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ON THE SECULAR ACCELERATION OF THE MOON'S MEAN MOTION.

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THE present Memoir exhibits a new method of taking account, in the Lunar Theory, of the variation of the eccentricity of the Sun's orbit. The approximation is carried to the same extent as in Prof. Adams' Memoir "On the Secular Variation of the Moon's Mean Motion" (*Phil. Trans.*, vol. CXLIII. (1853), pp. 397—406); and I obtain results agreeing precisely with his, viz., besides his new periodic terms in the longitude and radius vector, I obtain in the longitude the secular term

$$(-\tfrac{3}{2}m^2 + \tfrac{3771}{64}m^4)\int (e'^2 - E'^2)\, ndt,$$

and in the quotient radius, or radius vector divided by the mean distance, the secular term

$$(\tfrac{3}{4}m^2 - \tfrac{1973}{64}m^4)(e'^2 - E'^2),$$

which is, in fact, as will be shown, included implicitly in the results given in Professor Adams' Memoir. In quoting the foregoing results, I have written $e'^2 - E'^2$ in the place of $(e'+f't)^2 - e'^2 = 2e'f't$, which in the notation of the present Memoir it should have been; and I purposely refrain from here explaining the precise signification of the symbols: this is carefully done in the sequel. The method appears to me a very simple one in principle; and it possesses the advantage that it is not incorporated step by step with a lunar theory in which the eccentricity of the Sun's orbit is treated as constant; but it is added on to such a lunar theory, giving in the Moon's coordinates the supplementary terms which arise from the variation of the solar eccentricity, and thus serving as a verification of any process employed for taking account of such variation.

I have given the details of the work in a series of Annexes, 1 to 23: this appears to me the best course for presenting the investigation in a readable form.

I.

The inclination and eccentricity of the Moon's orbit, and, *à fortiori*, the variation of the position of the Ecliptic, and the Sun's latitude, are neglected; and the longitudes are measured from a fixed point in the Ecliptic. I write

n, the actual mean motion of the Moon at a given epoch;

viz., it is assumed that the mean longitude at the time t is $\epsilon + nt + n_2 t^2 +$ &c. where ϵ, n, n_2, &c. are absolute constants; and, moreover,

a, the calculated mean distance of the Moon;

that is, n^2a^3 is the sum of the masses of the Earth and Moon; a is therefore an absolute constant; and, in like manner,

n', the actual mean motion of the Sun at the same epoch,
a', the calculated mean distance of the Sun;

that is, if it were necessary to pay attention to the secular variation of the mean motion of the Sun, the assumption would be that the mean longitude was $\epsilon' + n't + n_2' t^2 +$ &c., ϵ', n', n_2', &c. being absolute constants, and $n'^2a'^3$ the sum of the masses of the Sun and Earth; a' would thus also be an absolute constant. But for the purpose of the present investigation the secular variation of the mean longitude of the Sun is neglected, or it is assumed that the mean longitude of the Sun is $\epsilon' + n't$, ϵ', n' being absolute constants; and that $n'^2a'^3$ is the sum of the masses of the Sun and Earth, a' being thus also an absolute constant.

I put also

m, the ratio of the mean motions of the Sun and Moon;

that is,

$$m = \frac{n'}{n}, \text{ or } n' = mn;$$

m is also an absolute constant.

The Sun is considered as moving in an elliptic orbit, the eccentricity whereof is $e' + \delta e'$ or $e' + f't$, e' and f' being absolute constants; the longitude of the Sun's perigee may be taken to be $\varpi' + (1 - c')n't$; so that the mean anomaly g' is $= n't + \epsilon - [\varpi' - (1 - c')n't] = c'n't + \epsilon' - \varpi'$; c', ϖ', being absolute constants; but c' is in fact treated as being $= 1$. Hence, if r', v' are the radius vector and longitude of the Sun, we have

$$r' = a' \operatorname{elqr}(e' + \delta e', g'),$$

$$v' = \varpi' + (1 - c')n't + \operatorname{elta}(e' + \delta e', g')$$
$$= n't + \epsilon + [\operatorname{elta}(e' + \delta e', g') - g'],$$

where

$$g' = c'n't + \epsilon' - \varpi'.$$

In the expression for the disturbing function the Sun's mass is taken to be $= n'^2a'^3$, or, what is the same thing, $= m^2n^2a'^3$.

Let r, v be the radius vector and longitude of the Moon; then, taking the usual approximate expression of the Disturbing Function, the equations of motion are

$$\frac{d}{dt}\frac{dr}{dt} - r\left(\frac{dv}{dt}\right)^2 + \frac{n^2a^3}{r^2} = m^2n^2a'^3\,\frac{r}{r'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right),$$

$$\frac{d}{dt}\frac{r^2dv}{dt} \qquad\qquad = m^2n^2a'^3\,\frac{r^2}{r'^3}\left(\quad - \tfrac{3}{2}\sin 2v - 2v'\right).$$

It will be convenient to assume

ρ, the quotient radius of the Moon's orbit,

ρ', the quotient radius of the Sun's orbit;

that is

$$r = \rho a, \quad r' = \rho' a'.$$

The equations of motion thus become

$$\frac{d}{dt}\frac{\delta\rho}{dt} - \rho\left(\frac{dv}{dt}\right)^2 + \frac{n^2}{\rho^3} = m^2n^2P,$$

$$\frac{d}{dt}\left(\rho^2\,\frac{dv}{dt}\right) \qquad = m^2n^2Q,$$

where for shortness

$$P = \frac{\rho}{\rho'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right),$$

$$Q = \frac{\rho^2}{\rho'^3}\left(\quad - \tfrac{3}{2}\sin 2v - 2v'\right),$$

in which

$$\rho' = \text{elqr}\,(e' + \delta e',\ g'),$$

$$v' = n't + \epsilon' + [\text{elta}\,(e' + \delta e',\ g') - g'].$$

I now change the notation by writing $\rho' + \delta\rho'$, $v' + \delta v'$, in the place of ρ', v', respectively, using henceforward ρ', v' to denote

$$\rho' = \qquad \text{elqr}\,(e',\ g'),$$

$$v' = n't + \epsilon' + [\text{elta}\,(e',\ g') - g'];$$

and I write also $\rho + \delta\rho$, $v + \delta v$, in the place of ρ, v, using henceforward ρ, v to denote the solutions of the equations obtained from the equations of motion by writing therein ρ', v' instead of the complete values $\rho' + \delta\rho'$, $v' + \delta v'$.

Suppose, in like manner, that the complete values of P, Q are denoted by $P + \delta P$, $Q + \delta Q$, where

$$\delta P = \frac{dP}{d\rho}\,\delta\rho + \frac{dP}{dv}\,\delta v + \frac{dP}{d\rho'}\,\delta\rho' + \frac{dP}{dv'}\,\delta v',$$

with a like value for δQ, the first powers of $\delta\rho$, δv, $\delta\rho'$, $\delta v'$ being alone attended to. Then, observing that the equations of motion are satisfied when $\delta\rho$, δv, $\delta\rho'$, $\delta v'$ are neglected, we have, it is clear,

$$\frac{d}{dt}\frac{d\delta\rho}{dt} - \delta\rho\left(\frac{dv}{dt}\right)^2 - 2\rho\frac{dv}{dt}\frac{d\delta v}{dt} - \frac{2n^2}{\rho^3}\delta\rho = m^2n^2\delta P,$$

$$\frac{d}{dt}\left(\rho^2\frac{d\delta v}{dt} + 2\rho\delta\rho\frac{dv}{dt}\right) = m^2n^2\delta Q.$$

The second of these equations gives

$$\rho^2\frac{d\delta v}{dt} + 2\rho\delta\rho\frac{dv}{dt} = m^2n^2\left(C + \int dQdt\right),$$

where the constant of integration, C, is to be so determined that δv may not contain any term of the form kt (for any such term is taken to be included in the term nt of $v+\delta v$). Multiplying the equation just obtained by $\frac{2}{\rho}\frac{dv}{dt}$, and adding it to the first equation, we have

$$\frac{d^2\delta\rho}{dt^2} + \left\{3\left(\frac{dv}{dt}\right)^2 - \frac{2n^2}{\rho^3}\right\}\delta\rho = m^2n^2\left(\delta P + \frac{2}{\rho}\frac{dv}{dt}\left(C + \int dQdt\right)\right),$$

which, with the above-mentioned integral equation, are the equations for the solution of the problem; but it will be convenient to write them under the slightly different form

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = \left\{n^2 + \frac{2n^2}{\rho^3} - 3\left(\frac{dv}{dt}\right)^2\right\}\delta\rho + m^2n^2\left\{\delta P + \frac{2}{\rho}\frac{dv}{dt}\left(C + \int dQdt\right)\right\},$$

$$\frac{d\delta v}{dt} = \qquad -\frac{2}{\rho}\frac{dv}{dt}\delta\rho + \frac{m^2n^2}{\rho^2}\left(C + \int \delta Qdt\right).$$

In these equations C is determined, as above, by the condition that $\frac{d\delta v}{dt}$ may contain no constant term; the values of ρ', v', $\delta\rho'$, $\delta v'$ are of course given by the theory of elliptic motion, and those of ρ, v are given by the ordinary lunar theory, in which the eccentricity of the solar orbit is treated as a constant; and then, $\delta\rho$, δv being obtained by integrating the equations, the radius vector and longitude of the Moon are $a(\rho+\delta\rho)$ and $v+\delta v$ respectively.

We have

$$P = \frac{\rho}{\rho'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right),$$

$$Q = \frac{\rho}{\rho'^3}\left(\quad -\tfrac{3}{2}\sin 2v - 2v'\right).$$

Moreover, by the lunar theory, observing that Plana's a is, or may be considered, identical with the a of the present Memoir, and putting also

$$\tau = nt + \epsilon - (n't + \epsilon'),$$

we have

$$\begin{aligned}
\frac{1}{\rho} = 1 + \tfrac{1}{6}\, m^2 - \tfrac{3}{4}\, m^2e'^2 & \\
- \tfrac{3}{2}\, m^2e' & \quad \cos g' \\
+ \quad m^2 - \tfrac{5}{2}\, m^2e'^2 & \quad \text{,,} \quad 2\tau \\
+ \tfrac{7}{2}\, m^2e' & \quad \text{,,} \quad 2\tau - g' \\
- \tfrac{1}{2}\, m^2e' & \quad \text{,,} \quad 2\tau + g' \\
- \tfrac{9}{4}\, m^2e'^2 & \quad \text{,,} \quad 2g' \\
+ \tfrac{17}{2}\, m^2e'^2 & \quad \text{,,} \quad 2\tau - 2g' \\
0 \quad m^2e'^2 & \quad \text{,,} \quad 2\tau + 2g',
\end{aligned}$$

$$\begin{aligned}
v = \quad nt + \epsilon & \\
- 3 \quad me' + 0 \quad m^2e' & \quad \sin g' \\
+ \tfrac{11}{8}\, m^2 \quad - \tfrac{55}{16}\, m^2e'^2 & \quad \text{,,} \quad 2\tau \\
+ \tfrac{77}{16}\, m^2e' & \quad \text{,,} \quad 2\tau - g' \\
- \tfrac{11}{16}\, m^2e' & \quad \text{,,} \quad 2\tau + g' \\
- \tfrac{9}{4}\, me'^2 + 0 \quad m^2e'^2 & \quad \text{,,} \quad 2g' \\
+ \tfrac{187}{16}\, m^2e'^2 & \quad \text{,,} \quad 2\tau - 2g' \\
0 \quad m^2e'^2 & \quad \text{,,} \quad 2\tau + 2g',
\end{aligned}$$

where the series are carried as far as m^2 and e'^2; the terms in e'^2 are given, as I shall have occasion to refer to them, but they are not used in the investigation, and, omitting them, the values are

$$\begin{aligned}
\frac{1}{\rho} = \quad 1 + \tfrac{1}{6}\, m^2 & \\
- \tfrac{3}{2}\, m^2e' & \quad \cos g' \\
+ \quad m^2 & \quad \text{,,} \quad 2\tau \\
+ \tfrac{7}{2}\, m^2e' & \quad \text{,,} \quad 2\tau - g' \\
- \tfrac{1}{2}\, m^2e' & \quad \text{,,} \quad 2\tau + g',
\end{aligned}$$

$$\begin{aligned}
v = \quad nt + \epsilon & \\
- 3 \quad me' & \quad \sin g' \\
+ \tfrac{11}{8}\, m^2 & \quad \text{,,} \quad 2\tau \\
+ \tfrac{77}{16}\, m^2e' & \quad \text{,,} \quad 2\tau - g' \\
- \tfrac{11}{8}\, m^2e' & \quad \text{,,} \quad 2\tau + g'
\end{aligned}$$

$$(g' = c'mnt + \text{const.}, \quad 2\tau = (2 - 2m)\, nt + \text{const.})$$

For the coordinates of the Sun we have

$$\begin{aligned} \frac{1}{\rho'} &= 1 \\ &\quad + e' \quad \cos g' \\ &\quad + e'^2 \quad ,, \ 2g', \end{aligned}$$

$$\begin{aligned} v' &= n't + \epsilon' \\ &\quad + 2\,e' \quad \sin g' \\ &\quad + \tfrac{5}{4}\,e'^2 \quad ,, \ 2g', \end{aligned}$$

the series being carried as far as e'^2; but the terms in e'^2 are only used for the formation of $\delta\rho'$, $\delta v'$; and, omitting them, we have

$$\begin{aligned} \frac{1}{\rho'} &= 1 \\ &\quad + e' \quad \cos g', \end{aligned}$$

$$\begin{aligned} v' &= n't + \epsilon' \\ &\quad + 2e' \quad \sin g'. \end{aligned}$$

