CHAPTER XXIII.

CHANGE OF THE VARIABLES IN A MULTIPLE INTEGRAL.

826. A NUMBER of cases have occurred in previous chapters in which the evaluation of an area or a volume has been much facilitated by a proper choice of coordinates, and changes have been made from one specific system of coordinates to another specific system, such, for example, as from Cartesians to polars, or to elliptic coordinates.

In particular, we have established the results, that in transforming from an x, y system, which may be regarded as Cartesian, to a u, v system, we have

$$\iint V\,dx\,dy = \iint V' \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv;$$

and when we change from a three-dimensional Cartesian x, y, z system to another system in terms of new variables u, v, w, we have

$$\iiint V\,dx\,dy\,dz = \iiint V' \frac{\partial(x, y, z)}{\partial(u, v, w)}\,du\,dv\,dw,$$

the symbol V' representing merely the value of V as expressed in terms of the new coordinate system.

These changes have been found very especially useful in the case where the *bounding curves or surfaces of the regions under consideration are themselves members of the three families,*

$$u = \text{const.}, \quad v = \text{const.}, \quad w = \text{const.}$$

This was the case in the typical example of Art. 793, viz. the evaluation of the area of a Carnot's cycle, bounded by isothermals $xy = a_1$, $xy = a_2$, and the adiabatics $xy^\gamma = \beta_1$, $xy^\gamma = \beta_2$; and it will be recalled that the region thus bounded was divided into elementary areas bounded by curves of the same types, viz.

$$xy=u, \qquad xy^{\gamma}=v,$$
$$xy=u+\delta u, \qquad xy^{\gamma}=v+\delta v.$$

Exactly the same course was followed in the three-dimension typical examples of Articles 797, 798.

827. **Further Examples.**

1. The quadrilateral bounded by the four parabolas

$$y^2=a^3x, \quad y^2=b^3x, \quad x^2=e^3y, \quad x^2=f^3y,$$

revolves round the axis of y; find the volume generated.

[Colleges a, 1890.]

If $\delta x\,\delta y$ be an elementary rectangle of this area, we have

$$V=\iint 2\pi x\,dx\,dy$$

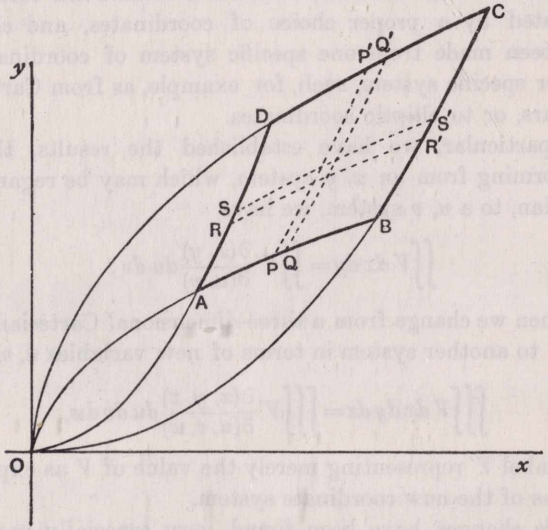

Fig. 295.

Now, instead of taking elements of rectangular shape such as $\delta x\,\delta y$, let us divide up the area by the families of parabolas

$$y^2=u^3x, \quad x^2=v^3y, \qquad \text{.................................(1)}$$

Then $u=a$ and $u=b$, $v=e$ and $v=f$ are the bounding parabolas of the region, and the elementary area enclosed by u, $u+\delta u$, v, $v+\delta v$ is $\pm J\delta u\,\delta v$.

From equations (1) $x=uv^2$, $y=u^2v$,

$$J\equiv\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} v^2, & 2uv \\ 2uv, & u^2 \end{vmatrix}=-3u^2v^2.$$

Hence
$$V=6\pi\int_a^b\int_e^f u^3v^4\,du\,dv$$
$$=\frac{6\pi}{4\,.\,5}\Big[u^4\Big]_a^b\Big[v^5\Big]_e^f$$
$$=\frac{3\pi}{10}(a^4-b^4)(e^5-f^5).$$

2. Evaluate the triple integral $\iiint\frac{dx\,dy\,dz}{xyz}$ taken through a volume bounded by six confocal quadrics, the semiaxes of the quadrics being

$$\left.\begin{matrix} a_1, & b_1, & c_1, \\ \text{and } a_1', & b_1', & c_1', \end{matrix}\right\} \quad \left.\begin{matrix} a_2, & b_2, & c_2, \\ \text{and } a_2', & b_2', & c_2', \end{matrix}\right\} \quad \left.\begin{matrix} a_3, & b_3, & c_3, \\ \text{and } a_3', & b_3', & c_3'. \end{matrix}\right\}$$

[Math. Trip., 1889.]

Taking a definite confocal a, b, c, let the three confocals through any point x, y, z of the region be

$$\frac{x^2}{a^2+\lambda}++=1,\quad \frac{x^2}{a^2+\mu}++=1,\quad \frac{x^2}{a^2+\nu}++=1,$$

and we have
$$x^2=\frac{(\lambda+a^2)(\mu+a^2)(\nu+a^2)}{(a^2-b^2)(a^2-c^2)},\text{ etc. (Art. 812);}$$

whence
$$\frac{2}{x}\frac{\partial x}{\partial\lambda}=\frac{1}{\lambda+a^2},\quad \frac{2}{x}\frac{\partial x}{\partial\mu}=\frac{1}{\mu+a^2},\text{ etc.,}$$

and
$$J\equiv\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)}=\frac{xyz}{8}\begin{vmatrix} \frac{1}{\lambda+a^2}, & \frac{1}{\mu+a^2}, & \frac{1}{\nu+a^2} \\ \frac{1}{\lambda+b^2}, & \frac{1}{\mu+b^2}, & \frac{1}{\nu+b^2} \\ \frac{1}{\lambda+c^2}, & \frac{1}{\mu+c^2}, & \frac{1}{\nu+c^2} \end{vmatrix}$$

Hence

$$\iiint\frac{dx\,dy\,dz}{xyz}=\frac{1}{8}\iiint\Sigma\frac{1}{\lambda+a^2}\left(\frac{1}{\mu+b^2}\cdot\frac{1}{\nu+c^2}-\frac{1}{\mu+c^2}\cdot\frac{1}{\nu+b^2}\right)d\lambda\,d\mu\,d\nu$$
$$=\tfrac{1}{8}\Sigma\,[\log(\lambda+a^2)]\,\{[\log(\mu+b^2)]\,[\log(\nu+c^2)]$$
$$-[\log(\mu+c^2)]\,[\log(\nu+b^2)]\},$$

and at one set of the boundaries

$$\lambda+a^2=a_1^2,\quad \lambda+b^2=b_1^2,\quad \lambda+c^2=c_1^2,$$
$$\mu+a^2=a_2^2,\quad \mu+b^2=b_2^2,\quad \mu+c^2=c_2^2,$$
$$\nu+a^2=a_3^2,\quad \nu+b^2=b_3^2,\quad \nu+c^2=c_3^2;$$

and for the other set,

$$\lambda+a^2=a_1'^2,\quad \lambda+b^2=b_1'^2,\text{ etc.}$$

Hence the limits for λ are from $a_1^2-a^2$ to $a_1'^2-a^2$,
for μ from $b_1^2-b^2$ to $b_1'^2-b^2$,
for ν from $c_1^2-c^2$ to $c_1'^2-c^2$.

Therefore

$$\iiint \frac{dx\,dy\,dz}{xyz} = \tfrac{1}{8}\sum \log\frac{a_1'^2}{a_1^2}\left(\log\frac{b_2'^2}{b_2^2}\log\frac{c_3'^2}{c_3^2} - \log\frac{b_3'^2}{b_3^2}\log\frac{c_2'^2}{c_2^2}\right)$$

$$= \begin{vmatrix} \log\frac{a_1'}{a_1}, & \log\frac{b_1'}{b_1}, & \log\frac{c_1'}{c_1} \\ \log\frac{a_2'}{a_2}, & \log\frac{b_2'}{b_2}, & \log\frac{c_2'}{c_2} \\ \log\frac{a_3'}{a_3}, & \log\frac{b_3'}{b_3}, & \log\frac{c_3'}{c_3} \end{vmatrix}$$

828. **Remarks on the Transformation.**

The usefulness of a change of variables is not, however, confined to the case in which the bounding curves or surfaces of the region considered *are particular cases of the families of curves or surfaces by which it has been deemed desirable to divide up the region* into elements and for which case the limits *are constants*.

The process of transformation is threefold:

(*a*) The transformation of the subject of integration into terms of the new variables.

(*b*) The determination of the new element of integration, which resolves itself into the calculation of J.

(*c*) The determination of the new limits.

Of these, (*a*) and (*b*) are merely algebraic processes, and give no trouble.

The determination of the new limits (*c*) however, often presents considerable difficulty to the student. And we cannot lay down explicit rules to be followed to suit all cases. Generally speaking, it is best to proceed, from geometrical considerations, first *forming a clear idea of the region which the original element of area or volume was made to traverse.* This will be clearly indicated by the limits of the integrals occurring in the expression to be transformed. Then the new limits for the transformed integral must be so chosen that the new *element of area or volume, as the case may be, traverses the same region, once and once only,* as was traversed by the original element in its march as defined by the limits of the original integral.

The student will require considerable practice in the assignment of the new limits, and therefore a number of illustrative examples are appended from which he may gather an idea of the course to be adopted.

And before proceeding to discuss them in detail the student is advised to note that at times, even *a change of order in the integration*, without any change in the variables, may be useful, and that in some cases an integration in different orders may lead to important conclusions. Some of the earlier examples are therefore confined to mere change of order with no change in the coordinates, and the necessary change in the limits will be the subject of main attention.

829. Change of Order of Integration.

Ex. 1. Consider $\int_a^b dx \int_c^d dy\, f(x, y)$, all the limits being known constants. Here the space bounded by $y=c$, $y=d$, $x=a$, $x=b$ is the region through which all products such as $f(x, y)\,\delta x\,\delta y$ are to be added, viz. the

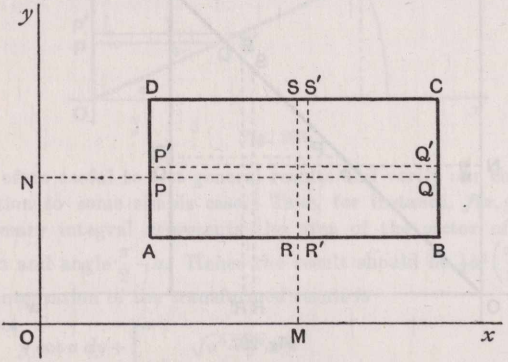

Fig. 296.

rectangle $ABCD$ in Fig. 296. In the integration as it stands we integrate first with regard to y, keeping x constant, thus adding up all elements in such a strip as $RSS'R'$ in the figure. Then all such strips are to be added in the operation $\int_a^b (\ )\,dx$.

If we wish to change the order of the operation and express it as

$$\int dy \int dx\, f(x, y)$$

we have to assign the new limits.

Clearly in this case the sum of such elements as we have considered, added up along such a strip as $PQQ'P'$ parallel to the x-axis, will be

$$\int_a^b f(x, y)\,dx,$$

and the sum of all these strips, from $y=c$ to $y=d$, will be

$$\int_c^d dy \int_a^b dx\, f(x, y).$$

Thus $$\int_a^b dx \int_c^d dy\, f(x, y) = \int_c^d dy \int_a^b dx\, f(x, y).$$

It appears therefore that in the case of constant limits no change is entailed by a change in the order of integration.

Ex. 2. Consider $\int_0^a \int_0^x f(x, y)\, dx\, dy.$

Here the limits for y are from $y=0$ to $y=x$, and for x from $x=0$ to $x=a$.

These indicate that the boundaries of the region for which the elements $f(x, y)\,\delta x\,\delta y$ are to be added are

the x-axis, the line $y=x$, the line $x=a$.

And if instead of taking strips parallel to the y-axis, we add up the elements in strips parallel to the x-axis, of which $PQQ'P'$ is a type

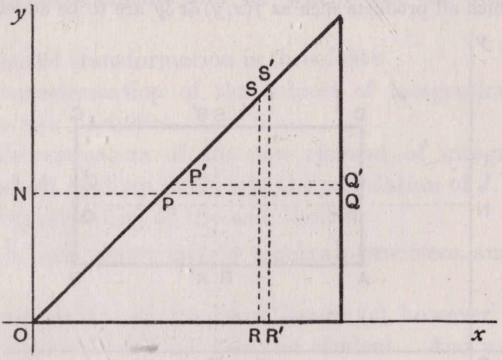

Fig. 297.

(Fig. 297), this summation is to be taken from $x=y$ to $x=a$, and $\int_y^a f(x, y)\, dx$ will be the sum for the strip $PQQ'P'$.

These strips are then to be added from $y=0$ to $y=a$, giving

$$\int_0^a \int_y^a f(x, y)\, dy\, dx$$

as the transformed result.

Ex. 3. Consider $\int_0^{a \cos \alpha} \int_{x \tan \alpha}^{\sqrt{a^2-x^2}} f(x, y)\, dx\, dy.$

The region of integration is bounded by the straight line $y=x \tan \alpha$, the circle $y=\sqrt{a^2-x^2}$, and the y-axis.

The present summation is that of strips parallel to the y-axis. If we change the order of the integration we must add up all elements in a strip parallel to the x-axis before adding the strips.

These strips change their character at the point where $y=a\sin\alpha$; from $y=0$ to $y=a\sin\alpha$, the length of a strip is bounded by the y-axis and the straight line $y=x\tan\alpha$; from $y=a\sin\alpha$ to $y=a$ the strip is terminated by the circle.

Hence the integration consists of two separate parts, viz.

$$\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x, y)\,dy\,dx+\int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\,dy\,dx.$$

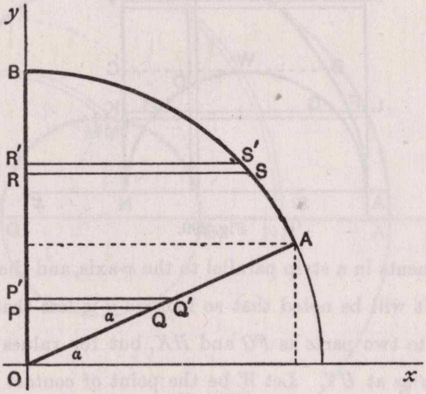

Fig. 298.

It is often useful to test general results and verify our conclusions by application to some simple case. Take, for instance, $f(x, y)=1$. Then the primary integral represents the area of the sector of a circle of radius a and angle $\frac{\pi}{2}-\alpha$. Hence the result should be $\frac{1}{2}a^2\left(\frac{\pi}{2}-\alpha\right)$.

The integration of the transformed result is

$$\int_0^{a\sin\alpha} y\cot\alpha\,dy+\int_{a\sin\alpha}^{a}\sqrt{a^2-y^2}\,dy$$

$$=\left[\frac{y^2}{2}\cot\alpha\right]_0^{a\sin\alpha}+\frac{1}{2}\left[y\sqrt{a^2-y^2}+a^2\sin^{-1}\frac{y}{a}\right]_{a\sin\alpha}^{a}$$

$$=\frac{a^2}{2}\sin\alpha\cos\alpha+\tfrac{1}{2}a^2\cdot\frac{\pi}{2}-\tfrac{1}{2}a^2\sin\alpha\cos\alpha-\frac{a^2}{2}\alpha=\frac{a^2}{2}\left(\frac{\pi}{2}-\alpha\right),$$

as it should be.

