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IMPORTANCE SAMPLING TECHNIQUES IN RELIABILITY CALCULATION

ABSTRACT

The importance sampling techniques have recently raised much attention among scientists working on the development and application of the calculation methods in reliability analysis of structures and structural systems. These wellknown variance reduction simulation techniques allow to combine the advantages of the Monte Carlo method and of the first order approximation that are still alternatively used in reliability analysis. Using the design point (β-point) as the center of an importance sampling probability distribution the sample size reduces drastically and the usual statistical assessement of the approximation error holds good. In the paper the importance sampling distribution parameters that assure the estimator of the failure probability to be most efficient are discussed and given in an approximate form which involves the curvatures of the limit condition in the β-point. The numerical effort for two sampling methods is examined. It is shown that several dozen samples can be sufficient to estimate even very small probabilities of failure.

1. INTRODUCTION

In structural design we have eventually to consider the problem that some inequalities have to be satisfied - maximal stress should be less than admissible stress value, load smaller than structural carrying capacity, crack shorter than its critical length etc. So long as all quantities which enter the equations are assumed to be deterministic

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the decision is "YES" or "NO" only. However, the deterministic approach may not be adequate anymore for new materials, advanced technology and modern structures. There is still more often required to account for variability of material parameters, loading, structural geometry. Since this variability usually has random nature the quantities which enter the equations in design procedure become random. Now, the answer the question whether an inequality is or is not satisfied cannot be anymore "YES" or "NO" only. Such an inequality in presence of the random quantities defines a random event and can always be satisfied but with some probability. This probability as a measure of structural safety is looked for in the reliability analysis of structure or structural system.

In the last decade the great effort was done in development of methods for reliability calculation. In the paper we briefly recall two methods: first order approximation at design point and Monte Carlo method. We especially call the attention to their advantages and weaknesses to point out all merits of another method called importance sampling which combines the advantages and avoids the weaknesses of the both. The main concept of the method and its various formulations are presented and discussed. The optimal choise of the formulation and parameters is suggested and illustrated on an example.

II. PROBLEM FORMULATION

In the beginning the boundary between the safe and unsafe (failure) regions is assumed to be given by an equality

$$g(x_1, x_2, \ldots, x_n) = 0 \qquad (2.1)$$

where x_i, $i=1,2,..n$, denote the samples of random variables (rv's), X_i. The rv's X_i describe random variability of material properties, dimensions, loads etc. The function $g(\underset{\sim}{x})$

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=0 is called the limit condition or limit surface. The structure or component fails if a sample $\underset{\sim}{x}$ of the random vector (rvec) $\underset{\sim}{X}$ belongs to the failure domain, D_F

$$D_F = (\underset{\sim}{x}: g(\underset{\sim}{x}) > 0) \tag{2.2}$$

Since the vector $\underset{\sim}{X}$ is random this domain is a random event and the purpose of the reliability analysis is to calculate the probability of this event - the failure probability, P_F

$$P_F = P[g(\underset{\sim}{X}) > 0] \tag{2.3}$$

In some simple cases, e.g. when the function $g(\cdot)$ is a linear combination of normal rv's or it is a product of log-normal rv's, the probability P_F can be determined due to the special properties of respective probability distributions. In the case of a few dimensions, n=2,3, the direct integration of the joint probability distribution function (jpdf), $f_X(\underset{\sim}{x})$, of the rvec $\underset{\sim}{X}$ over D_F migth be possible to carry out. In general, the function $g(\cdot)$ however does not have to be so simple and the dimensional size of practical problems is usually by far larger than a few dimensions. Therefore, the effort was done to propose approximate methods of calculation of the failure probability. In the following the main features of two such methods which are most often used in application are presented.

1. Monte Carlo Technique.

Early in the development of the structural reliability analysis the Monte Carlo method was tried to be applied to estimate the failure probability. In this method the samples, $\underset{\sim}{x}_i$, of the rvec, $\underset{\sim}{X}$, are generated by computer number generator from respective probability distributions of the rv's X_i. Those samples which belong to the failure domain ("successes") are added together and their proportion in the whole sample set yields approximately the probability of

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failure.

The estimation of the probability of a random event, F, from the Monte Carlo method is in fact the estimation of the mean of an indicator function, $\chi_F(\underset{\sim}{X})$, of the set F

$$\chi_F(\underset{\sim}{x}) = \begin{cases} 1 \text{ if } \underset{\sim}{x} \in F \\ 0 \text{ otherwise} \end{cases} \tag{2.4}$$

$\chi_F(\underset{\sim}{X})$ has a two-point distribution with the probabilities $P[\chi_F(\underset{\sim}{X})=1] = P_F$ and $P[\chi_F(\underset{\sim}{X})=0] = 1-P_F$, where P_F is the probability of the random event F. Thus, the mean and variance of this function are, respectively, $E[\chi_F(\underset{\sim}{X})]=P_F$ and $Var[\chi_F(\underset{\sim}{X})] = P_F(1-P_F)$.

The estimator of the mean of $\chi_F(\underset{\sim}{X})$ has the form

$$\chi_N^o = \frac{1}{N} \sum_{i=1}^{N} \chi_F(\underset{\sim}{X}_i) \tag{2.5}$$

where $\underset{\sim}{X}_i$'s are the independent rvec's with the same distribution as the rvec $\underset{\sim}{X}$. The mean and the variance of the estimator are, respectively, $E[\chi_N^o] = E[\chi_F(\underset{\sim}{X})] = P_F$ and $Var[\chi_N^o] = Var[\chi_F(\underset{\sim}{X})/N] = P_F(1-P_f)/N$. Hence, the coefficient of variation of the estimator is obtained as

$$\nu_{\chi_N^o} = \frac{\sqrt{Var[\chi_N^o]}}{E[\chi_N^o]} = \frac{1 - P_F}{\sqrt{N P_F}} \tag{2.5a}$$

For the efficient estimation the variation coefficient of the estimator should approach the zero with increasing sample set size, N, as fast as posible. From eq.(2.5) it is seen that for very small probabilities P_F the number of samples should be of order of magnitude about $P_F^{-2} \div P_F^{-4}$ at least to make the estimation meaningful. This explains the difficulties of the classical Monte Carlo method in calculation of failure probabilities which are usually very small.

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2. First Order Reliability Method (FORM).

Looking for more efficient methods another approach has been proposed - the first order reliability approximation. Due to the effort of many scientists and the completeness of their research - from theoretical investigations to the preparation of numerical procedures - this approach has become attractive for everyone who has to analyse the structures in presence of random design parameters. Hereafter, the main features of the method are only presented. One can find the details and discussion in the original papers, e.g. [1 ÷ 4], and review papers, [5 ÷ 7] say.

The first order reliability method is based on the linearization of the limit surface. The first question was now to choose the best point for linearization. The second problem assuming this point has been found and the linearization done was the calculation of the failure probability in general case of probability distributions of the rv's X_i. The intensive research in the last decade gave the answer to these questions.