If $e' + f't$ is written for e', then the value of $\delta e'$ is $= f't$; but as only the terms multiplied by the simple power f' are attended to, we may for convenience write $\delta e' = t$, the factor f' being restored in the final results: we thus have

$$\begin{aligned} \delta\frac{1}{\rho'} &= 1 \quad t\cos g' \\ &\quad + 2e' \quad ,, \ 2g', \end{aligned}$$

$$\begin{aligned} \delta v' &= 2 \quad t\sin g' \\ &\quad + \tfrac{5}{2}\,e' \quad ,, \ 2g', \end{aligned}$$

and we may add the equations

$$\begin{aligned} \frac{dv}{dt} &= \overbrace{\quad 1 \quad}^{n\,\times} \\ &\quad - 3\,m^2e' \quad \cos g' \\ &\quad + \tfrac{11}{4}\,m^2 \quad ,, \ 2\tau \\ &\quad + \tfrac{77}{8}\,m^2e' \quad ,, \ 2\tau - g' \\ &\quad - \tfrac{11}{8}\,m^2e' \quad ,, \ 2\tau + g', \end{aligned}$$

$$\begin{aligned} \delta\frac{1}{\rho'^3} &= 3\,e' \quad t \\ &\quad + 3 \quad t\cos\ g' \\ &\quad + \tfrac{9}{2}\,e' \quad ,, \ 2g', \end{aligned}$$

$$\begin{aligned} \frac{\delta v'}{\rho'^3} &= 2 \quad t\sin\ g' \\ &\quad + \tfrac{11}{2}\,e' \quad ,, \ 2g', \end{aligned}$$

which will be found useful.

II.

Proceeding now to the development of the solution, we have

$$\begin{aligned}\delta P = \; & \frac{1}{\rho'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right)\delta\rho \\ & + \frac{\rho}{\rho'^3}\left(\quad - \tfrac{3}{2}\sin 2v - 2v'\right)\delta v \\ & + \rho\left[\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right)\delta\frac{1}{\rho'^3} + \left(3\sin 2v - 2v'\right)\frac{\delta v'}{\rho'^3}\right],\end{aligned}$$

where the terms containing $\delta\rho$ and δv are (see Annex 1)

$$\left.\begin{array}{llll} & \tfrac{1}{2} & & \\ + & \tfrac{3}{2}\,e' & \cos & g' \\ + & \tfrac{3}{2} & \;\;" & 2\tau \\ + & \tfrac{21}{4}\,e' & \;\;" & 2\tau - g' \\ - & \tfrac{3}{4}\,e' & \;\;" & 2\tau + g' \end{array}\right|\;\delta\rho$$

$$\left.\begin{array}{llll} - & 3 & \sin & 2\tau \\ - & \tfrac{21}{2}\,e' & \;\;" & 2\tau - g' \\ + & \tfrac{3}{2}\,e' & \;\;" & 2\tau + g' \end{array}\right|\;\delta v,$$

and also

$$\begin{aligned}\delta Q = \; & \frac{\rho}{\rho'^3}\left(-3\sin 2v - 2v'\right)\delta\rho \\ & + \frac{\rho^2}{\rho'^3}\left(-3\cos 2v - 2v'\right)\delta v \\ & + \rho^2\left[\left(-\tfrac{3}{2}\sin 2v - 2v'\right)\delta\frac{1}{\rho'^3} + \left(3\cos 2v - 2v'\right)\frac{\delta v'}{\rho'^3}\right],\end{aligned}$$

where the terms containing $\delta\rho$ and δv are (see Annex 2)

$$\left.\begin{array}{llll} - & 3 & \sin & 2\tau \\ - & \tfrac{21}{2}\,e' & \;\;" & 2\tau - g' \\ + & \tfrac{3}{2}\,e' & \;\;" & 2\tau + g' \end{array}\right|\;\delta\rho$$

$$\left.\begin{array}{lll} -3 & \cos 2\tau \\ -\frac{21}{2}e' & ,, & 2\tau - g' \\ +\frac{3}{2}e' & ,, & 2\tau + g' \end{array}\right| \delta v,$$

but the additional term $\frac{171}{4}m^2e'\cos g'$ is ultimately added (see Annex 17) to the coefficient of δv.

Neglecting the terms in $\delta\rho$, δv, we have (see Annex 4)

$$\begin{array}{llll} \delta P = & \frac{3}{2}e' & t \\ & +\frac{3}{2} & t\cos g' \\ & -\frac{15}{2}e' & ,, & 2\tau \\ & +\frac{21}{4} & ,, & 2\tau - g' \\ & -\frac{3}{4} & ,, & 2\tau + g' \\ & +\frac{9}{2}e' & ,, & 2g' \\ & +\frac{51}{2}e' & ,, & 2\tau - 2g', \end{array}$$

and similarly (see Annex 5),

$$\begin{array}{llll} \delta Q = & \frac{15}{2}e' & t\sin 2\tau \\ & -\frac{21}{4} & ,, & 2\tau - g' \\ & +\frac{3}{4} & ,, & 2\tau + g' \\ & -\frac{51}{2}e' & ,, & 2\tau - 2g'. \end{array}$$

But in the foregoing expression of δP the terms belonging to the arguments g', $2g'$ give in $\delta\rho$, terms which rise by integration in δv; and in forming the expressions for δP, δQ, it is proper to take account of these terms. Attending only to the terms in question, we have

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = m^2n^2\delta P = \overbrace{\begin{array}{lll} \frac{3}{2}m^2 & t\cos g' \\ +\frac{9}{2}m^2e' & ,, & 2g'. \end{array}}^{n^2\times}$$

Now in general, if

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = n^2 \quad t\cos n\alpha t,$$

then

$$\delta\rho = \frac{2\alpha}{(1-\alpha^2)^2}\sin n\alpha t + \frac{1}{1-\alpha^2} \quad t\cos n\alpha t;$$

and hence the foregoing equation gives in $\delta\rho$ the terms

$$\begin{array}{llllll} 3m^3 & \sin g' & & +\frac{3}{2} & t\cos g' \\ +18m^3e' & ,, & 2g' & +\frac{9}{2}m^2e' & ,, & 2g'; \end{array}$$

or, neglecting the terms which contain m^3, the terms of $\delta\rho$ are

$$\begin{array}{lll} & \tfrac{3}{2}\,m^2 & t\cos g' \\ + & \tfrac{9}{2}\,m^2e' & \;\text{,,}\; 2g'. \end{array}$$

Substituting these terms in

$$\frac{d\delta v}{dt} = -\,2n\delta\rho,$$

we have

$$\frac{d\delta v}{dt} = n\times\overbrace{\begin{array}{ll} -\,3\,m^2 & t\cos g' \\ -\,9\,m^2e' & \;\text{,,}\; 2g'; \end{array}}$$

and since, in general,

$$\int t\cos n\alpha t\,dt = \frac{1}{n^2\alpha^2}\cos n\alpha t + \frac{1}{n\alpha}\,t\sin n\alpha t,$$

we obtain in δv the terms

$$n^{-1}\times\overbrace{\begin{array}{ll} -\,3 & \cos g' \\ -\,\tfrac{9}{4}\,e' & \;\text{,,}\; 2g' \end{array}} + \overbrace{\begin{array}{ll} -\,3\,m & t\sin g' \\ -\,\tfrac{9}{2}\,me' & \;\text{,,}\; 2g', \end{array}}$$

or for the present purpose the terms

$$n^{-1}\times\overbrace{\begin{array}{ll} -\,3 & \cos g' \\ -\,\tfrac{9}{4}\,e' & \;\text{,,}\; 2g'; \end{array}}$$

and these give in δP the additional terms (see Annex 6)

$$n^{-1}\times\overbrace{\begin{array}{lll} & \tfrac{27}{2} & \sin 2\tau \\ + & \tfrac{9}{2} & \;\text{,,}\; 2\tau - g' \\ + & \tfrac{9}{2} & \;\text{,,}\; 2\tau + g' \\ + & \tfrac{153}{8}\,e' & \;\text{,,}\; 2\tau - 2g' \\ + & \tfrac{9}{8}\,e' & \;\text{,,}\; 2\tau + 2g', \end{array}}$$

and in δQ the additional terms (see Annex 7)

$$n^{-1}\times\overbrace{\begin{array}{lll} & \tfrac{27}{2} & \cos 2\tau \\ + & \tfrac{9}{2} & \;\text{,,}\; 2\tau - g' \\ + & \tfrac{9}{2} & \;\text{,,}\; 2\tau + g' \\ + & \tfrac{153}{8}\,e' & \;\text{,,}\; 2\tau - 2g' \\ + & \tfrac{9}{8}\,e' & \;\text{,,}\; 2\tau + 2g'. \end{array}}$$

Combining the foregoing results, we have,

$$
\delta P = \begin{array}{llllll}
 & \frac{3}{2}e' & t & + & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{n^{-1}\times}} \\
+ & \frac{3}{2} & t\cos g' & & 0 & \sin g' \\
- & \frac{15}{2}e' & ,,\ 2\tau & + & \frac{27}{2}e' & ,,\ 2\tau \\
+ & \frac{21}{4} & ,,\ 2\tau-g' & + & \frac{9}{2} & ,,\ 2\tau-g' \\
- & \frac{3}{4} & ,,\ 2\tau+g' & + & \frac{9}{2} & ,,\ 2\tau+g' \\
+ & \frac{9}{2}e' & ,,\ 2g' & & 0 & ,,\ 2g' \\
+ & \frac{51}{2}e' & ,,\ 2\tau-2g' & + & \frac{153}{8}e' & ,,\ 2\tau-2g' \\
 & 0 & ,,\ 2\tau+2g' & + & \frac{9}{8}e' & ,,\ 2\tau+2g',
\end{array}
$$

and

$$
\delta Q = \begin{array}{llllll}
 & & & & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{n^{-1}\times}} \\
 & 0 & t\sin g' & + & 0 & \cos g' \\
+ & \frac{15}{2}e' & ,,\ 2\tau & + & \frac{27}{2}e' & ,,\ 2\tau \\
- & \frac{21}{4} & ,,\ 2\tau-g' & + & \frac{9}{2} & ,,\ 2\tau-g' \\
+ & \frac{3}{4} & ,,\ 2\tau+g' & + & \frac{9}{2} & ,,\ 2\tau+g' \\
 & 0 & ,,\ 2g' & & 0 & ,,\ 2g' \\
- & \frac{51}{2}e' & ,,\ 2\tau-2g' & + & \frac{153}{8}e' & ,,\ 2\tau-2g' \\
 & 0 & ,,\ 2\tau+2g' & + & \frac{9}{8}e' & ,,\ 2\tau+2g';
\end{array}
$$

whence also (see Annex 8)

$$
n\int \delta Q dt = \begin{array}{llllll}
 & & & & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{n^{-1}\times}} \\
 & 0 & t\cos g' & + & 0 & \sin g' \\
- & \frac{15}{4}e' & ,,\ 2\tau & + & \frac{69}{8}e' & ,,\ 2\tau \\
+ & \frac{21}{8} & ,,\ 2\tau-g' & + & \frac{15}{16} & ,,\ 2\tau-g' \\
- & \frac{3}{8} & ,,\ 2\tau+g' & + & \frac{39}{16} & ,,\ 2\tau+g' \\
 & 0 & ,,\ 2g' & & 0 & ,,\ 2g' \\
+ & \frac{51}{4}e' & ,,\ 2\tau-2g' & + & \frac{51}{16}e' & ,,\ 2\tau-2g' \\
 & 0 & ,,\ 2\tau+2g' & + & \frac{9}{16}e' & ,,\ 2\tau+2g'.
\end{array}
$$