Ex. 4. To change the order of integration in the integral

$$\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x, y)\,dx\,dy.$$

Here the region of integration is bounded by

(1) The parabola $y^2=ax$.

(2) The semicircle $x^2+y^2=ax$, which we may note is the circle of curvature at the vertex of the parabola, and lies entirely within the parabola.

(3) The straight line $x=a$; and this is a tangent to the circle.

Instead of adding up the quantities $f(x, y)\,\delta x\,\delta y$ along strips such as DE (Fig. 299) parallel to the y-axis, and then adding the strips, we have

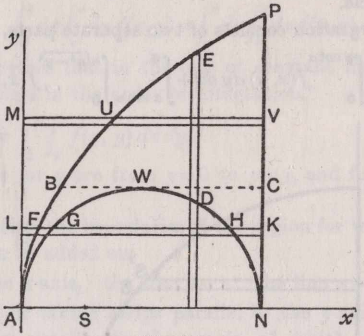

Fig. 299.

to add up elements in a strip parallel to the x-axis, and then add up these new strips. It will be noted that so long as y is less than $\frac{a}{2}$ such strips are broken into two parts as FG and HK, but for values of $y > \frac{a}{2}$ they are continuous as at UV. Let W be the point of contact of the tangent BC to the semicircle, which is parallel to the x-axis. The new integration must cover the three portions

(1) $AFBWGA$; (2) $WCKNHW$; (3) $BUPCWB$.

Referring to the figure in which the lines FK and UV parallel to the x-axis meet the y-axis at L and M respectively,

In region (1),

the limits for x are from LF to LG, and for y from 0 to NC.

In region (2),

the limits for x are from LH to LK, and for y from 0 to NC.

In region (3),

the limits for x are from MU to MV, and for y from NC to NP.

Hence the transformed result will be

$$\int_0^{\frac{a}{2}}\int_{\frac{y^2}{a}}^{\frac{a}{2}-\sqrt{\frac{a^2}{4}-y^2}} f(x, y)\,dy\,dx + \int_0^{\frac{a}{2}}\int_{\frac{a}{2}+\sqrt{\frac{a^2}{4}-y^2}}^{a} f(x, y)\,dy\,dx + \int_{\frac{a}{2}}^{a}\int_{\frac{y^2}{a}}^{a} f(x, y)\,dy\,dx.$$

Ex. 5. Change the order of integration in

$$\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r, \theta)\,r\,d\theta\,dr + \int_{\frac{\pi}{2}}^{\pi}\int_0^{a(1+\cos\theta)} f(r, \theta)\,r\,d\theta\,dr.$$

As the integral stands, integration is effected through a region bounded by the upper half cardioide $r=a(1+\cos\theta)$, the upper half circle $r=a\cos\theta$ and the intercepted portion of the initial line.

When the order of integration is changed we are to add elements along strips which are bounded by circular arcs as shown in Fig. 300, and then add all the strips. Let BC be the arc, with centre O, which touches the circle at B. Let MQ, $M'Q'$ be contiguous arcs with centres at O intercepted between the circle and the cardioide, and NP, $N'P'$ contiguous

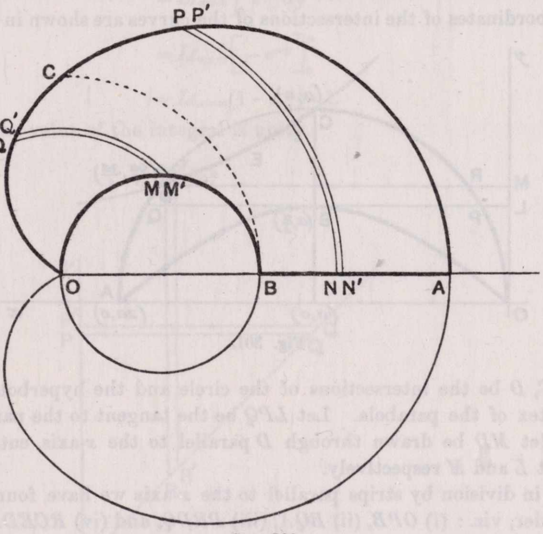

Fig. 300.

arcs with centres at O intercepted between the initial line and the cardioide. Then the new limits of integration are :

for θ, from $\theta = A\hat{O}M$ to $\theta = A\hat{O}Q$, for values of r from O to OB,

and for θ, from $\theta = 0$ to $\theta = A\hat{O}P$, for values of r from OB to OA.

The first of these accounts for the region $OMBCQO$.

The second accounts for the region $APCBA$.

And the transformed integral stands as

$$\int_0^a \int_{\cos^{-1}\frac{r}{a}}^{\cos^{-1}\frac{r-a}{a}} f(r, \theta)\, r\, dr\, d\theta + \int_a^{2a} \int_0^{\cos^{-1}\frac{r-a}{a}} f(r, \theta)\, r\, dr\, d\theta.$$

Ex. 6. Change the order of operation in the integration system

$$\int_0^a \int_{\frac{x}{2a}(2a-x)}^{\sqrt{2ax-x^2}} f(x, y)\, dx\, dy + \int_a^{\frac{9a}{5}} \int_{\frac{x}{2a}(2a-x)}^{\frac{2ax}{5x-3a}} f(x, y)\, dx\, dy$$
$$+ \int_{\frac{9a}{5}}^{2a} \int_{\frac{x}{2a}(2a-x)}^{\sqrt{2ax-x^2}} f(x, y)\, dx\, dy.$$

Here summation is effected by strips parallel to the y-axis within a region bounded by

(1) the parabola $2ay=x(2a-x)$,

(2) the semicircle $y^2=2ax-x^2$,

(3) the hyperbola $5xy=2ax+3ay$.

The coordinates of the intersections of the curves are shown in Fig. 301.

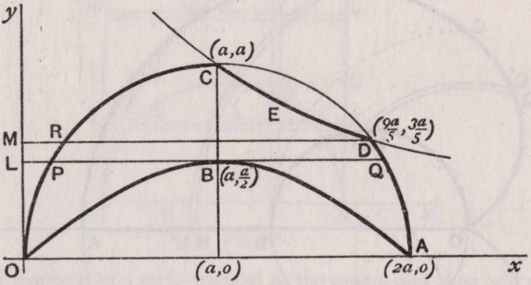

Fig. 301.

Let C, D be the intersections of the circle and the hyperbola, and B the vertex of the parabola. Let LPQ be the tangent to the parabola at B, and let MD be drawn through D parallel to the x-axis, cutting the y-axis at L and M respectively.

Then in division by strips parallel to the x-axis we have four regions to consider, viz.: (i) OPB, (ii) BQA, (iii) $PRDQ$, and (iv) $RCEDR$.

We then obtain for the transformed result,

$$\int_0^{\frac{a}{2}}\int_{a-\sqrt{a^2-y^2}}^{a-\sqrt{a^2-2ay}} f(x,y)\,dy\,dx+\int_0^{\frac{a}{2}}\int_{a+\sqrt{a^2-2ay}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dy\,dx$$

$$+\int_{\frac{a}{2}}^{\frac{3a}{5}}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dy\,dx+\int_{\frac{3a}{5}}^{a}\int_{a-\sqrt{a^2-y^2}}^{\frac{3ay}{5y-2a}} f(x,y)\,dy\,dx,$$

the several items of integration referring to the respective regions enumerated.

Ex. 7. Evaluate the integral $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$. [St. John's Coll., 1889.]

As the integral stands, summation is conducted over the infinite region bounded by the line $y=x$, the y-axis, and an infinite boundary, say $y=a$, where a is infinitely large, and along which the subject of integration $\frac{e^{-y}}{y}$ is ultimately zero, the strips being taken parallel to the y-axis.

Change the order of integration, taking strips parallel to the x-axis.

The new limits are: for x, from $x=0$ to $x=y$

and for y, from $y=0$ to $y=a$.

And the integral becomes $Lt_{a=\infty}\int_0^a\int_0^y \frac{e^{-y}}{y}\,dy\,dx$

$$=Lt_{a=\infty}\int_0^a \frac{e^{-y}}{y}\Big[\,x\,\Big]_0^y dy$$

$$=Lt_{a=\infty}\int_0^a e^{-y}\,dy$$

$$=Lt_{a=\infty}\Big[-e^{-y}\Big]_0^a$$

$$=Lt_{a=\infty}(1-e^{-a})=1.$$

Hence the value of the integral is unity.

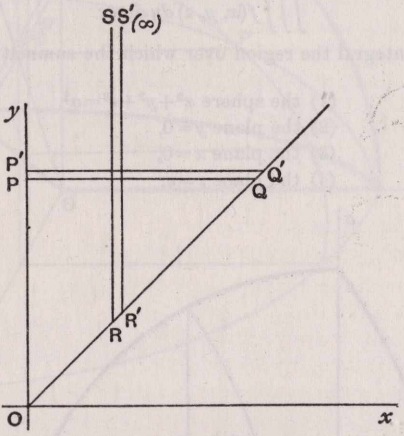

Fig. 302.

Ex. 8. Change the order of integration of the triple integral

$$\int_0^a\int_0^{a-x}\int_0^{a-x-y} f(x, y, z)\,dx\,dy\,dz$$

in all possible permutations of dx, dy, dz.

The integration referred to is evidently through the volume bounded by the three coordinate planes and the plane $x+y+z=a$.

The integration as it stands supposes this region divided into volume-elements $\delta x\,\delta y\,\delta z$ by means of slices or laminae parallel to the plane $x=0$, subdivided into tubes or prisms parallel to the z-axis, and these further subdivided into elementary cuboids by planes parallel to the plane $z=0$. The other modes of division and summation are obvious.

And the transformations are

$$\int_0^a\int_0^{a-x}\int_0^{a-z-x} f(x, y, z)\,dx\,dz\,dy,$$

$$\int_0^a\int_0^{a-y}\int_0^{a-y-z} f(x, y, z)\,dy\,dz\,dx,$$

$$\int_0^a \int_0^{a-y} \int_0^{a-x-y} f(x, y, z)\, dy\, dx\, dz,$$
$$\int_0^a \int_0^{a-z} \int_0^{a-z-x} f(x, y, z)\, dz\, dx\, dy,$$
$$\int_0^a \int_0^{a-z} \int_0^{a-y-z} f(x, y, z)\, dz\, dy\, dx.$$

Ex. 9. Express the integral

$$\int_0^a \int_0^{\frac{1}{\sqrt{2}}\sqrt{a^2-x^2}} \int_y^{\sqrt{a^2-x^2-y^2}} f(x, y, z)\, dx\, dy\, dz$$

as an integral of the form

$$\iiint f(x, y, z)\, dy\, dz\, dx.$$

In the first integral the region over which the summation is conducted is bounded by

(1) the sphere $x^2+y^2+z^2=a^2$,
(2) the plane $y=0$,
(3) the plane $x=0$,
(4) the plane $z=y$,

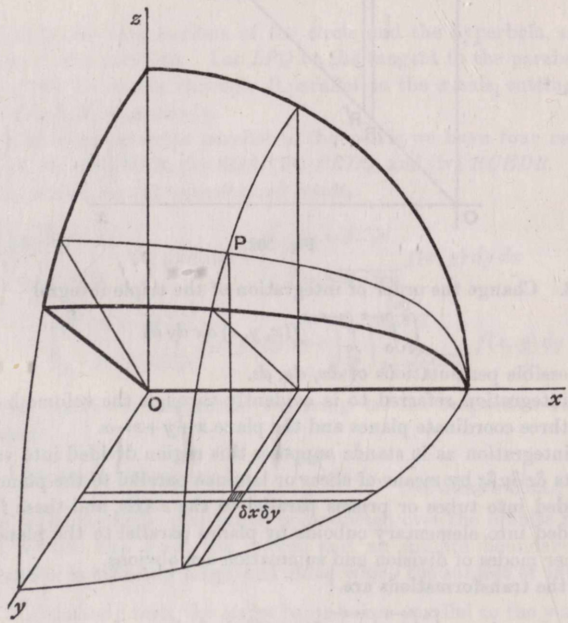

Fig. 303.

and the first integration was that of elementary cuboids in the tubes on $\delta x\, \delta y$ for base and parallel to the z-axis. The second with regard to y

added the tubes in a slice parallel to the plane $x=0$, and the third, integrated with regard to x, added up the slices.

We are now to construct tubes on $\delta y\,\delta z$ for base, and the limits for the first integration will be for x from 0 to $\sqrt{a^2-y^2-z^2}$.

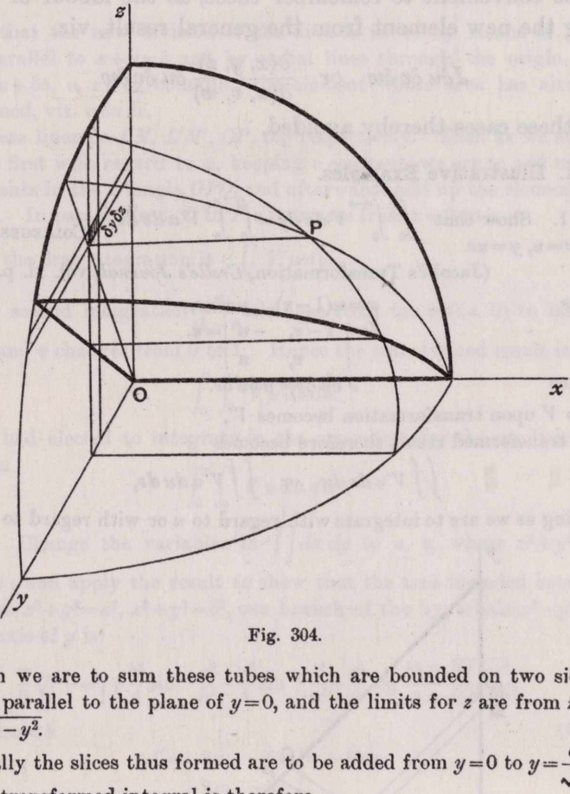

Fig. 304.

Then we are to sum these tubes which are bounded on two sides by planes parallel to the plane of $y=0$, and the limits for z are from $z=y$ to $z=\sqrt{a^2-y^2}$.

Finally the slices thus formed are to be added from $y=0$ to $y=\dfrac{a}{\sqrt{2}}$.

The transformed integral is therefore

$$\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}}\int_0^{\sqrt{a^2-y^2-z^2}} f(x, y, z)\,dy\,dz\,dx.$$

830. Examples of Change of the Variables.

We shall use the notation V for any function of the original variables and V' for the same function expressed in terms of the new variables.

In the case of change from Cartesians to Polars for two-dimension problems, the element of area $\delta x\,\delta y$ is replaced by $r\,\delta\theta\,\delta r$, and for three-dimension problems $\delta x\,\delta y\,\delta z$ is replaced by $r^2 \sin\theta\, \delta\theta\, \delta\phi\, \delta r$. In converting from three-dimension Cartesians to cylindrical coordinates $\delta x\, \delta y\, \delta z$ is replaced by the new element of volume $r\, \delta\theta\, \delta r\, \delta z$.