Firstly, the reliability problem has to be transformed from the original space of the rv's X_i into the space of standard, normal, independent rv's Y_i. Such a transformation, $\underset{\sim}{X}=\underset{\sim}{T}(\underset{\sim}{Y})$, was proposed in [8] and involves appropriate marginal and conditional probability distributions of the rv's X_i as follows

$$\begin{aligned}
\Phi(y_1) &= F_{X_1}(x_1) \\
\Phi(y_2) &= F_{X_2|X_1}(x_2|x_1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Phi(y_n) &= F_{X_n|X_1,\ldots,X_{n-1}}(x_n|x_1,x_2,\ldots,x_{n-1})
\end{aligned} \tag{2.6}$$

where $\Phi(\cdot)$ denotes the probability distribution function of standard normal random variable, i.e.

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$$\Phi(y) = \int_{-\infty}^{y} \varphi(z)dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-\frac{z^2}{2})dz \qquad (2.7)$$

The limit surface is transformed as well

$$g(\underset{\sim}{x}) = 0 \rightarrow h(\underset{\sim}{y}) = g[\underset{\sim}{T}(\underset{\sim}{y})] = 0 \qquad (2.8)$$

The function $h(\underset{\sim}{y})=0$ defines the boundary between the failure and safe domain in the space Y of the rv's Y_i. Now, the linearization procedure is done in the space Y for the limit surface $h(\underset{\sim}{y})=0$. The best linearization point is found as the point on the limit surface being the closest one to the origin of the coordinate system in the space Y. This point is called the β-point (design point). The distance between it and the origin is noted as β-distance. Due to the axial symmetry of the jpdf of the rvec $\underset{\sim}{Y}=[Y_1,Y_2,..,Y_n]$ (standard, normal, independent rv's Y_i) the following equality holds always true

$$P[l(\underset{\sim}{Y})>\underset{\sim}{0}] = \Phi(-\beta) \quad \text{provided that} \quad 0 \in (\underset{\sim}{y}: l(\underset{\sim}{y})<\underset{\sim}{0}) \qquad (2.9)$$

where $l(\underset{\sim}{y})=0$ defines a hyperplane being situated at a distance of β from the origin in the Y space, i.e.

$$l(\underset{\sim}{y}) = a_o + \underset{\sim}{a}^T\underset{\sim}{y} \qquad (2.10)$$

while the distance β and the directional cosines, α_i, of the hyperplane $l(\underset{\sim}{y})=0$ can be expressed by the constant a_o and the n-dimensional vector of coefficients, $\underset{\sim}{a}$, in the form

$$\beta = \frac{a_o}{\sqrt{\underset{\sim}{a}^T\underset{\sim}{a}}} \quad \text{and} \quad \alpha_i = \frac{a_i}{\sqrt{\underset{\sim}{a}^T\underset{\sim}{a}}} \qquad (2.11)$$

Thus, the failure probability (2.3) is estimated by

$$P_F = P[g(\underset{\sim}{X})>0] = P[h(\underset{\sim}{Y})>0] \approx P[l(\underset{\sim}{Y})>0] = \Phi(-\beta)$$

$$\text{provided that} \quad \underset{\sim}{0} \in (\underset{\sim}{y}: l(\underset{\sim}{y})<0) \qquad (2.12)$$

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$$P_F = P[g(\underset{\sim}{X})>0] = P[h(\underset{\sim}{Y})>0] \approx P[l(\underset{\sim}{Y})>0] = \Phi(\beta)$$

otherwise.

Unfortunately, in the first order reliability method it is in general impossible to estimate the error of the probability estimation. For a function $h(\underset{\sim}{y})=0$ with a great curvature in the β-point the linearization can lead to significant errors in the probability calculation. The magnitude of the error grows up when the dimension of the problem increases (the number of rv's X_i becomes larger). Nevertheless, the β-point, $\underset{\sim}{y}^{\beta}$, yields a very important information about the probability mass distribution within the failure domain. This point is always the maximum likelihood point. The definition of the β-distance

$$\beta = \sqrt{\sum_{i=1}^{n} y_i^2} = \min \quad \text{provided that } h(\underset{\sim}{y}) = 0 \tag{2.13}$$

and the fact that the probability distributions of the independent rv's Y_i are standard normal lead to the following equation

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}^{\beta}) = (2\pi)^{-n/2}\exp[-\frac{1}{2}\sum_{i=1}^{n}(y_i^{\beta})^2] = (2\pi)^{-n/2}\exp(-\frac{1}{2}\beta^2) = \max_{h(\underset{\sim}{y})=0} f_{\underset{\sim}{Y}}(\underset{\sim}{y}) \tag{2.14}$$

It means that the main part of the failure probability mass is concentrated around the β-point. This property of the β-point and some developments of the Monte Carlo technique led to the application of the importance sampling technique in reliability calculation.

III. PRINCIPLES OF THE IMPORTANCE SAMPLING.

The importance sampling, see e.g. [9], is a simulation technique that is based on the following equality

$$E_{\underset{\sim}{X}}[\chi_F(\underset{\sim}{X})] = \int_{R^n} \chi_F(\underset{\sim}{x})\, f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x} = \tag{3.1}$$

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$$= \int_{R^n} \chi_F(\underset{\sim}{w}) \frac{f_{\underset{\sim}{X}}(\underset{\sim}{w})}{g_{\underset{\sim}{W}}(\underset{\sim}{w})} g_{\underset{\sim}{W}}(\underset{\sim}{w}) d\underset{\sim}{w} = E_{\underset{\sim}{W}}\left[\chi_F(\underset{\sim}{W}) \frac{f_{\underset{\sim}{X}}(\underset{\sim}{W})}{g_{\underset{\sim}{W}}(\underset{\sim}{W})}\right]$$

where the following notation is used:

$f_{\underset{\sim}{X}}(\cdot)$ is the probability density function of a rvec $\underset{\sim}{X}$;

$g_{\underset{\sim}{W}}(\cdot)$ is the probability density function of another rvec $\underset{\sim}{W}$;

$E_{\underset{\sim}{X}}[\cdot]$ denotes the mean value operation with respect to $f_{\underset{\sim}{X}}(\cdot)$

$E_{\underset{\sim}{W}}[\cdot]$ denotes the mean value operation with respect to $g_{\underset{\sim}{W}}(\cdot)$

$\chi_F(\cdot)$ can be any function in R^n. If $\chi_F(\underset{\sim}{x})$ is in particular the indicator function of a set F, eq.(2.4), the both mean value operations in eq.(3.1) give the probability content of this set for the probability measure defined by the pdf $f_X(\cdot)$. If the set F denotes the failure domain in reliability calculation the mean value operations in eq.(3.1) define equivalently the failure probability P_F.

In the classical Monte Carlo method the estimator of the failure probability is given by eq.(2.5). It is seen that it can be substituted with another estimator according to the eq.(3.1) where the ratio of the pdf's means a weight, i.e.

$$\chi_N^{fg} = \frac{1}{N} \sum_{i=1}^{N} \chi_F(\underset{\sim}{X}_i) \frac{f_{\underset{\sim}{X}}(\underset{\sim}{X}_i)}{g_{\underset{\sim}{W}}(\underset{\sim}{X}_i)} \qquad (3.2)$$

The samples, $\underset{\sim}{x}_i$, are now generated according to the jpdf $g_W(\underset{\sim}{x})$ and put respectivly into expression (3.2) to get a realization of the estimator.

