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STRUCTURAL CLEARENCES AND INTERNAL FRICTION - SIMULATION BY EIGEN DISTORTIONS -

ABSTRACT

The variational principles and computational methods of analysis of initially slackened and stiffened structures are discussed.

The simulation of clearences or internal dry friction in the structural elements by eigen distortions is applied.

Presented considerations will be used in the problem of nonstandard design of structural setting (with clearences or friction in the structural joints) for load capacity maximization.

I. INTRODUCTION

The aim of this paper is to discuss the variational principles and computational procedures for the analysis of initially slackened and stiffened structures. The considerations are restricted to the truss structures. However, they can be easily generalized for any rod structure (eg. frame structure).

The initially slackened structure is composed of elements with mechanical characteristics shown on Fig.1a. The clearences $\bar{\varepsilon}$ introduced into the elements can be caused by some special design of joints in the truss structure - Fig.1b (or in the case of frame structure - Fig.1.c). On the other hand, the initially stiffened structure is composed of elements with mechanical characteristics shown on Fig.2.

The computational procedures of slackened and stiffened structure analysis will be very helpfull in the next stage of the planned research on optimal setting design [8]. The setting design problem deals with nonstandard design of joints (making use of clearences and friction) in order to obtain some demanded macrostiffness characteristics of the structure (eg.the case of supporting structure [7]).

Some examples of slackened elastic-plastic beams and frames treated as the problem of unilateral boundary conditios were considered by Gawęcki [6]. The unilateral problems in structural mechanics are mostly explored in application to the von Karman plates. Duvaunt and Lions [1] have presented the existence and uniqueness theorem of the problem. The duality in the case of obstacle problem have been developed by Bielski and Telega [2]. Numerical results for von Karman plates with unilateral displacement constraints have been described by Ohtake, Oden, and Kikuchi [3]. On the other hand, the existance and uniqueness of contact problem with friction have been discussed by Demkowicz and Oden [4]. The survey of variational inequality applications to unilateral problems in continua and structural mechanics can be found in [5].

In the presented paper the idea of simulation of unilateral constraints problems by eigen distortions is explored. Therefore consider an linear elastic body (in particular truss) subjected to an external body $\mathbf{X}$ and boundary $\mathbf{p}$ forces (Fig.3a). Assume that besides this loading there is also the eigen distortion field $\underline{\varepsilon}^{\circ}$ introduced into a structure in order to simulate a nonlinear constitutive characteristics. The resulting governing equations take the following form:

$$\mathfrak{X}(\underline{\sigma},\mathbf{X},\mathbf{p}) = \mathbf{0} \qquad \begin{cases} \operatorname{div}\underline{\sigma}+\mathbf{X}=\mathbf{0} & \text{in } V \\ \underline{\sigma}\mathbf{n} = \mathbf{0} & \text{on } A_p \end{cases}$$

$$\mathfrak{E}(\underline{\varepsilon}) = \mathbf{0} \qquad \begin{cases} \underline{\varepsilon} = \operatorname{grad}^{s}\mathbf{u} & \text{in } V \\ \mathbf{u} = \mathbf{0} & \text{on } A_u \end{cases} \tag{1}$$

$$\underline{\sigma} = \mathbf{A}(\underline{\varepsilon} - \underline{\varepsilon}^{\circ})$$

where:

$\underline{\sigma} = \underline{\sigma}^r + \underline{\sigma}^l$,

$\underline{\varepsilon} = \underline{\varepsilon}^r + \underline{\varepsilon}^l$,

$\underline{\sigma}^l$; $\underline{\varepsilon}^l$-states of stresses and deformations caused by external load,

$\underline{\sigma}^r$; $\underline{\varepsilon}^r$-states of stresses and deformations caused by eigen distortions $\underline{\varepsilon}^o$ (cf.Fig.3b illustrating the decomposition of the strain and stress states),

$\mathfrak{X}$ -the equilibrium operator,

$\mathcal{E}$ -the compatibility operator,

$\mathbf{A}$ -the elasticity (stiffness) matrix,

grad^s-the symmetric part of the gradient.

If the external load vanishes ($\mathbf{X}=\mathbf{0}$, $\mathbf{p}=\mathbf{0}$) equations (1) describe the selfequilibrated state of initial stresses $\sigma = \underline{\sigma}^r$ and the compatible state of initial deformations $\underline{\varepsilon} = \underline{\varepsilon}^r$ caused by the distortion state $\underline{\varepsilon}^o$:

$$\begin{aligned} &\mathfrak{X}(\underline{\sigma}^r) = \mathbf{0} \\ (2)\qquad &\mathcal{E}(\underline{\varepsilon}^r) = \mathbf{0} \\ &\underline{\sigma}^r = \mathbf{A}(\underline{\varepsilon}^r - \underline{\varepsilon}^o). \end{aligned}$$

Substituting constitutive relations $(1)_3$ to the virtual work principle:

$$(3)\qquad \int \underline{\sigma}\delta\underline{\varepsilon}dV = \int \mathbf{X}\delta\mathbf{u}dV + \int \mathbf{p}\delta\mathbf{u}dA_p$$

the modified principle of potential energy Π_ε

$$(4)\qquad \Pi_\varepsilon = \frac{1}{2}\int \underline{\varepsilon}\mathbf{A}\underline{\varepsilon}dV - \int \underline{\varepsilon}\mathbf{A}\underline{\varepsilon}^o dV - \int \mathbf{X}\mathbf{u}dV - \int \mathbf{p}\mathbf{u}dA_p$$

can be obtained [9]. The principle says that the deformation of the body subject to external load $\mathbf{X}$, $\mathbf{p}$ and eigen distortions $\underline{\varepsilon}^o$ takes the compatible form (satisfying $(1)_2$) which minimizes the functional (4). It means that the stationarity conditions of minimum of the functional (4) subject to the constraints $(1)_2$ take the form of equilibrium equations (1)₁.

