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RETARDATION EFFECT IN FATIGUE CRACK GROWTH UNDER STATIONARY STOCHASTIC LOADING

ABSTRACT

The crack propagation under gaussian stationary stochastic loading in presence of crack growth retardation effects is considered. The Wheeler retardation model is applied. The crack growth process is modeled by Markovian diffusion process. Some properties of envelope and clustering effect of the load process are taken into account in calculation of parameters of the Kolmogorov diffusion equation. The mean time to failure when the crack reaches its critical length is determined. The analysis of the mean life-time equation and results of an example show that the stochastic fluctuations of the stationary load process alone do not affect significantly random variations of the life-time. It allows to give a quasi-deterministic relation between the features of the fatigue fracture and some material and load parameters. It gives a potential to consider effectively the fatigue crack propagation with retardation effects as an additional failure mode in reliability analysis.

1. INTRODUCTION

Fatigue crack propagation laws for the crack growth under constant amplitude loading are usually expressed in the form of a differential equation:

$$\frac{da}{dn} = F(a, S_{max}, S_{min}) \tag{1.1}$$

that involves the actual crack length, a, and the maximum,

S_{max}, and minimum, S_{min}, of far-stress, S(t), in the n-th load cycle. Among many proposals for the function F(·) (see f.e. [1]) the proposal of Paris and Erdogan (PE) [2]

$$\frac{da}{dn} = C(\Delta K)^m \tag{1.2}$$

has been accepted to describe sufficiently accuratly the crack propagation rate in the most cases of practical importance. In Eq.(1.2) ΔK denotes the stress intensity range factor

$$\Delta K = Y(a)(S_{max} - S_{min})\sqrt{\pi a} = Y(a)\Delta S\sqrt{\pi a} \tag{1.3}$$

where Y(a) is a function depending on the geometry of the crack and specimen. ΔS is the difference between a maximum and the preceding minimum and denotes the amplitude of a stress cycle. C and m are material parameters. Some modifications of the PE equation accounting for the mean of stress cycles, $S_m = ½(S_{max}+S_{min})$, or for the effective stress amplitude, ΔS_{eff}, with an increased stress minimum instead of the actual S_{min} practically do not introduce principal changes in the approach.

The PE equation is also used in the analysis of the crack propagation under stochastically varying loading. The stress amplitude in each stress cycle is a random variable, ΔS_i. The solution of Eq.(1.2)

$$\int_0^{T_F} \frac{da}{\left[Y(a)\sqrt{\pi a}\right]^m} = C \sum_{i=0}^{N^+(T_F)} \Delta S_i^m \tag{1.4}$$

involves the random number, $N^+(T)$, of random stress amplitudes. The structural life-time, T_F, i.e. the time when the crack length reaches a critical (maximum admissible) value, is in principle a random value as well. The analysis for stationary stochastic loading [3] however shows that the coefficient of variation of T_F tends to zero for long life-time which is mostly of interest from the practical point of view. Thus, the random variations of the structural life-time due to the stationary stochastic load fluctuations may be neglected and the mean life-time, $\bar{T}_F$, should be eventually looked for in the analysis.

The form of the crack propagation laws suggests to look for a possibility to apply the theory of Markov processes to the crack growth problem in presence of stochastic loading. In order to overcome some difficulties connected with determination of the amplitude process, $\Delta S(t)$, the means or some equivalent values of arguments are assumed in function $F(\cdot)$ in Eq.(1.1) while the whole right-hand side of Eq.(1.1) is multiplied by a new stochastic process $X(t)$. If the process $X(t)$ is, say, a white noise [4] or a stationary pulse process [5] the analysis yields the solution, $A(t)$, in terms of Markov process. A suitable fitting of the parameters of $X(t)$ leads to very good agreement of the theoretical prediction of the fatigue crack length with experimental results. Though the derivation of the parameters of $X(t)$ solely from the stochastic load process without specific experimental results for this load process seems to be impossible this approach however shows that the Markovian approximation of the crack growth under stochastic loading can be succesfully used to describe the real crack propagation process.

Numerous experiments show rapid changes of the fatigue crack propagation rate after a load cycle with maximum of greater magnitude (overloading) than the subsequent maxima. The usual change in the mode I of crack propagation is a diminuation of the crack propagation rate after overloading (retardation of the crack growth). The duration of the retardation phase and the magnitude of the retardation effect depend on many factors including specimen geometry, environental effects, material properties and the magnitude of the overloading and of subsequent maxima. The physical nature of this phenomenon has not been completely explained, yet. One of the most convincing explanation is by existarce of a large plastic zone around the crack tip due to the overloading resulting in local changes of material properties (hardening) and compressive self-stresses. These facts are taken into account in phenomenological models of crack propagation in the ratardation phase. The modifications take the form of an additional factor, C_i in the right-hand side of the PE equation, [6], or lead to an appropriate reduction of the stress amplitude, [7].

An attempt to model the fatigue crack growth process with account for ocassional overloads was recently presented in [8] where a modified birth process was applied as a counting process for the number of crack length increments within and outside the retardation phases.

The results mentioned above, i.e. negligible scatter of the life-time, Markovian character of fatigue crack growth process, retardation effects of overloading, have set up a basis for the approach that is presented in this paper. Using the Wheeler model of the crack growth retardation and some properties of maxima of stochastic processes the parameters of the Kolmogorov diffusion equation are determined and the mean time at which the crack length reaches its critical value is calculated. In an example the calculation of the life-time for various band-widths of the load process and several values of the retardation parameters shows increasing but still moderate effect of the fatigue crack growth retardation for widening spectrum band of stationary stochastic load process.

The similar approach was presented by the author in his earlier paper [9]. The results there indicated that especially for the narrow-band loading it is not enough to consider single high maxima which can cause the retardation. The clustering of the high maxima has to be taken into account to give the proper description of fatigue crack growth in presence of the retardation effects. This modification is now introduced so that the proposed approach is valid for the wide-band as well as for the narrow-band stochastic load processes.

II. DETERMINISTIC CRACK GROWTH MODEL

It is assumed that the crack growth under constant amplitude loading is described by the PE equation (1.2). The retardation is taken into account by Wheeler's model [6]. In this model a retardation parameter (factor)

$$C_i = \left[\frac{r_{yi}}{a_o + r_{yo} - a_i} \right]^{m'} \tag{2.1}$$

is introduced. It allows to reduce the crack growth rate for every stress cycle following a high tensile load application. r_{yo} and r_{yi} in Eq.(2.1) denote the plastic zone sizes caused by the overloading with the maximum stress S_o^+ and by the subsequent maxima S_i^+, respectively. a_o and a_i are the crack lengths at S_o^+ and S_i^+, respectively. The exponent m' is an empirical shaping parameter. The plastic zone size for crack length a and a stress maximum S^+ is given as

$$r_y = \frac{1}{n\pi} \left[\frac{K_{max}}{S_y} \right]^2 = \frac{1}{n\pi} \left[\frac{Y(a) S^+ \sqrt{\pi a}}{S_y} \right]^2 \tag{2.2}$$

where S_y denotes the yield strength and n=2 or n=6 for plane stress or plane strain, respectively.

The duration of the retardation phase is as long as $C_i<1$, i.e. the following inequality must be then satisfied

$$a_o + r_{yo} > a_i + r_{yi} \tag{2.3}$$

when the boundary of the instantaneous plastic zone, $a_i + r_{yi}$, reaches the boundary of the plastic zone that had been caused by the overloading, $a_o + r_o$, the retardation effect either disappears (if the next maximum is not an overloading) or is again present with new a_o, r_o and subsequent a_i, r_i.

Since the variables in the modified PE equation remain separable the crack length, a_n, after n load cycles in the retardation phase given a_o and S_o^+ can be determined from the following equation

$$\int_{a_o}^{a_n} \frac{\left\{\left[2S_y^2 + S_o^2 Y^2(a_c)\right] a_o - 2S_y^2 a\right\}^{m'}}{\left[Y(a)\sqrt{a}\right]^{2m'+m}} da = \tag{2.4}$$

$$= C\pi^{m/2} \sum_{i=1}^{n} (S_i^+)^{2m'} (\Delta S_i)^m$$

For known load history, S_i^+ and ΔS_i, the recurrent application of the last equation with appropriate integral limits and $m'=0$ in the phases without retardation effects yields the whole history of fatigue crack propagation. Thus, the time or the number of load cycles when the crack length reaches its critical value, a_F, can be obtained.