In order to get the estimator, χ_N^{fg}, as effective as possible the jpdf of the rvec $\underset{\sim}{W}$ should assure the variance of χ_N^{fg} to be as closely to zero as possible. The jpdf $g_W(\underset{\sim}{x})$ should also satisfy the following conditions [11]

a) $\int_F g_{\underset{\sim}{W}}(\underset{\sim}{x}) d\underset{\sim}{x} = 1$ (3.3)

b) $g_{\underset{\sim}{W}}(\underset{\sim}{x}) = \lambda f_{\underset{\sim}{X}}(\underset{\sim}{x}) \chi_F(\underset{\sim}{x})$ almost everywhere in F. λ is a real number.

In other words the jpdf $g_W(\underset{\sim}{x})$ should be fully concentrated

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within the failure domain, F, and proportional there to the jpdf $f_{\underset{\sim}{X}}(\underset{\sim}{x})$.

Thought it is hardly possible to satisfy the both conditions, eqs.(3.3), for the jpdf $g_{\underset{\sim}{W}}(\underset{\sim}{x})$ in the failure probability calculation they give some indications how to choose this function in order to get more effective estimator than that one in the classical Monte Carlo method, eq.(2.5). Since it is practically not possible to concentrate the jpdf $g_{\underset{\sim}{W}}(\underset{\sim}{x})$ fully within the failure domain the main part of the probability mass which is defined by it should be mostly spread over the area which contributes the most significant share to the failure probability. This area stretches around the β-point as it was explained in the chapter II.2. The simplest choise is to define the normal rvec, $\underset{\sim}{W} \sim N(\underset{\sim}{y}^{\beta}, \underset{\sim}{I})$, in the space Y while $\underset{\sim}{y}^{\beta}$ denotes the β-point position which is now taken as the mean vector of $\underset{\sim}{W}$. $\underset{\sim}{I}$ denotes the unit covariance matrix for the components W_i, i=1,2..,n, of the rvec $\underset{\sim}{W}$.

The method described above is the simplest application of the importance sampling technique. It was proposed for the reliability analysis application in [11] and [12] where some advantages of it were shown making it interesting in some cases. The reliability calculation, however, due to its specific character and possible formulation in the space, Y, of standard, normal, independent rv's, Y_i, i=1,2,..,n, (see the chapter II.2), allows us to improve yet the sampling and make it approximately as effective as possible.

IV. IMPORTANCE SAMPLING IN RELIABILITY CALCULATION

Two ways can be distinguished in investigations of the importance sampling techniques in reliability calculation. One of them takes its origin from the following equation

$$P_F = P[\underset{\sim}{X} \in F] = \int_F \varphi_n(\underset{\sim}{y})\, d\underset{\sim}{y} = \tag{4.1}$$

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$$= \int_{\text{unit sphere}} \int_{r^2(\alpha)}^{\infty} f_{\chi_n^2}(r^2)\, dr^2 d\lambda(\underset{\sim}{\alpha}) = \int_{\text{unit sphere}} (1 - \chi_n^2 [r^2(\underset{\sim}{\alpha})])\, d\lambda(\underset{\sim}{\alpha})$$

where

$$\varphi_n(\underset{\sim}{\chi}) = \frac{1}{(2\pi)^{n/2}} \exp[-\frac{1}{2}(x_1^2 + \cdots + x_n^2)] \qquad (4.2)$$

is the n-dimensional standard normal distribution and the following notation is used:

$f_{\chi_n^2}(\cdot)$ and $\chi_n^2(\cdot)$ are, respectively, the probability density and cumulative probability function of the χ^2 random variable with n degree of freedom;

$\lambda(\underset{\sim}{\alpha})$ is the uniform probability measure on the unit sphere in the n-dimensional space;

$r(\underset{\sim}{\alpha})$ is the distance from the origin to the limit surface, $h(\underset{\sim}{\chi})=0$, in direction of the unit vector $\underset{\sim}{\alpha}$.

Eq.(4.1) can be alternatively written in the form

$$P_F = E_{\underset{\sim}{A}}[(1 - \chi_n^2[r^2(\underset{\sim}{A})])] \qquad (4.3)$$

where $E_{\underset{\sim}{A}}[\cdot]$ denotes the mean value operation with respect to the probability measure $\lambda(\underset{\sim}{\alpha})$. Thus, the estimator of the failure probability, P_F, in the Monte Carlo method is

$$P_F = \frac{1}{N} \sum_{i=1}^{N} (1 - \chi_n^2[r^2(\underset{\sim}{\alpha}_i)]) \qquad (4.4)$$

while the samples, $\underset{\sim}{\alpha}_i$, of the direction, $\underset{\sim}{A}$, can be obtained from the transformation

$$A_i = \frac{U_i}{|\underset{\sim}{U}|}, \quad i = 1,2,\ldots,n \quad \text{and} \quad |\underset{\sim}{U}| = (\sum_{i=1}^{n} U_i^2)^{1/2} \qquad (4.5)$$

where the random variables U_i are independent, standard and normal.

It is obvious that not all directions are equally significant for a failure domain with a β-point, $\underset{\sim}{\chi}^{\beta}$, and the β-distance, $\beta = r(\underset{\sim}{\alpha}^{\beta})$. The simulation should be concentrated ra-

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ther around the direction $\underset{\sim}{\alpha}^{\beta}$ where the limit surface is most closely to the origin and the terms in the estimator (4.4) take their greatest values. Such a simulation technique which uses the β-direction as a center of the direction samples is called the directional sampling. It was applied in [12 ÷ 15] for integration and in reliability analysis. The reader is refered to those papers for more technical details.

The main interest of this paper is focused on another method of the importance sampling, see [16÷18], which takes its origin in the following equation, [19], say:

$$P_F = \int_F \varphi_n(\underset{\sim}{\chi})d\underset{\sim}{\chi} = \int_{R^{n-1}} \int_{f_U(\underset{\sim}{u})}^{\infty} \varphi_{n-1}(\underset{\sim}{u})\ \varphi_1(u_n)d\underset{\sim}{u}\ du_n =$$

$$= \int_{R^{n-1}} \Phi[-f_U(\underset{\sim}{u})]\ \varphi_{n-1}(\underset{\sim}{u})d\underset{\sim}{u} = E_{\underset{\sim}{U}}(\Phi[-f_U(\underset{\sim}{U})]) \tag{4.6}$$

where

$$u_n = f_U(\underset{\sim}{u}) \tag{4.7}$$

is the solution of the limit state equation, $h_U(\underset{\sim}{u},u_n)=0$, with $\underset{\sim}{u}=[u_1,u_2,..,u_{n-1}]$ in a new coordinate system, $(\underset{\sim}{u},u_n)$, that is obtained by a simple rotation of the coordinate system, $(y_1,y_2,..,y_n)$, in the space Y so that in the new coordinate system the β-point lies on the u_n-axis. Such a transformation

$$\underset{\sim}{u} = \underset{\sim}{R}^T \cdot \underset{\sim}{\chi} \tag{4.8}$$

always exists where the last column of the $\underset{\sim}{R}$ matrix is the vector $\underset{\sim}{\chi}^{\beta}/\beta$ while the other columns may be found by one of the orthogonalization procedures, e.g.[20]. The transformation does not change the probability distribution and the respective rv's, U_i, $i=1,2,..,n$, remain standard, normal and independent.