The equation for $\delta\rho$ may be written,

$$
\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = m^2n^2\left(\delta P + 2n\int \delta Q dt\right),
$$

and we have, (see Annex 9)

$$\begin{array}{llll} \delta P + 2n\int \delta Q dt = & \tfrac{3}{2}\, e' & t & + \overbrace{\qquad\qquad\qquad}^{n^{-1}\times} \\ + \tfrac{3}{2} & t\cos g' & 0 & \sin g' \\ -15\, e' & ,,\ 2\tau & +\tfrac{123}{4}\, e' & ,,\ 2\tau \\ +\tfrac{21}{2} & ,,\ 2\tau - g' & +\tfrac{51}{8} & ,,\ 2\tau - g' \\ -\tfrac{3}{2} & ,,\ 2\tau + g' & +\tfrac{75}{8} & ,,\ 2\tau + g' \\ +\tfrac{9}{2}\, e' & ,,\ 2g' & 0 & ,,\ 2g' \\ +51\, e' & ,,\ 2\tau - 2g' & +\tfrac{51}{2}\, e' & ,,\ 2\tau - 2g' \\ 0 & ,,\ 2\tau + 2g' & +\tfrac{9}{4}\, e' & ,,\ 2\tau + 2g'. \end{array}$$

Hence observing that a term $n^2 t \cos n\alpha t$ in $\dfrac{d^2\delta\rho}{\delta t^2} + n^2\delta\rho$, gives in $\delta\rho$ the terms

$$\frac{1}{1-\alpha^2}\, t\cos n\alpha t \qquad + \frac{2\alpha}{(1-\alpha^2)^2}\,\frac{1}{n}\,\sin n\alpha t,$$

and a term $n \sin n\alpha t$ in $\dfrac{d^2\delta\rho}{dt^2} + n^2\delta\rho$, gives in $\delta\rho$ the term

$$\frac{1}{1-\alpha^2}\,\frac{1}{n}\,\sin n\alpha t,$$

we have (see Annex 10, but restoring the factor f'),

$\delta\rho = f'$ into as follows, viz.

$$\begin{array}{llll} \tfrac{3}{2}\, m^2 e' & t & & + \overbrace{\qquad\qquad\qquad}^{n^{-1}\times} \\ + \tfrac{3}{2}\, m^2 & t\cos g' & 3\, m^3 & \sin g' \\ +5\, m^2 e' & ,,\ 2\tau. & -\tfrac{203}{12}\, m^2 e' & ,,\ 2\tau \\ -\tfrac{7}{2}\, m^2 & ,,\ 2\tau - g' & +\tfrac{61}{24}\, m^2 & ,,\ 2\tau - g' \\ +\tfrac{1}{2}\, m^2 & ,,\ 2\tau + g' & -\tfrac{91}{24}\, m^2 & ,,\ 2\tau + g' \\ +\tfrac{9}{2}\, m^2 e' & ,,\ 2g' & -18\, m^3 e' & ,,\ 2g' \\ -17\, m^2 e' & ,,\ 2\tau - 2g' & +\tfrac{85}{6}\, m^2 e' & ,,\ 2\tau - 2g' \\ 0 & ,,\ 2\tau + 2g' & -\tfrac{3}{4}\, m^2 e' & ,,\ 2\tau + 2g'. \end{array}$$

The first column, containing the term in t and the terms $t\cos$ arg., shows that the constant term of ρ, and the terms involving the cosines of the same arguments, as obtained without attending to the variation of the solar eccentricity, are correct as regards the first power of t, when for the constant eccentricity e' we substitute $e'+f't$. In fact, the above-mentioned expression (correct to e'^2) of $\frac{1}{\rho}$ gives

$$\begin{aligned}
\rho = 1 - \tfrac{1}{6}m^2 + \tfrac{3}{4}\, m^2e'^2 & \\
+ \tfrac{3}{2}\, m^2e'^2 & \quad \cos g' \\
- \quad m^2 + \tfrac{5}{2}\, m^2e'^2 & \quad ,,\ 2\tau \\
- \tfrac{7}{2}\, m^2e' & \quad ,,\ 2\tau - g' \\
+ \tfrac{1}{2}\, m^2e' & \quad ,,\ 2\tau + g' \\
+ \tfrac{9}{4}\, m^2e'^2 & \quad ,,\ 2g' \\
- \tfrac{17}{2}\, m^2e'^2 & \quad ,,\ 2\tau - 2g' \\
0 & \quad ,,\ 2\tau + 2g',
\end{aligned}$$

and putting therein $e'+f't$ in the place of e', we have the first column of the foregoing expression of $\delta\rho$.

The second column, involving sin arg., contains the new periodic terms considered in Prof. Adams' Memoir of 1853, and the coefficients for the arguments g', 2τ, $2\tau - g'$, $2\tau + g'$, agree with his values; observing that his terms belong to $\delta\frac{1}{\rho} = -\frac{\delta\rho}{\rho^2} = -\delta\rho$, so that the signs are reversed; those for the remaining arguments $2g'$, $2\tau - 2g'$, $2\tau + 2g'$, are not given by him.

The equation for δv may be written,

$$\frac{d\delta v}{dt} = -2n\delta\rho + m^2n^2\int \delta Q dt,$$

and we have (see Annex 11)

$$\begin{aligned}
\frac{d\delta v}{dt} = \overbrace{- \ 3 \ m^2e' \quad t}^{n\,\times} & \\
- \ 3 \ m^2 \quad t\cos g' & \qquad - \ 6 \ m^3 \quad \sin g \\
+ \tfrac{55}{4}\, m^2e' \quad ,,\ 2\tau & \qquad + \tfrac{1019}{24}\, m^2e' \quad ,,\ 2\tau \\
+ \tfrac{77}{8}\, m^2 \quad ,,\ 2\tau - g' & \qquad - \tfrac{199}{48}\, m^2 \quad ,,\ 2\tau - g' \\
- \tfrac{11}{8}\, m^2 \quad ,,\ 2\tau + g' & \qquad + \tfrac{481}{48}\, m^2 \quad ,,\ 2\tau + g' \\
- \ 9 \ m^2e' \quad ,,\ 2g' & \qquad - \ 36 \ m^3e' \quad ,,\ 2g' \\
+ \tfrac{187}{8}\, m^2e' \quad ,,\ 2\tau - 2g' & \qquad - \tfrac{1207}{48}\, m^2e' \quad ,,\ 2\tau - 2g' \\
0 \quad ,,\ 2\tau + 2g' & \qquad + \tfrac{33}{16}\, m^2e' \quad ,,\ 2\tau + 2g',
\end{aligned}$$

whence, integrating by the formulæ

$$\int t\cos n\alpha t \ \ dt = \frac{1}{n\alpha}\, t\sin n\alpha t + \frac{1}{n^2\alpha^2}\cos n\alpha t,$$

$$\int \ \sin n\alpha t \ \ dt = \qquad\qquad - \frac{1}{n\alpha}\cos n\alpha t,$$

we have (see Annex 12, but restoring the factor f'),

$\delta v = f'$ into as follows, viz.

		$-\ \frac{3}{2}\ m^2ne't^2$		
		$n^{-1}\times$		
$-\ 3\ m$	$t\sin g'$	$+\ \overbrace{-3+6\,m^2}$	$\cos g'$	
$-\ \frac{55}{8}\ m^2e'$	„ 2τ	$-\ \frac{74}{3}\ m^2e'$	„ 2τ	
$+\ \frac{77}{16}\ m^2$	„ $2\tau-g'$	$+\ \frac{215}{48}\ m^2$	„ $2\tau-g'$	
$-\ \frac{11}{16}\ m^2$	„ $2\tau+g'$	$-\ \frac{257}{48}\ m^2$	„ $2\tau+g'$	
$-\ \frac{9}{2}\ me'$	„ $2g'$	$(-\frac{9}{4}+18\,m^2)\,e'$	„ $2g'$	
$+\ \frac{187}{8}\ m^2e'$	„ $2\tau-2g'$	$+\ \frac{2239}{96}\ m^2e'$	„ $2\tau-2g'$	
0	„ $2\tau+2g'$	$-\ \frac{33}{32}\ m^2e'$	„ $2\tau+2g'$.	

The first column, containing $t\sin$ arg., may be obtained from the before-mentioned expression (accurate to e'^2) of v, by substituting therein $e'+f't$ in the place of e'.

The term $-\frac{3}{2}m^2ne'f't^2$; or, as it may be written, $-\frac{3}{2}m^2\int n\,[(e'+f't)^2-e'^2]\,dt$, is the first term of the acceleration; the other terms of the second column are the new periodic terms in δv, considered by Prof. Adams; the coefficients for the arguments g', 2τ, $2\tau-g'$, $2\tau+g'$, agreeing with his values, but those for the remaining arguments $2g'$, $2\tau-2g'$, $2\tau+2g'$ not being given by him.

III.

Proceeding now to the calculation of the term in m^4 of the acceleration, we have,

$$\frac{d\delta v}{dt} = -\frac{2}{\rho}\frac{dv}{dt}\delta\rho + \frac{m^2n^2}{\rho^2}\left(C+\int\delta Q dt\right),$$

where the non-periodic part of $\delta\rho$ is of the form,

$$\delta\rho = (\tfrac{3}{2}m^2 + \Box m^4)\,e'f't,$$

and it is in the first place necessary to find the value of the numerical coefficient $\Box$, in fact to calculate the secular part of $\delta\rho$ as far as m^4. Reverting to the equation

$$\frac{d^2\delta\rho}{dt^2} + n^2\delta\rho = \left(n^2+\frac{2n^2}{\rho^3}-3\left(\frac{dv}{dt}\right)^2\right)\delta\rho + m^2n^2\left(\delta P + \frac{2}{\rho}\frac{dv}{dt}\left(C+\int\delta Q dt\right)\right),$$

and as before omitting in the process the factor f':

The part $\left(n^2+\dfrac{2n^2}{\rho^3}-3\left(\dfrac{dv}{dt}\right)^2\right)\delta\rho$ contains (see Annex 13), a term

$$= \tfrac{381}{8}\,m^4n^2e't.$$

The part of $m^2n^2\delta P$, which involves $\delta\rho$, contains (see Annex 14) a term

$$= -\tfrac{495}{32}\, m^4n^2e't.$$

The part of $m^2n^2\delta P$, which depends on δv, contains a term

$$= -\tfrac{15}{4}\, m^4n^2e't.$$

The part of $m^2n^2\delta P$, depending on $\delta\rho'$ and $\delta v'$, is found (see Annex 18) to contain, besides the term $\tfrac{3}{2}\, m^2ne't$ in m^2 already obtained, a new term

$$= -\tfrac{647}{32}\, m^4n^2e't.$$

And finally the part $m^2n^2\dfrac{2}{\rho}\dfrac{dv}{dt}\left(C+\int \delta Q dt\right)$ is found (see Annex 19) to contain a term

$$(2\,.\tfrac{-1455}{32} + \tfrac{675}{32} =) - \tfrac{2235}{32}\, m^4n^2e't,$$

where the component coefficient $-\tfrac{1455}{32}$, which arises from the new periodic terms of $\delta\rho$ and δv is separately calculated (see Annex 21).

Hence $\dfrac{d^2\delta\rho}{dt^2} + n^2\delta\rho$ contains the term

$$(\square = \tfrac{381}{8} - \tfrac{495}{32} - \tfrac{15}{4} - \tfrac{647}{32} - \tfrac{2235}{32} =) - \tfrac{1973}{32}\, m^4n^2e't,$$

and this gives in $\delta\rho$ the term

$$-\tfrac{1973}{32}\, m^4e't,$$

so that, restoring the term in m^2, and the common factor f', the complete secular term of $\delta\rho$ is

$$\delta\rho = (\tfrac{3}{2}\, m^2 - \tfrac{1973}{32}\, m^4)\, e'f't,$$

which, as will be shown, Art. IV., agrees with Prof. Adams' result.