For a few stress cycles the crack length increment is very small in comparsion to the plastic zone radius r_{yo},

$$a_i - a_o << r_{yo} \tag{2.5}$$

so that the retardation parameter, Eq.(2.1), takes an approximate form

$$C_i = \left[\frac{r_{yi}}{r_{yo}}\right]^{m'} \quad \text{for } a_i - a_o << r_{yo} \tag{2.6}$$

Hence, the small crack increment is estimated as follows

$$a_n - a_o = \frac{C\left[Y(a_o)\sqrt{\pi a_o}\right]^m}{(S_o^+)^{2m'}} \sum_{i=1}^{n} (S_i^+)^{2m'} (\Delta S_i)^m \qquad (2.7)$$

For random loading the load realizations are not known explicitly and a direct life-time calculation by Eq.(2.4) is not possible. Nevertheless, Eq.(2.4) is assumed to describe the crack propagation for every sample of the stochastic load process.

Experimental observations and the form of the crack growth law indicate that the actual crack length depends at most on the short parts of the load history and of the crack length path closely before to the actual stage. Very good agreement of experimental results with the theoretical predictions under Markovian assumption about the crack length process has been observed in [5]. Therefore, it everything makes sense to look for the solution of the stochastic crack growth problem with retardation effects in the class of Markov process as well.

The loading is assumed to be a stationary gaussian stochastic process which is twice differentiable in the mean square sense. In the crack growth process it is unlikely to expect great increments of the crack length within very short time intervals. This leads to the assumption that the crack length process may be considered as a diffusion Markov process. Hence, the statistical moments, $\bar{T}_F = E[T_F]$, $\bar{T}_{F2} = E[T_F^2]$ etc., of the life-time T_F can be recurrently calculated from the following differential equations

$$\eta(a)\,\frac{d\bar{T}_F(a)}{da} + \frac{1}{2}\,\sigma^2(a)\,\frac{d^2\bar{T}_F(a)}{da^2} = -1 \qquad (a)$$

$$(2.8)$$

$$\eta(a)\,\frac{d\bar{T}_{F2}(a)}{da} + \frac{1}{2}\,\sigma^2(a)\,\frac{d^2\bar{T}_{F2}(a)}{da^2} = -2\bar{T}_F(a) \qquad (b)$$

The coefficients $\eta(a)$ and $\sigma^2(a)$ are time independent due to the stationarity of the load process. Physically they are to be interpreted as the mean rate of crack propagation and rate of the second moment of crack increment, i.e.

$$\eta(a) = \lim_{t \to s} \frac{1}{t-s} E[A(t)-A(s) \mid A(s)=a] \quad (a)$$

$$\sigma^2(a) = \lim_{t \to s} \frac{1}{t-s} E[(A(t)-A(s))^2 \mid A(s)=a] \quad (b) \qquad (2.9)$$

The initial crack length, a_I, is assumed deterministic. The mean life-time, $\bar{T}_F(a_I, a_F)$, during which the crack length starting at $a=a_I$ is shorter than a given critical length a_F but longer than $a=0$ is found from Eq.(2.8a) with the boundary conditions

$$\bar{T}_F(a_F, a_F) = 0 \text{ and } \bar{T}_F(0, a_F) = 0 \qquad (2.10)$$

The first condition is trivial. The second condition expresses a mathematical property of the model and implies that the crack does not propagate if no initial length was assigned at t=0. Hence, the solution of Eq.(2.8a) takes the following form

$$\bar{T}_F(a_I, a_F) = \int_{a_I}^{a_F} \left\{ \int_{x_3}^{a_F} q(x_1) \exp\left[\int_{x_3}^{x_1} p(x)dx\right] dx_1 + C_1 \exp\left[\int_{x_3}^{a_F} p(x)dx\right] \right\} dx_3 + C_2 \qquad (2.11)$$

where

$$p(x) = \frac{2\eta(x)}{\sigma^2(x)} \quad \text{and} \quad q(x) = -\frac{2}{\sigma^2(x)} \tag{2.12}$$

and the integration constants by applying the boundary conditions are given as follows

$$C_1 = -\frac{\int\limits_0^{a_F}\int\limits_{x_3}^{a_F} q(x_1)\exp\left[\int\limits_{x_3}^{x_1} p(x)\,dx\right]dx_1\,dx_3}{\int\limits_0^{a_F}\exp\left[\int\limits_{x_3}^{a_F} p(x)\,dx\right]dx_3} \tag{2.13}$$

$$C_2 = 0$$

The solution for $\bar{T}_{F2}(a_I, a_F)$ can be found similarly with the appropriate functions

$$p(x) = \frac{2\eta(x)}{\sigma^2(x)} \quad \text{and} \quad q(x) = -\frac{4\bar{T}_F(x,a_F)}{\sigma^2(x)} \tag{2.14}$$

The determination of the parameters $\eta(a)$ and $\sigma^2(a)$ appears to be the crucial step in calculation of the fatigue life-time. These parameters $\eta(a)$ and $\sigma^2(a)$ define the rates of the first two moments of the crack growth increment given a crack length $A(s)=a$. The conditional moments for a time interval $[s,s+t]$ within which $N^+(T)$ load maxima occur given the maximum $S_0^+=u$ at $t=s$ and given the crack length $A(s)=a$ has to be determined at first. The expectation operation over them with respect to the random maximum S_0^+ will eventually produce the desired parameters $\eta(a)$ and $\sigma^2(a)$.

III. TIME INTERVAL AND NUMBER OF MAXIMA BETWEEN SUBSEQUENT RETARDATION PHASES

The retardation of the crack growth after a stress maximum S_0^+ at $t=s$ may be present at most during the time interval $[s,s+T_R]$ within which the subsequent maxima, S_i^+, are less than S_0^+. In general any maximum of a stochastic stress process can potentially cause the retardation effect. After a relatively low maximum however it is unlikely to expect a sequence of even lower maxima. Some analysis of the conditional mean of maxima occuring after a maximum S_0^+ at $t=s=0$ shows that this mean remains below the maximum S_0^+ within a time interval $[0,T]$ only if $S_0^+>\bar{S}^+$ where $\bar{S}^+$ denotes the mean of all maxima of the stochastic process $S(t)$. Otherwise this mean tends upwards to the mean $\bar{S}^+$. Thus, those maxima which are greater than the mean of maxima of the stochastic process are assumed to be the retardation relevant maxima. Their probability density function, $f_{S_R^+}(u)$, equal to the probability density function, $f_{S^+}(u)$, of all maxima S^+ given $S^+>\bar{S}^+$, i.e.

$$f_{S_R^+}(u) = \begin{cases} \dfrac{f_{S^+}(u)}{1 - F_{S^+}(\bar{S}^+)} & \text{for } u > \bar{S}^+ \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

The probability density, $f_{S^+}(\cdot)$, and the distribution function, $F_{S^+}(\cdot)$, together with other parameters of the maxima are given in Appendix 1.

Since the maxima tend to occur in clumps (especially for narrow-band processes) the time interval between two subsequent envelope maxima which begin two subsequent retardation phases is looked for. In the approach an approximate conditional mean of this interval given the first maximum $S_0^+=u$ is used. The half of the interval between qualified envelope upcrossings of the level u (see [10])

$$\bar{T}_E^+(u) = \frac{1}{2} \frac{\bar{T}_0^+(u)}{1 - \exp(-\sqrt{2\pi}\, qr)} \tag{3.2}$$

is assumed to estimate the conditional mean of the interval between beginnings of subsequent retardation phases. The following notation is used in Eq. (3.2):

$$\bar{T}_o^+(u) = \bar{T}_o \exp\left(\frac{r^2}{2}\right) \tag{3.3}$$

is the mean time interval between process upcrossings of the level u;

$$\bar{T}_o = 2\pi\sqrt{\frac{\sigma_S}{\sigma_{S'}}} \tag{3.4}$$

denotes the mean time interval between upcrossings of the mean $\bar{S}$, where $r=(u-\bar{S})/\sigma_S$. $\bar{S}$, σ_S^2 and $\sigma_{S'}^2$ are the mean and the variances of the process S(t) and its first derivative S'(t), respectively;

$$q^2 = 1 - \frac{\lambda_1^2}{\lambda_2} \tag{3.5}$$

is a unitless measure of the variability in frequency content of the process S(t), [10], where

$$\lambda_i = \int_0^\infty \omega^i G(\omega)\, d\omega / \sigma_S^2 \tag{3.6}$$

denotes the i-th normalized spectral moment of S(t) and $\lambda_0=1$, $\lambda_2=\sigma_{S'}^2/\sigma_S^2$, $\lambda_4=\sigma_{S''}^2/\sigma_S^2$. G(ω) is the spectral density function of S(t).

The conditional mean time interval, $\bar{T}_R(u)$, within which the process remains below the level u after the maximum $S_o^+=u$ is assumed to be a half of the interval between qualified envelope excursions above u, i.e.

$$\bar{T}_R(u) = \bar{T}_E^+(u) - \bar{T}_{PR}(u) \quad (3.7)$$

where

$$\bar{T}_{PR}(u) = \frac{1}{2}\,\bar{T}_C^+(u) = \frac{1}{2}\,\bar{T}_o\bar{N}_C \quad (3.8)$$

denotes a half of the mean duration of a clump with maxima above u and

$$\bar{N}_C(u) = \left[1 - \exp(-\sqrt{2\pi}\,qr)\right]^{-1} \quad (3.9)$$

is called the clump size and denotes the mean number of u upcrossings within a clump.