Thus, the estimator of the failure probability, P_F, in the Monte Carlo method takes the following form

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$$P_F = \frac{1}{N} \sum_{K=1}^{N} \Phi[-f_u(\underset{\sim}{u}_K)] \tag{4.9}$$

where the samples, $\underset{\sim}{u}_K$, K=1,2,..,N, of the rvec's, $\underset{\sim}{U}_K$, are generated from the pdf, $\varphi_{n-1}(\underset{\sim}{u})$. All random vector components, U_{Ki}, i=1,2,..,n-1, are independent, standard and normal, i.e. U_{Ki} - N(0,1) for K=1,2,...,N and i= 1,2,...,n-1. It is seen that the advantage from the maximum likelihood property of the β-point is incorporated in the estimator (4.9). The samples from the simulation procedure concentrate around this point on the hyperplane $u_n=\beta$. The question now arises whether it is possible to find a better importance sampling pdf which would minimize the variance of the estimator of P_F. In the forthcoming chapters an attempt is made to give the answer to this problem.

V. IMPORTANCE SAMPLING FOR SECOND ORDER APPROXIMATION

Let us assume the function $f_U(\underset{\sim}{u})$ is twice differentiable in the β-point, $\underset{\sim}{u}^{\beta}=\underset{\sim}{0}$. Due to the definition of the β-point where the function $f_U(\underset{\sim}{u})$ reaches its minimum the first derivatives of $f_U(\underset{\sim}{u})$ in $\underset{\sim}{u}^{\beta}$ are all equal to zero. Let us now approximate the function $f_U(\underset{\sim}{u})$ by a second order Taylor expansion in the β-point

$$f_U(\underset{\sim}{u}) \approx \beta + \frac{1}{2}\, \underset{\sim}{u}^T \cdot \underset{\sim}{H} \cdot \underset{\sim}{u} = t_U(\underset{\sim}{u}) \tag{5.1}$$

where $\underset{\sim}{H}$ denotes the matrix of the second derivatives of $f_U(\underset{\sim}{u})$ in the β-point, i.e.

$$\underset{\sim}{H} = \left[\frac{\partial^2 f_U(\underset{\sim}{u})}{\partial u_i \partial u_j}\right] \quad ; \quad i,j = 1,2,\ldots,n-1 \tag{5.2}$$

It is always possible (although not always uniquely) to rotate the coordinate system $(\underset{\sim}{u}, u_n)$ around the u_n-axis so that all mixed second derivatives in the new coordinate system $(\underset{\sim}{v}, v_n=u_n)$ vanish. In the transformation

$$\underset{\sim}{u} = \underset{\sim}{T} \cdot \underset{\sim}{v} \tag{5.3}$$

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$\underset{\sim}{T}$ is an orthogonal matrix, i.e. $\underset{\sim}{T}\cdot\underset{\sim}{T}^T=\underset{\sim}{I}$, that satisfies the equality

$$\underset{\sim}{H} = \underset{\sim}{T}\cdot\underset{\sim}{\Lambda}\cdot\underset{\sim}{T}^T \tag{5.4}$$

where $\underset{\sim}{\Lambda}=[\lambda_i]$ is the diagonal matrix of eigenvalues of $\underset{\sim}{H}$. The second order approximation of the limit condition (4.7) in the new coordinate system $(\underset{\sim}{v},v_n)$ takes the simple canonical form

$$f_v(\underset{\sim}{v}) = f_v(v_1,v_2,\ldots,v_{n-1}) \approx \beta + \frac{1}{2}\sum_{i=1}^{n-1}\kappa_i v_i^2 = t_v(\underset{\sim}{v}) \tag{5.5}$$

where $\underset{\sim}{\kappa}$ denotes the vector of main curvatures of the surface $v_n=f_v(\underset{\sim}{v})$ in the β-point, i.e.

$$\kappa_i = \frac{\partial^2 f_v(\underset{\sim}{v})}{\partial v_i^2} \qquad \text{for} \quad i = 1,2,\ldots,n-1 \tag{5.6}$$

The application of the transformation is presented in [17], [18], say. It is there used in the second order reliability method where the transformed limit condition, $h(\underset{\sim}{y})=0$, is approximated by a surface of the second order in the β-point. The noncentral χ^2-distribution eventually is applied to calculate the probability content of the approximate failure domain. In the asymptotic case when the β-distance goes to infinity, $\beta\to\infty$, it has been shown in [21] that the failure probability, P_F, can be calculated from the following expression

$$P_F = P[g(\underset{\sim}{X})>0] = P[f_v(\underset{\sim}{v})>v_n] \approx \Phi(-\beta)\prod_{i=1}^{n-1} c_i = \Phi(-\beta)\cdot c_\kappa \tag{5.7}$$

where

$$c_i = (1 + \Psi(-\beta)\kappa_i)^{-1/2} \tag{5.8}$$

and

$$\Psi(-\beta) = \frac{\varphi(\beta)}{\Phi(-\beta)} \qquad \text{while} \quad \Psi(-\beta) \to -\beta \text{ for } \beta \to \infty \tag{5.9}$$

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C_K is the correction factor that includes the main curvatures of the transformed limit surface in the β-point.

For not large β-distance and the limit surface which differs significantly from the parabolic approximation (5.5) even the corrected formula (5.7) gives the results that contain undefinable but likely significant errors. Therefore, it has been proposed in [16] to include the curvature information in the importance sampling procedure in order to make the failure probability calculation effective and at least approximately error free. It has been to some extent arbitrarily assumed the importance sampling pdf, $g_{\underset{\sim}{W}}(\underset{\sim}{w})$, to be of the following form

$$g_{\underset{\sim}{W}}(\underset{\sim}{w}) = \frac{1}{(2\pi)^{(n-1)/2} C_K} \exp\left[-\frac{1}{2}\sum_{i=1}^{n-1}\frac{w_i}{C_i^2}\right] = \varphi_{n-1}(\underset{\sim}{w};\underset{\sim}{0},\underset{\sim}{\Sigma}) \qquad (5.10)$$

where $\varphi_n(\cdot;\underset{\sim}{\mu},\underset{\sim}{\Sigma})$ denotes, in general, the joint probability density function of a n-dimensional rvec with the mean vector $\underset{\sim}{\mu}$ and the covariance matrix $\underset{\sim}{\Sigma}$. In the considered case the samples, $\underset{\sim}{w}_K$, K=1,2,..,N, in the importance sampling are generated independetly as the realizations of the (n-1) independent coordinates with zero means and variances equal to C_i^2, i=1,2,...,n-1, from eq.(5.8). The covariance matrix $\underset{\sim}{\Sigma}$ is diagonal with C_i^2 on the diagonal. The estimator of the failure probability is eventually proposed in the following form

$$P_F = \frac{1}{N}\sum_{K=1}^{N}\Phi[-f_v(\underset{\sim}{w}_K)]\; C_K \exp\left[\frac{1}{2}\Psi(-\beta)\sum_{i=1}^{n-1}\kappa_i w_i^2\right] \qquad (5.11)$$

The effectiveness of the estimator was not investigated. In particular the case of negative curvatures and large β-distance may raise doubts. The results of the examples in [17] and [18], however, where the importance sampling with various estimators was compared have shown that the estimator (5.11) yields very quickly (for small sample size) very

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good (with small coefficient of variation) estimate of the failure probability. This promising feature of estimators of that type originated the investigations presented in this paper.

