infty \frac{(x^2+b^2-c^2)\cos ax}{(x^2+b^2-c^2)^2+4b^2c^2}dx = \frac{1}{2}\pi e^{-ab}\frac{b\cos ac-c\sin ac}{b^2+c^2},$$

a, b, c being positive constants. [MATH. TRIPOS, PT. II., 1920.]

57. Show that $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\pi-4\theta)\tan\theta}{1-\tan\theta}d\theta = \pi\left(\log 2-\frac{\pi}{4}\right)$.

[TRIN. HALL and MAGD. COLL., 1881.]

so that even when the series for $F(x)$ ceases to be convergent when $x=1$, the final element of the summation indicated by the integration $\int_0^1 F(x)\left(\log\frac{1}{x}\right)^p dx$ will have no effect. Then we shall have, by putting $x=e^{-y}$,

$$I\equiv\int_0^1\left(\log\frac{1}{x}\right)^p F(x)\,dx=\int_0^\infty y^p e^{-y}F(e^{-y})\,dy$$

$$=\Gamma(p+1)\left(\frac{A_0}{1^{p+1}}+\frac{A_1}{2^{p+1}}+\frac{A_2}{3^{p+1}}+\dots\right),$$

and therefore I can be expressed in finite terms whenever $F(x)$ is such that this series is capable of summation.

An extensive class of definite integrals arises from this fact.

1073. It will be well to recount several previous results obtained. We have now used the symbol S_p to denote the complete series

$$S_p\equiv\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\dots\ ad\ inf.\quad (p>1),$$

and the numerical values of S_p up to S_{35} are tabulated in Art. 957.

Also, if $\sec x+\tan x=1+K_1\frac{x}{1!}+K_2\frac{x^2}{2!}+K_3\frac{x^3}{3!}+\dots$, then

$$K_n\cdot\frac{\pi^{n+1}}{2^{n+2}n!}=1+(-\tfrac{1}{3})^{n+1}+(\tfrac{1}{5})^{n+1}+(-\tfrac{1}{7})^{n+1}+(\tfrac{1}{9})^{n+1}+\dots\ ad\ inf.,$$

and rules were given (*Diff. Calc.*, Art. 573) for the calculation of K_n, the results being

$$K_1=1,\quad K_2=1,\quad K_3=2,\quad K_4=5,\quad K_5=16,$$

$$K_6=61,\quad K_7=272,\quad K_8=1385,\quad K_9=7936,\quad \text{etc.},$$

K_{2n} being the n^{th} "Eulerian" number $\equiv E_{2n}$; whilst K_{2n-1} is the n^{th} "Prepared Bernoullian" number $\equiv\frac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1}$, B_{2n-1} being the n^{th} Bernoullian number itself.

Also we have seen that

$$S_{2n}\equiv\frac{\pi^{2n}}{2(2n-1)!\,(2^{2n}-1)}K_{2n-1}\equiv\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\dots=\frac{(2\pi)^{2n}}{2(2n)!}B_{2n-1},$$

$$\frac{\pi^{2n+1}}{2^{2n+2}(2n)!}K_{2n}\equiv\frac{1}{1^{2n+1}}-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\dots=\frac{\pi^{2n+1}}{2^{2n+2}(2n)!}E_{2n},$$