It is convenient to remember these, as the labour of calculating the new element from the general result, viz.

$$J\delta u\, \delta v\, \delta w \quad \text{or} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)}\delta u\, \delta v\, \delta w$$

is in these cases thereby avoided.

831. Illustrative Examples.

Ex. 1. Show that $\int_0^c \int_0^{c-x} V\,dx\,dy = \int_0^1 \int_0^c V'u\,dv\,du$, [Colleges, 1881.] if $y+x=u$, $y=uv$.

(Jacobi's Transformation, *Crelle's Journal*, vol. xi. p. 307.*)

Here $x=u(1-v),\quad y=uv,$

$$J=\begin{vmatrix} 1-v, & -u \\ v, & u \end{vmatrix}=u.$$

Hence $J\,\delta u\,\delta v = u\,\delta u\,\delta v.$

Also V upon transformation becomes V'.

The transformed result therefore becomes

$$\iint V'u\,dv\,du \quad \text{or} \quad \iint V'u\,du\,dv,$$

according as we are to integrate with regard to u or with regard to v first.

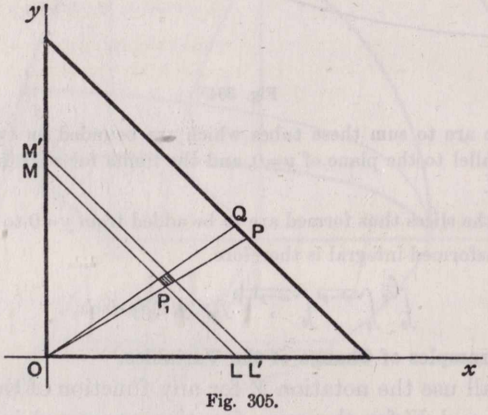

Fig. 305.

In our example the former is the case. We now have to determine the proper limits of integration.

In the original form the integration was for y from 0 to $c-x$ and for x from 0 to c.

* Gregory's *Examples*, p. 41.

The region through which the integration is to be conducted is then that bounded by the axes and the straight line $x+y=c$.

The transformation formulae

$$x+y=u, \quad y=\frac{v}{1-v}x$$

indicate that the new division of the area is to be by means of lines drawn parallel to $x+y=c$ and by radial lines through the origin, the lines u, $u+\delta u$, v, $v+\delta v$ bounding the element whose area has already been formed, viz. $u\,\delta u\,\delta v$.

Let these lines be LM, $L'M'$, OP, OQ respectively. Then as we are to integrate first with regard to u, keeping v constant, we are to add up all the elements in the triangle OPQ, and afterwards add up the elementary triangles. In passing from O to P u increases from $u=0$ to $u=c$.

Hence the first integration is $\int_0^c V'u\,du$.

In the second integration $\frac{v}{1-v}$ changes from $\tan 0$ (*i.e.* 0) to $\tan 90°$ (*i.e.* ∞), and v changes from 0 to 1. Hence the transformed result is

$$\int_0^1\int_0^c V'u\,dv\,du.$$

If we had elected to integrate in the opposite order the result would have been

$$\int_0^c\int_0^1 V'u\,du\,dv.$$

Ex. 2. Change the variables in $\iint dx\,dy$ to u, v, where $x^2+y^2=u$, $x^2-y^2=v$; and apply the result to show that the area included between the circles $x^2+y^2=a^2$, $x^2+y^2=b^2$, one branch of the hyperbola $x^2-y^2=c^2$ and the axis of y is

$$\frac{\pi}{8}(b^2-a^2)+\frac{b^2}{4}\sin^{-1}\frac{c^2}{b^2}-\frac{a^2}{4}\sin^{-1}\frac{c^2}{a^2}+\frac{c^2}{4}\log\frac{b^2+\sqrt{b^4-c^4}}{a^2+\sqrt{a^4-c^4}},$$

where $c<a<b$. (R.P.)

Here
$$J'=\begin{vmatrix} 2x, & 2y \\ 2x, & -2y \end{vmatrix}=-8xy,$$

and therefore
$$J=-\frac{1}{8}\frac{1}{xy}=-\frac{1}{4}\frac{1}{\sqrt{u^2-v^2}},$$

and the transformed integral is $-\frac{1}{4}\iint\frac{du\,dv}{\sqrt{u^2-v^2}}$, where it remains to assign the proper limits.

The region over which summation is to be conducted is the portion $ABECDFA$ of Fig. 306.

If OFE be the asymptote of the rectangular hyperbola, the area of the portion $FECD$ is plainly $\frac{1}{8}(\pi b^2-\pi a^2)$. We have then to turn our attention to the portion $ABEF$. And for this the line FE is a case of rectangular hyperbola, viz. $v=0$. Hence for this region the limits are constant, viz. $u=a^2$ and $u=b^2$, $v=0$ to $v=c^2$, and with this assignment of limits we may omit the $-$ sign and take

$$\text{Area } ABEF=\frac{1}{4}\int_{a^2}^{b^2}\int_0^{c^2}\frac{du\,dv}{\sqrt{u^2-v^2}}$$

$$=\frac{1}{4}\int_{a^2}^{b^2}\left[\sin^{-1}\frac{v}{u}\right]_{v=0}^{v=c^2}du$$

$$=\frac{1}{4}\int_{a^2}^{b^2}\sin^{-1}\frac{c^2}{u}\,du$$

$$=\frac{1}{4}\left[u\sin^{-1}\frac{c^2}{u}\right]_{a^2}^{b^2}+\frac{1}{4}\int_{a^2}^{b^2}\frac{c^2}{\sqrt{u^2-c^4}}\,du$$

$$=\frac{1}{4}\left[u\sin^{-1}\frac{c^2}{u}+c^2\cosh^{-1}\frac{u}{c^2}\right]_{a^2}^{b^2}$$

$$=\frac{b^2}{4}\sin^{-1}\frac{c^2}{b^2}-\frac{a^2}{4}\sin^{-1}\frac{c^2}{a^2}+\frac{c^2}{4}\log\frac{b^2+\sqrt{b^4-c^4}}{a^2+\sqrt{a^4-c^4}}.$$

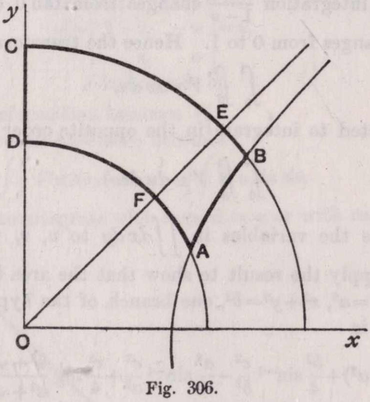

Fig. 306.

Hence adding the portion $FECD$ already found, we have

Area of $ABECDFA$

$$=\frac{\pi}{8}(b^2-a^2)+\frac{b^2}{4}\sin^{-1}\frac{c^2}{b^2}-\frac{a^2}{4}\sin^{-1}\frac{c^2}{a^2}+\frac{c^2}{4}\log\frac{b^2+\sqrt{b^4-c^4}}{c^2+\sqrt{a^4-c^4}}.$$

Ex. 3. Show by transforming to polar coordinates that

$$\int_0^{a\tan\alpha}\int_0^{a\tan\beta}\frac{dx\,dy}{(x^2+y^2+a^2)^2}$$

$$=\frac{1}{2a^2}\{\sin\alpha\tan^{-1}(\tan\beta\cos\alpha)+\sin\beta\tan^{-1}(\tan\alpha\cos\beta)\}.$$

[Colleges, 1887.]

Putting $x=r\cos\theta$, $y=r\sin\theta$ and remembering that the element of area $\delta x\,\delta y$ is replaced in polars by $r\,\delta\theta\,\delta r$, we have $\iint\frac{r\,d\theta\,dr}{(r^2+a^2)^2}$; and it remains to assign the limits for r and θ.

The region of integration is the rectangle bounded by $x=0$, $x=a\tan\alpha$, $y=0$, $y=a\tan\beta$. If γ be the angle which the diagonal through the origin makes with the x-axis, $\tan\gamma=\dfrac{\tan\beta}{\tan\alpha}$.

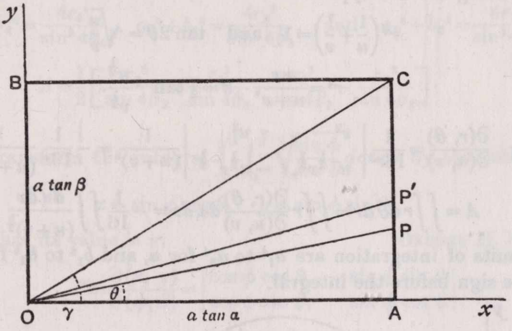

Fig. 307.

The whole integration consists of two parts, viz.

$$\int_0^\gamma\int_0^{a\tan\alpha\sec\theta}\frac{r\,d\theta\,dr}{(r^2+a^2)^2}+\int_\gamma^{\frac{\pi}{2}}\int_0^{a\tan\beta\operatorname{cosec}\theta}\frac{r\,d\theta\,dr}{(r^2+a^2)^2},$$

the first referring to the portion of the rectangle between the diagonal and the x-axis, and the second to the part between the diagonal and the y-axis.

This is clearly

$$\frac{1}{2}\int_0^\gamma\left[-\frac{1}{r^2+a^2}\right]_0^{a\tan\alpha\sec\theta}d\theta+\frac{1}{2}\int_\gamma^{\frac{\pi}{2}}\left[-\frac{1}{r^2+a^2}\right]_0^{a\tan\beta\operatorname{cosec}\theta}d\theta$$

$$=\frac{1}{2a^2}\int_0^\gamma\left(1-\frac{\cos^2\theta}{\cos^2\theta+\tan^2\alpha}\right)d\theta+\frac{1}{2a^2}\int_\gamma^{\frac{\pi}{2}}\left(1-\frac{\sin^2\theta}{\sin^2\theta+\tan^2\beta}\right)d\theta$$

$$=\frac{1}{2a^2}\int_0^\gamma\frac{\tan^2\alpha\,d\theta}{\sec^2\alpha\cos^2\theta+\tan^2\alpha\sin^2\theta}+\frac{1}{2a^2}\int_\gamma^{\frac{\pi}{2}}\frac{\tan^2\beta\,d\theta}{\sec^2\beta\sin^2\theta+\tan^2\beta\cos^2\theta}$$

$$=\frac{1}{2a^2}\int_0^\gamma\frac{\sec^2\theta\,d\theta}{\operatorname{cosec}^2\alpha+\tan^2\theta}+\frac{1}{2a^2}\int_\gamma^{\frac{\pi}{2}}\frac{\operatorname{cosec}^2\theta\,d\theta}{\operatorname{cosec}^2\beta+\cot^2\theta}$$

$$=\frac{1}{2a^2}\Big[\sin\alpha\tan^{-1}(\sin\alpha\tan\theta)\Big]_0^\gamma+\frac{1}{2a^2}\Big[\sin\beta\tan^{-1}(\sin\beta\cot\theta)\Big]_{\frac{\pi}{2}}^\gamma$$

$$=\frac{1}{2a^2}\sin\alpha\tan^{-1}(\cos\alpha\tan\beta)+\frac{1}{2a^2}\sin\beta\tan^{-1}(\cos\beta\tan\alpha).$$

Ex. 4. Two lemniscates whose equations are $r^2=a_1^2\cos 2\theta$ and $r^2=b_1^2\sin 2\theta$ respectively, are drawn through a point P, and two others whose respective equations are $r^2=a_2^2\cos 2\theta$ and $r^2=b_2^2\sin 2\theta$ are drawn through Q. P and Q are both in the first quadrant. The remaining intersections of the four curves in the first quadrant are R and S. The coordinates of these points are respectively (r_1, θ_1), (r_2, θ_2), (r_3, θ_3), (r_4, θ_4).

It is required to show that the curvilinear quadrilateral thus enclosed has an area
$$\frac{1}{2}\left\{\left(\frac{r_3^2}{\sin 4\theta_3}+\frac{r_4^2}{\sin 4\theta_4}\right)-\left(\frac{r_1^2}{\sin 4\theta_1}+\frac{r_2^2}{\sin 4\theta_2}\right)\right\}.$$

Considering the two types $r^2=u^{\frac{1}{2}}\cos 2\theta$, $r^2=v^{\frac{1}{2}}\sin 2\theta$, we obtain
$$r^4\left(\frac{1}{u}+\frac{1}{v}\right)=1 \quad\text{and}\quad \tan 2\theta=\sqrt{\frac{u}{v}},$$

i.e.
$$r^4=\frac{uv}{u+v},\quad \theta=\tfrac{1}{2}\tan^{-1}\frac{u^{\frac{1}{2}}}{v^{\frac{1}{2}}}.$$

Hence $\dfrac{\partial(r,\theta)}{\partial(u,v)}=\dfrac{1}{16r^3}\begin{vmatrix} v^2, & u^2 \\ u^{-\frac{1}{2}}v^{\frac{1}{2}}, & -u^{\frac{1}{2}}v^{-\frac{1}{2}}\end{vmatrix}\dfrac{1}{(u+v)^3}=-\dfrac{1}{16r}\,\dfrac{1}{(u+v)^{\frac{3}{2}}}.$

Also $A=\iint r\,d\theta\,dr=\iint r\dfrac{\partial(r,\theta)}{\partial(u,v)}\,du\,dv=-\dfrac{1}{16}\iint\dfrac{du\,dv}{(u+v)^{\frac{3}{2}}}$

The limits of integration are a_1^4 to a_2^4 for u, and b_1^4 to b_2^4 for v taking a positive sign before the integral.

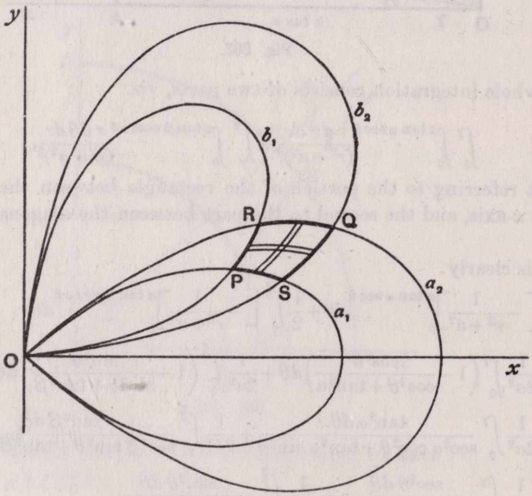

Fig. 308.