VI. THE BEST IMPORTANCE SAMPLING DISTRIBUTION

Let us assume the second order form, $v_n = t_v(\underset{\sim}{y})$, eq.(5.5), of the transformed limit surface, $h(\underset{\sim}{y})=0$, really corresponds with the original limit condition, $g(\underset{\sim}{x})=0$, transformed into the space Y with the standard, normal probability measure and where the coordinate system is given by $(\underset{\sim}{y}, v_n)$. Let us assume moreover the estimator, X_N^{Φ}, of the probability distribution, P_F, in the following form

$$P_F \approx X_N^{\Phi} = \frac{1}{N} \sum_{k=1}^{N} \Phi[-t_v(\underset{\sim}{y}_k)] \frac{\varphi_{n-1}(\underset{\sim}{y}_k; \underset{\sim}{0}, \underset{\sim}{I})}{\varphi_{n-1}(\underset{\sim}{y}_k; \underset{\sim}{0}, \underset{\sim}{\Sigma})} = \frac{1}{N} \sum_{k=1}^{N} \Xi_k \qquad (6.1)$$

where

$$t_v(\underset{\sim}{y}) = \beta + \frac{1}{2} \sum_{i=1}^{n-1} \kappa_i y_i^2 \qquad (6.2)$$

and the covariance matrix, $\underset{\sim}{\Sigma}$, in the importance sampling pdf is diagonal with the variances, σ_i^2, $i=1,2,\ldots,n-1$, on the diagonal, i.e.

$$\underset{\sim}{\Sigma} = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_{n-1}^2] \qquad (6.3)$$

The purpose now is to calculate the variances, σ_i^2, so that the estimator (6.1) becomes most effective.

It is easily seen that the mean and the second moment of the estimator are equal, respectively, to

$$E_{\Sigma}[X_N^{\Phi}] = \frac{1}{N} N E_{\Sigma}[\Xi] = P_F \qquad (a)$$

$$E_{\Sigma}[(X_N^{\Phi})^2] = \frac{1}{N} \left\{ E_{\Sigma}[\Xi^2] + (N-1) P_F^2 \right\} \qquad (b) \qquad (6.4)$$

where $E_{\Sigma}[\cdot]$ denotes the avaraging operation with respect to

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the jpdf $\varphi_{n-1}(\underset{\sim}{y};\underset{\sim}{0},\underset{\sim}{\Sigma})$. $E_\Sigma[\Xi]=E_\Sigma[\Xi_K]$ and $E_\Sigma[\Xi^2]=E_\Sigma[\Xi_K^2]$ for every $K=1,2,..,N$. Hence, the variance of the estimator is equal to

$$\mathrm{Var}[X_N^\Phi] = \frac{1}{N}\,(E_\Sigma[\Xi^2] - P_F^2) \tag{6.5}$$

Minimizing the variance (6.5) means to look for a minimum of the second moment of Ξ with respect to the variances σ_i^2, $i=1,2,..,n-1$. The second moment, $E_\Sigma[\Xi^2]$, has the explicite form as follows

$$E_\Sigma[\Xi^2] = \int_{R^{n-1}} \Phi^2[-t_v(\underset{\sim}{y})]\,\frac{\varphi^2_{n-1}(\underset{\sim}{y};\underset{\sim}{0},\underset{\sim}{I})}{\varphi_{n-1}(\underset{\sim}{y};\underset{\sim}{0},\underset{\sim}{\Sigma})}\,d\underset{\sim}{y} =$$

$$= \frac{1}{(2\pi)^{(n-1)/2}} \int_{R^{n-1}} \Phi^2[-t_v(\underset{\sim}{y})] \prod_{i=1}^{n-1} \sigma_i \exp\left[-\frac{v_i^2}{2\sigma_i^2}(2\sigma_i^2 - 1)\right] d\underset{\sim}{y} \tag{6.6}$$

The necessary condition for the minimum, $\partial E_\Sigma[\Xi^2]/\partial\sigma_i=0$ for $i=1,2,..,n-1$, leads to the system of equations as follows

$$\frac{\partial E_\Sigma[\Xi^2]}{\partial\sigma_i} = \int_{R^{n-1}} \Phi^2[-t_v(\underset{\sim}{y})](\sigma_i^2 - v_i^2) \prod_{j=1}^{n-1} \exp\left[-\frac{v_j^2}{2\sigma_j^2}(2\sigma_j^2 - 1)\right] d\underset{\sim}{y} = 0$$

$$\text{for } i = 1,2,\ldots,n-1 \tag{6.7}$$

Since the search for an analytical solution of eqs.(6.7) did not success and the possible numerical solution would be impractical in application an approximate solution is proposed. The second order Taylor expansion of the function $\ln[\Phi(-\beta-s)]$ at $s=0$

$$\ln[\Phi(-\beta-s)] = \ln[\Phi(-\beta)] - s\Psi(-\beta) \tag{6.8}$$

leads to the following approximate expression for $\Phi^2[-t_v(\underset{\sim}{y})]$

$$\Phi^2[-t_v(\underset{\sim}{y})] \approx \Phi^2(-\beta)\exp\left[-\Psi(-\beta)\sum_{i=1}^{n-1}\kappa_i v_i^2\right] \tag{6.9}$$

Eqs.(6.7) with the approximation (6.9) take the simple in-

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tegral form

$$\int_0^{\infty} (\sigma_i^2 - v_i^2) \exp\left\{-\frac{v_i^2}{2\sigma_i^2}\left[2\sigma_i^2[1 + \Psi(-\beta)\kappa_i] - 1\right]\right\} dv_i = 0 \qquad \text{for } i = 1,2,\ldots,n-1 \tag{6.10}$$

from which the variances, σ_i^2, i=1,2..,n-1, are obtained as

$$\sigma_i^2 = \frac{1}{1 + \Psi(-\beta)\kappa_i} \qquad \text{for } i = 1,2,\ldots,n-1 \tag{6.11}$$

Thus, the estimator (6.1) can be explicitly written as

$$P_F \approx \chi_N^{\varphi} = \frac{1}{N} \sum_{k=1}^{N} \Phi[-t_v(\underset{\sim}{v}_k)]\, C_k \exp[\tfrac{1}{2} \Psi(-\beta) \sum_{i=1}^{n-1} \kappa_i v_i^2] \tag{6.12}$$

with C_k and $\Psi(-\beta)$ from eqs.(5.8) and (5.9), respectively. Though different reasoning the estimator (6.12) has exactly the same eventual form as that one from eq.(5.11) which has been proposed in [16]. An advantage of the actual derivation lies in the fact that the effectivness of the estimator here was the choice criterion and eq.(6.12) approximately defines the most effective estimator of the failure probability in the class of estimators given by the folmula (6.1). The adverb "approximately" has to be added due to the approximation (6.9). The limitations and the influence of this approximation are discussed hereafter.