The mean number of cycles, $\bar{N}_{PR}(u)$, within the time interval when the maxima are above a level u and before the beginning of the next retardation phase is assumed to be equal to a half of the clump size, $\bar{N}_C(u)$, i.e.

$$\bar{N}_{PR}(u) = \frac{1}{2}\,\bar{N}_C(u) \quad (3.10)$$

In this way only the cycles with maxima above and minima below u enter the calculation within this time interval so that the cycles of very small amplitude there are neglected. The mean number of cycles, $\bar{N}_R(u)$, within the time interval between the maximum $S_o^+=u$ and the subsequent envelope excursion is obtained as the difference

$$\bar{N}_R(u) = \nu^+\bar{T}_E^+(u) - \bar{N}_{PR}(u) \quad (3.11a)$$

where

$$\nu^{+} = \frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}} \tag{3.11b}$$

denotes the mean rate of maxima.

IV. SPECIFIC SOLUTION OF THE LIFE-TIME EQUATION

The equality $C_i=1$, see Eq.(2.1), determines the necessary crack length increment, $\Delta a_{CR}(u)=a_{CR}(u)-a_o$, to terminate the retardation phase which has been initilized by a maximum S_o^+. The conditional crack length increment, $\Delta a_{CR}(u)$, given $S_o^+=u$ and $S_i^+<u$ can be roughly estimated as

$$\Delta a_{CR}(u) = a_o\left[\frac{2S_y^2 + u^2Y(a_o)}{2S_y^2 + E^2[S^+|S^+<u]Y^2(a_o)} - 1\right] \tag{4.1}$$

where $E[S^+|S^+<u]$ denotes the conditional mean of maxima given $S^+<u$ (see Appendix 1). Eq.(4.1) is not very exact but it gives at least the order of magnitude of the increment $\Delta a_{CR}(u)$.

The number of stress cycles between two subsequent retardation relevant envelope maxima is usually small for stationary stress process. Even for extremly narrow-band processes there occur at most several dozen stress maxima that eventually produce very small increment of the crack length. The increment due to the small maxima, $S_i^+<S_o^+$, following a retardation relevant maximum $S_o^+>\bar{S}$ appears to be so small that the current plastic zone, r_{yi}, does not reach the boundary of the great plastic zone, r_{yo}, before a stress maximum does not exceed the level S_o^+. Hence, the duration of the retardation phase is assumed to be equal to the time interval T_R within which the stress maxima remain below the retardation relevant maximum S_o^+ preceding them. The post-retardation phase then follows until the next envelope maximum occurs.

Since the crack length increment between two retardation relevant envelope maxima is very small Eq.(2.7) may be used to determine the first and the second conditional moment of the increment given the crack length $a=a_o$ at $t=s$. The mean of the increment between the maximum $S_o^+=u$ at $t=s=0$ and a subsequent envelope maximum greater than u is calculated for the mean numbers of stress maxima $\bar{N}_R(u)$ and $\bar{N}_{PR}(u)$, Eqs.(3.11) and (3.10), as follows

$$E\left[A(\bar{T}_E(u))-a_o|u\right]=C\left[Y(a_o)\sqrt{\pi a_o}\right]^m\left\{\frac{\bar{N}_R(u)}{u^{2m'}}E\left[(S^+)^{2m'}(\Delta S)^m\middle|S^+<u\right]+\right.$$
$$\left.+\ \bar{N}_{PR}(u)E\left[(\Delta S)^m\middle|S^+>u\right]\right\} \qquad (4.2)$$

The second moment of the increment is calculated either for full- or zero-correlated stress cycles within $T_R(u)$ and $T_{PR}(u)$ while the retardation and post-retardation phases are assumed to be independent. The randomness of the cycle numbers is neglected and their conditional means enter only the following equations:

$$E\left[\left[A(\bar{T}_E(u))-a_o\right]^2\middle|u\right]=$$
$$C^2\left[Y(a_o)\sqrt{\pi a_o}\right]^{2m}\left\{\frac{\bar{N}_R^2(u)}{u^{4m'}}E\left[(S^+)^{4m'}(\Delta S)^{2m}\middle|S^+<u\right]+\right.$$
$$+\frac{2\bar{N}_{PR}(u)\bar{N}_R(u)}{u^{2m'}}E\left[(S^+)^{2m'}(\Delta S)^m\middle|S^+<u\right]E\left[(\Delta S)^m\middle|S^+>u\right]+ \qquad (4.3a)$$
$$\left.+\ \bar{N}_{PR}^2(u)E\left[(\Delta S)^{2m}\middle|S^+>u\right]\right\}$$

for full-correlated stress cycles and

$$E\left[\left[A(\bar{T}_E(u))-a_o\right]^2\middle|u\right] =$$

$$c^2\left[Y(a_o)\sqrt{\pi a_o}\right]^{2m}\left\{\frac{\bar{N}_R(u)}{u^{4m'}}E\left[(S^+)^{4m'}(\Delta S)^{2m}\middle|S^+<u\right] +\right.$$

$$+\frac{2\bar{N}_{PR}(u)\bar{N}_R(u)}{u^{2m'}}E\left[(S^+)^{2m'}(\Delta S)^m\middle|S^+<u\right]E\left[(\Delta S)^m\middle|S^+>u\right] + \tag{4.3b}$$

$$+\bar{N}_{PR}(u)E\left[(\Delta S)^{2m}\middle|S^+>u\right] + \bar{N}_{PR}(u)\left[\bar{N}_{PR}(u)-1\right]E^2\left[(\Delta S)^m\middle|S^+>u\right] +$$

$$\left.+\frac{\bar{N}_R(u)[\bar{N}_R(u)-1]}{u^{4m'}}E^2\left[(S^+)^{2m'}(\Delta S)^m\middle|S^+<u\right]\right\}$$

for zero-correlated stress cycles.

Eqs. (4.2) and (4.3) define the conditional moments of the crack length increment given the stress maximum $S_o^+=u$ at the beginning of the retardation phase. The conditional parameters $\eta(a|u)$ and $\sigma^2(a|u)$ are the rates of these moments. The time intervals $T_E(u)$ are so small in comparsion with the whole life-time that the parameters may be estimated as follows

$$\eta(a|u) = \frac{E\left[A[\bar{T}_E(u)]-a|u\right]}{\bar{T}_E(u)} \tag{a}$$

(4.4)

$$\sigma^2(a|u) = \frac{E\left[\left[A[\bar{T}_E(u)]-a\right]^2\middle|u\right]}{\bar{T}_E(u)} \tag{b}$$

If the crack and specimen geometry dependent function, Y(a), is a constant, Y(a)=Y=const, Eqs. (4.4) can be expressed in the following form:

$$\eta(a|u) = \eta_o(C,m,m'|u)a^{m/2} \quad (a)$$
$$(4.5)$$
$$\sigma^2(a|u) = \sigma_o^2(C,m,m'|u)a^m \quad (b)$$

where $\eta_o(C,m,m'|u)$ and $\sigma_o^2(C,m,m'|u)$ are conditioned by the stress maximum u at the beginning of the retardation phase and depend on the material constants, C, m, and retardation shaping parameter, m', only.

In order to remove the condition $S_o^+=u$ the first three terms of the Taylor expansion of $\eta_o(\cdot|u)$ and $\sigma_o^2(\cdot|u)$ around the mean of retardation relevant maxima, Eq.(A1.9a)

$$\bar{s}^+_{ret} = E\left[s^+ \middle| s^+ > \bar{s}^+\right] \quad (4.6)$$

are averaged. It yields the expressions

$$\eta_o(C,m,m') = \eta_o(C,m,m'|\bar{s}^+_{ret}) + \frac{1}{2}\frac{\partial^2\eta_o(\cdot|u)}{\partial^2 u} Var\left[s^+ \middle| s^+ > \bar{s}^+\right] \quad (a)$$
$$(4.7)$$
$$\sigma_o^2(C,m,m') = \sigma_o^2(C,m,m'\left|\bar{s}^+_{ret}\right.) + \frac{1}{2}\frac{\partial^2\sigma_o^2(\cdot|u)}{\partial^2 u} Var\left[s^+ \middle| s^+ > \bar{s}^+\right] \quad (b)$$

which lead to the unconditional parameters of the life-time equation

$$\eta(a) = \eta_o(C,m,m')a^{m/2} \quad (a)$$
$$(4.8)$$
$$\sigma^2(a) = \sigma_o^2(C,m,m')a^m \quad (b)$$

for Y(a)=Y=const.