Hence $A=\dfrac{1}{16}\displaystyle\int_{a_1^4}^{a_2^4}\int_{b_1^4}^{b_2^4}\frac{du\,dv}{(u+v)^{\frac{3}{2}}}$

$$=\frac{1}{16}\int_{a_1^4}^{a_2^4}\left[-\frac{2}{(u+v)^{\frac{1}{2}}}\right]_{b_1^4}^{b_2^4}du$$

$$=\frac{1}{8}\int_{a_1^4}^{a_2^4}\left\{\frac{1}{(b_1^4+u)^{\frac{1}{2}}}-\frac{1}{(b_2^4+u)^{\frac{1}{2}}}\right\}du$$

$$=\frac{1}{4}\left[(b_1^4+u)^{\frac{1}{2}}-(b_2^4+u)^{\frac{1}{2}}\right]_{a_1^4}^{a_2^4}$$

$$=\tfrac{1}{4}\left[(b_1^4+a_2^4)^{\frac{1}{2}}-(b_1^4+a_1^4)^{\frac{1}{2}}-(b_2^4+a_2^4)^{\frac{1}{2}}+(b_2^4+a_1^4)^{\frac{1}{2}}\right]$$

Now the curves a_1, b_1 intersect at r_1, θ_1, and

$$a_1^4+b_1^4=\frac{r_1^4}{\cos^2 2\theta_1}+\frac{r_1^4}{\sin^2 2\theta_1}=\frac{4r_1^4}{\sin^2 4\theta_1}.$$

Similarly,

$$a_1^4+b_2^4=\frac{4r_4^4}{\sin^2 4\theta_4},\quad a_2^4+b_1^4=\frac{4r_3^4}{\sin^2 4\theta_3},\quad \text{and}\quad a_2^4+b_2^4=\frac{4r_2^4}{\sin^2 4\theta_2}.$$

Hence $$A=\frac{1}{2}\left[\frac{r_3^2}{\sin 4\theta_3}+\frac{r_4^2}{\sin 4\theta_4}-\frac{r_1^2}{\sin 4\theta_1}-\frac{r_2^2}{\sin 4\theta_2}\right].$$

Ex. 5. Transform the integral $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin\phi}{\sin\theta}}\,d\phi\,d\theta$ by the substitution

$$x=\sin\phi\cos\theta,\quad y=\sin\phi\sin\theta,$$

and show that its value is π. [Oxford II. P., 1880.]

Here $$J'\equiv\frac{\partial(x,y)}{\partial(\phi,\theta)}=\begin{vmatrix}\cos\phi\cos\theta, & -\sin\phi\sin\theta\\ \cos\phi\sin\theta, & \sin\phi\cos\theta\end{vmatrix}$$

$$=\sin\phi\cos\phi$$

and $$\iint\sqrt{\frac{\sin\phi}{\sin\theta}}\,d\phi\,d\theta=\iint\frac{1}{\sin\phi\cos\phi}\sqrt{\frac{\sin\phi}{\sin\theta}}\,dy\,dx$$

$$=\iint\frac{1}{\sqrt{y}}\,\frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx.$$

The original limits were $\theta=0$ to $\theta=\frac{\pi}{2}$ and $\phi=0$ to $\phi=\frac{\pi}{2}$.

Now $x^2+y^2=\sin^2\phi$ and $\frac{y}{x}=\tan\theta$.

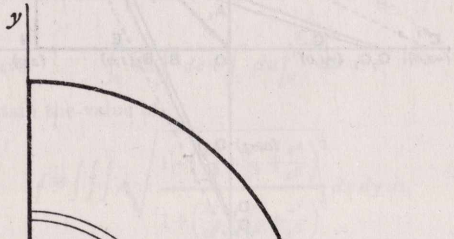

Fig. 309.

We may then regard the integration as extending through the positive quadrant of the circle $x^2+y^2=1$. The limits for x will then be from $x=0$ to $x=\sqrt{1-y^2}$, and for y from $y=0$ to $y=1$.

Keeping y constant

$$\int_0^1\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{y}}\frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx$$

$$=\int_0^1 \frac{1}{\sqrt{y}}\left[\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{1-y^2}} dy$$

$$=\int_0^1 \frac{1}{\sqrt{y}}\cdot\frac{\pi}{2}\,dy=\frac{\pi}{2}\Big[\,2\sqrt{y}\,\Big]_0^1=\pi.$$

Ex. 6. Show that if $x=u(1+v)$ and $y=v(1+u)$,

$$\int_0^2\int_0^x\{(x-y)^2+2(x+y)+1\}^{-\frac{1}{2}}\,dx\,dy=\int_0^1\int_v^{\frac{2}{1+v}} dv\,du,$$

and prove the identity by finding the value of each integral.

[Oxford II. P., 1889.]

Here $J=\begin{vmatrix}1+v, & u\\ v, & 1+u\end{vmatrix}=1+u+v$

and $(x-y)^2+2(x+y)+1=(u-v)^2+2(u+v)+4uv+1=(u+v+1)^2.$

Hence $\iint\{(x-y)^2+2(x+y)+1\}^{-\frac{1}{2}}\,dx\,dy=\iint dv\,du.$

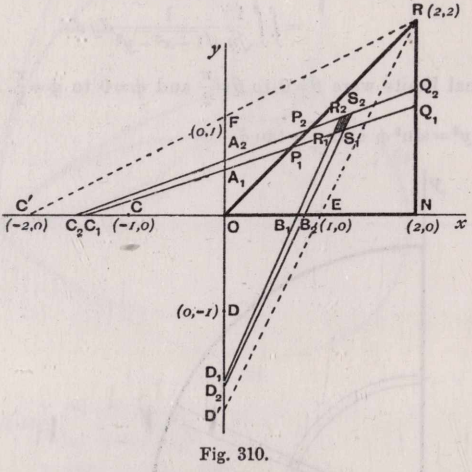

Fig. 310.

Next consider the limits. The region through which the summation in the first integral is to be effected is that bounded by the x-axis, the line $y=x$, and the ordinate $x=2$; *i.e.* the triangle ONR in the accompanying figure (Fig. 310).

The loci $u=$const., $v=$const. are respectively the lines

$$\frac{x}{u}-\frac{y}{1+u}=1,\quad \frac{y}{v}-\frac{x}{1+v}=1.$$

We are to integrate first with regard to u, keeping v constant, *i.e.* along a strip formed by the lines v, $v+\delta v$. These lines, represented by $C_1A_1P_1Q_1$ and $C_2A_2P_2Q_2$ respectively in the figure, form a strip of gradually widening breadth in passing from P to Q, for, as the intercept OC_1 on the x-axis increases (negatively), the line rotates counterclockwise. It begins its rotation, as far as our triangle is concerned, with coincidence with ON, for which $v=0$, and ends its rotation when $v=1$, when the line is $\frac{y}{1}-\frac{x}{2}=1$, and passes through $R(2, 2)$, taking the position $C'R$. Now along the whole length of OR, *i.e.* $y=x$, we have $u=v$, and along the whole length of NR, *i.e.* $x=2$, we have $2=u(1+v)$, *i.e.* $u=\frac{2}{1+v}$.

Hence, in integrating along the strip $P_1Q_1Q_2P_2$, keeping $v=$constant u changes from $u=v$ at P_1 to $u=\frac{2}{1+v}$ at Q_1.

Hence the limits for u are v and $\frac{2}{1+v}$, and for v, 0 and 1.

Hence $$\int_0^2\int_0^x \{(x-y)^2+2(x+y)+1\}^{-\frac{1}{2}}\,dx\,dy=\int_0^1\int_v^{\frac{2}{1+v}} dv\,du.$$

The student may show without difficulty that each side of the identity takes the value $2\log 2-\frac{1}{2}$.

If, however, the integration had been conducted in the reverse order, integrating first for strips along which u is constant, it is to be noted that the character of such strips changes when the line $D_1B_1R_1$ passes through $E(1, 0)$, the strips being terminated by OE $(v=0)$ and OR $(v=u)$ for the portion OER and by EN $(v=0)$ and NR $\left(v=\frac{2}{u}-1\right)$ for the second part.

We then have $$\int_0^1 du\int_0^u dv+\int_1^2 du\int_0^{\frac{2}{u}-1} dv.$$

Ex. 7. Obtain the value of

$$I\equiv\iiint\sqrt{\frac{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}}{1+\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}}}\,dx\,dy\,dz,$$

the integral being taken for all values of x, y, z, such that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}<1.$$

We shall divide up the ellipsoidal volume into a set of thin homoeoidal shells, that is shells bounded by ellipsoidal surfaces, concentric, similar and similarly situated with the bounding surface. Let a typical member of this family of surfaces be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\rho^2,$$

ρ lying between 0 and 1.

Then the volume of the shell bounded by ρ and $\rho+\delta\rho$ is

$$\delta\{\tfrac{4}{3}\pi(a\rho)(b\rho)(c\rho)\}=4\pi abc\rho^2\,\delta\rho,$$

and the value of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$ at points between the boundaries of the shell differs from ρ^2 by an infinitesimal only.

Hence $$I=\int_0^1\sqrt{\frac{1-\rho}{1+\rho}}\cdot 4\pi abc\rho^2\,d\rho.$$

Write $\rho=\cos\phi$.

Then $$I=\int_0^{\frac{\pi}{2}}\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}\cdot 4\pi abc\,.\cos^2\phi\sin\phi\,d\phi$$

$$=4\pi abc\int_0^{\frac{\pi}{2}}(1-\cos\phi)\cos^2\phi\,d\phi$$

$$=4\pi abc\left(\frac{1}{2}\frac{\pi}{2}-\frac{2}{3}\right)$$

$$=\frac{1}{3}\pi abc(3\pi-8).$$

Ex. 8. If $xu+yv=a^2$ and $xv-yu=0$, prove that

$$\iint V\,dx\,dy=-\iint\frac{V'a^4\,du\,dv}{(u^2+v^2)^2}.$$

And if the limits in the former integral are $y=0$ to $y=\sqrt{a^2-x^2}$ and $x=0$ to $x=a$, investigate the limits in the latter. [St. John's, 1885.]

Here $$x=\frac{a^2u}{u^2+v^2},\quad y=\frac{a^2v}{u^2+v^2},$$

and $$J=\frac{a^4}{(u^2+v^2)^4}\begin{vmatrix} v^2-u^2, & -2uv \\ -2uv, & u^2-v^2\end{vmatrix}=-\frac{a^4}{(u^2+v^2)^2};$$

whence $$\iint V\,dx\,dy=-\iint\frac{V'a^4\,du\,dv}{(u^2+v^2)^2},$$

where V' is what V becomes after substitution for x and y in terms of u and v.

Next, as to the limits. In $\int_0^a\int_0^{\sqrt{a^2-x^2}}V\,dx\,dy$ the integration is over the region bounded by the positive quadrant of the circle $x^2+y^2=a^2$.

Eliminating v and u alternately, we have

$$x^2+y^2-\frac{a^2}{u}x=0,\quad x^2+y^2-\frac{a^2}{v}y=0,$$

and the curves $u=$const., $v=$const., are orthogonal circles touching the axes at the origin. Let us integrate first with regard to v, then with regard to u. Whilst integrating with regard to v, the element $J\,\delta u\,\delta v$ is bounded always by the two complete semicircles u and $u+\delta u$, so long as this ring lies entirely within the circle $x^2+y^2=a^2$, and the limits for v are from the case where the v-curve is a circle of infinite radius coinciding with the x-axis, to the case where it is a point circle at the origin. The radius is $\frac{a^2}{2v}$. Hence the limits for v are from $v=0$ to $v=\infty$. And the u-circle has a radius $\frac{a^2}{2u}$, and changes from a circle of radius $\frac{a}{2}$ to a circle of radius zero, *i.e.* u changes from $u=a$ to $u=\infty$.

When the u-circle has a radius in excess of $\frac{a}{2}$, the limits for v will be from the value of v for which the u-circle cuts the a-circle, viz. at P, in Fig. 311, to the value of v for which the v-circle becomes a point-circle at the origin, *i.e.* when $v=\infty$.

Now at P we have

$$\frac{a^2}{v}y=x^2+y^2=a^2 \text{ and } \frac{a^2}{u}x=a^2,$$

i.e. at that point $x=u$ and $y=v$, whence $v^2=a^2-u^2$.

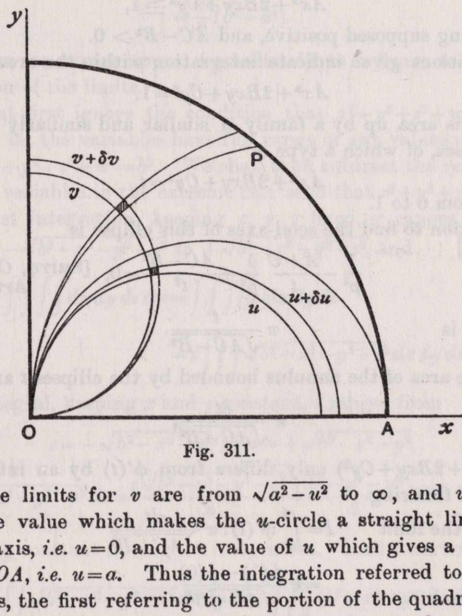

Fig. 311.

Hence the limits for v are from $\sqrt{a^2-u^2}$ to ∞, and u now varies between the value which makes the u-circle a straight line coincident with the y-axis, *i.e.* $u=0$, and the value of u which gives a semicircle on the radius OA, *i.e.* $u=a$. Thus the integration referred to divides into two portions, the first referring to the portion of the quadrant included in a semicircle on OA for diameter, and the other to the remainder of the quadrant.

Thus

$$\int_0^a\int_0^{\sqrt{a^2-x^2}} V\,dx\,dy=a^4\int_a^\infty\int_0^\infty \frac{V'\,du\,dv}{(u^2+v^2)^2}+a^4\int_0^a\int_{\sqrt{a^2-u^2}}^\infty \frac{V'\,du\,dv}{(u^2+v^2)^2}.$$

It may be observed that the transformation formulae $x=\frac{a^2u}{u^2+v^2}$, $y=\frac{a^2v}{u^2+v^2}$ indicate an inversion from the Cartesian coordinates x, y of a point within the circle, with a for the constant of inversion, to a point whose coordinates are u, v, which lies without the circle. Hence as (x, y) is to traverse the *interior* of the quadrant of the circle, (u, v) is to traverse the portion of the first quadrant of space which lies *outside* the quadrant of the circle, and therefore, the circle having equation $u^2+v^2=a^2$ in the new coordinates, the limits must be

$$v=\sqrt{a^2-u^2} \text{ to } v=\infty \text{ from } u=0 \text{ to } u=a,$$

and $\quad v=0 \quad$ to $v=\infty$ from $u=a$ to $u=\infty$,

which agrees with the result stated.

Ex. 9. Obtain the value of the integral

$$I\equiv\iint\phi'(Ax^2+2Bxy+Cy^2)\,dx\,dy,$$

extended to all values of x, y which satisfy the condition

$$Ax^2+2Bxy+Cy^2\leqq 1,$$

A and C being supposed positive, and $AC-B^2>0$.

The conditions given indicate integration within the area bounded by the ellipse

$$Ax^2+2Bxy+Cy^2=1.$$

Divide this area up by a family of similar and similarly situated concentric ellipses, of which a type is

$$Ax^2+2Bxy+Cy^2=t,$$

t varying from 0 to 1.

The equation to find the semi-axes of this ellipse is

$$\frac{1}{\rho^4}-\frac{A+C}{t}\frac{1}{\rho^2}+\frac{AC-B^2}{t^2}=0,$$ [SMITH, *Conic Sections*, Art. 171.]

and its area is

$$\pi\frac{t}{\sqrt{AC-B^2}}.$$

Hence the area of the annulus bounded by the ellipses t and $t+\delta t$ is

$$\pi\frac{\delta t}{\sqrt{AC-B^2}},$$

and $\phi'(Ax^2+2Bxy+Cy^2)$ only differs from $\phi'(t)$ by an infinitesimal at any point of this ring.