Resuming now the equation,

$$\frac{d\delta v}{dt} = -\frac{2}{\rho}\frac{dv}{dt}\delta\rho + \frac{m^2n^2}{\rho^2}\left(C + \int \delta Q dt\right),$$

the part $-\dfrac{2}{\rho}\dfrac{dv}{dt}\delta\rho$ contains (see Annex 22) the term

$$(\tfrac{1973}{16} + \tfrac{275}{8} =)\, \tfrac{2523}{16}\, m^4ne't,$$

and the part $\dfrac{m^2n^2}{\rho}\left(C + \int \delta Q dt\right)$ contains (see Annex 23) the term

$$(\tfrac{45}{8} - \tfrac{1455}{32} =) - \tfrac{1275}{32}\, m^4ne't,$$

so that we have in $\dfrac{d\delta v}{dt}$ the term

$$(\tfrac{2523}{16} - \tfrac{1275}{32} =)\, \tfrac{3771}{32}\, m^4ne't,$$

giving in δv the term

$$\tfrac{3771}{64}\, m^4 n e' t^2,$$

or, restoring the term in m^2 and the common factor f', the complete secular term of δv is

$$\delta v = (-\tfrac{3}{2}\, m^2 + \tfrac{3771}{64}\, m^4)\, n e' f' t^2,$$

which agrees with the value obtained by Prof. Adams. It is right to remark that the m of Prof. Adams is different from that of the present Memoir; we have in fact,

$$m\,(\text{Adams}) = m\,\{1 + (\tfrac{3}{2}\, m^2 - \tfrac{3771}{64}\, m^4)\, 2e' f' t\}$$

in the notation of the present Memoir; but as f'^2 is throughout neglected, we may in the foregoing expression of the secular part of δv substitute the m of Prof. Adams. And then if the term be written in the form

$$\delta v = (-\tfrac{3}{2}\, m^2 + \tfrac{3771}{64}\, m^2) \int [(e' + f't)^2 - e'^2]\, n dt,$$

the two results are seen to agree together. But as regards the before-mentioned secular term,

$$\delta \rho = (\tfrac{3}{2}\, m^2 - \tfrac{1973}{32}\, m^4)\, e' f' t,$$

the identification is less easy, and I shall consider it in the following article.

IV.

It will be convenient to write M, N, A, E', in place of the m, n, a, e', of the foregoing part of the present Memoir, and to now use m, n, e', in the significations in which they are employed by Prof. Adams; E' (the constant part of the solar eccentricity) is his E', and his e' is $E' + f't$. As to his symbols a, $a_{,}$, these, I think, ought to have been represented, and I shall here represent them by a, $\mathrm{a}_{,}$ [1]. And I take a such that $n^2 a^3 =$ Sum of the masses of the Earth and Moon; or, taking this to be unity, we have $N^2 A^3 = 1$, $n^2 a^3 = 1$.

The formulæ of Prof. Adams' memoir, which it will be necessary to make use of, may be written

$$\begin{array}{lll} \dfrac{\mathrm{a}}{r} = 1 - \tfrac{11}{8}\, m^2 - \tfrac{201}{16}\, m^4 e'^2 & & \\ \quad - \tfrac{3}{2}\, m^2 e' & \cos g' & \\ \quad + \quad m^2 - \tfrac{5}{2}\, m^2 e'^2 & \text{,,}\ 2\tau & \\ \quad + \tfrac{7}{2}\, m^2 e' & \text{,,}\ 2\tau - g' & \\ \quad - \tfrac{1}{2}\, m^2 e' & \text{,,}\ 2r + g' & \end{array}$$

[1] Plana, in his Lunar Theory, uses the three letters a, $\mathrm{a}_{,}$, a; his a and n being such that $n^2 a^3 =$ Sum of the masses of the Earth and Moon. There is an obvious inconvenience in writing a, $a_{,}$, in the place of his a, $\mathrm{a}_{,}$.

the sine terms being disregarded,

$$n = N\{1 + (-\tfrac{3}{2}m^2 + \tfrac{3771}{64}m^4)\,2E'f't\},$$

(whence also

$$m = M\{1 + (\tfrac{3}{2}m^2 - \tfrac{3771}{64}m^4)\,2E'f't\}$$

since $m = \dfrac{n'}{n}$, $M = \dfrac{N'}{N}$).

$$\frac{1}{n} = \mathrm{a}_{,}^{\frac{3}{2}}\{1 + m^2 - \tfrac{197}{64}m^4 + (\tfrac{3}{2}m^2 - \tfrac{3867}{64}m^4)\,e'^2\},$$

$$1 = \frac{\mathrm{a}}{\mathrm{a}_{,}}\{1 - \tfrac{1}{2}m^2 + \tfrac{13}{4}m^4 + (-\tfrac{3}{4}m^2 + \tfrac{3201}{64}m^4)\,e'^2\},$$

which formulæ $\left(\text{observing that } \dfrac{1}{\rho + \delta\rho} = \dfrac{A}{r}\right)$ lead to the value of $\delta\rho$, and we should obtain for the secular part the foregoing expression, which will now be

$$\delta\rho = (\tfrac{3}{2}M^2 - \tfrac{1973}{32}M^4)\,E'f't.$$

We in fact have

$$\frac{1}{n} = a^{\frac{3}{2}} = \mathrm{a}_{,}^{\frac{3}{2}}\{1 + m^2 - \tfrac{197}{64}m^4 + (\tfrac{3}{2}m^2 - \tfrac{3867}{64}m^4)\,e'^2\},$$

and thence

$$\begin{aligned} a = \mathrm{a}_{,}\,&\{1 + \tfrac{2}{3}m^2 - \tfrac{197}{96}m^4 + (m^2 - \tfrac{3867}{96}m^4)\,e'^2 \\ &\quad - \tfrac{1}{9}(m^4 + 3m^4e'^2)\} \\ = \mathrm{a}_{,}\,&\{1 + \tfrac{2}{3}m^2 - \tfrac{623}{288}m^4 + (m^2 - \tfrac{3899}{96}m^4)\,e'^2\}\,; \end{aligned}$$

but

$$1 = \frac{\mathrm{a}}{\mathrm{a}_{,}}\{1 - \tfrac{1}{2}m^2 + \tfrac{13}{4}m^4 + (-\tfrac{3}{4}m^2 + \tfrac{3201}{64}m^4)\,e'^2\},$$

and therefore

$$\begin{aligned} a = \mathrm{a}\,&\{1 + \tfrac{2}{3}m^2 - \tfrac{623}{288}m^4 + (m^2 - \tfrac{3899}{96}m^4)\,e'^2 \\ &\quad - \tfrac{1}{2}m^2 + \tfrac{13}{4}m^4 \qquad - \tfrac{1}{2}m^4e'^2 \\ &\quad - \tfrac{1}{3}m^4 + (-\tfrac{3}{4}m^2 + \tfrac{3201}{64}m^4)\,e'^2 \\ &\quad - \tfrac{1}{2}m^4e'^2\} \\ = \mathrm{a}\,&\{1 + \tfrac{1}{6}m^2 + \tfrac{217}{288}m^4 + (\tfrac{1}{4}m^2 + \tfrac{1613}{192}m^4)\,e'^2\}\,; \end{aligned}$$

and hence, since for the non-periodic part

$$\frac{\mathrm{a}}{r} = 1 - \tfrac{11}{8}m^2 - \tfrac{201}{16}m^4e'^2,$$

we find

$$\frac{a}{r} = 1 + \tfrac{1}{6} m^2 + \tfrac{217}{288} m^4 + (\tfrac{1}{4} m^2 + \tfrac{1613}{192} m^4) e'^2$$
$$- \tfrac{11}{8} m^4 \qquad - \tfrac{201}{16} m^4 e'^2$$
$$= 1 + \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) e'^2$$
$$= 1 + \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) E'^2 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4)\, 2E'f't.$$

But

$$\frac{A}{a} = \left(\frac{n}{N}\right)^{\frac{2}{3}} = 1 + (-m^2 + \tfrac{3771}{96} m^4)\, 2E'f't;$$

and therefore

$$\frac{A}{r} = 1 + \tfrac{1}{6} m^2 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) E'^2 + (\ \tfrac{1}{4} m^2 - \tfrac{799}{192} m^4)\, 2E'f't$$
$$+ (-\ m^2 + \tfrac{3771}{96} m^4)\, 2E'f't$$
$$- \tfrac{1}{6} m^4 .\, 2E'f't$$
$$= 1 + \tfrac{1}{6} m^4 - \tfrac{179}{288} m^4 + (\tfrac{1}{4} m^2 - \tfrac{799}{192} m^4) E'^2 + (-\tfrac{3}{4} m^2 + \tfrac{2237}{64} m^4)\, 2E'f't.$$

But we have

$$m = M\{1 + (\tfrac{3}{2} M^2 - \tfrac{3771}{64} M^4)\, 2E'f't\};$$

and thence in the foregoing expression

$$m^2 = M^2 + 3M^4 .\, 2E'f't,$$
$$m^4 = M^4;$$

and therefore

$$\frac{A}{r} = 1 + \tfrac{1}{6} M^2 - \tfrac{179}{288} M^4 + (\tfrac{1}{4} M^2 - \tfrac{799}{192} M^4) E'^2 + (-\tfrac{3}{4} M^2 + \tfrac{2237}{64} M^4)\, 2E'f't$$
$$+ \ \tfrac{1}{2}\ M^4 .\, 2E'f't$$
$$= 1 + \tfrac{1}{6} M^2 - \tfrac{179}{288} M^4 + (\tfrac{1}{4} M^2 - \tfrac{799}{192} M^4) E'^2 + (-\tfrac{3}{4} M^2 + \tfrac{2269}{64} M^4)\, 2E'f't;$$

and observing that in the periodic terms we may write A, in the place of a, and neglect the sine terms, we have

$$\frac{A}{r} = \frac{1}{\rho + \delta\rho} = 1 + \tfrac{1}{6} M^2 - \tfrac{179}{288} M^4 + (\tfrac{1}{4} M^2 - \tfrac{799}{192} M^4) E'^2$$
$$+ (-\tfrac{3}{4} M^2 + \tfrac{2269}{64} M^4)\, 2E'f't$$
$$- \tfrac{3}{2} m^2 e' \qquad \cos g'$$
$$+ \ m^2 (1 - \tfrac{5}{2} e'^2) \qquad ,, \quad 2\tau$$
$$+ \tfrac{7}{2} m^2 e' \qquad ,, \quad 2\tau - g'$$
$$- \tfrac{1}{2} m^2 e' \qquad ,, \quad 2\tau + g',$$

say $\dfrac{1}{\rho+\delta\rho}=1+X$; and thence $\rho+\delta\rho=\dfrac{1}{1+X}=1-X+X^2$, the non-periodic part whereof is

$$\begin{aligned}1-\tfrac{1}{6}M^2+\tfrac{179}{128}M^4&+(-\tfrac{1}{4}M^2+\tfrac{179}{192}M^4)E'^2+(\tfrac{3}{4}M^2-\tfrac{2269}{64}M^4)2E'f't\\ +\tfrac{1}{36}M^4&+\tfrac{1}{12}M^4E'^2+\tfrac{1}{4}M^4.2E'f't\\ +\tfrac{1}{2}\cdot\tfrac{9}{4}\,m^4e'^2&\\ +\tfrac{1}{2}\,m^4(1-5e'^2)&\\ +\tfrac{1}{2}\cdot\tfrac{49}{4}\,m^4e'^2&\\ +\tfrac{1}{2}\cdot\tfrac{1}{4}\,m^4e'^2,&\end{aligned}$$

where the terms in m^4 and $m^4e'^2$ are

$$=\tfrac{1}{2}m^4+(\tfrac{9}{8}-\tfrac{5}{2}+\tfrac{49}{8}+\tfrac{1}{8}=)\,\tfrac{39}{8}m^4e'^2,$$

which are

$$=\tfrac{1}{2}M^4\qquad+\tfrac{39}{8}M^4E'^2\qquad+\tfrac{39}{8}M^4.2E'f't;$$

so that the foregoing expression of the non-periodic part of $\rho+\delta\rho$ is

$$\begin{aligned}=1&-\tfrac{1}{6}M^2+(\tfrac{179}{128}+\tfrac{1}{36}+\tfrac{1}{2}=)\ \tfrac{2219}{1152}M^4\\ &+(-\tfrac{1}{4}M^2+(\tfrac{179}{192}+\tfrac{1}{12}+\tfrac{39}{8}=)\ \tfrac{1131}{192}M^4)E'^2\\ &+(\tfrac{3}{4}M^2+(-\tfrac{2269}{64}-\tfrac{1}{4}+\tfrac{39}{8}=)-\tfrac{1973}{64}M^4)2E'f't;\end{aligned}$$

or the secular term of $\delta\rho$ is

$$=(\tfrac{3}{4}M^2-\tfrac{1973}{64}M^4)\,2E'f't,$$

which is the required formula.