Analogously, substituting to equation (3) the deformation state defined by (1)₂ the modified principle of complementary energy Π_σ:

$$\Pi_\sigma = \frac{1}{2} \int \underline{\sigma} \mathbf{A}^{-1} \underline{\sigma} dV + \int \underline{\sigma} \underline{\varepsilon}^\circ dV \qquad (5)$$

can be obtained. The principle says that the stress state of the body subject to external load **X**, **p** (**u=0** on Au) and eigen distortions $\underline{\varepsilon}^\circ$ takes the form equilibring the external load and minimizing the functional (5). It means that the stationarity conditions of minimum of the functional (5) subject to the constraints (1)₁ take the form of compatibility conditions (1)₂.

The variational principle and computational method for the slackened structure analysis will be discussed in Sections II and III while the problem of initially stiffened structures will be considered in Section IV. Finally, in Section V the application of the presented method to an optimal structural design problem will be discussed.

II. VARIATIONAL PRINCIPLES FOR SLACKENED STRUCTURES

A slackened structure can be defined as a structure with the following constitutive relations (written for the one-dimensional case - cf.Fig.1a):

$$\begin{aligned} \sigma &= 0 && \text{for} \quad |\varepsilon| < \bar{\varepsilon} \\ \sigma &= A\,(\varepsilon - \bar{\varepsilon}) && \text{for} \quad \varepsilon \geq \bar{\varepsilon} \\ \sigma &= A\,(\varepsilon + \bar{\varepsilon}) && \text{for} \quad \varepsilon \leq \bar{\varepsilon} \end{aligned} \qquad (6)$$

where $\bar{\varepsilon} > 0$ describes the initial clearance in the considered element.

Generalizing the concept of slackened structures for an elastic body, the local clearances can be simulated by a field of eigen distortions $\underline{\varepsilon}^\circ$. The variational principle of maximization of the complementary energy Π_σ (Fig.1a) of the body with eigen distortions and relation of the stationarity conditions obtained in this way to the constitutive equations for the slackened structure will

be discussed. In such a way a general description of the constitutive relations for the slackened body (structure) and the corresponding variational principle will be obtained.
Therefore consider the following:

(7) $$\max_{\varepsilon^{o}} \left[\frac{1}{2} \int (\underline{\sigma}\mathbf{A}\underline{\sigma} + \underline{\sigma}\underline{\varepsilon}^{o}) dV \right]$$

subject to the constraints (1)₁ and:

(8) $$K(\underline{\varepsilon}^{o}) \leq \bar{\varepsilon}$$

where the scalar function K describes the local energy measure of distortion field, while $\bar{\varepsilon}$ describes the maximal admissible value of local distortions. The above problem can be reformulated making use of the principle of minimum of the complementary energy (5):

(9) $$\max_{\varepsilon^{o}} . \min_{\sigma} \left[\frac{1}{2} \int (\underline{\sigma}\mathbf{A}\underline{\sigma} + \underline{\sigma}\underline{\varepsilon}^{o}) dV \right]$$

subject to (1)₂ and (8).
Let us introduce now the augmental functional:

(10) $$J(\underline{\sigma},\underline{\varepsilon}^{o},\underline{\lambda},\psi) = \int \{ \frac{1}{2} \underline{\sigma}\mathbf{A}^{-1}\underline{\sigma} + \underline{\sigma}\underline{\varepsilon}^{o} + \underline{\lambda}(\mathrm{div}\underline{\sigma} + \mathbf{X}) + \psi[\bar{\varepsilon} - K(\underline{\varepsilon}^{o})] \} dV + \int \underline{\lambda}(\underline{\sigma}\mathbf{n} - \mathbf{p}) dA_{p}$$

where $\underline{\lambda}$, ψ are vector and scalar Lagrange multipliers.
Substituting the relation:

(11) $$\underline{\lambda}\,\mathrm{div}\underline{\sigma} = \mathrm{div}(\underline{\lambda}\underline{\sigma}) - (\mathrm{grad}^{o}\underline{\lambda})\underline{\sigma}$$

and making use of the Ostrogradsky's theorem the functional J can be expressed as follows:

(12) $$J(\underline{\sigma},\underline{\varepsilon}^{o},\underline{\lambda},\psi) = \int \{ \frac{1}{2} \underline{\sigma}\mathbf{A}^{-1}\underline{\sigma} + \underline{\sigma}\underline{\varepsilon}^{o} - (\mathrm{grad}^{o}\underline{\lambda})\underline{\sigma} + \underline{\lambda}\mathbf{X} + \psi[\bar{\varepsilon} - K(\underline{\varepsilon}^{o})] \} dV + \int \underline{\lambda}(\underline{\sigma}\mathbf{n} - \mathbf{p}) dA_{p} + \int \underline{\lambda}\underline{\sigma}\mathbf{n} dA.$$