Hence in the limit

$$I=\int_0^1\phi'(t)\,.\,\pi\frac{dt}{\sqrt{AC-B^2}}$$

$$=\pi\frac{\phi(1)-\phi(0)}{\sqrt{AC-B^2}}.$$

Ex. 10. Prove that $\iint du\,dv$ over a portion of the surface $w=0$ is

$$\iint\frac{\partial(u, v, w)}{\partial(x, y, z)}\frac{dS}{\left\{\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right\}^{\frac{1}{2}}},$$

u, v, w being functions of x, y, z.

Let x, y, z be a point on the surface $w=0$ at which an element of the normal is δn. Then $\delta n=\dfrac{\delta w}{h}$, where $h^2=w_x{}^2+w_y{}^2+w_z{}^2$ (Art. 789).

Also $\delta S \,.\, \delta n$ is an element of volume, and may be replaced in volume-integration by

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}\,\delta u\,\delta v\,\delta w \quad \text{(Art. 794)},$$

i.e. $\delta S \,.\, \dfrac{\delta w}{h}$ may be replaced by $\dfrac{1}{\dfrac{\partial(u, v, w)}{\partial(x, y, z)}}\,\delta u\,\delta v\,\delta w$

and $$\iint du\,dv = \iint \frac{\partial(u, v, w)}{\partial(x, y, z)}\,\frac{dS}{h}.$$

Ex. 11. Prove that $I \equiv \iiiint dx\,dy\,dz\,dw$ for all values of the variables for which $x^2+y^2+z^2+w^2$ is not less than a^2 and not greater than b^2 is

$$= \frac{\pi^2}{2}(b^4 - a^4).$$

In this case we cannot appeal immediately to a figure to help in the determination of the limits.

We may at first ignore the condition that $x^2+y^2+z^2+w^2$ is not less than a^2, and let the variables have full range of any values up to such as will make $x^2+y^2+z^2+w^2=b^2$. We shall then subtract the result for such as make the variables in the extreme case such that $x^2+y^2+z^2+w^2 \not> a^2$.

In the first integration, keeping x, y, z fixed, w ranges through all values from $-\sqrt{b^2-x^2-y^2-z^2}$ to $+\sqrt{b^2-x^2-y^2-z^2}$, and

$$\iiiint dx\,dy\,dz\,dw = \iiint [w]\,dx\,dy\,dz$$

$$= 2\iiint \sqrt{b^2-x^2-y^2-z^2}\,dx\,dy\,dz.$$

In this integral, keeping x and y constant, z ranges from

$$z = -\sqrt{b^2-x^2-y^2} \text{ to } z = +\sqrt{b^2-x^2-y^2},$$

and $$\int \sqrt{b^2-x^2-y^2-z^2}\,dz = \frac{z\sqrt{b^2-x^2-y^2-z^2}}{2} + \frac{b^2-x^2-y^2}{2}\sin^{-1}\frac{z}{\sqrt{b^2-x^2-y^2}},$$

x and y being constant during the integration. And inserting the limits,

$$\iiint \sqrt{b^2-x^2-y^2-z^2}\,dx\,dy\,dz = \iint \frac{\pi}{2}(b^2-x^2-y^2)\,dx\,dy.$$

We have now reduced $\iiiint dx\,dy\,dz\,dw$ to $2\,.\,\dfrac{\pi}{2}\iint (b^2-x^2-y^2)\,dx\,dy$; and now we are to integrate with regard to y, keeping x constant, and the limits for y are from $-\sqrt{b^2-x^2}$ to $+\sqrt{b^2-x^2}$.

Also $$\int (b^2-x^2-y^2)\,dy = (b^2-x^2)y - \frac{y^3}{3},$$

and $$= 2\left[\tfrac{2}{3}(b^2-x^2)^{\frac{3}{2}}\right]$$

when the limits are taken.

We have now arrived at $\frac{4}{3}\pi\int(b^2-x^2)^{\frac{3}{2}}dx$, the limits for x being from $-b$ to $+b$. Put $x=b\sin\theta$. The integral then becomes

$$2\cdot\frac{4}{3}\pi\int_0^{\frac{\pi}{2}} b^3\cos^3\theta\, b\cos\theta\, d\theta \quad\text{or}\quad \frac{8}{3}\pi b^4\frac{3}{4}\frac{1}{2}\frac{\pi}{2},\quad \textit{i.e. } \frac{\pi^2}{2}b^4.$$

Now, in exactly the same way we may see, as is indeed obvious at once, that the amount included in excess by giving the variables free play up to the case $x^2+y^2+z^2+w^2=b^2$ instead of excluding those values which make $x^2+y^2+z^2+w^2<a^2$ is $\frac{\pi^2}{2}a^4$.

Hence the summation of the cases from

$$x^2+y^2+z^2+w^2=a^2 \text{ to } x^2+y^2+z^2+w^2=b^2$$

is

$$\frac{\pi^2}{2}(b^4-a^4).$$

It is clear also that after the first integration with regard to w had been completed we might for the remainder have illustrated the triple integral

$$\iiint\sqrt{b^2-x^2-y^2-z^2}\,dx\,dy\,dz$$

by integration through a spherical volume, the summation being that of $\sqrt{b^2-x^2-y^2-z^2}$ throughout the sphere $x^2+y^2+z^2=b^2$.

Then writing $x^2+y^2+z^2=r^2$, we have

$$I=2\int_0^{2\pi}\int_0^{\pi}\int_0^{b}\sqrt{b^2-r^2}\,r^2\sin\theta\,d\theta\,d\phi\,dr$$

$$=8\pi\int_0^b r^2\sqrt{b^2-r^2}\,dr=8\pi\, b^4\int_0^{\frac{\pi}{2}}\sin^2\chi\cos^2\chi\,d\chi,\quad (r=b\sin\chi)$$

$$=8\pi\, b^4\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{2\Gamma(3)}=\frac{\pi^2b^4}{2}, \text{ as before.}$$

832. **Case of an Implicit Relation between Two Sets of Variables.**

In our previous work and in the typical examples discussed, we have regarded the transformation formulae to be such that each of the one set of variables is expressed, or easily expressible, as an explicit function of the variables of the new group. If this be not so, we can still form the Jacobian by the rules of Arts. 543 and 544, *Diff. Calculus.*

For in the case when

$$f_1(x, y, u, v)=0,\quad f_2(x, y, u, v)=0$$

are the connecting equations, we have

$$\frac{\partial(f_1, f_2)}{\partial(x, y)}\cdot\frac{\partial(x, y)}{\partial(u, v)}=\frac{\partial(f_1, f_2)}{\partial(u, v)};$$

and when

$$f_1(x, y, z, u, v, w)=0,$$
$$f_2(x, y, z, u, v, w)=0,$$
$$f_3(x, y, z, u, v, w)=0,$$

are the connecting formulae,

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3 \frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)};$$

and generally, if there be n connecting equations,

$$f_1=0,\ f_2=0,\ f_3=0, \dots f_n=0,$$

between $2n$ variables,

$$u_1, u_2, \dots u_n \quad \text{and} \quad x_1, x_2, \dots x_n,$$

$$\frac{\partial(f_1, f_2, \dots f_n)}{\partial(x_1, x_2, \dots x_n)} \cdot \frac{\partial(x_1, x_2, \dots x_n)}{\partial(u_1, u_2, \dots u_n)} = (-1)^n \frac{\partial(f_1, f_2, \dots f_n)}{\partial(u_1, u_2, \dots u_n)}.$$

Hence for a double integration

$$\iint V\,dx\,dy = \iint V' \frac{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}\,du\,dv,$$

and for a triple integration

$$\iiint dx\,dy\,dz = -\iiint V' \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}\,du\,dv\,dw,$$

and so on.

Digression on Jacobians. Jacobi's and Bertrand's Definitions.

833. Jacobi's Definition.

If $f_1, f_2, f_3, \dots f_n$ be any function of the n variables

$$x_1, x_2, x_3, \dots x_n,$$

the determinant

$$J \equiv \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1}, & \dfrac{\partial f_1}{\partial x_2}, & \dfrac{\partial f_1}{\partial x_3}, \dots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1}, & \dfrac{\partial f_2}{\partial x_2}, & \dfrac{\partial f_2}{\partial x_3}, \dots & \dfrac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial f_n}{\partial x_1}, & \dfrac{\partial f_n}{\partial x_2}, & \dfrac{\partial f_n}{\partial x_3}, \dots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $f_1, f_2, f_3, \dots f_n$ with regard to $x_1, x_2, \dots x_n$. Jacobi in one of his memoirs pointed out the strong analogy which the properties of this function bears to those of a differential coefficient of a function of a single variable. This resemblance of results, rather than of demonstrations, has already been mentioned (*Diff. Calculus*, Articles 542 onwards). It was by starting from the form of this determinant that Jacobi's investigation proceeded.

834. **Bertrand's System of Increments.**

A different standpoint was suggested by M. J. Bertrand in a memoir to the Académie des Sciences (1851), which has many advantages, and Jacobi's results may be deduced from M. Bertrand's new definitions almost as corollaries.

Let $f_1, f_2, \dots f_n$ be n functions of the n independent variables $x_1, x_2, \dots x_n$.

Let us give to these independent variables the following n systems of increments, viz.

$$\left.\begin{matrix} d_1x_1, & d_1x_2, & d_1x_3, \dots d_1x_n \\ d_2x_1, & d_2x_2, & d_2x_3, \dots d_2x_n \\ & \text{etc.,} & \\ d_nx_1, & d_nx_2, & d_nx_3, \dots d_nx_n \end{matrix}\right\}, \quad \dots\dots\dots\dots\dots\dots\text{(A)}$$

and let the corresponding increments in the several functions be

$$\left.\begin{matrix} d_1f_1, & d_1f_2, & d_1f_3, \dots d_1f_n \\ d_2f_1, & d_2f_2, & d_2f_3, \dots d_2f_n \\ & \text{etc.,} & \\ d_nf_1, & d_nf_2, & d_nf_3, \dots d_nf_n \end{matrix}\right\}, \quad \dots\dots\dots\dots\dots\text{(B)}$$

i.e. d_rf_s is the increment of f_s when x_1, x_2, etc., increase to $x_1+d_rx_1$, $x_2+d_rx_2$, etc.

These several increments d_1x_1, d_2x_1, d_3x_1, etc., though increments of the same variable, are arbitrary and independent, and there is reserved to us the power of making them equal later, or of assuming any such relations between them as we may subsequently choose.

It is clear that we have the n^2 relations of which

$$d_rf_s=\frac{\partial f_s}{\partial x_1}d_rx_1+\frac{\partial f_s}{\partial x_2}d_rx_2+\dots+\frac{\partial f_s}{\partial x_n}d_rx_n \quad \dots\dots\dots\dots\text{(C)}$$

is a type, it being unnecessary in the partial differential coefficients occurring to specify which of the particular increments we choose when we proceed to the limit in their formation.

835. **Bertrand's Definition of a Jacobian.**

M. Bertrand's definition of a Jacobian is that it is the ratio of the determinant formed by the increments of Group B to the determinant formed of the increments in Group A.

Now

$$\begin{vmatrix} d_1x_1, & d_1x_2, & d_1x_3, & \dots, & d_1x_n \\ d_2x_1, & d_2x_2, & d_2x_3, & \dots, & d_2x_n \\ \dots & \dots & \dots & \dots & \dots \\ d_nx_1, & d_nx_2, & d_nx_3, & \dots, & d_nx_n \end{vmatrix} \times \begin{vmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \dots, & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1}, & \frac{\partial f_2}{\partial x_2}, & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1}, & \frac{\partial f_n}{\partial x_2}, & \dots, & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} d_1f_1, & d_1f_2, & d_1f_3, & \dots, & d_1f_n \\ d_2f_1, & d_2f_2, & d_2f_3, & \dots, & d_2f_n \\ \dots & \dots & \dots & \dots & \dots \\ d_nf_1, & d_nf_2, & d_nf_3, & \dots, & d_nf_n \end{vmatrix},$$

by the rule of multiplication of determinants and by virtue of the equations of Group C.

Hence Bertrand's definition agrees with that of Jacobi. We *have, however, gained command over the increments of the independent variables.*

If we adopt the notation Df and Dx for the determinants

$$\begin{vmatrix} d_1f_1, & d_1f_2, & \dots \\ d_2f_1, & \dots, & \dots \\ \dots & \dots & \dots \\ d_nf_1, & \dots, & \dots \end{vmatrix} \text{ and } \begin{vmatrix} d_1x_1, & d_1x_2, & \dots \\ d_2x_1, & \dots, & \dots \\ \dots & \dots & \dots \\ d_nx_1, & \dots, & \dots \end{vmatrix},$$

respectively, we have $J = \frac{Df}{Dx}$.