V.

It is interesting to see how the coefficient $\frac{3771}{64}$ is made up. In Prof. Adams' Memoir we have

$$\begin{aligned}\tfrac{3771}{64}=&\\ =&-\tfrac{3}{2}-\tfrac{3}{4} &&(=-\tfrac{9}{4})\\ &+\tfrac{15}{4}+\tfrac{135}{64}-\tfrac{117}{8}+\tfrac{495}{128}-\tfrac{285}{16}-\tfrac{147}{16}-\tfrac{3}{16}-\tfrac{1323}{256}-\tfrac{27}{256} &&(=-\tfrac{2391}{64})\\ &+\tfrac{9}{2}-\tfrac{27}{4}-\tfrac{45}{2}-\tfrac{45}{2}-\tfrac{15}{2}-\tfrac{495}{32}+\tfrac{285}{4}-\tfrac{147}{4}-\tfrac{3}{4}+\tfrac{441}{8}+\tfrac{441}{8}+\tfrac{9}{8}+\tfrac{9}{8} &&(=+\tfrac{3153}{32}),\end{aligned}$$

where it may be remarked that the terms

$$\tfrac{495}{128}-\tfrac{285}{16}=(-\tfrac{1785}{128})$$

and

$$-\tfrac{495}{32}+\tfrac{285}{4}=(+4\cdot\tfrac{1785}{128})$$

make together $3\cdot\frac{1785}{128}=\frac{5355}{128}$, and that it is in fact by the addition of these terms that Plana's coefficient $\frac{2187}{128}$ is changed into $\frac{3771}{64}$.

But in the present Memoir the coefficient $\frac{3771}{64}$ is obtained by means of an entirely different set of component numbers, viz. we have

$$\tfrac{3771}{64} = -\tfrac{381}{8} + \tfrac{495}{32} + \tfrac{15}{4} + \tfrac{647}{32} + \tfrac{1455}{16} - \tfrac{675}{32} + \tfrac{275}{16} + \tfrac{45}{16} - \tfrac{1455}{64}.$$

I had imagined, from the way in which the numbers $\frac{1455}{16} - \frac{1455}{64}$ presented themselves, that, if they were omitted, Plana's value $\frac{2187}{128}$ would have been obtained; but the result shows that this is not so.

As just deduced from the formula of Prof. Adams, the number $\frac{1973}{64}$ is obtained as follows, viz.

$$\begin{aligned}\tfrac{1973}{64} &= (-\tfrac{3771}{96} - 1) - \tfrac{1}{3} - \tfrac{1}{2} + (\tfrac{3153}{64} + \tfrac{3}{4}) - \tfrac{1}{2} - \tfrac{201}{16} + \tfrac{3771}{96} - \tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{39}{8} \\ &= -1 - \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{3}{4} - \tfrac{1}{2} - \tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{4} \qquad (=-1) \\ &\quad + \tfrac{3153}{64} - \tfrac{201}{16} - \tfrac{39}{8},\end{aligned}$$

where *ut suprà*

$$\tfrac{3153}{64} = \tfrac{1}{2}\left\{\tfrac{9}{2} - \tfrac{27}{4} - \tfrac{45}{2} - \tfrac{45}{2} - \tfrac{15}{2} - \tfrac{495}{32} + \tfrac{285}{4} - \tfrac{147}{4} - \tfrac{3}{4} + \tfrac{441}{8} + \tfrac{441}{8} + \tfrac{9}{8} + \tfrac{9}{8}\right\};$$

$-\frac{201}{16}$ is a number occurring in his Memoir, and which is in effect obtained irrespectively of the new periodic terms, and $-\frac{39}{8}$ is a number obtained as above, irrespectively of the new periodic terms. According to the method of the present Memoir, the number $\frac{1973}{64}$ was obtained in the form

$$\tfrac{1973}{64} = -\tfrac{1}{2}\left(\tfrac{381}{8} - \tfrac{495}{32} - \tfrac{15}{4} - \tfrac{647}{32} - \tfrac{1455}{16} + \tfrac{675}{32}\right).$$

VI.

If the investigation were pursued further, a question would arise as to the proper form to be given to the arguments; for in these, $nt + \epsilon$ seems to stand in the place of v, the value whereof is

$$v = nt + \epsilon - (\tfrac{3}{2}m^2 - \tfrac{3771}{64}m^4)\, ne'f't^2,$$

say $v = nt + \epsilon + kne'f't^2$, and it might be considered that in the arguments $nt + \epsilon$ should be changed into $nt + \epsilon + kne'f't^2$, or, what is the same thing, that τ should be changed into $\tau + kne'f't^2$, but that g' should remain unaltered (this assumes that there is not in the Sun's longitude any term corresponding to the acceleration). The arguments, instead of being of the simple form kt, would thus be of the form $kt + k_2 f't^2$. But this would not only increase the difficulty of integration, but would be inconsistent with the general plan of the solution; and it would seem to be the proper course to imagine the cosine or sine of such an argument to be developed $\left(\begin{matrix}\cos\\ \sin\end{matrix}\, kt + k_2 f't^2 = \begin{matrix}\cos\\ \sin\end{matrix}\, kt \mp k_2 f't^2 \begin{matrix}\sin\\ \cos\end{matrix}\, kt\right)$ in such manner as to bring the secular part of the argument outside the cos or sin; this is, in fact, the form which the solution takes when the arguments are left throughout in their original form, for the terms of the form $f't^2 \begin{matrix}\cos\\ \sin\end{matrix}$ arg. would present themselves in the subsequent approximations. But I shall not at present further examine the question.

Annexes containing the Details of the Calculation.

Annex 1.

Calculation of part of δP.

$$\delta P = \frac{1}{\rho'^3}\left(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v'\right)\delta\rho$$

$$+ \frac{\rho}{\rho'^3}\left(\ \ -3\sin 2v - 2v'\right)\delta v.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v' =$			$\dfrac{1}{\rho'^3} =$
$\tfrac{1}{2}$			1
$+\tfrac{3}{2}$	cos	2τ	$+3e' \cos g'$.
$+3e'$	„	$2\tau - g'$	
$-3e'$	„	$2\tau + g'$	

Product is =

$$\begin{array}{lll} \tfrac{1}{2} & & +\tfrac{3}{2}e' \cos g' \\ +\tfrac{3}{2} & \cos\ 2\tau & +\tfrac{9}{2}e'\left(\tfrac{1}{2}\cos 2\tau - g' \quad + \tfrac{1}{2}\cos 2\tau + g'\right) \\ +3e' & ,,\ \ 2\tau - g' & \\ -3e' & ,,\ \ 2\tau + g' & \end{array}$$

$$\begin{array}{rll} = & \tfrac{1}{2} & \\ & +\tfrac{3}{2}e' & \cos\ g' \\ & +\tfrac{3}{2} & ,,\ \ 2\tau \\ (+3+\tfrac{9}{4}=) & +\tfrac{21}{4}e' & ,,\ \ 2\tau - g' \\ (-3+\tfrac{9}{4}=) & -\tfrac{3}{4}e' & ,,\ \ 2\tau + g', \end{array}$$

which is the coefficient of $\delta\rho$.

And

$-3\sin 2v - 2v' =$			$\dfrac{1}{\rho'^3} =$
-3	sin	2τ	1
$-6e'$	„	$2\tau - g'$	$+3e' \cos g'$.
$+6e'$	„	$2\tau + g'$	

Product is =

$$\begin{array}{lll} -3 & \sin 2\tau & -9e'(\tfrac{1}{2}\sin 2\tau - g' + \tfrac{1}{2}\sin 2\tau + g') \\ -6e' & ,, \; 2\tau - g' & \\ +6e' & ,, \; 2\tau + g' & \end{array}$$

$$\begin{array}{rlll} = & -3 & \sin 2\tau \\ (-6-\tfrac{9}{2}=) & -\tfrac{21}{2}e' & ,, \; 2\tau - g' \\ (+6-\tfrac{9}{2}=) & +\tfrac{3}{2}e' & ,, \; 2\tau + g'; \end{array}$$

or, since $\rho = 1$, this is the coefficient of δv.

Annex 2.

Calculation of part of δQ.

$$\delta Q = \frac{\rho}{\rho'^3}(-3\sin 2v - 2v')\,\delta\rho$$

$$+ \frac{\rho^2}{\rho'^3}(-3\cos 2v - 2v')\,\delta v.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$$\begin{array}{lll|l} -3\sin 2v - 2v' = & & & \dfrac{1}{\rho'^3} = \\ -3 & \sin 2\tau & & 1 \\ -6e' & ,, \; 2\tau - g' & & +3e' \cos g'. \\ -6e' & ,, \; 2\tau + g' & & \end{array}$$

Product is =

$$\begin{array}{lll} -3 & \sin 2\tau & -9e'(\tfrac{1}{2}\sin 2\tau - g' + \tfrac{1}{2}\sin 2\tau + g') \\ -6e' & ,, \; 2\tau - g' & \\ -6e' & ,, \; 2\tau + g' & \end{array}$$

$$\begin{array}{rlll} = & -3 & \sin 2\tau \\ (-6-\tfrac{9}{2}=) & -\tfrac{21}{2}e' & ,, \; 2\tau - g' \\ (+6-\tfrac{9}{2}=) & +\tfrac{3}{2}e' & ,, \; 2\tau + g' \end{array}$$

or, since $\rho = 1$, this is the coefficient of $\delta\rho$.

$$\begin{array}{lll|l} -3\cos 2v - 2v' = & & & \dfrac{1}{\rho'^3} = \\ -3 & \cos 2\tau & & 1 \\ -6e' & ,, \; 2\tau - g & & +3e' \cos g'. \\ +6e' & ,, \; 2\tau + g & & \end{array}$$

Product is =

$$\begin{array}{lll} -\ 3 & \cos\ 2\tau & -9e'(\tfrac{1}{2}\cos 2\tau - g' + \tfrac{1}{2}\cos 2\tau + g') \\ -\ 6\,e' & ,,\ \ 2\tau - g' & \\ +\ 6\,e' & ,,\ \ 2\tau + g' & \end{array}$$

$$\begin{array}{rlll} = & -\ 3 & \cos\ 2\tau \\ (-6-\tfrac{9}{2}=) & -\tfrac{21}{2}\,e' & ,,\ \ 2\tau - g' \\ (+6-\tfrac{9}{2}=) & +\tfrac{3}{2}\,e' & ,,\ \ 2\tau + g'\,; \end{array}$$

or, since $\rho^2 = 1$, this is the coefficient of δv.

Annex 3.

Calculation of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, 2v - 2v'$.

$$v - v' = \tau - 2e' \sin g'$$

$$2v - 2v' = 2\tau - e' \sin g'.$$

$$\begin{array}{rl} \cos 2v - 2v' = & \cos\ 2\tau \\ & +\ \sin\ 2\tau\,.\,4e' \sin g' \\ = & \cos\ 2\tau \\ & +2e'\ ,,\ \ 2\tau - g' \\ & -2e'\ ,,\ \ 2\tau\ \ g' \end{array}$$

$$\begin{array}{rl} \sin 2v - 2v' = & \sin\ 2\tau \\ & -\ \ \cos\ 2\tau\,.\,4e' \sin g' \\ = & \sin\ 2\tau \\ & +2e'\ ,,\ \ 2\tau - g' \\ & -2e'\ ,,\ \ 2\tau + g'. \end{array}$$

The expressions are calculated (*post,* Annex 16) as far as m^2.

Annex 4.