First of all it has to be pointed out that the smallest value, β, of the function

$$t_v(\underset{\sim}{v}) = \beta + \frac{1}{2} \sum_{i=1}^{n-1} \kappa_i v_i^2 \tag{6.13}$$

may be reached in the point $\underset{\sim}{v}=\underset{\sim}{0}$ only if

$$\kappa_i \geq -\frac{1}{\beta} \qquad \text{for } i = 1,2,\ldots,n-1 \tag{6.14}$$

Otherwise, there exists always a point on the paraboloid $t(\underset{\sim}{v})$ that is closer to the origin than that one in $\underset{\sim}{v}=\underset{\sim}{0}$. It means that the equalities in eqs.(6.14) define the minimum

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(or maximum negative) curvatures of a paraboloid to be a limit surface with the β-point in $\underset{\sim}{y}=\underset{\sim}{0}$.

Secondly, due to the inequality, $\Psi(-\beta)>\beta$, the variance (6.11) would become negative for $-\beta^{-1}<\kappa_i<-[\Psi(-\beta)]^{-1}$. Such a case should be excluded while the variance has to be defined properly.

Finally, the approximation (6.9) is acceptable for the nonnegative curvatures, $\kappa_i\geq 0$, i=1,2,..,n-1, only. It is seen that for a negative curvature the function $\phi^2[-t_V(\underset{\sim}{y})]$ goes to the one while the approximation tends to the infinite. Due to the infinite integration limits in (6.7) for every component v_i this property of the approximation makes it unreliable and thereby useless at all for negative curvatures.

Therefore, the case of negative curvatures should be investigated separately with a special care. It is difficult to find another approximation of $\phi^2[-t_V(\underset{\sim}{y})]$ that would lead to an analytical solution of the maximum search problem (6.7). A possible second order expansion at $\underset{\sim}{y}=\underset{\sim}{0}$

$$\ln(1-\phi^2[-t_V(\underset{\sim}{y})]) \approx \approx \ln[1-\phi^2(-\beta)] + \frac{\phi^2(-\beta)}{1-\phi^2(-\beta)}\Psi(-\beta)\sum_{i=1}^{n-1}\kappa_i v_i^2 \tag{6.15}$$

leads to the approximation

$$\phi^2[-t_V(\underset{\sim}{y})]) \approx \approx 1-[1-\phi^2(-\beta)]\exp\left[\frac{\phi^2(-\beta)}{1-\phi^2(-\beta)}\Psi(-\beta)\sum_{i=1}^{n-1}\kappa_i v_i^2\right] \tag{6.16}$$

Though the approximation (6.16) coincides with the function $\phi^2[-t_V(\underset{\sim}{y})]$ for $\kappa_i<0$, i=1,2,..,n-1, in the limits, $v_i=0$ and $v_i=\infty$, it underestimates it too far from below within the argument intervals which are significant in integration in eqs.(6.7). Therefore, another method has to be chosen to estimate the best variances of the importance sampling distribution in eq.(6.1), i.e. $\varphi_{n-1}(\underset{\sim}{y};\underset{\sim}{0},\underset{\sim}{\Sigma})$.

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For the both approximations, eqs.(6.9) and (6.16), the (n-1)-dimensional integral in every equation from the system (6.7) is expressed as a product of the (n-1) one-dimensional integrals depending, respectively, on the variances σ_i^2, i=1,2,..,n-1. Following this feature it is assumed the system (6.7) to be approximated by a set of the following equations

$$\frac{\partial E_{\Sigma}[\Xi^2]}{\partial \sigma_i} \approx \tag{6.17}$$

$$\approx \Theta_i \int_0^{\infty} \Phi^2(-\beta-\frac{1}{2}\kappa_i v_i^2)(\sigma_i^2 - v_i^2) \exp\left[-\frac{v_i^2}{2\sigma_i^2}(2\sigma_i^2 - 1)\right] dv_i = 0$$

$$\text{for } i = 1,2,\ldots,n-1$$

where

$$\Theta_i = \int_{R^{n-2}} \vartheta_i(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}) dv_1 \cdots dv_{i-1} dv_{i+1} \cdots dv_{n-1} \tag{6.18}$$

In eq.(6.17) it is admitted that the function $\Phi^2[-t_V(\underset{\sim}{y})]$ may be approximated by a product form

$$\Phi^2[-t_V(\underset{\sim}{y})] \approx \tag{6.19}$$

$$\approx \vartheta_i(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}) \Phi^2(-\beta - \frac{1}{2}\kappa_i v_i^2)$$

The functions $\vartheta_i(\cdot)$ additionally depend on β, σ_K and κ_K, $K \neq i$, but their integrals over R^{n-2} can be considered as constants in the minimum search algorithm. The minimum conditions (6.17) are accurate for two-dimensional case, n=2, or for a paraboloidal cylinder, $\kappa_K = 0$, $K \neq i$.

The solution of eq.(6.17) with respect to the i-th variance, σ_i^2, has to be found numerically. The results, σ_i^2, for various β-distances, β=0.5,2,4,6,8, within the curvature interval $\kappa_i \in [-10,10]$ are shown in Fig.1. A comparsion of this results with the approximation (6.11) for nonnegative curvatures, $\kappa_i \geq 0$, shows a very good coincidence of the both

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solutions, Fig.2.

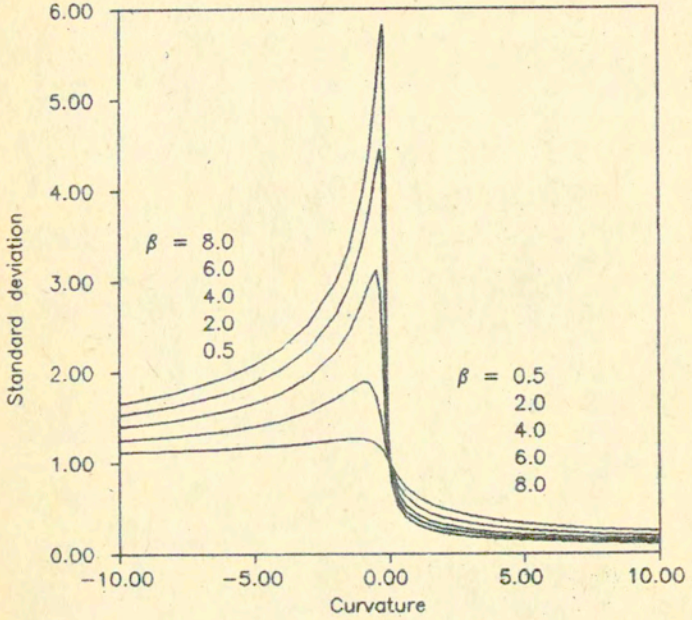

Fig.1. Standard deviation versus curvature for various β−distances.