836. **Corollaries.**

1. It follows at once that if $F_1, F_2, \dots F_n$ be functions of $f_1, f_2, \dots f_n$, and $f_1, f_2, \dots f_n$ be functions of $x_1, x_2, \dots x_n$, then, since

$$\frac{DF}{Dx} = \frac{DF}{Df} \cdot \frac{Df}{Dx},$$

we have

$$\left\{\begin{matrix}\text{Jacobian of } F_1, F_2, \dots \\ \text{with regard to } x_1, x_2, \dots\end{matrix}\right\} = \left\{\begin{matrix}\text{Jacobian of } F_1, F_2, \dots \\ \text{with regard to } f_1, f_2, \dots\end{matrix}\right\} \times \left\{\begin{matrix}\text{Jacobian of } f_1, f_2, \dots \\ \text{with regard to } x_1, x_2, \dots\end{matrix}\right\}.$$

2. Also, since $\frac{Df}{Dx}\times\frac{Dx}{Df}=1$, we have

$$\left\{\begin{matrix}\text{Jacobian of } f_1, f_2, \dots \\ \text{with regard to } x_1, x_2, \dots\end{matrix}\right\}\times\left\{\begin{matrix}\text{Jacobian of } x_1, x_2, \dots \\ \text{with regard to } f_1, f_2, \dots\end{matrix}\right\}=1.$$

3. Again, if $F_1=0$, $F_2=0, \dots F_r=0 \dots$, $F_n=0$ be n independent equations connecting n variables $u_1, u_2, \dots u_n$, and n other variables $x_1, x_2, \dots x_n$, then, since

$$\frac{\partial F_r}{\partial x_1}d_s x_1+\frac{\partial F_r}{\partial x_2}d_s x_2+\dots+\frac{\partial F_r}{\partial x_n}d_s x_n$$
$$+\frac{\partial F_r}{\partial u_1}d_s u_1+\frac{\partial F_r}{\partial u_2}d_s u_2+\dots+\frac{\partial F_r}{\partial u_n}d_s u_n=0,$$

we have

$$\frac{\partial F_r}{\partial x_1}d_s x_1+\dots+\frac{\partial F_r}{\partial x_n}d_s x_n=-\left(\frac{\partial F_r}{\partial u_1}d_s u_1+\dots+\frac{\partial F_r}{\partial u_n}d_s u_n\right),$$

which may be abbreviated into

$$d_{s,x}F_r=-d_{s,u}F_r, \quad\dots\dots\dots\dots\dots\dots(a)$$

the suffix x being attached to indicate those partial differential coefficients in which $u_1, u_2, \dots$ are regarded as constant whilst $x_1, x_2, \dots$ vary and *vice versâ.*

Now D_xF and D_uF are the respective determinants

$$\begin{vmatrix} d_{1,x}F_1, & d_{1,x}F_2, & \dots & d_{1,x}F_n \\ d_{2,x}F_1, & d_{2,x}F_2, & \dots & d_{2,x}F_n \\ \dots & \dots & \dots & \dots \\ d_{n,x}F_1, & d_{n,x}F_2, & \dots & d_{n,x}F_n \end{vmatrix} \text{ and } \begin{vmatrix} d_{1,u}F_1, & d_{1,u}F_2, & \dots & d_{1,u}F_n \\ d_{2,u}F_1, & d_{2,u}F_2, & \dots & d_{2,u}F_n \\ \dots & \dots & \dots & \dots \\ d_{n,u}F_1, & d_{n,u}F_2, & \dots & d_{n,u}F_n \end{vmatrix},$$

and by virtue of equations (a) the constituents of the one only differ from the corresponding constituents of the other by a negative sign, whence

$$D_xF=(-1)^n D_uF,$$

that is

$$\frac{Du}{Dx}=(-1)^n\frac{\frac{D_xF}{Dx}}{\frac{D_uF}{Du}}.$$

Hence in the case of *implicit* connections amongst the $2n$ variables $u_1, u_2, \dots u_n$; $x_1, x_2, \dots x_n$, by virtue of n equations $F_1=0$, $F_2=0, \dots F_n=0$, connecting them,

$$\left\{\begin{matrix}\text{The Jacobian of } u_1, u_2, \dots u_n \\ \text{with regard to } x_1, x_2, \dots x_n\end{matrix}\right\}$$

$$=(-1)^n \frac{\left\{\begin{matrix}\text{Jacobian of } F_1, F_2, \dots \text{ with regard to } x_1, x_2, \dots, \\ \text{treating } u_1, u_2, \dots \text{ as constants}\end{matrix}\right\}}{\left\{\begin{matrix}\text{Jacobian of } F_1, F_2, \dots \text{ with regard to } u_1, u_2, \dots, \\ \text{treating } x_1, x_2, \dots \text{ as constants}\end{matrix}\right\}}.$$

The substance of this and the immediately preceding articles on M. Bertrand's treatment of Jacobians was communicated to the author many years ago by his former tutor, the late Dr. E. J. Routh. The reader may consult Bertrand's *Calcul Différentiel*, pages 62-70, and *Calcul Intégral*, pages 465-469.

837. **Advantage of Bertrand's Definition.**

It will be seen that M. Bertrand's definition leads to simpler proofs of the fundamental properties of Jacobians than those given in Arts. 540, 544 of the author's *Differential Calculus*, and retains a command of the several increments which we shall find useful for subsequent work in the transformation of a multiple integral.

838. **Bertrand's Method of Calculating the Jacobian Determinant.**

Let there be $2n$ variables, in two groups, viz. $x_1, x_2, \dots x_n$ and $u_1, u_2, \dots u_n$, connected by n independent implicit relations $F_1=0, F_2=0, F_3=0, \dots F_n=0$. Then n of the $2n$ variables are independent. If increments be given to each, these $2n$ increments are connected by n homogeneous linear equations, and if $n-1$ of the increments be chosen to be zero, the ratios of the remaining $n+1$ are determinate by the n connecting equations.

Consider the n incremental systems,

$$\left\{\begin{matrix} d_1u_1, & d_1u_2, & d_1u_3, & \dots, & d_1u_n \\ 0, & d_2u_2, & d_2u_3, & \dots, & d_2u_n \\ 0, & 0, & d_3u_3, & \dots, & d_3u_n \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots & d_nu_n \end{matrix}\right\} \left\{\begin{matrix} d_1x_1, & 0, & 0, & \dots & 0 \\ d_2x_1, & d_2x_2, & 0, & \dots & 0 \\ d_3x_1, & d_3x_2, & d_3x_3, & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d_nx_1, & d_nx_2, & d_nx_3, & \dots & d_nx_n \end{matrix}\right\},$$

that is systems in which

increments $d_1u_1, d_1u_2, \dots, d_1u_n$ give rise to an increment d_1x_1 in x_1, but make no change in $x_2, x_3, \dots, x_n$, and increments $d_2u_2, d_2u_3, \ldots, d_2u_n, d_2x_1$ give rise to a change d_2x_2 in x_2, but make no change in $u_1, x_3, x_4, \ldots, x_n$, and so on.

Let J be the Jacobian of $x_1, x_2, \ldots, x_n$ with regard to $u_1, u_2, \ldots, u_n$. Then forming J according to Bertrand's definition, each of the determinants of the increments, the one formed from the x-increments, the other from the u-increments, reduces to its diagonal term, and

$$J = Lt\,\frac{d_1x_1 \,.\, d_2x_2 \,.\, d_3x_3 \ldots d_nx_n}{d_1u_1 \,.\, d_2u_2 \,.\, d_3u_3 \ldots d_nu_n} = \frac{\partial x_1}{\partial u_1} \cdot \frac{\partial x_2}{\partial u_2} \cdot \frac{\partial x_3}{\partial u_3} \cdots \frac{\partial x_n}{\partial u_n},$$

where $\dfrac{\partial x_r}{\partial u_r}$ is the limit of the infinitesimal change in x_r to that in u_r when $u_1, u_2, \ldots u_{r-1}, x_{r+1}, x_{r+2} \ldots x_n$ are regarded as constants.

839. It is necessary for the use of this rule to consider the several connecting equations reduced to such form that

(1) x_1 is a function of $u_1, x_2, x_3, \ldots, x_n$; u_1 only varying;

(2) x_2 is a function of $u_1, u_2, x_3, \ldots, x_n$; u_2 only varying;

(3) x_3 is a function of $u_1, u_2, u_3, x_4, \ldots, x_n$; u_3 only varying;

....

(n) x_n is a function of $u_1, u_2, u_3, \ldots, u_n$; u_n only varying.

The calculation of J will then be reduced to the multiplication of the several partial differential coefficients derived therefrom.

840. Illustrative Examples.

Ex. 1. If $x = r\cos\theta$, $y = r\sin\theta$, write

$x = \sqrt{r^2 - y^2}$, containing *one* of the new variables;

$y = r\sin\theta$, containing *two* and no x.

Then $J = \dfrac{r}{\sqrt{r^2 - y^2}} \,.\, r\cos\theta = r.$

Ex. 2. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, write

$x = \sqrt{r^2 - y^2 - z^2}$, containing *one* of the new variables;

$z = r\cos\theta$, containing *two* and no x;

$y = r\sin\theta\sin\phi$, containing *three* and no x or z.

Then $J = \dfrac{\partial x}{\partial r} \cdot \dfrac{\partial z}{\partial \theta} \cdot \dfrac{\partial y}{\partial \phi} = \dfrac{r}{x} \,.\, (-r\sin\theta)(r\sin\theta\cos\phi) = -r^2\sin\theta.$

Ex. 3. If $x+y+z=u$, $y+z=uv$, $z=uvw$, we have

$x=u-y-z$, containing *one* new variable,

$y=uv-z$, containing *two* and no x,

$z=uvw$, containing *three* and no x or y;

and $$J=\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial w}=1\,.\,u\,.\,uv=u^2v.$$

Ex. 4. If $x_1=r\sin\theta\cos\phi$, $x_3=r\sin\theta\sin\phi$,

$x_2=r\cos\theta\cos\psi$, $x_4=r\cos\theta\sin\psi$,

we have $x_1=\sqrt{r^2-x_2{}^2-x_3{}^2-x_4{}^2}$, containing r, x_2, x_3, x_4;

$x_2=\sqrt{r^2\cos^2\theta-x_4{}^2}$, containing r, θ, x_4;

$x_3=r\sin\theta\sin\phi$, containing r, θ, ϕ;

$x_4=r\cos\theta\sin\psi$, containing r, θ, ψ;

and
$$J=\frac{\partial x_1}{\partial r}\frac{\partial x_2}{\partial\theta}\frac{\partial x_3}{\partial\phi}\frac{\partial x_4}{\partial\psi}=\frac{r}{x_1}\cdot\frac{-r^2\sin\theta\cos\theta}{x_2}\,.\,r\sin\theta\cos\phi\,.\,r\cos\theta\cos\psi$$
$$=-r^3\sin\theta\cos\theta.$$

Ex. 5. If
$$x_1=r\cos\theta_1,$$
$$x_2=r\sin\theta_1\cos\theta_2,$$
$$x_3=r\sin\theta_1\sin\theta_2\cos\theta_3,$$
$$x_4=r\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4,$$
$$x_5=r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\cos\theta_5,$$
$$x_6=r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5,$$
we have
$$x_6=\sqrt{r^2-x_1{}^2-x_2{}^2-x_3{}^2-x_4{}^2-x_5{}^2},$$
$$x_1=r\cos\theta_1,$$
$$x_2=r\sin\theta_1\cos\theta_2,$$
$$x_3=r\sin\theta_1\sin\theta_2\cos\theta_3,$$
$$x_4=r\sin\theta_1\sin\theta_2\sin\theta_3\cos\theta_4,$$
$$x_5=r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\cos\theta_5;$$
and
$$J=\frac{r}{x_6}(-r\sin\theta_1)(-r\sin\theta_1\sin\theta_2)(-r\sin\theta_1\sin\theta_2\sin\theta_3)$$
$$\times(-r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4)(-r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5)$$
$$=(-1)^5r^5\sin^4\theta_1\sin^3\theta_2\sin^2\theta_3\sin\theta_4,$$
a result which can obviously be generalised.

841. Change of the Variables in any Multiple Integral. General Theorem.

Let the integral in question be
$$I=\iiint\dots\int V\,dx_1\,dx_2\dots dx_n,$$
there being n integration signs, and V any function of the variables x_1, x_2, ... x_n. Let the new system of variables be u_1, u_2, ... u_n, there being n independent connecting relations
$$F_1=0,\ F_2=0,\dots F_n=0,$$

between the two groups of variables, either set forming a group in which there is no interdependence. That is, the group $x_1, x_2, \dots x_n$ forms a set of n independent variables, as also does the group $u_1, u_2, \dots u_n$. When a further relation is assigned, say $\phi(x_1, x_2, \dots x_n)=0$, to be satisfied at the boundaries of the region of integration, an interdependence of the x-group is created, and one of the x-group of variables is dependent upon the others. Integration is then to be conducted for the domain or region bounded by the specific limitation $\phi=0$. There will then be a corresponding relation amongst the u-group of coordinates, and a specific limitation will be implied for the new definition of the domain of integration when I has been referred to its new coordinates.

842. In the transformation of I three separate considerations are to be attended to. As has already been pointed out in the case of double and triple integration, we have to consider

(1) the determination of the new form of V, which is merely an algebraic matter of substitution or elimination;

(2) the assignment of the new limits which is also an algebraic matter, materially assisted in the case of double and triple integration by geometrical considerations;

(3) the determination of the new element of integration which is to replace $dx_1\,dx_2\,dx_3 \dots dx_n$.

As regards the assignment of new limits it is not possible to give a general rule, but it must be *such as will cause the march of the new element as described in the new system of variables to traverse the same domain once and once only as was traversed in the march of the original element, which domain was defined by the limits of integration in the original system of variables.*

Let us imagine that the connecting equations have been thrown into the forms

$x_1=f_1(u_1, x_2, x_3, \dots x_n)$ ......(1), *i.e.* $u_2, u_3, \dots u_n$ eliminated;
$x_2=f_2(u_1, u_2, x_3, \dots x_n)$ ......(2), *i.e.* $x_1, u_3, u_4, \dots u_n$ „
$x_3=f_3(u_1, u_2, u_3, x_4, \dots x_n)$ ...(3), etc.;
etc.,
$x_n=f(u_1, u_2, u_3, \dots u_n)$ ......(n) etc.

We have seen in earlier articles and examples, that in a given multiple integral the order of integration may be changed, provided a suitable change be made in the limits.

Then, first, suppose we attempt to replace integration with regard to x_1 by integration with regard to u_1.

Change the order of integration in

$$I \equiv \iiint \dots \int V \, dx_1 \, dx_2 \dots dx_n,$$

so that dx_1 stands last with the suitable change in the limits. We then have to perform the operation

$$I \equiv \int \left[\iiint \dots \int V \, dx_2 \, dx_3 \dots dx_n \right] dx_1,$$

and in this operation $x_2, x_3, \dots x_n$ are to be regarded as constants, and equation (1) gives $dx_1 = \frac{\partial f_1}{\partial u_1} du_1$.

And since $\int U \, dx_1 = \int U \frac{\partial x_1}{\partial u_1} du_1$, we have as x_1 and u_1 are the only varying quantities

$$I = \int \left[\iint \dots \int V_1 \, dx_2 \, dx_3 \dots dx_n \right] \frac{\partial f_1}{\partial u_1} du_1,$$

where V_1 is what V becomes when $f_1(u_1, x_2, x_3, \dots x_n)$ has been substituted for x_1, that is, V_1 is the value of V expressed in terms of $u_1, x_2, x_3, \dots x_n$.

We have now arrived at

$$I = \iiint \dots \int V_1 \frac{\partial f_1}{\partial u_1} dx_2 \, dx_3 \dots dx_n \, du_1.$$

Let us repeat the process.

By change of order of integration with a suitable change in the limits, transfer dx_2 so that it stands last.

$$I = \iiint \dots \int V_1 \frac{\partial f_1}{\partial u_1} dx_3 \, dx_4 \dots dx_n \, du_1 \, dx_2$$

or

$$\int \left[\iint \dots \int V_1 \frac{\partial f_1}{\partial u_1} dx_3 \, dx_4 \dots dx_n \, du_1 \right] dx_2,$$

and in this operation $x_3, x_4, \dots x_n, u_1$ are to be regarded as constants, and equation (2) gives $dx_2 = \frac{\partial f_2}{\partial u_2} du_2$.

Whence again applying the theorem $\int U' dx_2 = \int U' \frac{\partial x_2}{\partial u_2} du_2$, and x_2, u_2 being the only varying quantities, we have

$$I = \int\left[\iint \dots \int V_2 \frac{\partial f_1}{\partial u_1} dx_3 \, dx_4 \dots dx_n \, du_1\right] \frac{\partial f_2}{\partial u_2} du_2,$$

where V_2 is what V_1 becomes when $f_2(u_1, u_2, x_3, \dots x_n)$ is substituted for x_2, that is V_2 is the value of V expressed in terms of $u_1, u_2, x_3, \dots x_n$; and we have now arrived at

$$I = \iiint \dots \int V_2 \frac{\partial f_1}{\partial u_1} \frac{\partial f_2}{\partial u_2} dx_3 \, dx_4 \dots dx_n \, du_1 \, du_2.$$

Continuing this process of changing the order of integration so that dx_3 is transferred to the end, and then exchanging the variable x_3 for u_3, etc., we finally arrive at

$$I = \iiint \dots \int V_n \frac{\partial f_1}{\partial u_1} \cdot \frac{\partial f_2}{\partial u_2} \cdot \frac{\partial f_3}{\partial u_3} \dots \frac{\partial f_n}{\partial u_n} du_1 \, du_2 \dots du_n,$$

where V_n is the value of V when all letters of the x-group in V have been replaced by letters of the u-group, that is $V_n \equiv V'$, say.