The solution, Eq.(2.11) for the mean life-time with the parameters (4.8) can be found as follows

$$\bar{T}_F(a_l, a_F) = -\frac{2}{m-2}\frac{q_o a_l}{p_o}\left\{1 - \left[\frac{a_l}{a_F}\right]^{m/2-1} - \right. \tag{4.9}$$

$$\left. - \frac{m}{2}\left[\frac{m-2}{2p_o a_l}\right]^{\frac{m}{m-2}} a_l \int_{a_l}^{a_F} \exp\left[\kappa(x, a_l)\right] \gamma\left[\frac{m}{m-2}, \kappa(x, a_l)\right] dx\right\}$$

where

$$p_o = \frac{2\eta_o(C, m, m')}{\sigma_o^2(C, m, m')} \; ; \qquad q_o = -\frac{2}{\sigma_o^2(C, m, m')} \tag{a}$$

(4.10)

$$\kappa(x, a) = \frac{2p_o}{m-2}\left[\frac{a}{x}\right]^{m/2} x \tag{b}$$

and $\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt$ denotes the incomplete Γ-function.

V. MOMENTS OF STRESS MAXIMA AND AMPLITUDES

The conditional parameters $\eta_o(\cdot|u)$ and $\sigma_o^2(\cdot|u)$, Eqs.(4.4), involve the first and second conditional moments of product of the 2m'-th (or 4m'-th) power of the stress maximum and of the m-th (or 2m-th) power of the stress amplitude, Eqs.(4.2) and (4.3). In the calculation of these moments within retardation as well as within post-retardation phases the two-dimensional joint probability distribution of the maximum $S^+<u$ and $S^+>u$, respectively, and of the subsequent minimum has to be known. According to the model of conditional stochastic process $S_u(t)$, see [11], the determination of this probability distribution requires the five-dimensional joint probability distribution of $S^+ = S_u(t_i)$, $S^- = S_u(t_i + t_{ex})$, $S^{+''} = S_u''(t_i)$, $S^{-''} = S_u''(t_i + t_{ex})$ and T_{ex} to be known. T_{ex} is the random time interval between the neighbouring extremes. It is however very difficult to determine the probability density function of T_{ex} even for stationary gaussian processes without the actual condition about the maximum $S_0^+=u$ at $t=t_i$. For the crack growth without retardation effects the assumption of the mean value of T_{ex}

$$T_{ex} \approx \bar{T}_{ex} = \frac{1}{\nu_{ex}} = \pi \left[\frac{\lambda_2}{\lambda_4}\right]^{1/2} \tag{5.1}$$

has led to satisfactory results [3]. In the present case the mean time interval (5.1) is assumed as well when performing the expectation operations within retardation and post-retardation phase. In the remaining four-dimensional integral the integration with respect to S^- nd $S^{-"}$ can be carried out analytically for natural values of the material parameter m. The calculation of respective mean values which enter Eqs.(4.2) and (4.3) requires numerical computation of the two-dimensional integral only

$$\Xi^-(u;2m',m)=E\left[(S^+)^{2m'}(\Delta S)^m \middle| S^+<u\right]=E\left[(S^+)^{2m'}(S^+-S^-)^m \middle| S^+<u\right] =$$

$$= \int_{-\infty}^{u} ds^+ \int_0^{\infty} (s^+)^{2m'} u_m(s^+,z^+) f_{S^+Z^+}(s^+,z^+) dz^+ / F_{S^+}(u) \tag{5.2a}$$

for the condition $S^+<u$ or the following integral

$$\Xi^+(u;2m',m)=E\left[(S^+)^{2m'}(\Delta S)^m \middle| S^+>u\right]=E\left[(S^+)^{2m'}(S^+-S^-)^m \middle| S^+>u\right] =$$

$$= \int_{u}^{\infty} ds^+ \int_0^{\infty} (s^+)^{2m'} u_m(s^+,z^+) f_{S^+Z^+}(s^+,z^+) dz^+ / [1 - F_{S^+}(u)] \tag{5.2b}$$

for the opposite condition, $S^+>u$.

The joint probability density function $f_{S^+Z^+}(s,z)$ of the maximum, S^+ and the negative second derivative $Z^+=-S"$ is

$$f_{S^+Z^+}(s,z) = \frac{z\ \exp\left\{-[\lambda_4(s-\bar{S})^2-2\lambda_2 z(s-\bar{S})+z^2]/[2\sigma_S^2(\lambda_4-\lambda_2^2)]\right\}}{\sigma_S^3\sqrt{2\pi\lambda_4(\lambda_4-\lambda_2^2)}} \qquad (5.3)$$

where $\bar{S}$ and σ_S^2 denote the mean and variance of the stress process S(t). λ_i is the i-th normalized spectral moment of S(t), Eq.(3.6). The formulae for $u_m(s^+,z^+)$ are given in Appendix 2 for various natural values of the exponent, m'=1,2,3, 4,5,6.

Eqs.(5.2) complete the set of equations that are necessary to determine the parameters $\eta(a)$ and $\sigma^2(a)$ and eventually to calculate the moments of the structural life-time.

VI. EXAMPLE

A crack with the initial length a_i=0.001 m is assumed to propagate in an infinite metal-sheet so that Y(a) = 1.0 = const. The material parameters, C, m, S_y, are kept constant:

$$C = 10^{-14}\ ; \qquad m = 3.0 \quad ; \qquad S_y = 500\ \text{MPa} \qquad (6.1)$$

The shaping parameter, m', of the Wheeler retardation model, Eq.(2.1) varies from m'=0 (no retardation effects are present) to m'=5 in the example. The stochastic stress process, S(t), is assumed to be stationary gaussian with the mean and variance, respectively:

$$\bar{S} = 300\ \text{MPa} \quad \text{and} \quad \sigma_S^2 = 3600\ \text{MPa}^2 \qquad (6.2)$$

The calculation is carried out for two types of covariance function (CF):

- type I

$$K_I(\tau) = \sigma_S^2 \rho_I(\tau) = \sigma_S^2 \exp(-\alpha^2 \tau^2) \cos(\beta\tau) \qquad (6.3a)$$

- type II

$$K_{II}(\tau) = \sigma_S^2 \rho_{II}(\tau) = \sigma_S^2 \frac{\sin\alpha\tau}{\alpha\tau} \cos(\beta\tau) \qquad (6.3b)$$

with $\beta=10.0\ \mathrm{sec}^{-1}$. In the both cases the band-width of the stress process depends on the parameter α. For $\alpha\to0$ the band-width parameter

$$\gamma^2 = 1 - \frac{\lambda_2^2}{\lambda_4} \qquad (6.4)$$

tends to zero and for $\alpha\to\infty$ it is $\gamma^2=0.816$ for the type I and $\gamma^2 = 0.667$ for the type II of covariance function.

For the material parameter m=3 the solution (4.9) can be expressed without γ-function and it takes the following form

$$\bar{T}_F = \bar{T}_F(a_I, a_F) = -\frac{2q_o a_I}{p_o}\left\{1 - R_{IF}^{1/2} - \right. \qquad (6.5)$$

$$\left. - \frac{3}{8}\frac{1}{p_o a_I}\left[(p_o a_I)^{-2}(1 - R_{IF}^{-1}) + 4(p_o a_I)^{-1}(1 - R_{IF}^{-1/2}) + 2\ln R_{IF}\right]\right\}$$

where $R_{IF}=a_I/a_F$.

For given m=3 and $C=10^{-11}$ the factors $\eta_0(c,m,m')$ and $\sigma_0^2(c,m,m')$, Eqs.(4.7), depend on the band-width parameter γ (or the parameter of covariance function α). Their values for the both types of covariance function are given in Tables 1 and 2 for various γ (α) and m'.

TABLE 1a.

Parameter $\eta_{oI}(C,m,m')*10^8$ for type I of CF.

τ	0.07	0.14	0.27	0.47	0.60	0.73	0.80
(α)	(0.25)	(0.50)	(1.00)	(2.00)	(3.00)	(5.00)	(10.0)
m'							
0.0	1.84	1.83	1.84	1.70	1.45	1.22	1.27
1.0	1.74	1.73	1.73	1.53	1.28	1.06	1.11
2.0	1.66	1.65	1.65	1.40	1.17	0.95	1.01
3.0	1.59	1.58	1.58	1.31	1.08	0.87	0.93
4.0	1.54	1.53	1.53	1.25	1.02	0.81	0.88
5.0	1.50	1.49	1.48	1.19	0.97	0.77	0.83

TABLE 1b.

Parameter $\eta_{oII}(C,m,m')*10^8$ for type II of CF.

τ	0.06	0.11	0.22	0.32	0.47	0.67
(α)	(0.50)	(1.00)	(2.00)	(3.00)	(5.00)	(10.0)
m'						
0.0	1.84	1.89	1.99	1.97	1.76	1.34
1.0	1.73	1.78	1.83	1.79	1.56	1.17
2.0	1.65	1.68	1.72	1.66	1.43	1.05
3.0	1.59	1.61	1.63	1.56	1.33	0.96
4.0	1.53	1.55	1.56	1.48	1.25	0.90
5.0	1.49	1.51	1.50	1.42	1.19	0.85

TABLE 2a.