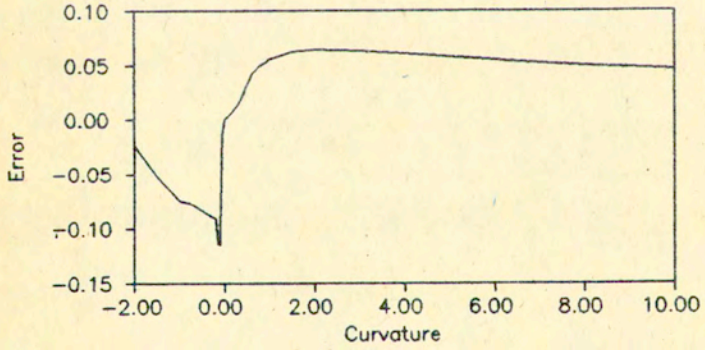

Fig.2. Extreme difference between the standard deviations from the approximations (6.11) or (6.20) and from the solution (6.7)

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As it was mentioned before the curvatures greater than $-1/\beta$ in the β-point are only of interest in the paraboloidal limit surface case. The analysis of the results from eq.(6.17) within the range $-1/\beta \leq \kappa_i \leq 0$ leads us to the following approximation of the importance sampling variance

$$\sigma_i^2 = \frac{1}{1 + \frac{\kappa_i \Psi(-\beta)}{1 + \kappa_i \omega}} \qquad \text{for } \kappa_i \in [-\frac{1}{\beta}, 0] \qquad (6.20)$$

where

$$\omega = \beta - \frac{\Psi(-\beta)}{\beta} (1 + \beta)$$

The maximum difference between the results from eq.(6.17) and this approximation is shown in Fig.2 as well. It has to be pointed out that the approximation (6.20) is rather guessed to fit as well and simply as possibly the solutions from eq.(6.17) than to be derived according to a rigourus mathematical estimate procedure. Nevertheless, the approximation differs satisfactory small to be applied in numerical calculation of the failure reliability.

VII. APPLICATION IN RELIABILITY ANALYSIS

It seems to be not possible and not necessary in this general presentation to analyse in detail all cases which one might meet in application of the importance sampling method to practical problems. It is easily seen that the problems that make the greatest difficulties in reliability analysis, e.g. multimode failure, series, parallel or mixed systems, more than one β-point (local minima) etc., can be succesfully and, moreover, approximately exactly solved by importance sampling method. In this chapter some of these problems are discussed and their solutions are suggested.

One of the most typical situation in reliability analysis is that a structure may fail if any of m possible failure mechanisms becomes active - a series system. The fai-

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lure domain, F_G, then is an union of the single failure domains, F_j, $j=1,2,..,m$, and the failure probability can be written as

$$P[F_G] = P[\bigcup_{j=1}^{m} F_j] = P[\bigcup_{j=1}^{m} h_j(\underset{\sim}{\chi}) \geq 0] \tag{7.1}$$

where $h_j(\underset{\sim}{\chi})=0$ is the j-th limit surface in the $\underset{\sim}{Y}$ space with the β_j-distance to the origin and the $\underset{\sim}{\kappa}_j$ main curvature vector in the β_j-point. Every $\underset{\sim}{\kappa}_j$ vector gives an appropriate importance sampling distribution variances, $(\underset{\sim}{\sigma}^2_i)_j$. The importance sampling procedure can be successfully applied by introducing an additional random variable, J, that indicates which β_j-point and variances should be taken into account for choosing the orthogonal transformation of the coordinate system and importance sampling distribution in current simulation step. The significance of the β_j-points is measured by the probabilities, p_j, which are assessed from the first order approximation in the following form, e.g. [11],

$$p_j = \frac{\Phi(-\beta_j)}{\sum_{j=1}^{m} \Phi(-\beta_j)} \tag{7.2}$$

Firstly, the sample, j, of the rv J is generated according to its probability distribution (7.2). The β_j-point and the $\underset{\sim}{\sigma}^2_j$ variance vector then are chosen for the transformation and importance sampling. The sample $\underset{\sim}{v}_{jk}$ is generated and the value $v_{nj}=f_{Gj}(v_{1j},v_{2j},\cdots,v_{(n-1)j})$ is calculated where the global limit condition for the j-th transformation is the minimum envelope of the transformed individual limit conditions as described in chapter V, eqs.(5.2÷5.5).

Another situation that is often met in reliability analysis is the parallel system where the intersection of individual failure domain constitutes the global failure domain for the system, i.e.

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$$P[F] = P[\bigcap_{j=1}^{m} F_j] = P[\bigcap_{j=1}^{m} h_j(\underset{\sim}{Y}) \leq 0] \tag{7.3}$$

It is here reasonable to look for the minimum distance between the origin and the failure domain F and consider it as the β-distance. The β-point for the intersection lies usually on the edge where two or more limit surfaces cut themselves. The global limit surface is not differentiable in the β-point and the Taylor expansion as described in eq.(5.5) has no sense more. Nevertheless, the transformation to the pseudocanonical form is possible by calculating the finite second differences instead of the derivatives in eq.(5.2). The finite second differences are also used in calculating the main curvatures and the variances for importance sampling eventually. Further calculation proceed in the similar way as described in chapter 6.

VIII. EXAMPLE

The same example as used in [17] and [18] is here considered to illustrade the efficiency of the importance sampling method. The limit condition is assumed linear

$$g(\underset{\sim}{X}) = \pm \sum_{i=1}^{n} x_i \mp c = 0 \tag{8.1}$$

while the rv's X_i are assumed to be independently and identically exponentially distributed with the distribution function parameter λ. After transformation (2.6) the transformed limit condition becomes strongly non-linear

$$h(\underset{\sim}{Y}) = \pm \frac{1}{\lambda} \sum_{i=1}^{n} \ln[\Phi(-y_i)] \mp c = 0 \tag{8.2}$$

The exact solution of the reliability problem

$$P_F = P[h(\underset{\sim}{Y}) > 0] \tag{8.3}$$

is still attainable because the sum of logarithms of stan-

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dard normal rv's has the gamma probability distribution.

The computation effort and efficiency of two sampling methods that were described in chapter III and VI, respectively, are compared for three probability levels: $P_F = 10^{-2}$, 10^{-4}, 10^{-6}, and four dimensions of $\underset{\sim}{X}$-vector, n=2,5,10,20. The distribution parameter λ is addmited λ=1.0.