Now it has been seen that

$$\frac{\partial f_1}{\partial u_1} \cdot \frac{\partial f_2}{\partial u_2} \cdot \frac{\partial f_3}{\partial u_3} \dots \frac{\partial f_n}{\partial u_n} \equiv J,$$

the Jacobian of $x_1, x_2, \dots x_n$ with regard to $u_1, u_2, \dots u_n$; and

$$J = (-1)^n \begin{vmatrix} \frac{\partial F_1}{\partial u_1}, & \frac{\partial F_1}{\partial u_2}, & \dots & \frac{\partial F_1}{\partial u_n} \\ \frac{\partial F_2}{\partial u_1}, & \frac{\partial F_2}{\partial u_2}, & \dots & \frac{\partial F_2}{\partial u_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial F_n}{\partial u_1}, & \frac{\partial F_n}{\partial u_2}, & \dots & \frac{\partial F_n}{\partial u_n} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial F_1}{\partial x_1}, & \frac{\partial F_1}{\partial x_2}, & \dots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1}, & \frac{\partial F_2}{\partial x_2}, & \dots & \frac{\partial F_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial F_n}{\partial x_1}, & \frac{\partial F_n}{\partial x_2}, & \dots & \frac{\partial F_n}{\partial x_n} \end{vmatrix}$$

or

$$(-1)^n \frac{\dfrac{\partial(F_1, F_2, F_3, \dots F_n)}{\partial(u_1, u_2, u_3, \dots u_n)}}{\dfrac{\partial(F_1, F_2, F_3, \dots F_n)}{\partial(x_1, x_2, x_3, \dots x_n)}},$$

where in forming the numerator all letters of the x-group are considered constant, and in the denominator all letters of the u-group are considered constant.

Hence, we have finally,

$$\iiint \cdots \int V\,dx_1\,dx_2\,dx_3 \ldots dx_n$$

$$=(-1)^n \iiint \cdots \int V' \frac{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(u_1, u_2, \ldots u_n)}}{\dfrac{\partial(F_1, F_2, \ldots F_n)}{\partial(x_1, x_2, \ldots x_n)}} du_1\,du_2\,du_3 \ldots du_n.$$

843. Ex. If $\left.\begin{array}{l} xu+yv=a^2, \\ xv-yu=0, \end{array}\right\}$ be the connecting equations,

$$J=\frac{\begin{vmatrix} x, & y \\ -y, & x \end{vmatrix}}{\begin{vmatrix} u, & v \\ v, & -u \end{vmatrix}}=-\frac{x^2+y^2}{u^2+v^2}=-\frac{a^4}{(u^2+v^2)^2}.$$

Compare the process of Ex. 8, Art. 831.

844. The Vanishing of J.

It may be noted that the vanishing of J would imply that when $x_1, x_2, \ldots x_n$ are regarded as functions of $u_1, u_2, u_3, \ldots$, there would be some identical relation amongst the members of the x-group of variables; and if J were infinite, we should have $J'=0$, and there would be some identical relation amongst the values of $u_1, u_2, \ldots u_n$ as expressed in terms of $x_1, x_2, \ldots x_n$, (Art. 547, *Differential Calculus*). We have, however, assumed all our several connecting equations $F_1=0, F_2=0, \ldots F_n=0$, to be independent relations, so that no such identical relation can occur amongst either set of variables.

845. Remarks.

It may be useful to call attention to the fact that in the geometrical treatment of Arts. 792 and 794 for double and triple integrals respectively, the new element of integration was formed and the variables were changed to the new group *all together*. In the general proof of Art. 842, the original variables were exchanged for the new variables *one at a time*. When a geometrical method of determining the new limits is not available, this consideration will often be useful for their proper assignment, and may be used when other means are wanting. But the process followed out in detail is generally tedious, as every change in order of an integration as well as every exchange of a new variable for an old one necessitates in general a readjustment of the limits of each integration.

846. **Examples in which Multiple Integrals of Order higher than the Third occur in Physics.**

Multiple integrals occur frequently in researches of physical nature, of higher degree of multiplicity than the third. For instance, in the problem of the illumination of one surface by another, the two surfaces being such that every point of the one can be seen from each point of the other, the quantity to be evaluated is the quadruple integral *

$$\iiiint \frac{\cos\phi \cos\phi'}{r^2}\, dS\, dS',$$

where dS, dS' are the elements of the two surfaces; ϕ, ϕ' the angles which the outward normals make with r, the distance between dS and dS', and the integration is to be conducted over each surface. In such case, the limits form two separate groups, the one referring to surface S, the other to surface S', and if any transformation of variables be required, a new assignment of limits being required, they will be available from geometrical conditions for each group.

Another illustration from Physics is in the mutual potential of two attracting systems, which for a continuous distribution of matter in regions P, Q has for its expression the sextuple integral

$$W_{PQ} \equiv \iiiiiint \frac{\rho_p \rho_q}{r_{pq}}\, d\tau_p d\tau_q,$$

where ρ_p is the volume density at a point p of the region P;
ρ_q the volume density at a point q of the region Q;
$d\tau_p$, $d\tau_q$ elements of volume at p and q, and r_{pq} the distance from p to q.

In this case also the system of limits will be two separate systems, the one ensuring summation through the region P and the other through the region Q. And if any change of variable be required to facilitate integration, necessitating a new assignment of limits, they will be available as in the former case from the geometrical conditions for each group.

* See Herman, *Geometrical Optics*, Art. 157.

847. Case of Implicit Relations.

If in Art. 839 Equations (1), (2), ... (n) had not been supposed to express

x_1 explicitly as a function of $u_1, x_2, x_3, \dots x_n$,

x_2 explicitly as a function of $u_1, u_2, x_3, \dots x_n$,

etc.,

but had been given as implicit relations, viz.

$\phi_1(u_1, x_1, x_2, \dots x_n) \quad =0 \dots(1)$, in which $u_2, u_3, \dots u_n$ are eliminated,

$\phi_2(u_1, u_2, x_2, x_3, \dots x_n)=0 \dots(2)$, in which $x_1, u_3, \dots u_n$ are eliminated,

$\phi_3(u_1, u_2, u_3, x_3, \dots x_n)=0 \dots(3)$, etc.,

etc.,

$\phi_n(u_1, u_2, u_3, \dots u_n, x_n)=0 \dots(n)$ etc.,

we have in the subsequent work, from equation (1), considering $x_2, x_3, \dots x_n$ as constants,

$$dx_1 = -\frac{\dfrac{\partial\phi_1}{\partial u_1}}{\dfrac{\partial\phi_1}{\partial x_1}}\,du_1\,;$$

and from equation (2), considering $u_1, x_3, x_4, \dots x_n$ as constants,

$$dx_2 = -\frac{\dfrac{\partial\phi_2}{\partial u_2}}{\dfrac{\partial\phi_2}{\partial x_2}}\,du_2,$$

and so on.

And we finally obtain in the same way as before,

$$\iint\dots\int V\,dx_1\,dx_2\dots dx_n$$

$$=(-1)^n\iint\dots\int V'\,\frac{\dfrac{\partial\phi_1}{\partial u_1}\cdot\dfrac{\partial\phi_2}{\partial u_2}\cdot\dfrac{\partial\phi_3}{\partial u_3}\dots\dfrac{\partial\phi_n}{\partial u_n}}{\dfrac{\partial\phi_1}{\partial x_1}\cdot\dfrac{\partial\phi_2}{\partial x_2}\cdot\dfrac{\partial\phi_3}{\partial x_3}\dots\dfrac{\partial\phi_n}{\partial x_n}}\,du_1\,du_2\dots du_n.$$

848. For example, taking

$$\phi_1 \equiv r^2 - x^2 - y^2 - z^2 \;\; =0 \text{ (containing } z, y, x, r),$$
$$\phi_2 \equiv r^2\sin^2\theta - x^2 - y^2 = 0 \text{ (containing } y, x, r, \theta),$$
$$\phi_3 \equiv r\sin\theta\cos\phi - x = 0 \text{ (containing } x, r, \theta, \phi).$$

Then we have

$$\iiint V\,dx\,dy\,dz = -\iiint V' \frac{\dfrac{\partial\phi_1}{\partial r}\cdot\dfrac{\partial\phi_2}{\partial\theta}\cdot\dfrac{\partial\phi_3}{\partial\phi}}{\dfrac{\partial\phi_1}{\partial z}\cdot\dfrac{\partial\phi_2}{\partial y}\cdot\dfrac{\partial\phi_3}{\partial x}}\,dr\,d\theta\,d\phi$$

$$= -\iiint V' \frac{2r\,.\,2r^2\sin\theta\cos\theta(-r\sin\theta\sin\phi)}{(-2z)(-2y)(-1)}\,dr\,d\theta\,d\phi$$

$$= -\iiint V' \frac{r^4\sin^2\theta\cos\theta\sin\phi}{r\sin\theta\sin\phi\,.\,r\cos\theta}\,dr\,d\theta\,d\phi$$

$$= -\iiint V' r^2\sin\theta\,dr\,d\theta\,d\phi,$$

as we should expect; see Ex. 2, Art. 840, and elsewhere.

849. Example of Assignment of Limits.

Ex. As an example of the assignment of limits in a multiple integral, let us take two squares of sides $2a$ in parallel planes at distance c apart, the squares being placed so that they form the ends of a rectangular parallelepiped of square section, and let us find *the mean value of the squares of the distances of points on the one square from points on the other*. By a mean or average value we shall suppose to be meant that each square is divided up into equal small elements, and the sum of the squares of the distances apart is to be divided by their number, *i.e.* if there be n such elements, and r_{PQ} be the distance between two of them at P and at Q respectively, $\dfrac{\Sigma r_{PQ}^2}{n}$, or, which is the same thing, $\dfrac{\Sigma r_{PQ}^2\,\delta S_P\,\delta S_Q}{\Sigma\delta S_P\,\delta S_Q}$ if δS_P and δS_Q be the elements at P and Q; and in the limit, when n becomes infinitely large, we have

$$\frac{\iiiint r_{PQ}^2\,dS_P\,dS_Q}{\iiiint dS_P\,dS_Q}.$$ (See Chapter XXXVI., Art. 1657.)

Let O, O' be the centres of the squares, and take O for origin and axes of x and y parallel to the sides of the squares.

Divide up each square by families of lines parallel to the axes, and let $(x, y, 0)$, (x', y', c) be the respective coordinates of P and Q. Then the Mean Value required is

$$M = \frac{\iiiint[(x-x')^2+(y-y')^2+c^2]\,dx'\,dy'\,dx\,dy}{\iiiint dx'\,dy'\,dx\,dy}.$$

Now keeping the position of Q fixed, we may add up all the elements $r_{PQ}^2\,\delta x\,\delta y$ in a strip between x and $x+\delta x$, by varying y from $-a$ to $+a$, keeping x', y', x constant. Then, still keeping x', y' constants, we may add up all the strips in the square $ABCD$ which lies in the x-y plane, by integrating with regard to x from $x=-a$ to $x=+a$. We have then completed the summation of all such quantities as $r_{PQ}^2\,dx'\,dy'$ for all positions of P in the square $ABCD$. In the same way we may add up the results of these integrations for various points of the square $A'B'C'D'$, by integrating with regard to y' from $-a$ to $+a$, keeping x' constant to add up the elements in a strip between x' and $x'+\delta x'$. And finally integrating with regard to x' from $-a$ to $+a$ will add up the results for all the strips in the square $A'B'C'D'$ and will complete the integration.

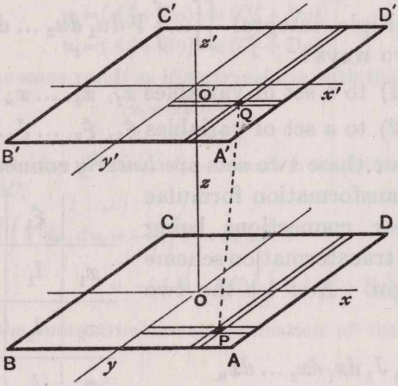

Fig. 312.

And the same with the denominator. The result for the denominator is obviously the product of the two areas, *i.e.* $4a^2 \times 4a^2$ or $16a^4$.

The numerator is

$$\iiiint (x^2+y^2+x'^2+y'^2-2xx'-2yy'+c^2)\,dx'\,dy'\,dx\,dy,$$

and it will save some trouble to observe:

(1) That for every term $xx'\,\delta x'\,\delta y'\,\delta x\,\delta y$, there is another term

$$x(-x')\,\delta x'\,\delta y'\,\delta x\,\delta y.$$

Hence such a term contributes nothing to the value of the integral, and the same with the yy' term.

(2) That obviously

$$\Sigma x^2\,dS\,dS' = \Sigma y^2\,dS\,dS' = \Sigma x'^2\,dS\,dS' = \Sigma y'^2\,dS\,dS'.$$

Hence it will be sufficient to attend to the value of one of them, and quadruple the result.

Now

$$\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} x^2\,dx'\,dy'\,dx\,dy = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} 2ax^2\,dx'\,dy'\,dx$$

$$= \int_{-a}^{a}\int_{-a}^{a} (2a)\left(\frac{2a^3}{3}\right) dx'\,dy' = (2a)^3 \cdot \frac{2a^3}{3}.$$

Hence the value of the numerator is

$$4\left(\tfrac{16}{3}a^6\right) + c^2 \,.\, 16a^4,$$

and $$M = \frac{4a^2+3c^2}{3}.$$

It follows that the mean of the squares of the distances from any point of a square to any other point of the same square is $\frac{4a^2}{3}$, by putting $c=0$. [Also see Art. 1657 and Art. 1658, Ex. 2.]

850. **A Consideration useful for the Simplification of some Transformation Formulae.**

Let a multiple integral $\iint\dots\int V\,du_1\,du_2\dots du_n$ be transformed in two ways:

(1) to a set of variables $x_1, x_2, \dots x_n$;

(2) to a set of variables $\xi_1, \xi_2, \dots \xi_n$.