Parameter $\sigma^2_{oI}(C,m,m')*10^{16}$ for type I of CF.
(upper values - full-correlated cycles
lower values - zero-correlated cycles)

m'	γ 0.07 (α) (0.25)	0.14 (0.50)	0.27 (1.00)	0.47 (2.00)	0.60 (3.00)	0.73 (5.00)	0.80 (10.0)
0.0	75.7 53.1	40.3 29.6	23.7 18.1	13.8 10.4	7.7 5.9	4.1 3.1	3.2 2.3
1.0	67.2 46.1	35.7 25.7	20.8 15.8	10.9 8.4	5.8 4.7	2.9 2.3	2.3 1.8
2.0	60.7 41.0	32.1 22.9	18.6 14.0	9.1 7.1	4.7 3.9	2.3 1.9	1.8 1.5
3.0	55.7 37.1	29.4 20.8	16.9 12.7	7.8 6.2	4.0 3.4	1.9 1.6	1.5 1.3
4.0	51.7 34.1	27.3 19.1	15.6 11.8	7.0 5.7	3.5 3.1	1.7 1.5	1.3 1.2
5.0	48.5 31.8	25.6 17.9	14.6 11.0	6.3 5.2	3.2 2.8	1.5 1.4	1.2 1.1

TABLE 2b.

Parameter $\sigma^2_{oII}(C,m,m')*10^{16}$ for type II of CF.
(upper values - full-correlated cycles
lower values - zero-correlated cycles)

m'	γ 0.06 (α) (0.50)	0.11 (1.00)	0.22 (2.00)	0.32 (3.00)	0.47 (5.00)	0.67 (10.0)
0.0	92.3 63.6	55.3 37.6	38.5 25.3	29.7 19.3	16.9 11.7	6.2 4.5
1.0	81.6 54.9	48.0 32.1	32.0 21.0	23.8 15.6	12.9 9.2	4.5 3.4
2.0	73.5 48.6	42.7 28.2	27.6 18.1	20.0 13.3	10.5 7.7	3.5 2.8
3.0	67.2 43.9	38.7 25.4	24.4 16.0	17.4 11.7	8.9 6.7	2.9 2.5
4.0	62.3 40.2	35.6 23.2	22.1 14.5	15.5 10.5	7.8 6.1	2.5 2.2
5.0	58.4 37.4	33.1 21.6	20.3 13.4	14.1 9.7	7.1 5.6	2.3 2.1

It is seen that $\sigma^2_{0i}(C,m,m') \ll \eta_{0i}(C,m,m')$ for $i = I, II$. The similar strong inequality are also valid for the parameters p_c and q_0, Eq.(4.10a), i.e.

$$q_0 \gg p_0 \quad ; \quad p_0 \gg 1 \tag{6.6}$$

Thus, the order of magnitude of the last term within the large bracket in Eq.(6.5) is nearly equal to $\ln(R_{IF})/(p_0 a_I)$. This is a few order less than one for all initial and critical crack length values which are of practical interest. It leads to the simplified mean life-time equation that takes the following form

$$\bar{T}_F = \bar{T}_F(a_I, a_F) = \frac{2a_I}{\eta_0(C,m,m')}\left\{1 - \left[\frac{a_I}{a_F}\right]^{1/2}\right\} \tag{6.7}$$

The mean life-time curves as functions of the bandwidth parameter $\dot{\gamma}$ (or α) are plotted in Figs.1 for various Wheeler's shaping parameter values m' and for the both types of covariance function, Eqs.(6.3). The calculation shows that the difference between the results from Eq.(6.5) and (6.7) does not exceed 1% of the mean life-time. The effect of the stress cycle correlation is also hardly observable. Thus, Eq.(6.7) yields sufficiently accurate approximation of the fatigue life-time of structure under stochastic stationary loading.

The negligence of the parameter $\sigma^2_0(C,m,m')$ in Eq.(6.7) leads to the quasi-deterministic relation for the fatigue life-time. It agrees with results in [3] where the fatigue crack growth without retardation effects was considered. There was shown that the random variations of the life-time are not significant in the fatigue crack growth problem if the structure is subjected to stationary stochastic loading.

The above results can be verified by comparsion with the solution obtained due to the direct integration of the crack growth equation (2.4) at $m'=0$ (no retardation effects are present). The number, $\bar{N}_F$, of stress cycles to failure is calculated as

$$\bar{N}_F^+ = \frac{1}{C(\pi a_I)^{m/2}\Xi^+(-\infty;0,m)}\frac{2a_I}{m-2}\left\{1-\left[\frac{a_I}{a_F}\right]^{m/2-1}\right\} \qquad (6.8)$$

where $\Xi^+(-\infty;0,m)$ is given by Eq.(5.2b). The mean life-time is now the product of the mean cycle number, $\bar{N}_F^+$, and the mean cycle duration $\bar{T}=1/\nu^+$, Eq.(3.11b), i.e.

$$\bar{\bar{T}}_F^0 = \bar{T}_F = \frac{\bar{N}_F^+}{\nu^+} \qquad (6.9)$$

In Tables 3 the results from Eq.(6.7) at $m'=0$ and results from Eq.(6.9) are given for the initial and critical crack length $a_I=0.001$ m and $a_F=0.1$ m, respectively.

TABLE 3a.
Life-time $\bar{T}_{FI}\times10^{-3}$, Eq.(6.7) at $m'=0$ and $\bar{\bar{T}}_{FI}^0\times10^{-3}$, Eq.(6.9), for type I of CF.

γ	0.07	0.14	0.27	0.47	0.60	0.73	0.80
(α)	(0.25)	(0.50)	(1.00)	(2.00)	(3.00)	(5.00)	(10.0)
$\bar{T}_{FI}$	98.0	98.3	97.7	106.1	124.0	147.7	141.3
$\bar{\bar{T}}_{FI}^0$	98.0	98.3	97.7	106.3	124.3	147.0	139.0

TABLE 3b.
Life-time $\bar{T}_{FII}*10^{-3}$, Eq.(6.7) at m'=0 and $\bar{T}^{o}_{FII}*10^{-3}$, Eq.(6.9), for type II of CF.

γ	0.06	0.11	0.22	0.32	0.47	0.67
(α)	(0.50)	(1.00)	(2.00)	(3.00)	(5.00)	(10.0)
$\bar{T}_{FII}$	97.9	95.0	90.5	91.3	102.3	133.9
$\bar{T}^{o}_{FII}$	97.7	95.1	90.8	91.6	103.6	133.6

The life-time solutions without retardation effects, m'=0, from the present approach, Eq.(6.7), and by the simple integration of Eq.(2.4) with m'=0, Eq.(6.9), practically yield the same results for any band-width of the stress process.

VII. REMARKS AND CONCLUSIONS

Direct application of the present result, say Eq.(6.7), in reliability analysis would be rather inconvenient due to the Taylor expansion, Eq.(4.7a), in calculation of the parameter $\eta_0(C,m,m')$. The expansion requires the second derivative of $\eta_0(\cdot|u)$ at the mean of retardation relevant maxima, $u=\bar{S}_{ret}$. The calculati n of a single value of $\eta_0(\cdot|u)$ for a set of arguments involves two double integrals, $\Xi^-(u;2m',m)$ and $\Xi^+(u;0,m)$, Eqs.(5.2), that have to be numerically computed. For the second derivative of $\eta_0(\cdot|u)$ three values of $\eta_0(\cdot|u)$ are needed at three values of u. This would be the most time consuming step in any reliability algorithm. Therefore, the importance of the second term in Eq.(4.7a) has been investigated. In Figs.2 the bounds of the relative error between the life-time, $\bar{T}_F$, with the parameter $\eta_0(C,m,m')$ from Eq.(4.4a) and the life-time, $\bar{T}_{FS}$, with the simplified parameter

$$\eta_{oS}(C,m,m') = \eta_o(C,m,m'|\bar{S}^+_{ret}) \tag{7.1}$$

are shown for the both types of correlation function. It is seen that the negligence of the second term in the Taylor expansion of $\eta_o(C,m,m')$, Eq.(4.7a), yields a very small error which never exceeds a few percents. Comparing Eqs.(4.2), (4.4), (4.8a), (5.2) and taking into account the above fact the life-time equation (6.7) can be expressed in the following form

$$\bar{T}_{FS} = \frac{2a_i}{(m-2)\eta_{oS}(C,m,m')}\left\{1 - \left[\frac{a_i}{a_F}\right]^{m/2-1}\right\} \tag{7.2a}$$

where

$$\eta_{oS}(C,m,m') = \frac{C(Y^2\pi)^{m/2}}{\bar{T}_E(U)}\left[\frac{\bar{N}_R(U)}{U^{2m'}}\Xi^-(U;2m',m) + \bar{N}_{PR}(U)\Xi^+(U;0,m)\right] \tag{7.2b}$$

and $U=\bar{S}^+_{ret}$, Eq.(4.6).