Calculation of a part of δP.

$$\delta P = \rho\left\{(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v')\,\delta\,\frac{1}{\rho'^3} + (3\sin 2v - 2v')\frac{\delta v'}{\rho'^3}\right\}.$$

For $\cos 2v - 2v'$, $\sin 2v - 2v'$, see Annex 3.

$$\begin{array}{llll} \multicolumn{4}{l}{\tfrac{1}{2}+\tfrac{3}{2}\cos 2v-2v'=} \\ & \tfrac{1}{2} & & \\ + & \tfrac{3}{2} & \cos & 2\tau \\ + & 3\,e' & \text{,,} & 2\tau-g' \\ - & 3\,e' & \text{,,} & 2\tau+g' \end{array} \qquad\Bigg|\qquad \begin{array}{llll} \multicolumn{4}{l}{\delta\,\dfrac{1}{\rho'^3}=} \\ & 3\,e' & t & \\ + & 3 & t\cos & g' \\ + & 9\,e' & \text{,,} & 2g'. \end{array}$$

Product is =

$$\begin{array}{l} \tfrac{3}{2}\,e'\quad t \\ +\tfrac{9}{2}\,e'\quad t\cos 2\tau \\ +\tfrac{3}{2}\quad \text{,,}\quad g' \\ +\tfrac{9}{2}\;(\tfrac{1}{2}\,t\cos 2\tau-g' \;+\tfrac{1}{2}\,t\cos 2\tau+g'\;) \\ +\,9\,e'\,(\tfrac{1}{2}\;\text{,,}\;2\tau-2g' \;+\tfrac{1}{2}\;\text{,,}\;2\tau\;) \\ -\,9\,e'\,(\tfrac{1}{2}\;\text{,,}\;2\tau \;+\tfrac{1}{2}\;\text{,,}\;2\tau+2g') \\ +\tfrac{9}{2}\,e'\quad t\cos 2g' \\ +\tfrac{27}{2}\,e'\,(\tfrac{1}{2}\,t\cos 2\tau-2g' \;+\tfrac{1}{2}\,t\cos 2\tau+2g'), \end{array}$$

which is =

$$\begin{array}{rlll} & \tfrac{3}{2}\,e' & t & \\ & +\tfrac{3}{2} & t\cos & g' \\ (\tfrac{9}{2}+\tfrac{9}{2}-\tfrac{9}{2}=) & +\tfrac{9}{2}\,e' & \text{,,} & 2\tau \\ & +\tfrac{9}{4} & \text{,,} & 2\tau-g' \\ & +\tfrac{9}{4} & \text{,,} & 2\tau+g' \\ & +\tfrac{9}{2}\,e' & \text{,,} & 2g' \\ (\tfrac{9}{2}+\tfrac{27}{4}=) & +\tfrac{45}{4}\,e' & \text{,,} & 2\tau-2g' \\ (-\tfrac{9}{2}+\tfrac{27}{4}=) & +\tfrac{9}{4}\,e' & \text{,,} & 2\tau+2g' \end{array}$$

$$\begin{array}{llll} \multicolumn{4}{l}{3\sin 2v-2v'=} \\ & 3 & \sin & 2\tau \\ + & 6\,e' & \text{,,} & 2\tau-g' \\ - & 6\,e' & \text{,,} & 2\tau+g' \end{array} \qquad\Bigg|\qquad \begin{array}{llll} \multicolumn{4}{l}{\dfrac{\delta v'}{\rho'^3}=} \\ & 2 & t\sin & g' \\ + & \tfrac{11}{2}\,e' & \text{,,} & 2g'. \end{array}$$

Product is =

$$\begin{array}{l} \quad 6\quad(\tfrac{1}{2}\,t\cos 2\tau-g' \;-\tfrac{1}{2}\,t\cos 2\tau+g'\;) \\ +\,12\,e'\,(\tfrac{1}{2}\;\text{,,}\;2\tau-2g' \;-\tfrac{1}{2}\;\text{,,}\;2\tau\;) \\ -\,12\,e'\,(\tfrac{1}{2}\;\text{,,}\;2\tau \;-\tfrac{1}{2}\;\text{,,}\;2\tau+2g') \\ +\tfrac{33}{2}\,e'\,(\tfrac{1}{2}\;\text{,,}\;2\tau-2g' \;-\tfrac{1}{2}\;\text{,,}\;2\tau+2g'), \end{array}$$

which is =

$$\begin{array}{rlll}
(-6-6=) & -12\,e' & t\cos & 2\tau \\
 & +3 & \text{,,} & 2\tau-g' \\
 & -3 & \text{,,} & 2\tau+g' \\
(6+\frac{33}{4}=) & +\frac{57}{4}\,e' & \text{,,} & 2\tau-2g' \\
(6-\frac{33}{4}=) & -\frac{9}{4}\,e' & \text{,,} & 2\tau+2g',
\end{array}$$

whence, adding the two products, and observing that $\rho^2=1$, the required terms are

$$\begin{array}{rlll}
= & \frac{3}{2}\,e' & t & \\
 & +\frac{3}{2} & t\cos & g' \\
(\frac{9}{2}-12=) & -\frac{15}{2}\,e' & \text{,,} & 2\tau \\
(\frac{9}{4}+3=) & +\frac{21}{4} & \text{,,} & 2\tau-g' \\
(\frac{9}{4}-3=) & -\frac{3}{4} & \text{,,} & 2\tau+g' \\
 & +\frac{9}{2}\,e' & \text{,,} & 2g' \\
(\frac{45}{4}+\frac{57}{4}=) & +\frac{51}{2}\,e' & \text{,,} & 2\tau-2g' \\
(\frac{9}{4}-\frac{9}{4}=) & 0 & \text{,,} & 2\tau+2g'.
\end{array}$$

Annex 5.

Calculation of a part of δQ, viz.

$$\delta Q=\rho^2\left[(-\tfrac{3}{2}\sin 2v-2v')\,\delta\frac{1}{\rho'^3}+(3\cos 2v-2v')\frac{\delta v'}{\rho'^3}\right],$$

$$\begin{array}{lll|lll}
-\frac{3}{2}\sin 2v-2v'= & & & \delta\dfrac{1}{\rho'^3}= & & \\
-\frac{3}{2} & \sin & 2\tau & 3\,e' & t & \\
-3\,e' & \text{,,} & 2\tau-g' & +3 & t\cos & g' \\
+3\,e' & \text{,,} & 2\tau+g' & +9\,e' & \text{,,} & 2g'.
\end{array}$$

Product is

$$\begin{array}{l}
-\frac{9}{2}\,e'\quad t\sin 2\tau \\
-\frac{9}{2}\quad(\frac{1}{2}\,t\sin\ 2\tau+g' \quad +\frac{1}{2}\,t\sin\ 2\tau-g'\ ) \\
-9\,e'\,(\frac{1}{2}\ \text{,,}\ 2\tau \quad +\frac{1}{2}\ \text{,,}\ 2\tau-2g') \\
+9\,e'\,(\frac{1}{2}\ \text{,,}\ 2\tau+2g' \quad +\frac{1}{2}\ \text{,,}\ 2\tau\ ) \\
-\frac{27}{2}\,e'\,(\frac{1}{2}\ \text{,,}\ 2\tau+2g' \quad +\frac{1}{2}\ \text{,,}\ 2\tau-2g'),
\end{array}$$

which is =

$$\begin{array}{rlll}
(-\tfrac{9}{2}-\tfrac{9}{2}+\tfrac{9}{2}=) & -\tfrac{9}{2}\,e' & t\sin & 2\tau \\
 & -\tfrac{9}{4} & \;\;,, & 2\tau-g' \\
 & -\tfrac{9}{4} & \;\;,, & 2\tau+g' \\
(-\tfrac{9}{2}-\tfrac{27}{4}=) & -\tfrac{45}{4}\,e' & \;\;,, & 2\tau-2g' \\
(+\tfrac{9}{2}-\tfrac{27}{4}=) & -\tfrac{9}{4}\,e' & \;\;,, & 2\tau+2g';
\end{array}$$

and

$$\begin{array}{lll|lll}
\cos 2v-2v'= & & & \dfrac{\delta v'}{\rho'^3}= & & \\
3 & \cos & 2\tau & 2 & t\sin & g' \\
+6\,e' & \;,, & 2\tau-g' & & & \\
-6\,e' & \;,, & 2\tau+g' & +\tfrac{11}{2}\,e' & \;,, & 2g'.
\end{array}$$

Product is

$$\begin{array}{rlll}
6 & (\tfrac{1}{2}\,t\sin\ 2\tau+2g' & -\tfrac{1}{2}\,t\sin\ 2\tau-g' &) \\
+12\,e' & (\tfrac{1}{2}\;\;,,\;\;2\tau & -\tfrac{1}{2}\;\;,,\;\;2\tau-2g' &) \\
-12\,e' & (\tfrac{1}{2}\;\;,,\;\;2\tau+2g' & -\tfrac{1}{2}\;\;,,\;\;2\tau &) \\
+\tfrac{33}{2}\,e' & (\tfrac{1}{2}\;\;,,\;\;2\tau+2g' & -\tfrac{1}{2}\;\;,,\;\;2\tau-2g' &),
\end{array}$$

which is =

$$\begin{array}{rlll}
(6+6=) & 12\,e' & t\sin & 2\tau \\
 & -3 & \;\;,, & 2\tau-g' \\
 & +3 & \;\;,, & 2\tau+g' \\
(-6-\tfrac{33}{4}=) & -\tfrac{57}{4}\,e' & \;\;,, & 2\tau-2g' \\
(-6+\tfrac{33}{4}=) & +\tfrac{9}{4}\,e' & \;\;,, & 2\tau+2g'.
\end{array}$$

Adding the two products together, and observing that $\rho^2=1$, the required terms are

$$\begin{array}{rlll}
(-\tfrac{9}{2}+12=) & \tfrac{15}{2}\,e' & t\sin & 2\tau \\
(-\tfrac{9}{4}-3=) & -\tfrac{21}{4} & \;\;,, & 2\tau-g' \\
(-\tfrac{9}{4}+3=) & +\tfrac{3}{4} & \;\;,, & 2\tau+g' \\
(-\tfrac{45}{4}-\tfrac{57}{4}=) & -\tfrac{51}{2}\,e' & \;\;,, & 2\tau-2g' \\
(-\tfrac{9}{4}+\tfrac{9}{4}=) & 0 & \;\;,, & 2\tau+2g'.
\end{array}$$

Annex 6.

Calculation of terms in δP, viz.

$$\begin{array}{lll|lll}
 & & & \multicolumn{3}{c}{\overbrace{\qquad n^{-1}\times\qquad}} \\
-3 & \sin & 2\tau & -3 & \cos & g' \\
-\tfrac{21}{2}\,e' & \;,, & 2\tau-g' & -\tfrac{9}{4}\,e' & \;,, & 2g'; \\
+\tfrac{3}{2}\,e' & \;,, & 2\tau+g' & & &
\end{array}$$

the product of which is

$$\begin{array}{llllll}
9 & (\tfrac{1}{2} & \sin\, 2\tau + g' & + \tfrac{1}{2} & \sin\, 2\tau - g' &) \\
+ \tfrac{63}{2} e' & (\tfrac{1}{2} & ,,\ \ 2\tau & + \tfrac{1}{2} & ,,\ \ 2\tau - 2g' &) \\
- \tfrac{9}{2} e' & (\tfrac{1}{2} & ,,\ \ 2\tau + 2g' & + \tfrac{1}{2} & ,,\ \ 2\tau &) \\
+ \tfrac{27}{4} e' & (\tfrac{1}{2} & ,,\ \ 2\tau + 2g' & + \tfrac{1}{2} & ,,\ \ 2\tau - 2g' &),
\end{array}$$

which is =

$$\begin{array}{llll}
 & \multicolumn{3}{c}{\overbrace{\hspace{12em}}^{n^{-1} \times}} \\
(\tfrac{63}{4} - \tfrac{9}{4} =) & \tfrac{27}{2} e' & \sin & 2\tau \\
 & + \tfrac{9}{2} & ,, & 2\tau - g' \\
 & + \tfrac{9}{2} & ,, & 2\tau + g' \\
(\tfrac{63}{4} + \tfrac{27}{8} =) & + \tfrac{153}{8} e' & ,, & 2\tau - 2g' \\
(-\tfrac{9}{4} + \tfrac{27}{8} =) & + \tfrac{9}{8} e' & ,, & 2\tau + 2g'.
\end{array}$$

Annex 7.