And suppose these two sets are *linearly* connected with each other, the transformation formulae for the linear connections being given by the transformation scheme in the margin. And let the two results be

	ξ_1	ξ_2	ξ_3	...
x_1	l_1	m_1	n_1	...
x_2	l_2	m_2	n_2	...
x_3	l_3	m_3	n_3	...
...	...	...	...	...

$$\iint\dots\int V_1 J_1\,dx_1\,dx_2\dots dx_n$$

and $\iint\dots\int V_2 J_2\,d\xi_1\,d\xi_2\dots d\xi_n.$

Then, the Jacobian is a covariant of $u_1, u_2, \dots u_n$; we have

$$J_2=J_1\begin{vmatrix} l_1, & l_2, & \dots \\ m_1, & m_2, & \dots \\ \dots & \dots & \dots \end{vmatrix}=\mu J_1$$

(*Diff. Calc.*, Art. 546),

μ being the transformation modulus. And that the above expressions are equal may be seen by transforming directly, for

$$\iint\dots\int V_1 J_1\,dx_1\,dx_2\dots dx_n$$

$$=\iint\dots\int V_2 J_1\frac{\partial(x_1, x_2, \dots x_n)}{\partial(\xi_1, \xi_2, \dots \xi_n)}\,d\xi_1\,d\xi_2\dots d\xi_n$$

$$=\iint\dots\int V_2 J_1\,\mu\,d\xi_1\,d\xi_2\dots d\xi_n$$

$$=\iint\dots\int V_2 J_2\,d\xi_1\,d\xi_2\dots d\xi_n,$$

and the results are identical, as might have been expected.

It follows that if a transformation be proposed to a set of variables $\xi_1, \xi_2, \xi_3, \dots$, a transformation to another set

$x_1, x_2, x_3, \ldots$ may be substituted for the former, where a suitable choice of linear connection between the former and the latter sets may sometimes be made to simplify the working.

851. For example, if the transformation formulae proposed be

$$u_1 = (A\xi + B\eta)\sin(C\xi + D\eta),$$
$$u_2 = (A\xi + B\eta)\cos(C\xi + D\eta),$$

we shall have the same result as if we transform with the easier formulae

$$\left.\begin{aligned} u_1 &= x\sin y, \\ u_2 &= x\cos y, \end{aligned}\right\}$$

for which the Jacobian is obviously $-x$, and multiply the result by the modulus $AD - BC$.

Thus $$\iint V\,du_1\,du_2 = -\iint V_1 x\,dx\,dy$$
$$= -(AD - BC)\iint V_2(A\xi + B\eta)\,d\xi\,d\eta,$$

thus avoiding the more troublesome evaluation of the Jacobian with regard to ξ, η.

852. Speaking of the result

$$\iiint V\,dx\,dy\,dz = \iiint V' \frac{\partial(x, y, z)}{\partial(u, v, w)}\,du\,dv\,dw,$$

Lacroix* remarks: "Ce resultat a été donnée pour la première fois par Lagrange en 1773. Mais Legendre, en 1788, en a fait des applications que Lagrange n'avoit point indiquées." This application referred in part to the analytical proof of a theorem with regard to the attraction of a spheroid.

The corresponding result for a double integral had been employed by Euler in 1769.

Many references with regard to the history of the subject are given by Todhunter, *Integral Calculus*, Art. 251. There is a valuable table of references in Lacroix's *Calc. Diff. et Int.*, vol. ii., prefixed to the volume, which may be useful to students interested in the subject and desiring to consult early writers.

* Lacroix, *Calcul. Diff. et Int.*, vol. ii., p. 206.

PROBLEMS.

1. If the rectangular coordinates of a point are

$$x = \alpha + e^{\beta} \cos \alpha, \quad y = \beta + e^{\beta} \sin \alpha,$$

show that the area included between the curves α_1, β_1, α_2, β_2 is

$$\tfrac{1}{2}(\alpha_1 - \alpha_2)(2\beta_1 - e^{2\beta_1} - 2\beta_2 + e^{2\beta_2}).$$

[MATH. TRIP., 1873.]

2. Integrate $\iint x^2\,dx\,dy$ over the space enclosed by the four parabolas $\quad y^2 = 4ax, \quad y^2 = 4bx, \quad x^2 = 4cy, \quad x^2 = 4dy.$

[TRINITY COLL., 1882.]

3. The four curves $y = ax^2$, $y = bx^2$, $y = cx^3$, $y = dx^3$ intersect in four points, excluding the origin, and thus form a curvilinear quadrilateral; prove that its area is

$$\frac{1}{12}(a^4 \sim b^4)\left(\frac{1}{c^3} \sim \frac{1}{d^3}\right).$$

[OXFORD II. P., 1901.]

4. An area is bounded by those portions of the four rectangular hyperbolae $xy = a^2$, $xy = a'^2$, $x^2 - y^2 = c^2$, $x^2 - y^2 = c'^2$, which lie in the first quadrant. Every element of the area is multiplied by the square of its distance from the centre. Prove that the sum of all such products is

$$\tfrac{1}{2}(a^2 \sim a'^2)(c^2 \sim c'^2).$$

[J. M. SCH., OXF., 1904.]

5. If the surface density σ of the area in the first quadrant bounded by

$$x^m y^n = a_1^{m+n}, \quad x^p y^q = b_1^{p+q},$$
$$x^m y^n = a_2^{m+n}, \quad x^p y^q = b_2^{p+q},$$

be given by $\sigma xy = k$, show that the mass is

$$k\frac{(m+n)(p+q)}{mq - np}\log\frac{a_1}{a_2}\cdot\log\frac{b_1}{b_2}.$$

6. Change the variables from x and y to u and v in the double integral

$$\int_0^a \int_x^{\frac{a^2}{x}} \phi(x, y)\,dx\,dy,$$

where $xy = u^2$, $x^2 + y^2 = v^2$.

[ST. JOHN'S, 1882.]

7. Show that in $\int_{-a}^{a}\int_{-b}^{b} f(x, y)\,dx\,dy$ all terms in $f(x, y)$ may be omitted which contain an odd power of x or y.

Find $\int_0^a \int_{-x}^{x} (x + y)\cos(mx + ny)\,dx\,dy.$

[TRINITY COLL., 1881.]

8. Transform $\int_0^{\infty}\int_0^{\sqrt{2ax}} \frac{a^2\,dx\,dy}{(x^2 + y^2 + a^2)^2}$ by the substitution

$$x/\xi = y/\eta = \sqrt{x^2 + y^2 + a^2}/a,$$

and show that its value is $\pi/4\sqrt{2}$.

[OXFORD II. P., 1903.]

9. Change the order of integration in

$$\int_0^{\frac{a}{2}}\int_{\frac{x^2}{a}}^{x-\frac{x^2}{a}} V\,dx\,dy.$$

[St. John's, 1889.]

10. If $xy=\xi$, $x^2-y^2=\eta$ transform $\int_0^1\int_0^1 V\,dx\,dy$ so that in the result we integrate first with regard to ξ and then with regard to η.

[R. P.]

11. Change the order of integration in the expression

$$\int_0^{c\sqrt{\frac{h}{k}}}\int_{k\frac{x^2}{c^2+x^2}}^{h\frac{c^2}{c^2+x^2}} V\,dx\,dy\,;$$

also, *change the variables* to ξ and η where $x^2+y^2=\eta$, $\xi x=cy$, without assigning the new limits. (It may be assumed that k is greater than h.)

[St. John's, 1888.]

12. Prove that

$$\iint\left(\frac{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}\right)^{\frac{1}{2}} dx\,dy=\frac{\pi}{4}\left(\frac{\pi}{2}-1\right)ab,$$

the integral being taken for all positive values of x and y such that

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}<1.$$

[Colleges, 1886.]

13 Express $\iint f(x, y)\,dx\,dy$ in terms of r and θ, where $x=r\cos\theta$, $y=r\sin\theta$.

Change the order of integration in

$$\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{ax}} f(x, y)\,dx\,dy.$$

[Colleges α, 1883.]

14. Change the order of integration in

$$\int_0^{\frac{ab}{\sqrt{a^2+b^2}}}\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} f(x, y)\,dy\,dx.$$

[St. John's, 1892.]

15. Change the order of integration in

$$\int_0^{\frac{\pi}{3}}\int_{a\sec^2\frac{\theta}{2}}^{\frac{8}{3}a\cos\theta} f(r, \theta)\,d\theta\,dr.$$

16. Change the variables from x, y to u, v, where $x^2+y^2=u$, $xy=v$, and find the limits in the new integral when integration is extended over the positive quadrant of the circle $x^2+y^2=a^2$.

[St. John's, 1881.]

17. Change the order of integration in the integral

$$\int_c^a \int_{\frac{b}{a}\sqrt{a^2-x^2}}^b V\,dx\,dy,$$

where c is less than a. [COLLEGES α, 1888.]

18. Change the order of integration in

$$\int_0^a \int_{\frac{1}{2}\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} U\,dx\,dy,$$

U being a function of x and y.

Express the same integral in polar coordinates. [COLLEGES α, 1886.]

19. Show that

$$\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dx\,dy = \int_a^{2a} \int_{\eta-a}^{a} \frac{2x}{y} V\,d\eta\,d\xi,$$

when $$\xi = \frac{y^2}{2x}, \quad \eta = \frac{x^2+y^2}{2x};$$

and change the order of integration in the latter integral.

[COLLEGES β, 1889.]

20. If the density of a plate be $\dfrac{\mu}{x^2+y^2}$, show that the mass of the part enclosed by the curves $x^2-y^2=\alpha$, $x^2-y^2=\beta$, $xy=\gamma$, $xy=\delta$ is

$$\frac{\mu}{2}\int_\alpha^\beta \int_\gamma^\delta \frac{du\,dv}{u^2+4v^2}.$$

Show whether this gives the mass of one of the areas between the two curves, or of both. [COLLEGES α, 1883.]

21. Change the variables from (x, y) to (u, v) in the double integral $\iint \phi(x, y)\,dx\,dy$, where $x^2+y^2=u$, $xy=v$, and the integration extends over the area bounded by the straight lines

$$y=x, \quad x+y=1, \quad y=0,$$

obtaining the new limits on the supposition that the order of integration is first u and then v. [COLLEGES α, 1870.]

Verify your result by evaluation of the integral for the case when $\phi(x, y)\equiv 1$.

22. Change the variables from x and y to ξ and η in the expression $\iint V\,dx\,dy$, having given $\phi(x, y, \xi, \eta)=0$ and $\psi(x, y, \xi, \eta)=0$.

Show, by transforming to polar coordinates, that

$$c\int_0^{\frac{c}{\sqrt{2}}\tan\alpha} \int_0^{\frac{c}{\sqrt{2}}\tan\alpha} \frac{dx\,dy}{(x^2+y^2+c^2)^{\frac{3}{2}}} = \tan^{-1}\frac{\sec\alpha-\cos\alpha}{2}.$$

[TRINITY, 1882.]

23. If r, r' be the distances of a point in the plane of reference from two fixed points at a distance $2c$ apart on the axis of x, then between corresponding limits of integration

$$\iint 2cy\,dx\,dy = \iint rr'\,dr\,dr'.$$

[OXFORD II., 1886.]

24. Prove that

$$\int_0^l dx \int_0^x dy\, F(x, y) = \int_0^l dx \int_0^x dy\, F(l-y, l-x),$$

and hence deduce that

$$\int_0^\pi d\theta \int_0^\theta d\theta' (\sin\theta \sin\theta')^{2\iota-1} \sin(\theta-\theta') = \frac{1.3.5\ldots(4\iota-1)}{2.4.6\ldots 4i} \cdot \frac{\pi}{\iota}.$$

[SYLVESTER.]

25. Prove that

$$\int_0^x dx \int_0^x dz\, f'(z)\, \phi(x-z) = \int_0^x dz \{f(z) - f(0)\}\, \phi(x-z).$$

[ST. JOHN'S, 1885.]

26. Transform the integral $\int V\,dx\,dy$ by the substitution

$$x = c\cos\xi\cosh\eta, \quad y = c\sin\xi\sinh\eta.$$

[COLLEGES γ, 1890.]

27. If $u + v\sqrt{-1} = \phi(x + y\sqrt{-1})$, prove that

$$\iint\left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2\right] dx\,dy = \iint\left[\left(\frac{\partial V'}{\partial u}\right)^2 + \left(\frac{\partial V'}{\partial v}\right)^2\right] du\,dv,$$

when V' is the result of substituting for x, y in terms of u, v in V.

[COLLEGES α, 1881.]

28. If $x = a\sin\alpha\cos\xi\cosh\eta$ and $y = a\sin\alpha\sin\xi\sinh\eta$, transform

$$\int_0^a \int_0^{\cos\alpha\sqrt{a^2-x^2}} \{(x - a\sin\alpha)^2 + y^2\}^{-\frac{1}{2}}\, dx\,dy$$

into an integral in terms of ξ and η, and evaluate the new integral.

29. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $S = \iint dx\,dy\sqrt{1 + p^2 + q^2}$, transform the variables in the integral to θ, ϕ, where

$$x = a\sin\theta\cos\phi, \quad y = b\sin\theta\sin\phi.$$

[IVORY, *Phil. Trans.*, 1809.]

30. Prove that the assumptions

$$x_1 = r\cos\theta_1,$$
$$x_2 = r\sin\theta_1\cos\theta_2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{n-1} = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\cos\theta_{n-1},$$
$$x_n = r\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-2}\sin\theta_{n-1},$$

will transform the integral $\iiint \ldots V\,dx_1\,dx_2\,dx_3 \ldots dx_n$ into

$$\pm \iiint \ldots V' r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \ldots \sin\theta_{n-2}\,dr\,d\theta_1 \ldots d\theta_{n-1}.$$

[Clare, etc., 1881; Todhunter, *Int. Calc.*, p. 241.]

31. Show that

$$48 \iiint (x^2+y^2+z^2)\,dx\,dy\,dz = 5\pi a^5$$

for positive values of x, y, z limited by $x^2+y^2 \nless az$ and $\ngtr a^2$.

[Oxford II. P., 1889.]

32. Prove that

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{(x^2+y^2+z^2)^{\frac{m}{2}}}{(x^2+y^2+z^2+a^2)^{\frac{m+5}{2}}}\,dx\,dy\,dz = \frac{\pi}{2(m+3)} \cdot \frac{1}{a^2}.$$

[Colleges γ, 1882.]

33. Two given rectangular hyperbolae have the same asymptotes; two other given rectangular hyperbolae have also common asymptotes, one of which coincides with an asymptote of the first pair, while the other is parallel to their other asymptote. Show that the area of the curvilinear quadrangle formed by the four hyperbolae is the same, whatever the distance between the pair of parallel asymptotes. [Math. Tripos, 1895.]

34. Transform the double integral

$$\iint x^{m-1} y^{n-1}\,dy\,dx$$

by the formulae $x+y=u$, $y=uv$, showing that the transformed result is

$$\iint u^{m+n-1}(1-v)^{m-1} v^{n-1}\,du\,dv.$$

[Jacobi, *Crelle's Journal*, tom. xi.]

35. If $u_1 x = u_2 u_3, \quad u_2 y = u_3 u_1, \quad u_3 z = u_1 u_2,$

prove that $\iiint V\,dx\,dy\,dz$

is transformed into $4 \iiint V_1\,du_1\,du_2\,du_3.$

[Oxford II. P., 1885.]

36. Show that

$$\int_a^{a\sqrt{2}} dx \int_0^{\sqrt{2a^2-x^2}} \frac{dy}{(x^2+y^2)^{\frac{3}{2}}} = \left(1-\frac{\pi}{4}\right)\frac{1}{a\sqrt{2}},$$

and both from geometrical considerations and by direct evaluation, show that this integral is equal to the integral

$$\int_0^a dy \int_a^{\sqrt{2a^2-y^2}} \frac{dx}{(x^2+y^2)^{\frac{3}{2}}}.$$

[Oxford I. P., 1912.]