Eqs.(7.2) determines a relation between the material parametrs and the features of fatigue failure (life-time, crack length). The integrals $\Xi^-(\cdot)$ and $\Xi^+(\cdot)$ moreover depend on the load parameters (mean, variance, band-width parameter). If any or a few of these quantities is random Eqs.(7.2) or their appropriate transformed forms can be used in reliability analysis to find the failure probability, design point, life-time or crack length characteristics with account for the random properties of the load and structural material.

The present approach and the results of the example allow to draw some general conclusions on the fatigue crack propagation with retardation effects under stationary stochastic loading. The observations can be summarized as follows:

1. the fatigue life-time is not significantly subjected to the random variations due to the stationary random fluctuations of the stress process alone. It allows to consider the life-time as a deterministic function of the material and load parameters;
2. in the crack propagation process two phases occur alternatively: the retardation phase after a retardation relevant envelope maximum and the post-retardation phase between the first maximum which exceeds the prior retardation relevant maximum and the next one that begins the new retardation phase. It is worth to notice that the crack length increment within a retardation phase is a few order of magnitude less than the increment, Eq.(4.1), which would be necessary for the crack and associated plastic zone to reach the boundary of the large plastic zone that has been produced by the preceding retardation relevant maximum;
3. the fatigue life-time mainly depends on the band-width of the stress process and is not significantly affected by the shape of correlation (or spectral density) function alone. In the example the results are quite similar for two different stress correlation fuctions at equal band-width parameter values. It is connected with a direct dependence of the clump size of maxima on the band-width and with the fact that the crack length increment within the post-retardation phase (during clump appearence) when the retardation effect is absent and cycle amplitudes are great is much greater than that in the retardation phase.
4. the effect of the retardation parameter, m', Figs.3, appears to be more significant for wide-band stress processes than for narrow-banded. The crack length incremets within post-retardation phases dominate the whole crack growth process. Therefore, for narrow-band processes for which the clump size is much greater than for wide-band processes the retardation effect becomes less important due to the large crack length increments during the

clump appearances;

5. the relative simple relation between the features of fatigue failure and some material and load parameters gives a chance for successlully applications of the present results in reliability analysis where the random nature of the material and load characteristics is usually assumed;
6. the present approach can be a starting-point for consideration of more complicated load processes, say a series of blocks of different stationary stochastic processes. Random duration of the block, random parameters of the processes which compose the series involve many interesting problems that are waiting to be solved.

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APPENDIX 1. DISTRIBUTION AND MOMENTS OF MAXIMA

The probability distribution function of maxima of a stationary stochastic process S(t) is given in an integral form as

$$f_{S^+}(u) = -\bar{T}_0^+ \int_{-\infty}^{0} s'' f_{SS'S''}(u,0,s'')\,ds'' \qquad (A1.1)$$

where $\bar{T}_0$ denotes the mean time interval between maxima and

$$\nu_0^+ = \frac{1}{\bar{T}_0^+} = -\int_{-\infty}^{0} s'' f_{S'S''}(0,s'')\,ds'' \qquad (A1.2)$$

is the mean rate of maxima.

If the process S(t) is gaussian Eq.(A1.1) takes the form

$$f_{S^+}(u) = \frac{\tau}{\sigma_S}\,\varphi\left(\frac{u_\sigma}{\tau}\right) + \sqrt{2\pi(1-\tau^2)}\;\frac{u_\sigma}{\sigma_S}\,\varphi(u_\sigma)\Phi(\tau_1 u_\sigma) \qquad (A1.3)$$

If $\bar{S}$ and σ_S^2 denote the mean and variance of S(t) and λ_i the i-th not normalized spectral moment then u_σ, τ and τ_1 are given by

$$u_\sigma = \frac{u-\bar{S}}{\sigma_S}; \quad \tau^2 = 1 - \frac{\lambda_2^2}{\lambda_0\lambda_4}; \quad \tau_1 = \frac{\sqrt{1-\tau^2}}{\tau} \qquad (A1.4)$$

The mean rate of maxima is now

$$\nu_0^+ = \frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}} \qquad (A1.5)$$

The probability distribution of maxima results from integration of Eq.(A1.1) and is given as

$$F_{S^+}(u) = \int_{-\infty}^{u} f_{S^+}(x)dx = \Phi(\frac{u_\sigma}{\gamma}) - \sqrt{2\pi(1-\gamma^2)}\ \varphi(u_\sigma)\Phi(\gamma_1 u_\sigma) \qquad (A1.6)$$

The mean number of maxima $a<S^+<b$ within a time interval $[0,T]$ is

$$\bar{N}^+_{ab}(T) = \nu^+_0 T\ [F_{S^+}(b) - F_{S^+}(a)] \qquad (A1.7)$$

The mean and variance of all maxima are, respectively

$$\bar{S}^+ = \int_{-\infty}^{\infty} x f_{S^+}(x)dx = \bar{S} + \sigma_S\sqrt{\pi(1-\gamma^2)/2} \qquad (a)$$

(A1.8)

$$\sigma^2_{S^+} = \int_{-\infty}^{\infty} (x-\bar{S}^+)^2 f_{S^+}(x)dx = \sigma^2_S[1 - (1-\gamma^2)(\frac{\pi}{2} - 1)] \qquad (b)$$

The conditional mean, second moment and variance of maxima given $S^+>u$ are, respctively

$$E[S^+|S^+>u] = \int_u^{\infty} x f_{S^+}(x)dx/[1-F_{S^+}(u)] =$$

$$= \Big\{\sigma_S\gamma\varphi(\frac{u_\sigma}{\gamma}) + \bar{S}\Phi(-\frac{u_\sigma}{\gamma}) + \sqrt{2\pi(1-\gamma^2)}\Big[u_\sigma\varphi(u_\sigma)\Phi(\gamma_1 u_\sigma) + \qquad (a)$$

$$+ \sigma_S(\frac{1}{4} + \frac{1}{2\pi}\mathrm{arctg}\gamma_1 - \int_0^{u_\sigma}\varphi(x)\Phi(\gamma_1 x)dx)\Big]\Big\}/[1-F_{S^+}(u)] ;$$

(A1.9)

$$E[S^{+2}|S^{+}>u] = \int_u^{\infty} x^2 F_{S^+}(x)dx/[1-F_{S^+}(u)] =$$

$$= \Big\{\Phi(-\frac{u_\sigma}{\tau})[\sigma_S^2(2-\tau^2)+\bar{S}^2] + \varphi(\frac{u_\sigma}{\tau})\sigma_S\tau(\sigma_S u_\sigma+2\bar{S}) + \qquad (b)$$

$$+ \sqrt{2\pi(1-\tau^2)}\varphi(u_\sigma)\Phi(\tau_1 u_\sigma)[\sigma_S^2(u_\sigma^2+2) + 2\sigma_S\bar{S}u_\sigma + \bar{S}^2] +$$

$$+ 2\sqrt{2\pi(1-\tau^2)}\sigma_S\bar{S}\Big[\frac{1}{4} + \frac{1}{2\pi}\mathrm{arctg}\tau_1 - \int_0^{u_\sigma}\varphi(x)\Phi(\tau_1 x)dx\Big]\Big\}/[1-F_{S^+}(u)]$$

$$\mathrm{Var}[S^+|S^+>u] = E[S^{+2}|S^+>u] - E^2[S^+|S^+>u] \qquad (c)$$

For the opposite condition, $S^+<u$, the respective equations can be obtained from Eqs.(A1.8) and (A1.9) and constitude the following set

$$E[S^+|S^+<u] = \Big\{\bar{S}^+ - E[S^+|S^+>u][1 - F_{S^+}(u)]\Big\}/F_{S^+}(u) \qquad (a)$$

(A1.10)

$$E[S^{+} \; |S^+<u] = \Big\{\sigma_S^2 + \bar{S}^{+2} - E[S^{+2}|S^+>u][1 - F_{S^+}(u)]\Big\}/F_{S^+}(u)$$

(b)

$$\mathrm{Var}[S^+|S^+<u] = E[S^{+2}|S^+<u] - E^2[S^+|S^+<u] \qquad (c)$$

APPENDIX 2. CONDITIONAL MEAN OF THE M-TH POWER OF AMPLITUDE

Let S(t) be a stationary gaussian process with the mean $\bar{S}$ and covariance function r(t), $r(0)=\sigma_S^2=\lambda_0$, $r''(0)=\lambda_2$, $r^{iv}(0)=\lambda_4<\infty$. For a given time interval between extremes, T_{ex}, the double-conditioned process, $S_{u'u''}(t)$, has a local maximum, $S_0^+=u'$ at t=0, and a local minimum, $S_T^-=u''$, at $t=T_{ex}$. The two-dimensional density function of S_0^+ and S_T^- involves a product of conditional density functions as follows

$$f_{S_0^+S_T^-}(u',u'') = \tag{A2.1}$$

$$= f_{S_0^+S_0''}(u',s_0'')\, f_{S_T''|S_0^+S_0''}(s_T''|u',s_0'')\, f_{S_T^-|S_0^+S_0''S_T''}(u''|u',s_0'',s_T'')$$