Now the variation of the functional can be written:

$$\delta J = \int [\delta \underline{\sigma} (\mathbf{A}^{-1} \underline{\sigma} + \underline{\varepsilon}^{o} - \mathrm{grad}^{s} \underline{\lambda}) + \delta \underline{\varepsilon}^{o} (\underline{\sigma} - \psi \frac{\partial K(\varepsilon^{o})}{\partial \varepsilon^{o}}) + \tag{13}$$

$$+ \delta \underline{\lambda} (\mathrm{div}\, \underline{\sigma} + \mathbf{X}) + \delta \psi [\bar{\varepsilon} - K(\underline{\varepsilon}^{o})]] dV +$$

$$+ \int [\delta \underline{\lambda} (\underline{\sigma}\mathbf{n} - \mathbf{p}) + \underline{\lambda} \mathbf{n} \delta \underline{\sigma}] dA_p + \int \delta \underline{\sigma} \mathbf{n} \underline{\lambda} dA.$$

The stationarity condition $\delta J = 0$ is assured by the following relation within the domain V:

$$\underline{\sigma} = \mathbf{A}(\mathrm{grad}^{s} \underline{\lambda} - \underline{\varepsilon}^{o})$$

$$\underline{\sigma} = \psi \frac{\partial K(\varepsilon^{o})}{\partial \varepsilon^{o}}$$

$$\psi [\bar{\varepsilon} - K(\underline{\varepsilon}^{o})] = 0 \tag{14}$$

$$\bar{\varepsilon} - K(\underline{\varepsilon}^{o}) \geq 0 , \qquad \psi \geq 0$$

$$\mathrm{div}\, \underline{\sigma} + \mathbf{X} = \mathbf{0}$$

and the equation for the boundary integrals:

$$\int [\delta \underline{\lambda} (\underline{\sigma}\mathbf{n} - \mathbf{p}) + \underline{\lambda} \delta \underline{\sigma} \mathbf{n}] dA_p + \int \underline{\lambda} \delta \underline{\sigma} \mathbf{n} dA = 0. \tag{15}$$

The leatest equation leads for any variations $\delta\underline{\lambda}$ and $\delta\underline{\sigma}$ to the boundary conditions:

$$\underline{\sigma}\mathbf{n} = \mathbf{p} \quad \text{on } A_p \qquad \text{and} \qquad \underline{\lambda} = \mathbf{0} \quad \text{on } A_u. \tag{16}$$

As $\underline{\lambda}$ can be identified with the displacement field $\mathbf{u}$, the conditions $(14)_{1-5}$ provide the relation:

$$\underline{\sigma} = \mathbf{A}(\underline{\varepsilon} - \underline{\varepsilon}^{o}) = \begin{cases} \psi \dfrac{\partial K(\varepsilon^{o})}{\partial \varepsilon^{o}}, & K(\underline{\varepsilon}^{o}) = \bar{\varepsilon} \\ 0, & K(\underline{\varepsilon}^{o}) < \bar{\varepsilon} \end{cases} \tag{17}$$

that is the stress field is generated by the gradient rule associated with the constraint surface $K(\underline{\varepsilon}^{o})-\bar{\varepsilon}=0$ where ψ is a scalar multiplier expressing the external load intensity. The constitutive relation (17) determines the eigen distortions modeling the internal clearences and can be written in the following form:

$$\underline{\varepsilon} = \underline{\varepsilon}^{o}; \quad \underline{\sigma} = \mathbf{0} \qquad \text{for } K(\underline{\varepsilon}^{o}) < \bar{\varepsilon}$$

(18)

$$K(\underline{\varepsilon}^{o}) = \bar{\varepsilon}, \quad \Delta\underline{\sigma} = \mathbf{A}\,\Delta\underline{\varepsilon} \qquad \text{for } K(\underline{\varepsilon}^{o}) \geq \bar{\varepsilon}.$$

The above constitutive law generalizes the one-dimensional case of the slackened body (6).

From computing point of view it is important to have another equivalent formulation of the variational principle (7). Decomposing the stress state $\underline{\sigma} = \underline{\sigma}^{r} + \underline{\sigma}^{l}$ (where $\underline{\sigma}^{l}$ describes the stresses due to external load) and taking into account the orthogonality between the selfequilibrated stress states $\underline{\sigma}^{r}$ and the compatible deformations $\underline{\varepsilon}^{l}$ ($\int \underline{\sigma}^{r}\underline{\varepsilon}^{l}dV = 0$) one can express the functional (7) in the form:

(19) $$J = \frac{1}{2}\int[(\underline{\sigma}^{r}+\underline{\sigma}^{l})\mathbf{A}^{-1}(\underline{\sigma}^{r}+\underline{\sigma}^{l}) + (\underline{\sigma}^{r}+\underline{\sigma}^{l})\underline{\varepsilon}^{o}]dV =$$

$$+\frac{1}{2}\int[\underline{\sigma}^{l}\mathbf{A}^{-1}\underline{\sigma}^{l} + \underline{\sigma}^{r}\mathbf{A}^{-1}\underline{\sigma}^{r} + 2\underline{\sigma}^{l}\underline{\varepsilon}^{o}_{\varrho} + 2\underline{\sigma}^{r}\underline{\varepsilon}^{o}_{\chi}]dV$$

where: $\underline{\varepsilon}^{o}_{\chi} = -\mathbf{A}\underline{\sigma}^{r}$ –the part of distortion field associated with selfequilibrated stresses $\underline{\sigma}^{r}$,

$\underline{\varepsilon}^{o}_{\varrho} = \text{grad}^{s}\mathbf{u}^{o}$ –the part of compatible and stressless distortions.