In Table I the estimates of the probabilities and coefficients of variation of the estimates, (COV), are given for the both methods. The number of samples for simulation around the β-point in the n-dimensional Y space (chapter III) is N=100,200,400,500 and the respective resultes are given in parenthesis. The missing values in the Tables mean that the simulation was successless because no sample occured in the failure domain. It results from the very strong curvature of the transformed limit condition and great variances (equal to 1.0) of the normal distributions used in simulation. An improvement might be is reached if the variances according to the Eqs.(6.17) and (6.20) have been appropriately applied. Such an approach, however, involves the curvatures or quasi-curvatures of the limit condition in the β-point and the results for comparsion were obtained from such a simulation in [17] and [18].

The results for the second method (chapter VI) are obtained at the sample sizes N=10,20,30,40,50. Though the second method is much more than ten times more efficient in sample size than the first one the profit in computational effort is not exactly of the same order. In the first method the one function value calculation is only required for every sample. In the second one an iteration algorithm for function zero search has to be in general applied for every sample. Such an algorithm always involves a number of function calls. The convergence, however, is usually satisfactory after 5÷7 calls already. Therefore, the method described in chapter VI is for all that much more efficient in computation and should be recommended for application.

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TABLE 1.

Probability of failure - $P_F = P[\sum_{i=1}^{n} X_i - c \geq 0] = 10^{-2}$

Sample size	%	n=2	n=5	n=10	n=20
(100) 10	Relat. error	(0) 6	(1) -3	(12) 0	(20) 22
	COV	(10) 6	(19) 5	(27) 7	(40) 10
(300) 30	Relat. error	(-8) 1	(5) 0	(25) -1	(27) 0
	COV	(6) 2	(11) 3	(15) 4	(22) 9
(500) 50	Relat. error	(-6) 1	(17) 2	(20) -4	(20) 0
	COV	(5) 1	(8) 2	(12) 4	(18) 6

Probability of failure - $P_F = P[-\sum_{i=1}^{n} X_i + c \geq 0] = 10^{-2}$

Sample size	%	n=2	n=5	n=10	n=20
(100) 10	Relat. error	(-27) -1	(41) 2	(-19) 6	(-68) 15
	COV	(18) 4	(26) 6	(27) 16	(32) 27
(300) 30	Relat. error	(-26) 7	(-2) -4	(-13) -3	(-12) -16
	COV	(11) 2	(15) 5	(17) 9	(27) 19
(500) 50	Relat. error	(-26) 2	(1) -4	(-12) -2	(-19) -2
	COV	(8) 1	(13) 4	(14) 8	(19) 13

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TABLE 1 (continued).

Probability of failure - $P_F = P[\sum_{i=1}^{n} X_i - c \geq 0] = 10^{-4}$

Sample size	%	n=2	n=5	n=10	n=20
(100) 10	Relat. error	(10) 3	(8) 3	(42) 10	(-) 18
	COV	(12) 7	(30) 11	(49) 6	(-) 18
(300) 30	Relat. error	(-4) -1	(-3) 0	(6) 4	(-) -2
	COV	(8) 3	(18) 6	(33) 4	(-) 10
(500) 50	Relat. error	(-1) -4	(14) -6	(6) 1	(-18) 2
	COV	(6) 2	(13) 4	(25) 4	(58) 8

Probability of failure - $P_F = P[-\sum_{i=1}^{n} X_i + c \geq 0] = 10^{-4}$

Sample size	%	n=2	n=5	n=10	n=20
(100) 10	Relat. error	(-37) 4	(-20) -7	(-30) 7	(-76) 13
	COV	(23) 2	(25) 10	(45) 19	(50) 32
(300) 30	Relat. error	(-28) -6	(-32) -2	(4) -5	(-55) -20
	COV	(16) 5	(16) 6	(29) 11	(23) 22
(500) 50	Relat. error	(-29) 1	(3) -4	(-19) -2	(-52) -4
	COV	(12) 3	(26) 4	(23) 9	(21) 15

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TABLE 1 (continued).

Probability of failure - $P_F = P[\sum_{i=1}^{n} X_i - c \geq 0] = 10^{-6}$

Sample size	%	n=2	n=5	n=10	n=20
(100) 10	Relat. error	(11) 3	(22) 8	(-) 2	(-) -17
	COV	(14) 8	(40) 7	(-) 15	(-) 28
(300) 30	Relat. error	(2) -2	(2) 2	(-) 5	(-) 31
	COV	(8) 5	(25) 6	(-) 8	(-) 21
(500) 50	Relat. error	(3) -5	(34) -3	(-39) 5	(-) 9
	COV	(6) 3	(17) 5	(58) 7	(-) 16

Probability of failure - $P_F = P[-\sum_{i=1}^{n} X_i + c \geq 0] = 10^{-6}$

Sample size	%	n=2	n=5	n=10	n=20
(100) 10	Relat. error	(-50) 2	(-46) 9	(-24) 13	(-77) 11
	COV	(24) 3	(28) 17	(61) 28	(60) 36
(300) 30	Relat. error	(-44) -2	(-42) 4	(-42) 14	(-68) -23
	COV	(15) 5	(21) 10	(34) 16	(27) 24
(500) 50	Relat. error	(-39) 0	(-33) -3	(-48) -2	(-70) -6
	COV	(12) 3	(19) 8	(24) 14	(21) 17

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IX. CONCLUDING REMARKS

In all importance sampling techniques which are applied in structural reliability calculation the information about the β-point is of the greatest importance. The sampling concentrates around the β-point and the failure probability estimation is possible even for small probabilities. An additional information about the curvatures of the limit surface in the β-point allows to estimate the variances of the normal importance sampling distributions that lead to the approximately most efficient estimator of the failure probability for this class of importance sampling distributions. It should be pointed out that for the sake of importance sampling the limit surface needs not necessarily to be smooth in the β-point. Second order finite differences suffice to approximately calculate the best variances of the importance sampling distributions.

It has already been shown in [18] that the imformation about the main curvatures radically accelerates the estimator convergence. In the matter of fact any curvature information, not necessarily about the main curvatures, can be applied. The transformation of the coordinate system $\underset{\sim}{Y} \to (U_1, \underset{\sim}{U}) \to (V_1, \underset{\sim}{V})$ in order to eliminate the mixed second derivatives in the Taylor expansion of the limit condition in the β-point and to eventually obtain the main curvatures - see eq. (5.3) - additionally requires at least $n(n-1)/2$ function value calculations each of them involving the function zero search algorithm with 5÷7 function calls, say. Instead it is suggested to use the simple second derivatives or finite differences in the β-point in the rotated coordinate system $(U_1, \underset{\sim}{U})$, only - see eq. (4.8). This simplification slightly worsens the efficiency of the estimator but allows us to avoid some numerical difficulties. Since one transformation, moreover, is absent the function value calculation is faster and the algorithm may be even more efficient than that one with the most efficient estimator.

The importance sampling can be applied to calculation

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of any probability and, in fact, to eventually estimation of the whole probability distribution of a function of random variables. The method provides all statistical features usually required, say confidence level, confidence interval. The numerical effort is reasonable and the solution has the statistical background with attainable error estimation. The further extension to stochastic processes, outcrossing problems are possible as well and will be reported elsewhere.

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