The conditional probability density function of the minimum (the last function in the above equation) is gaussian with the mean and variance, respectively

$$\bar{S}_T^- = \bar{S} + \underset{\sim}{T}_\rho \underset{\sim}{T}^{-1}(u'-\bar{S},0,0,s_0'',s_T'') \qquad \text{(a)}$$

$$\sigma_{S^-}^2 = \sigma_S^2 - \underset{\sim}{T}_\rho \underset{\sim}{T}^{-1} \underset{\sim}{T}_\rho' \qquad \text{(b)} \tag{A2.2}$$

where the matrix $\underset{\sim}{T}$ and the vector $\underset{\sim}{T}_\rho$ involve the covariances of the process and its derivatives as follows

$$\underset{\sim}{T} = \left[\begin{array}{ccc|cc} 1 & 0 & r' & -\lambda_2 & r'' \\ 0 & \lambda_2 & -r'' & 0 & -r''' \\ r'' & -r'' & \lambda_2 & r''' & 0 \\ \hline -\lambda & 0 & r''' & \lambda_4 & r^{iv} \\ r'' & -r''' & 0 & r^{iv} & \lambda_4 \end{array}\right] = \left[\begin{array}{c|c} \underset{\sim}{T}_{11} & \underset{\sim}{T}_{12} \\ \hline \underset{\sim}{T}_{21} & \underset{\sim}{T}_{22} \end{array}\right] \tag{A2.3}$$

$$\underset{\sim}{T}_\rho' = [r,-r',0,r'',-\lambda_2] \tag{A2.4}$$

and all $r, r', \ldots, r^{iv}(\tau)$ are calculated at $t=T_{ex}$.

The conditional mean, $E[(S_0^+ - S_T^-)^m | u', s_0'', s_T'']$, is obtained from intgration of $(u'-u'')^m$ with the density (A2.1) with respect to u''. The integration can be carried out analytically for natural exponent m and gives the following expressions

$$E[(S_0^+ - S_T^-)^m | u', s_0'', s_T''] =$$

$$\begin{aligned}
\text{for } m=1 \quad &- = (u' - \bar{s}_T^-) \\
m=2 \quad &- = (u' - \bar{s}_T^-)^2 + \sigma_{s^-}^2 \\
m=3 \quad &- = (u' - \bar{s}_T^-)^3 + 3(u' - \bar{s}_T^-)\,\sigma_{s^-}^2 \\
m=4 \quad &- = (u' - \bar{s}_T^-)^4 + 6(u' - \bar{s}_T^-)^2\sigma_{s^-}^2 + 3\sigma_{s^-}^4 \\
m=5 \quad &- = (u' - \bar{s}_T^-)^5 + 10(u' - \bar{s}_T^-)^3\sigma_{s^-}^2 + 15(u' - \bar{s}_T^-)\,\sigma_{s^-}^4 \\
m=6 \quad &- = (u' - \bar{s}_T^-)^6 + 10(u' - \bar{s}_T^-)^4\sigma_{s^-}^2 + 45(u' - \bar{s}_T^-)^2\sigma_{s^-}^4 + 15\sigma_{s^-}^2
\end{aligned} \tag{A2.5}$$

Since the variable s_T'' enters linearly the mean $\bar{s}_T^-$ the difference $u' - \bar{s}_T^-$ can be rewritten in the form

$$u' - \bar{s}_T^- = c_T[\alpha(u', s_0^+) - s_T^{-''}] = c_T(\alpha - s_T'') \tag{A2.6}$$

where c_T and $\alpha(u', s_0'')$ result from an appropriate transformation of $\bar{s}_T^-$.

The conditional density function of the second derivative S_T'' at $t=T_{ex}$ given $S_0^+=u'$, $S_0''=s_0''$ at $t=0$ is given by

$$f_{S_T''|S_0^+ S_0''}(s_T''|u', s_0'') = \frac{s_T''\varphi(\zeta_1 - \mu s_T'')}{\Psi(\zeta_1)}\mu^2 \tag{A2.7}$$

where

$$\zeta_1 = [m_2 - (m_1 + s_o^*) t_{12}/t_{11}]\nu \; ; \quad \nu = \sqrt{t_{11}/|T_{2.1}|}$$

$$\Psi(\zeta) = \varphi(\zeta) + \zeta\Phi(\zeta)$$

while m_1, m_2, t_{11}, t_{12} are the components of the vector $\underset{\sim}{m}$ and the matrix $\underset{\sim}{T}_1$ given as follows

$$\underset{\sim}{m} = \underset{\sim}{T}_{21}\underset{\sim}{T}_{11}^{-1}(s^+ - \bar{S}, 0, 0)' = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \qquad \text{(a)}$$

(A2.8)

$$\underset{\sim}{T}_{2.1} = \underset{\sim}{T}_{22} - \underset{\sim}{T}_{21}\underset{\sim}{T}_{11}^{-1}\underset{\sim}{T}_{12} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}; \quad |\underset{\sim}{T}_{2.1}| = \det[\underset{\sim}{T}_{2.1}] \qquad \text{(b)}$$

Thus, the integration for any natural power of S_T^* with the density function (A2.7) can be carried out analytically and leads to the following formulae

$$S_m = E[(S_T^{-*})^m | u', s_o^*] =$$

$$\text{for } m=1 - = [\zeta_1 + \Phi(\zeta_1)/\Psi(\zeta_1)]/\nu$$

$$\text{for } m=2 - = [\zeta_1^2 + 2 + \zeta_1\Phi(\zeta_1)/\Psi(\zeta_1)]/\nu^2$$

$$\text{for } m=3 - = [\zeta_1^3 + 5\zeta_1 + (\zeta_1^2 + 3)\Phi(\zeta_1)/\Psi(\zeta_1)]/\nu^3$$

$$\text{for } m=4 - = [\zeta_1^4 + 9\zeta_1^2 + 8 + (\zeta_1^3 + 7\zeta_1)\Phi(\zeta_1)/\Psi(\zeta_1)]/\nu^4 \qquad \text{(A2.9)}$$

$$\text{for } m=5 - = [\zeta_1^5 + 14\zeta_1^3 + 33\zeta_1 + (\zeta_1^4 + 12\zeta_1^2 + 15)\Phi(\zeta_1)/\Psi(\zeta_1)]/\nu^5$$

$$\text{for } m=6 - = [\zeta_1^6 + 20\zeta_1^4 + 87\zeta_1^2 + 48 + (\zeta_1^5 + 18\zeta_1^3 + 57\zeta_1)\Phi(\zeta_1)/\Psi(\zeta_1)]/\nu^6$$

Applying the expectation operation with respect to S_T^* in Eq.(A2.5) and taking into account (A2.7) and (A2.9) the double-conditioned means of the m-th power of amplitude given a maximum $S_0^+ = u'$ and the second derivative $S_0^{+''} = s_0''$ at t=0 take the following forms

$$u_m(u', s_0'') = E[(S_0^+ - S_T^-) \mid u', s_0''] =$$

$$\begin{aligned}
\text{for } m=1 \; - &= c_T(\alpha - S_1) \\
m=2 \; - &= c_T^2(\alpha^2 - 2\alpha S_1 + S_2) \\
m=3 \; - &= c_T^3(\alpha^3 - 3\alpha^2 S_1 + 3\alpha S_2 + S_3) \\
m=4 \; - &= c_T^4(\alpha^4 - 4\alpha^3 S_1 + 6\alpha^2 S_2 - 4\alpha S_3 + S_4) \\
m=5 \; - &= c_T^5(\alpha^5 - 5\alpha^4 S_1 + 10\alpha^3 S_2 - 10\alpha^2 S_3 + 5\alpha S_4 - S_5) \\
m=6 \; - &= c_T^6(\alpha^6 - 6\alpha^5 S_1 + 15\alpha^4 S_2 - 20\alpha^3 S_3 + 15\alpha^2 S_4 - 6\alpha S_5 + S_6)
\end{aligned} \qquad (A2.10)$$

CAPTIONS TO THE FIGURES

Fig.1. Mean life- time, $\bar{T}_{F_i}$, i=I,II, as function of band-with parameter, γ, for various retardation parameter values, m'. a) type I of CF; b) type II of CF.

Fig.2. Envelopes of the relative error of approximation (7.2) for $0 \leq m' \leq 5$. a) type I of CF: b) type II of CF.

Fig.3. Mean life-time, $\bar{T}_{F_i}(m')$, as function of retardation parameter, m', (normalized by the mean life-time without retardation effects, $\bar{T}_{F_i}(0)$, i=I,II), for various band-width parameter values, γ. a) type I of CF; b) type II of CF.