Calculation of terms in δQ, viz.

$$\begin{array}{lll|lll}
 & & & \multicolumn{3}{c}{\overbrace{\hspace{8em}}^{n^{-1} \times}} \\
-3 & \cos & 2\tau & -3 & \cos & g' \\
-\tfrac{21}{2} e' & ,, & 2\tau - g' & -\tfrac{9}{4} e' & ,, & 2g'; \\
+\tfrac{3}{2} e' & ,, & 2\tau + g' & & &
\end{array}$$

the product of which is

$$\begin{array}{llllll}
9 & (\tfrac{1}{2} & \cos\, 2\tau - g' & + \tfrac{1}{2} & \cos\, 2\tau + g' &) \\
+ \tfrac{63}{2} e' & (\tfrac{1}{2} & ,,\ \ 2\tau - 2g' & + \tfrac{1}{2} & ,,\ \ 2\tau &) \\
- \tfrac{9}{2} e' & (\tfrac{1}{2} & ,,\ \ 2\tau & + \tfrac{1}{2} & ,,\ \ 2\tau + 2g' &) \\
+ \tfrac{27}{4} e' & (\tfrac{1}{2} & ,,\ \ 2\tau - 2g' & + \tfrac{1}{2} & ,,\ \ 2\tau + 2g' &),
\end{array}$$

which is =

$$\begin{array}{llll}
 & \multicolumn{3}{c}{\overbrace{\hspace{12em}}^{n^{-1} \times}} \\
(\ \tfrac{63}{4} - \tfrac{9}{4} =) & \tfrac{27}{2} e' & \cos & 2\tau \\
 & + \tfrac{9}{2} & ,, & 2\tau - g' \\
 & + \tfrac{9}{2} & ,, & 2\tau + g' \\
(\ \tfrac{63}{4} + \tfrac{27}{8} =) & + \tfrac{155}{8} e' & ,, & 2\tau - 2g' \\
(\ \tfrac{9}{4} + \tfrac{27}{8} =) & + \tfrac{9}{8} e' & ,, & 2\tau + 2g'.
\end{array}$$

Annex 8.

Calculation of $n\int \delta Q dt$.

We have

$$n\int \sin n\alpha t\, dt = -\frac{1}{\alpha} t \cos n\alpha t + \frac{1}{n\alpha}\sin n\alpha t,$$

$$\int \cos n\alpha t\, dt = \qquad \frac{1}{n\alpha}\sin n\alpha t,$$

and in all the arguments α is taken $= 2$.

					$n^{-1}\times$	
$n\int \delta Q dt = -\frac{15}{4}e'$	t cos	2τ	$(+\frac{15}{8}+\frac{27}{4}=)$	$+\frac{69}{8}e'$	sin	2τ
$+\frac{21}{8}$	,,	$2\tau-g'$	$(-\frac{21}{16}+\frac{9}{4}=)$	$+\frac{15}{16}$	,,	$2\tau-g'$
$-\frac{3}{8}$	,,	$2\tau+g'$	$(+\frac{3}{16}+\frac{9}{4}=)$	$+\frac{39}{16}$	,,	$2\tau+g'$
$+\frac{51}{4}e'$	,,	$2\tau-2g'$	$(-\frac{51}{8}+\frac{153}{16}=)$	$+\frac{51}{16}e'$	,,	$2\tau-g'$
0	,,	$2\tau+2g'$	$(0+\frac{9}{16}=)$	$+\frac{9}{16}$	,,	$2\tau+2g'$.

Annex 9.

Calculation of $\delta P + 2n\int \delta Q\, dt$; viz. this is

						$n^{-1}\times$	
	$\frac{3}{2}e'$	t					
	$\frac{3}{2}$	t cos	g'				
$(-\frac{15}{2}-\frac{15}{2}=)$	$-15e'$	,,	2τ	$(+\frac{27}{2}+\frac{69}{4}=)$	$+\frac{123}{4}e'$	sin	2τ
$(+\frac{21}{4}+\frac{21}{4}=)$	$+\frac{21}{2}$	,,	$2\tau-g'$	$(+\frac{9}{2}+\frac{15}{8}=)$	$+\frac{51}{8}$	,,	$2\tau-g'$
$(-\frac{3}{4}-\frac{3}{4}=)$	$-\frac{3}{2}$	,,	$2\tau+g'$	$(+\frac{9}{2}+\frac{39}{8}=)$	$+\frac{75}{8}$	,,	$2\tau+g'$
$(+\frac{9}{2}+0=)$	$+\frac{9}{2}e'$	,,	$2g'$	$(0+0=)$	0	,,	$2g'$
$(+\frac{51}{2}+\frac{51}{2}=)$	$+51e'$	,,	$2\tau-2g'$	$(+\frac{153}{8}+\frac{51}{8}=)$	$+\frac{51}{2}e'$	,,	$2\tau-2g'$
$(0+0=)$	0	,,	$2\tau+2g'$	$(+\frac{9}{8}+\frac{9}{8}=)$	$+\frac{9}{4}e'$	,,	$2\tau+2g'$.

Annex 10.

Calculation of $\delta\rho$ from the equation

$$\frac{d^2\delta\rho}{dt^2} + n^2 d\rho = m^2n^2\left(\delta P + 2n\int \delta Q\, dt\right).$$

In $\dfrac{d^2\delta\rho}{dt^2}+n^2\delta\rho$, a term $n^2t\cos n\alpha t$, gives in $\delta\rho$,

$$\frac{1}{1-\alpha^2}\,t\cos n\alpha t+\frac{2\alpha}{(1-\alpha^2)^2}\,\frac{1}{n}\sin n\alpha t\,;$$

and a term $n\sin n\alpha t$, gives in $\delta\rho$,

$$\frac{1}{1-\alpha^2}\,\frac{1}{n}\sin n\alpha t,$$

and $\alpha=2$, and therefore $\dfrac{1}{1-\alpha^2}=-\tfrac{1}{3}$, for all the args. except g', $2g'$; for these, $\alpha=m$ or $2m$, and therefore $\dfrac{1}{1-\alpha^2}=1$.

$\delta\rho=$					$n^{-1}\times$			
$\frac{3}{2}\ m^2e'$	t							
$+\frac{3}{2}\ m^2$	$t\cos$	g'	$(+3\ =)$	$+3$	m^3	$\sin$	g'
$+5\ m^2e'$	,,	2τ	$(-\frac{20}{3}-\frac{41}{4}=)$	$-\frac{203}{12}$	m^2e'	,,	2τ
$-\frac{7}{2}\ m^2$	,,	$2\tau-g'$	$(+\frac{14}{3}-\frac{17}{8}=)$	$+\frac{61}{24}$	m^2	,,	$2\tau-g'$
$+\frac{1}{2}\ m^2$	,,	$2\tau+g'$	$(-\frac{2}{3}-\frac{25}{8}=)$	$-\frac{91}{24}$	m^2	,,	$2\tau+g'$
$+\frac{9}{2}\ m^2e'$	,,	$2g'$	$(+18+0=)$	$+18$	m^3e'	,,	$2g'$
$-17\ m^2e'$	,,	$2\tau-2g'$	$(+\frac{68}{3}+\frac{17}{2}=)$	$+\frac{85}{6}$	m^2e'	,,	$2\tau-2g'$
0	,,	$2\tau+2g'$	$(\ 0-\frac{3}{4}=)$	$-\frac{3}{4}$	m^2e'	,,	$2\tau+2g'$.

Annex 11.

Calculation of $\dfrac{d\delta v}{dt}$; viz. this is

$$=-2n\,\delta\rho+m^2n^2\int\delta Q\,dt.$$

$\dfrac{d\delta v}{dt}=$

	$n\times$										
$(-3\ =)$	-3	$m^2e'\ t$								
$(-3\ =)$	-3	m^2	$t\cos$	g'	$(-6\ =)$	-6	m^3	$\sin$	g'
$(-10-\frac{15}{4}=)$	$-\frac{55}{4}$	m^2e'	,,	2τ	$(+\frac{203}{6}+\frac{69}{8}=)$	$+\frac{1019}{24}$	m^2e'	,,	2τ
$(+7+\frac{21}{8}=)$	$+\frac{77}{8}$	m^2	,,	$2\tau-g'$	$(-\frac{61}{12}+\frac{15}{16}=)$	$-\frac{199}{48}$	m^2	,,	$2\tau-g'$
$(-1-\frac{3}{8}=)$	$-\frac{11}{8}$	m^2	,,	$2\tau+g'$	$(+\frac{91}{12}+\frac{39}{16}=)$	$+\frac{481}{48}$	m^2	,,	$2\tau+g'$
$(-9+0=)$	-9	m^2e'	,,	$2g'$	$(-36+0=)$	-36	m^3e'	,,	$2g'$
$(+34+\frac{51}{4}=)$	$+\frac{187}{4}$	m^2e'	,,	$2\tau-2g'$	$(-\frac{85}{3}+\frac{51}{16}=)$	$-\frac{1207}{48}$	m^2e'	,,	$2\tau-2g'$
$(\ 0+0\ )$	0		,,	$2\tau+2g'$	$(+\frac{3}{2}+\frac{9}{16}=)$	$+\frac{33}{16}$	m^2e'	,,	$2\tau+2g'$.

Annex 12.

Calculation of δv from the foregoing value of $\dfrac{d\delta v}{dt}$.

We have

$$n\int \cos n\alpha t\, dt = \frac{1}{\alpha}\, t \sin n\alpha t + \frac{1}{n\alpha^2} \cos n\alpha t,$$

$$\int \sin n\alpha t\, dt = \qquad - \frac{1}{n\alpha} \cos n\alpha t\,;$$

$\alpha = 2$ for all the arguments, except only $\alpha = m,\ 2m$, for the arguments $g',\ 2g'$, respectively.

$\delta v =$					
				$-\ \frac{3}{2} m^2ne'$	t^2
				$+\ n^{-1}\times$	
$-3\ m$	$t \sin g'$		$\left(-\dfrac{3}{m^2} + 6 =\right)$	$-3 + 6\, m^2$	$\cos g'$
$-\frac{55}{8}\ m^2e'$	„	2τ	$(-\frac{55}{16} - \frac{1019}{48} =)$	$-\frac{74}{3}\ m^2 e'$	„ 2τ
$+\frac{77}{16}\ m^2$	„	$2\tau - g'$	$(+\frac{77}{32} + \frac{199}{96} =)$	$+\frac{215}{48}\ m^2$	„ $2\tau - g'$
$-\frac{11}{16}\ m^2$	„	$2\tau + g'$	$(-\frac{11}{32} - \frac{481}{96} =)$	$-\frac{257}{48}\ m^2$	„ $2\tau + g'$
$-\frac{9}{2}\ me'$	„	$2g'$	$\left(-\dfrac{9}{4m^2} + 18 =\right)$	$(-\frac{9}{4} + 18\, m^2)\, e'$	„ $2g'$
$+\frac{187}{8}\ m^2e'$	„	$2\tau - 2g'$	$(+\frac{187}{16} + \frac{1207}{96} =)$	$+\frac{2329}{96}\ m^2 e'$	„ $2\tau - 2g'$
0	„	$2\tau + 2g'$	$(\ 0 - \frac{33}{32} =)$	$-\frac{33}{32}\ m^2 e'$	„ $2\tau + 2g'$.

The remaining Annexes relate to the determination of the non-periodic or secular terms of the order m^4, in $\delta\rho$ and δv respectively.

Annex 13.

Calculation of term of $\left(n^2 + \dfrac{2n^2}{\rho^3} - 3\left(\dfrac{dv}{dt}\right)^2\right)\delta\rho$.

We have

$n^2 + \dfrac{2n^2}{\rho^3} - 3\left(\dfrac{dv}{dt}\right)^2 =$				$\delta\rho =$		
	$n^2\times$					
$(1 + 2 + m^2 - 3 =)$	m^2			$\frac{3}{2} m^2e'$	t	
$(-9 + 18 =)$	$+\ 9\ m^2e'$	$\cos g'$		$+\frac{3}{2} m^2$	$t \cos g'$	
$(+6 - \frac{33}{2} =)$	$-\frac{21}{2}\ m^2$	„	2τ	$+5\, m^2e'$	„	2τ
$(+21 - \frac{231}{4} =)$	$-\frac{147}{4}\ m^2e'$	„	$2\tau - g'$	$-\frac{7}{2} m^2$	„	$2\tau - g'$
$(-3 + \frac{33}{4} =)$	$+\frac{21}{4}\ m^2e'$	„	$2\tau + g'$	$+\frac{1}{2} m^2$	„	$2\tau + g'$
				$+$ &c.		

where, in the second factor, the arguments not occurring in the first factor are omitted, as not giving rise to any non-periodic term; and so in other similar cases. Hence term of product is

$$\begin{array}{llll} m^4n^2e't & & 1 & .\ \tfrac{3}{2} \\ & +\tfrac{1}{2}\ . & 9 & .\ \tfrac{3}{2} \\ & +\tfrac{1}{2}\ . & -\tfrac{21}{2} & .\ 5 \\ & +\tfrac{1}{2}\ . & -\tfrac{147}{4} & .\ -\tfrac{7}{2} \\ & +\tfrac{1}{2}\ . & \tfrac{21}{4} & .\ \tfrac{1}{2} \end{array}$$

$$= \left(\tfrac{3}{2} + \tfrac{27}{4} - \tfrac{105}{4} + \tfrac{1029}{16} + \tfrac{21}{16} =\right) + \tfrac{381}{8}\, m^4n^2e't.$$

Annex 14.