The first component $\frac{1}{2}\int \underline{\sigma}^{l}\mathbf{A}^{-1}\underline{\sigma}^{l}dV$ is constant, therefore the considered variational principle can be equivalently expressed by the functional:

(20) $$J = \int(\underline{\sigma}^{l}\underline{\varepsilon}^{o} + \frac{1}{2}\underline{\sigma}^{r}\mathbf{A}^{-1}\underline{\sigma}^{r} - \underline{\sigma}^{r}\underline{\varepsilon}^{o}_{\chi})dV.$$

Expressing $\underline{\varepsilon}^{o}_{\chi}$ by $\underline{\sigma}^{r}$ and $\underline{\varepsilon}^{o}_{\varrho}$ by $\mathbf{u}^{o}$ and using the Clapeyron's theorem, one can finally formulate the variational principle equivalent to (7), (8) as follows:

$$(21) \qquad \max \left(\int \mathbf{X} \mathbf{u}^{\circ} dV + \int \mathbf{p} \mathbf{u}^{\circ} dA_p - \frac{1}{2} \int \underline{\sigma}^T \mathbf{A}^{-1} \underline{\sigma}^T dV \right)$$

subject to the constraints $(2)_1$, (8).

III. COMPUTATIONAL PROCEDURE FOR SLACKENED STRUCTURE ANALYSIS

Applying the constitutive conditions (17) to a truss structure under increasing loading process one can see that each element remains in the unloaded state ($\sigma = 0$) up to the moment when its deformation exceedes the value $\bar{\varepsilon}$. Then the element reacts purely elasticly for further load increments ($\Delta\sigma = E\,\Delta\varepsilon$).

The analysis of the slackened structure includes in the general case the first stage, when the structure behaves like a geometrically variable system. The corresponding state of deformation is equal to a geometrically compatible state of eigen distortions $\underline{\varepsilon} = \underline{\varepsilon}^{\circ}_{\ell}$ (cf. $(18)_1$), which describes the solution of the variational problem (21), $(1)_1$, (8). When the capacity of stressless deformation of the structure is exceeded then the first substructure with all clearences taken out and able to carry the external load can be defined. The evolution of the loaded substructure can be analyzed following the increments of external load and taking into account the constitutive law (18).

The program MOLO presented below analyzes the stress redistribution in the slackened trusses (with clearences in joints) while the external load increases. From computing point of view it is easier however to analyze the problem during the unloading process. Therefore the algorithm starts when the structure is fully loaded. It means that all clearences are reduced to zero and the internal forces appeared in all elements. The algorithm of the program can be described as follows.

{a} Determination of the stress state $\underline{\sigma}^{L}$ caused in the structure without clearences by the external load.

{b} Introducing of the eigen distortions modeling clearences in all elements "i":

(22) $\varepsilon_i^o = \bar{\varepsilon}_i \operatorname{sgn}(\sigma_i^L)$

where $\bar{\varepsilon}_i$ – the clearence available in the rod "i" .

(c) Determination of the stress state in the slackened structure:

(23) $\sigma_i = \sigma_i^L + \sum_j E_i (D_{ij} - \delta_{ij}) \varepsilon_j^o$

where: E_i –the modulus of elasticity

D_{ij} –the influence matrix describing the deformation ε_i^r caused in the member "i" by the unit distortion $\varepsilon_j^o = 1$ forced into the element "j"

δ_{ij} –the Kronecker's symbol.

(d) If there is an element "i" that $\sigma_i^L \sigma_i < 0$, then the scaling of the external load has to be done. The scaling means multiplication of the stress state $\underline{\sigma}^L$ by the following coefficient: $\alpha = (\sigma_i^L - \sigma_i) / \sigma_i^L$.

(e) Initialization of the $\mathcal{A}$ set ($\mathcal{A}$ = {0}).

(f) Determination of the stress state $\underline{\sigma}' = \underline{\sigma}^L$ describing the proportions of stress decreasing in the unloading process.

(g) Determination of the element "i" (or the set of elements) with the minimal value of the coefficient $\beta_i = \sigma_i / \sigma_i'$ and inclusion of this element to the set $\mathcal{A}$. The element defined above will be eliminated from the structure as the first one during unloading process.

(h) Reduction of the external load: $\underline{\sigma}^L := B\, \underline{\sigma}^L$, where $B = 1 - \beta_i$.