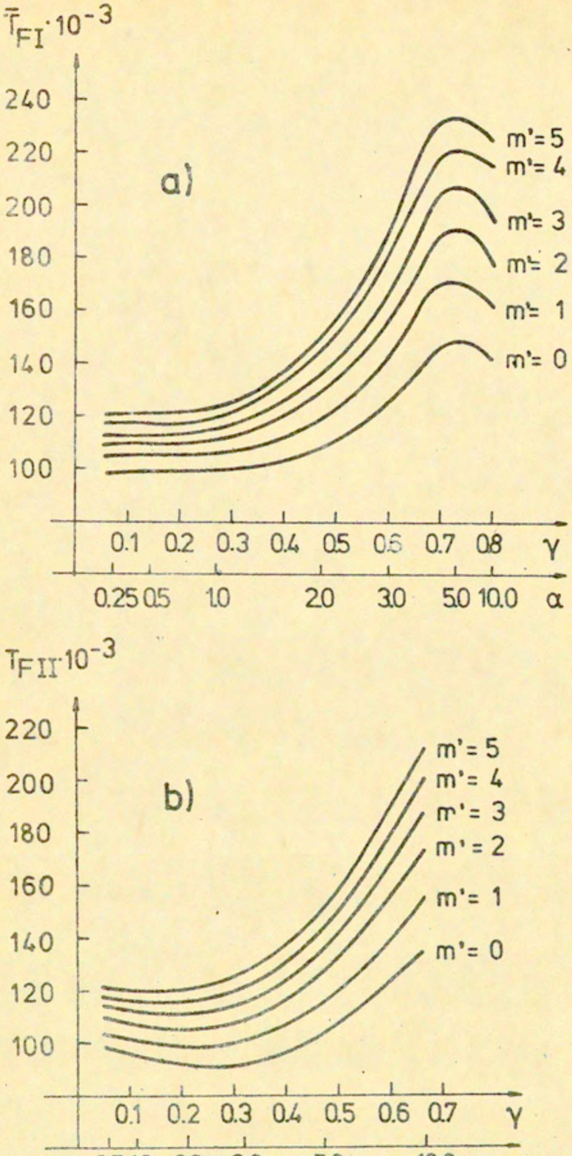
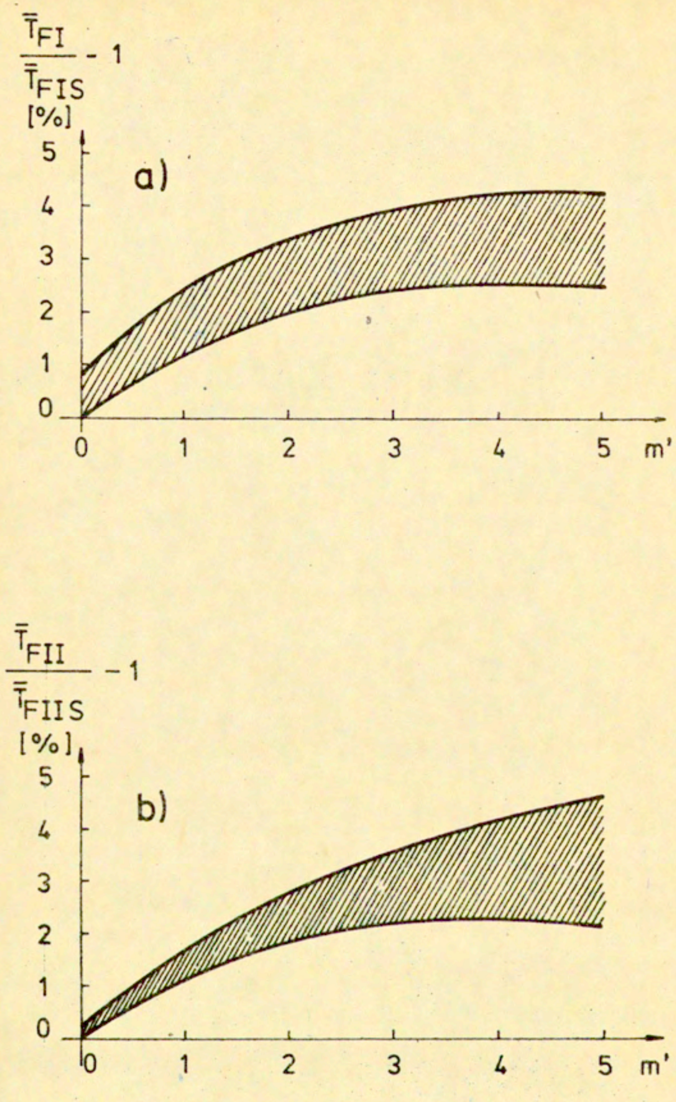

$\bar{T}_{FI} \cdot 10^{-3}$
240
220
200
180
160
140
120
100
a)
m'=5
m'=4
m'=3
m'=2
m'=1
m'=0
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 γ
0.25 0.5 1.0 2.0 3.0 5.0 10.0 α
$T_{FII} \cdot 10^{-3}$
220
200
180
160
140
120
100
b)
m'=5
m'=4
m'=3
m'=2
m'=1
m'=0
0.1 0.2 0.3 0.4 0.5 0.6 0.7 γ
0.5 1.0 2.0 3.0 5.0 10.0 α

Fig. 1.



$\frac{\bar{T}_{FI}}{\bar{T}_{FIS}} - 1$
[%]
5
4
3
2
1
0
a)
0
1
2
3
4
5
m'
$\frac{\bar{T}_{FII}}{\bar{T}_{FIIS}} - 1$
[%]
5
4
3
2
1
0
b)
0
1
2
3
4
5
m'

Fig. 2.

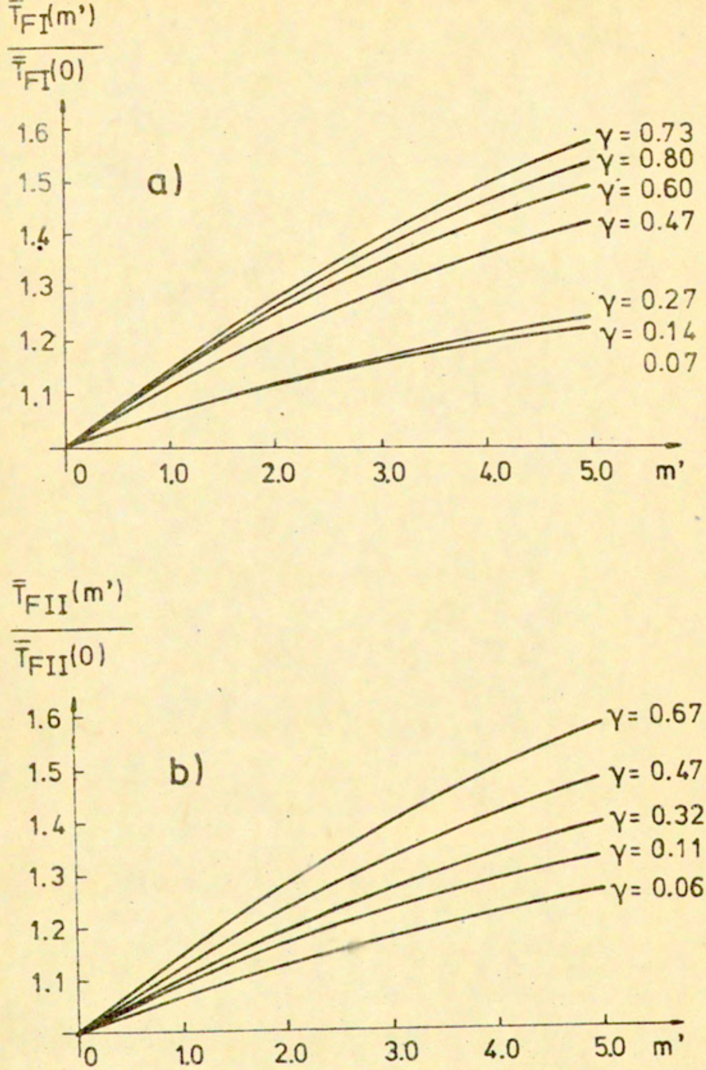

$\bar{T}_{FI}(m')$ / $\bar{T}_{FI}(0)$
1.6
1.5
1.4
1.3
1.2
1.1
a)
γ = 0.73
γ = 0.80
γ = 0.60
γ = 0.47
γ = 0.27
γ = 0.14
0.07
0
1.0
2.0
3.0
4.0
5.0
m'
$\bar{T}_{FII}(m')$ / $\bar{T}_{FII}(0)$
1.6
1.5
1.4
1.3
1.2
1.1
b)
γ = 0.67
γ = 0.47
γ = 0.32
γ = 0.11
γ = 0.06
0
1.0
2.0
3.0
4.0
5.0
m'

Fig. 3.