(i) Determination of the stress state in the set $\mathcal{A}$ caused by eigen distortions of the elements outside of the set $\mathcal{A}$ (the eigen distortions in the elements of the set $\mathcal{A}$ are asumed to vanish):

(24) $\sigma_i^r = E_i D_{ij} \varepsilon_j^o \qquad i \in \mathcal{A}\,, \quad j \notin \mathcal{A}$.

{j} Determination of the distortions in the elements of the set λ from the following equations:

$$E_i (D_{ij} - \delta_{ij}) \varepsilon_j^o = - \sigma_i^r - \sigma_i^l \qquad i,j \in \lambda. \tag{25}$$

These distortions simulate the total unloading of the elements from the set λ (elimination of the set λ from the structure).

{k} If the distortions determined above increased too rapidly it means that the structure starts to be a kinematic chain (after elimination of the set λ of elements). Adding the last elemnt "i" joined to the set λ to this kinematic chain, the isostatic sub-structure is obtained. Determination of this substructure ends the program. If the substructure remained after elimination of the set λ is still geometricaly well definesd, go to the step {l}.

{l} Calculation of the modified stress state of the structure with the set λ eliminated:

$$\sigma_i = \sigma_i^l + E_i (D_{ij} - \delta_{ij}) \varepsilon_j^o \qquad i,j \text{ - all elements of the structure} \tag{26}$$

and determination of the modified state of the comparative stresses $\underline{\sigma}'$:

$$\sigma_i' = \sigma_i^l + E_i (D_{ij} - \delta_{ij}) \varepsilon_j^o \qquad j \in \lambda, \ i \text{ - all elements of the structure.} \tag{27}$$

Then return to the step {g}.

In the general case, for the n-redundant structure one can obtain up to n stages of the structure degradation. Each stage of the degradation eliminates a set of elements and redistributes the state of stresses. The last stage of degradation determines an isostatic substructure.

The unloading process for the slackened structure shown on the Fig.4 was calculated and exposed on the Fig.5. The succesive stages of the structure degradation are merked by the bold line on the Fig.5a-f respectively. It is possible now to interprete the behaviour of the structure during the loading process as well. If the intensity of the external load is less than $\alpha = 0.034P$, the substucture carrying this load is isostatic (Fig.5f). When the load increases (for $\alpha \in \langle 0.034P, 0.049P \rangle$) the active substructure grows up to another isostatic truss (Fig.5e). When the external load increases more, some new elements are succesively included to the active substructure because the clearences vanish in this process. Therefore the stages of growth of the structure are shown on the Fig.5d,c,b,a for the load intensities $\alpha \in \langle 0.049P, 0.078P\rangle$, $\langle 0.078P, 0.096P\rangle$, $\langle 0.096P, 0.235P\rangle$, $\langle 0.235P, 0.995P\rangle$, $\langle 0.995P, P\rangle$ respectively.

IV INITIALLY STIFFENED STRUCTURES

An initially stiffened structure (structure with internal dry fiction) can be defined as a structure with the following constitutive relations (written for the one-dimensional case-cf.Fig.2a):

$$\varepsilon = 0 \qquad \text{for} \quad |\sigma| < \bar{\sigma}$$

$$\varepsilon = A^{-1}(\sigma - \bar{\sigma}) \qquad \text{for} \quad \sigma \geq \bar{\sigma} \tag{28}$$

$$\varepsilon = A^{-1}(\sigma + \bar{\sigma}) \qquad \text{for} \quad \sigma \leq -\bar{\sigma}$$

where $\bar{\sigma} > 0$ describes the maximal stress carried by dry friction in the considered element.

Dually to the discussion from the Section II, simulating the internal friction by the selfequilibrated state of stresses σ^r, let us determine the eigen distortion field $\underline{\varepsilon}^\circ$ which maximizes the potentiel energy π_ε of the body (Fig.2a). The variational principle defined here leads to the general description of the initially stiffened structure. Therefore, define the distortion state $\underline{\varepsilon}^\circ$ satisfying the following conditions:

$$\max_{\varepsilon^\circ} \left(\frac{1}{2} \int (\underline{\varepsilon}\mathbf{A}\underline{\varepsilon} - \underline{\varepsilon}\underline{\sigma}^\circ - \mathbf{X}\mathbf{u})dV - \int \mathbf{p}\mathbf{u}dA_p \right. \tag{29}$$

subject to the constraints $(1)_2$ and:

$$K(\underline{\sigma}^\circ) \leq \bar{\sigma} \tag{30}$$

where $\underline{\sigma}^\circ = \mathbf{A}\,\underline{\varepsilon}^\circ$.

Taking into account the minimum of the potential energy principle (4) and discussing the problem analogously like in the Section I, the following stationarity conditions can be obtained:

$$(31)\qquad \underline{\varepsilon} = \mathbf{A}^{-1}\underline{\sigma} + \underline{\varepsilon}^{o} = \begin{cases} -\psi \dfrac{\partial K(\underline{\sigma}^{o})}{\partial \underline{\sigma}^{o}} & K(\underline{\sigma}^{o}) = \bar{\sigma} \\ \mathbf{0} & K(\underline{\sigma}^{o}) \le \bar{\sigma} \end{cases}$$

Conditions (31) describe the deformation field generated by the gradient rule associated with the constraint surface $K(\underline{\sigma}^{o}) - \bar{\sigma} = 0$ and the coefficient ψ is a scalar multiplier expressing the load intensity. The constitutive relation (31) determines the eigen distortions simulating the internal frictions and can be written in the form which is a generalization of the one-dimensional case (28):

$$(32)\qquad \begin{aligned} &\underline{\sigma} = -\underline{\sigma}^{o}, \quad \underline{\varepsilon} = \mathbf{0} && \text{for } K(\underline{\sigma}) < \bar{\sigma} \\ &K(\underline{\sigma}^{o}) = \bar{\sigma}, \quad \Delta\underline{\sigma} = \mathbf{A}\,\Delta\underline{\varepsilon} && \text{for } K(\underline{\sigma}) \ge \bar{\sigma}, \end{aligned}$$

Similarly to the considerations from the Section I the alternative form of the variational principle (29),(30) can be formulated. Making use of the orthogonality between selfequilibrated ($\underline{\sigma}^{r}$) and compatible ($\underline{\varepsilon}^{r}$) states and taking into account the Clapeyron's theorem the following variational principle equivalent to (29), (30) can be obtained:

$$(33)\qquad \max\left(-\frac{1}{2}\int \underline{\varepsilon}^{r}\mathbf{A}\underline{\varepsilon}^{r}dV - \int \mathbf{X}\mathbf{u}^{o}dV - \int \mathbf{p}\mathbf{u}^{o}dA_{p}\right) =$$

$$= \min\left(\frac{1}{2}\int \underline{\varepsilon}^{r}\mathbf{A}\underline{\varepsilon}^{r}dV + \int \mathbf{X}\mathbf{u}^{o}dV + \int \mathbf{p}\mathbf{u}^{o}dA_{p}\right)$$

subject to the constraints $(2)_2$,(30).
The first stage of evolution (during loading) of the stress state defines some selfequilibrated state $\underline{\sigma} = -\underline{\sigma}^{o}_{\chi}$ (cf. $(32)_1$), which describes the solution of the variational problem (33),$(2)_2$,(30).

The program MOSF presented below analyzes the stress state evolution analogously to the program MOLO from the Section III. Analyzing the problem during unloading process the procedure should start from the step {1} of the following algorithm.

(a) Determination of the deformation state $\underline{\varepsilon}^{L}$ caused in the structure without internal friction by the external load.

(b) Introducing of the eigen stresses $\sigma_i^o = A\varepsilon_i^o$ modeling the internal friction in all elements "i":

$$\sigma_i^o = \bar{\sigma}_i \, \mathrm{sgn}(\sigma_i^L) \tag{34}$$

where: $\bar{\sigma}_i$ – the yield value of the internal friction in the rod "i"

(c) Determination of the deformations in the initially stiffened structure:

$$\varepsilon_i = \varepsilon_i^L + \sum_j D_{ij}\varepsilon_j^o \; . \tag{35}$$

(d) If there is an element "i" that $\sigma_i^L\sigma_i < 0$ than the scaling of the external load has to be done. The scaling means multiplication of the deformation state $\underline{\varepsilon}^L$ by the coefficient $\alpha = (\varepsilon_i^L - \varepsilon_i)/\varepsilon_i^L$.

(e) Initialization of the $\mathcal{A}$ set of rigid elements ($\mathcal{A} = \{0\}$).

(f) Determination of the deformation state $\underline{\varepsilon}' = \underline{\varepsilon}^L$ describing the proportions of the deformation decreasing during the unloading process.

(g) Determination of the element "i" (or the set of elements) with the minimal value of the coefficient $\beta_i = \varepsilon_i \; / \; \varepsilon_i'$ and inclusion of this element to the set $\mathcal{A}$. The member defined above will change for the undeformable, rigid element as the first one during the unloading process.

(h) Reduction of the external load: $\underline{\varepsilon}^L := B\,\underline{\varepsilon}^L$, where $B = 1 - \beta_i$.

(i) Determination of the deformation state in the set $\mathcal{A}$ caused by the eigen distortions of the elements outside of the set $\mathcal{A}$ (the eigen distortions inside the set $\mathcal{A}$ are asumed to vanish):

$$\varepsilon_i^r = D_{ij}\varepsilon_j^o \qquad i \in \mathcal{A} \, , \quad j \notin \mathcal{A} \; . \tag{36}$$

{j} Determination of the eigen distortions in the elements of the set $\mathcal{A}$ from the following equations:

(37) $$D_{ij}\varepsilon_j^o = -\varepsilon_i^r - \varepsilon_i^l \qquad i,j \in \mathcal{A} .$$

These distortions simulate the total stiffening of the element from the set $\mathcal{A}$ (the substructure $\mathcal{A}$ is changed for the set of rigid bodies).

{k} If all elements of the structure are included into the set $\mathcal{A}$ it means that the whole structure has changed for the rigid body with the selfequilibrated stress state $\underline{\sigma} = -\underline{\sigma}_R^o = -\mathbf{A}\underline{\varepsilon}^o$ stored inside and that the first stage of undeformed structure is reached. If there are elements of the structure not included into the set $\mathcal{A}$ the procedure is not finished. Then go to the step {l}.

{l} Calculation of the modified deformation state of the structure with the rigid substructure $\mathcal{A}$:

(38) $$\varepsilon_i = \varepsilon_i^l + D_{ij}\varepsilon_j^o \qquad i,j \text{ -all elements of the structure}$$

and determination of the modified state of comparative deformations $\underline{\varepsilon}'$:

(39) $$\varepsilon_i' = \varepsilon_i^l + D_{ij}\varepsilon_j^o \qquad j \in \mathcal{A},\ i\text{-all elements of the structure.}$$

Then return to the step {g}.

In the general case for the n-redundant structure one can obtain up to n stages of the structure stiffening. Each stage of stiffening includes a set of elements into the rigid part $\mathcal{A}$ of the structure and redistributes the state of deformations.

An example of initially stiffened cantilever and the stages of its deformation development are presented on Fig.6.

V DISCUSSION

The standard computational method of analysis of initially slackened and stiffened structures is based on following of the external load increments. Everytime when the local clearences or friction is drained, the global stiffness matrix should be actualized and the static analysis repeated. The method of simulation by eigen distortions presented in the paper allows to calculate the modifications of local stiffness characteristics without numerically costly renewed global analysis of the structure.

Analysing examples of slackened and stiffened structures (eg. the cantilever presented on Fig.6) one can define the macrostiffness characteristic describing the relation between the external force P and the displacement of the point A where this force is acting. For the truss (once redundant) with clearences the macrostiffness characteristic takes the form presented by the line a on Fig.7 while for the initially stiffened case the macrostiffness is described by the line b. The line c describes the case of linear behaviour of the elastic structure without clearences and friction.

Now, a new problem of modeling of macrostiffness characteristic can be formulated. It is obvious that playing with clearences $\bar{\varepsilon}$ or friction capacities $\bar{\sigma}$ in elements of the structure, the macrostiffness p-u relation can be modified. This nonstandard approach to structural design can be called the setting design and is applicable for example to supporting structures [7]. There is a wide class of engineering structures that can be decompose into the main structure and the supporting structure. The problem of maximization of load capacity of the main structure (for variable external load and local stresses constrained) requires some defined macrostiffness characteristic of the supporting structure. In the second stage of the optimal setting design problem the state of clearences $\bar{\varepsilon}$ and friction capacities $\bar{\sigma}$ to be introduced into the structure (in order to create the macrostiffness characteristic defined above) is analyzed.

The problem of application of initial slackening and stiffening of structures to optimal setting design will be discussed in details in the next paper [8].

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FIGURE CAPTIONS

1. Mechanical characteristics of elements with clearences.
2. Mechanical characteristics of elements with internal dry friction.
3. Decomposition of strain and stress states.
4. An example of truss structure.
5. The stages of stress state evolution in the initially slackened structure.
6. An example of initially stiffened structure.
7. Modifications of mascostiffness characteristic.

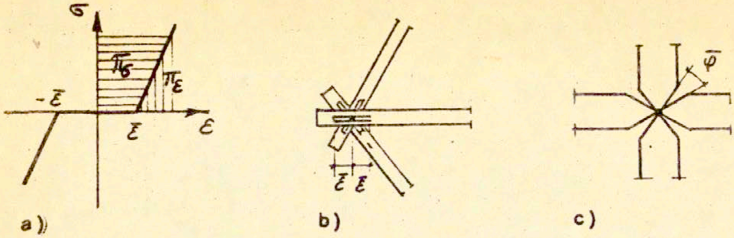


Fig.1

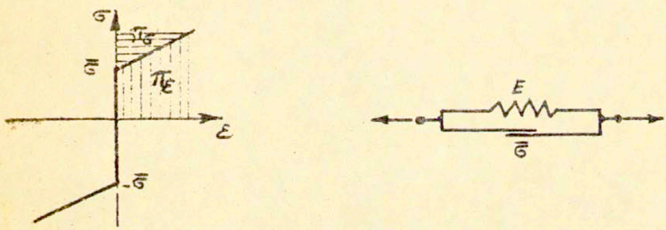


Fig.2

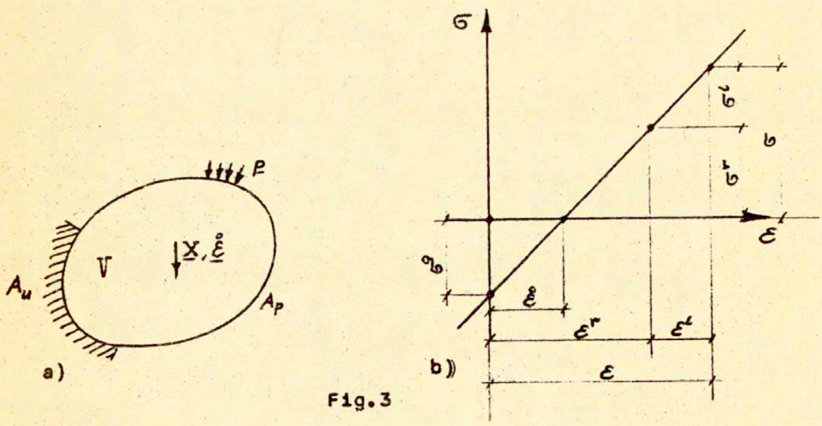


Fig.3

$\bar{\varepsilon}_1 = \ldots = \bar{\varepsilon}_{24} = 8 \cdot 10^{-5}$

$P = 500$ KG

$E = 2{,}1 \cdot 10^6$ KG/cm^2

$A_1 = \ldots A_{24} = 1$ cm^2

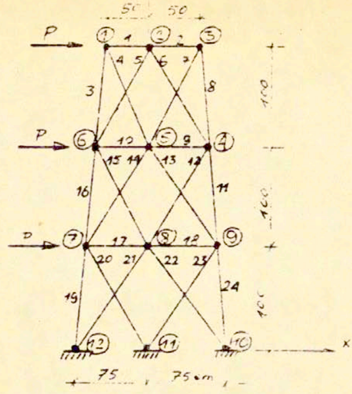


Fig.4

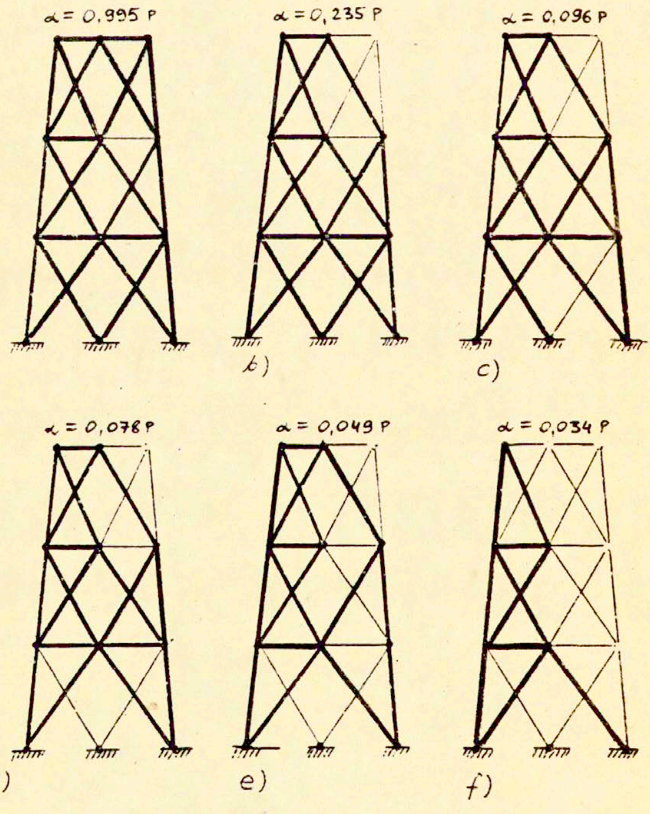
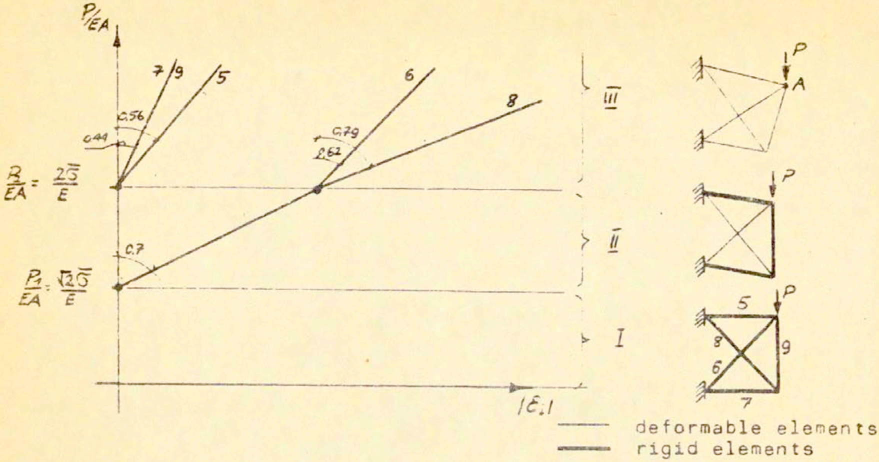
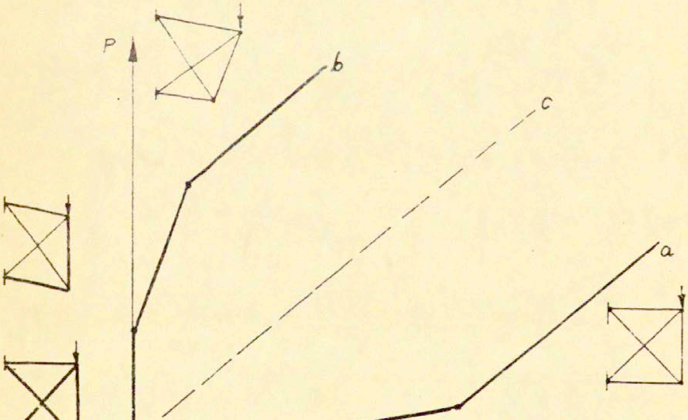


Fig.5



P/EA
7/9
5
6
8
0.56
0.44
0.79
0.62
0.7
$\frac{P_2}{EA} = \frac{2\overline{\sigma}}{E}$
$\frac{P_1}{EA} = \frac{\sqrt{2}\overline{\sigma}}{E}$
III
II
I
|ε₁|
P
A
5
8
9
6
7
deformable elements
rigid elements

Fig.6



P
b
c
a
u

Fig.7