Calculation of term of $m^2n^2\delta P$.

$m^2n^2\delta P$, the part involving δv is

$m^2n^2 \times$		$\delta v =$		
$-\ 3$	$\sin 2\tau$	$-\ 3\ m$	$t \sin g'$
$-\ \frac{21}{2}\ e'$	„ $2\tau - g'$	$-\ \frac{55}{8}\ m^2e'$	„ 2τ
$+\ \frac{3}{2}\ e'$	„ $2\tau + g'$	$+\ \frac{77}{16}\ m^2$	„ $2\tau - g'$
		$-\ \frac{11}{16}\ m^2$	„ $2\tau + g'$
		+ &c.	

and term of the product is

$$\begin{array}{llll} m^4n^2e't & \tfrac{1}{2}\ . & -\ 3 & .\ -\tfrac{55}{8} \\ & +\tfrac{1}{2}\ . & -\tfrac{21}{2} & .\ \tfrac{77}{16} \\ & +\tfrac{1}{2}\ . & \tfrac{3}{2} & .\ -\tfrac{11}{16} \end{array}$$

$$= \left(\tfrac{165}{16} - \tfrac{1617}{64} - \tfrac{33}{64} =\right) - \tfrac{495}{32}\, m^4n^2e't.$$

Annex 15.

Calculation of term of $m^2n^2\delta P$.

$m^2n^2\delta P$, the term involving $\delta\rho$ is

$m^2n^2 \times$		$\delta\rho =$	
$\frac{1}{2}$		$\frac{3}{2}\ m^2e'$	t
$+\ \frac{3}{2}\ e'$	$\cos g'$	$+\ \frac{3}{2}\ m^2$	$t \cos g'$
$+\ \frac{3}{2}$	„ 2τ	$+\ 5\ m^2e'$	„ 2τ
$+\ \frac{21}{4}\ e'$	„ $2\tau - g'$	$-\ \frac{7}{2}\ m^2$	„ $2\tau - g$
$-\ \frac{3}{4}\ e'$	„ $2\tau + g'$	$+\ \frac{1}{2}\ m^2$	„ $2\tau + g'$
		+ &c.	

and term of the product is

$$
\begin{array}{llll}
m^4n^2e't & \frac{1}{2}\,. & \frac{3}{2} \\
+\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\
+\frac{1}{2}\,. & \frac{3}{2}\,. & 5 \\
+\frac{1}{2}\,. & \frac{21}{4}\,. & -\frac{7}{2} \\
+\frac{1}{2}\,. & -\frac{3}{4}\,. & \frac{1}{2}
\end{array}
$$

$$= \left(\tfrac{3}{4} + \tfrac{9}{8} + \tfrac{15}{4} - \tfrac{147}{16} - \tfrac{3}{16} = \right) - \tfrac{15}{4}\, m^4n^2e't.$$

Annex 16.

Calculation of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\ 2v-2v'$, as far as m^2.

$$\cos 2v - 2v' = \cos 2\tau + X = \cos 2\tau - X\sin 2\tau - \tfrac{1}{2}X^2\cos 2\tau,$$

$$\sin 2v - 2v' = \sin 2\tau + X = \sin 2\tau + X\cos 2\tau\ \ \tfrac{1}{2}X^2\sin 2\tau,$$

where

$$
\begin{array}{llll}
X = & -(4+6m)\,e' & \sin & g' \\
& +\frac{11}{4}\,m^2 & ,, & 2\tau \\
& +\frac{77}{8}\,m^2e' & ,, & 2\tau-g' \\
& -\frac{11}{8}\,m^2e' & ,, & 2\tau+g',
\end{array}
$$

and thence

$$
\begin{array}{lllll}
X\sin 2\tau = & -(4+\ 6\ m)\,e' & (\frac{1}{2}\cos 2\tau - g' & -\frac{1}{2}\cos 2\tau+g') \\
& +\frac{11}{4}\,m^2 & (\frac{1}{2} & -\frac{1}{2}\ ,,\ 4\tau\quad) \\
& +\frac{77}{8}\,m^2e' & (\frac{1}{2}\cos g' & -\frac{1}{2}\ ,,\ 4\tau-g') \\
& -\frac{11}{8}\,m^2e' & (\frac{1}{2}\ ,,\ g' & -\frac{1}{2}\ ,,\ 4\tau+g'),
\end{array}
$$

which is =

$$
\begin{array}{llll}
& \frac{11}{8}\,m^2 & & \\
(\frac{77}{16}-\frac{11}{16}=) & +\frac{33}{8}\,m^2e' & \cos & g' \\
& -(2+\ 3\ m)\,e' & ,, & 2\tau-g' \\
& +(2+\ 3\ m)\,e' & ,, & 2\tau+g' \\
& -\frac{11}{8}\,m^2 & ,, & 4\tau \\
& -\frac{77}{16}\,m^2e' & ,, & 4\tau-g' \\
& +\frac{11}{16}\,m^2e' & ,, & 4\tau+g';
\end{array}
$$

$$
\begin{aligned}
X\cos 2\tau = -(4+6m)e' & \left(\tfrac{1}{2}\sin 2\tau+g' - \tfrac{1}{2}\sin 2\tau-g'\right)\\
+\tfrac{11}{4}m^2 & \left(\tfrac{1}{2} \text{ „ } 4\tau \right)\\
+\tfrac{77}{8}m^2e' & \left(\tfrac{1}{2} \text{ „ } 4\tau-g' - \tfrac{1}{2} \text{ „ } g'\right)\\
-\tfrac{11}{8}m^2e' & \left(\tfrac{1}{2} \text{ „ } 4\tau+g' - \tfrac{1}{2} \text{ „ } g'\right),
\end{aligned}
$$

which is =

$$
\begin{array}{lll}
(-\tfrac{77}{16}-\tfrac{11}{16}=) & \tfrac{11}{2}m^2e' & \sin g'\\
 & +(2+3m)e' & \text{ „ } 2\tau-g'\\
 & -(2+3m)e' & \text{ „ } 2\tau-g'\\
 & +\tfrac{11}{8}m^2 & \text{ „ } 4\tau\\
 & +\tfrac{77}{16}m^2e' & \text{ „ } 4\tau-g'\\
 & -\tfrac{11}{16}m^2e' & \text{ „ } 4\tau+g';
\end{array}
$$

we have, moreover,

$$
\begin{aligned}
X^2 &= -2(4+6m)e'\sin g' \,.\, \tfrac{11}{4}m^2\sin 2\tau\\
&= -22\,m^2e'\left(\tfrac{1}{2}\cos 2\tau-g' - \tfrac{1}{2}\cos 2\tau+g'\right)\\
&= -11\,m^2e' \quad \cos 2\tau-g'\\
&\quad +11\,m^2e' \quad \text{ „ } 2\tau-g',
\end{aligned}
$$

and thence

$$
\begin{aligned}
X^2\cos 2\tau &= -11\,m^2e'\left(\tfrac{1}{2}\cos 4\tau-g'+\tfrac{1}{2}\cos g'\right)\\
&\quad +11\,m^2e'\left(\tfrac{1}{2} \text{ „ } 4\tau+g'+\tfrac{1}{2} \text{ „ } g'\right)\\
&= (-\tfrac{11}{2}+\tfrac{11}{2}=) \quad 0 \quad \cos g'\\
&\quad -\tfrac{11}{2}\,m^2e' \quad \text{ „ } 4\tau-g'\\
&\quad +\tfrac{11}{2}\,m^2e' \quad \text{ „ } 4\tau+g';
\end{aligned}
$$

and

$$
\begin{aligned}
X^2\sin 2\tau &= -11\,m^2e'\left(\tfrac{1}{2}\sin 4\tau-g'+\tfrac{1}{2}\sin g'\right)\\
&\quad -11\,m^2e'\left(\tfrac{1}{2} \text{ „ } 4\tau+g'+\tfrac{1}{2} \text{ „ } g'\right)\\
&= (-\tfrac{11}{2}-\tfrac{11}{2}=) -11\,m^2e' \quad \sin g'\\
&\quad -\tfrac{11}{2}\,m^2e' \quad \text{ „ } 4\tau-g'\\
&\quad +\tfrac{11}{2}\,m^2e' \quad \text{ „ } 4\tau+g';
\end{aligned}
$$

and thence

$$
\begin{array}{llll}
\cos 2v-2v' = & & -\tfrac{11}{8}m^2 & \\
 & & +1 & \cos 2\tau\\
 & & -\tfrac{33}{8}m^2e' & \text{ „ } g'\\
 & & +(2+3m)e' & \text{ „ } 2\tau-g'\\
 & & -(2+3m)e' & \text{ „ } 2\tau+g'\\
 & & +\tfrac{11}{8}m^2 & \text{ „ } 4\tau\\
 & (+\tfrac{77}{16}+\tfrac{11}{4}=) & +\tfrac{121}{16}m^2e' & \text{ „ } 4\tau-g'\\
 & (-\tfrac{11}{16}-\tfrac{11}{4}=) & -\tfrac{55}{16}m^2e' & \text{ „ } 4\tau+g';
\end{array}
$$

and

$$\sin 2v - 2v' = \quad 1 \quad \sin\ 2\tau$$
$$(-\tfrac{11}{2}+\tfrac{11}{2}=) \quad 0\ m^2 e' \quad ,,\ g'$$
$$+(2+\ 3\ m)e' \quad ,,\ 2\tau - g'$$
$$-(2+\ 3\ m)e' \quad ,,\ 2\tau + g'$$
$$+\tfrac{11}{8} m^2 \quad ,,\ 4\tau$$
$$(\ \tfrac{77}{16}+\tfrac{11}{4}=) \quad +\tfrac{121}{16} m^2 e' \quad ,,\ 4\tau - g'$$
$$(-\tfrac{11}{16}-\tfrac{11}{4}=) \quad -\tfrac{55}{16} m^2 e' \quad ,,\ 4\tau + g'.$$

Annex 17.

Calculation of term in δQ.

The part of δQ containing δv is $\frac{\rho^2}{\rho'^3}(-3\cos 2v-2v')\,\delta v$; and it is necessary to find in $\frac{\rho^2}{\rho'^3}(-3\cos 2v-2v')$ the coefficient of $\cos g'$ as far as m^2; this is in fact required, *post* Annex 21, for the calculation of $m^2n^2\left(C+\int\delta Q\,dt\right)$.

$$\rho^2 = 1 - \tfrac{1}{3}\ m^2$$
$$+\ 3\ m^2 e' \quad \cos\ g'$$
$$-\ 2\ m^2 \quad ,,\ 2\tau$$
$$-\ 7\ m^2 e' \quad ,,\ 2\tau - g'$$
$$+\ 1\ m^2 e' \quad ,,\ 2\tau + g'$$

$$\frac{1}{\rho'^3} = \quad 1$$
$$+\ 3\ e' \quad \cos\ g'$$

and thence

$$\frac{\rho^2}{\rho'^3} =$$
$$1 - \tfrac{1}{3}\ m^2$$
$$+(3+\ 2\ m^2)e' \quad \cos\ g'$$
$$-\ 2\ m^2 \quad ,,\ 2\tau$$
$$(-7-\tfrac{1}{2}.2.3=) \quad -\ 10\ m^2 e' \quad ,,\ 2\tau - g'$$
$$(+1-\tfrac{1}{2}.2.3=) \quad -\ 2\ m^2 e' \quad ,,\ 2\tau + g'$$

$$-\ 3\ \cos 2v - 2v' =$$
$$\tfrac{33}{8}\ m^2$$
$$+\tfrac{99}{8}\ m^2 e' \quad \cos\ g'$$
$$-\ 3 \quad ,,\ 2\tau$$
$$-(6+\ 9\ m)e' \quad ,,\ 2\tau - g'$$
$$+(6+\ 9\ m)e' \quad ,,\ 2\tau + g'$$
&c.

where, in the second column, the omitted terms have arguments containing 4τ, and consequently, do not, by combination with the first column, give rise to any term with the argument g'. The term arg. g' arises from the combinations

$$\begin{array}{l}
\tfrac{33}{8}\, m^2 \;.\; 3e' \quad \cos\; g' \\
+\tfrac{99}{8}\, m^2e' \,.\, 1 \quad ,,\; g' \\
-\;3 \quad .-10\, m^2e' \,(\cos 2\tau \cos 2\tau - g' = \tfrac{1}{2}\cos 4\tau - g' + \tfrac{1}{2}\cos g') \\
-\;3 \quad .-\;2\; m^2e' \,(\;,,\; 2\tau \;,,\; 2\tau + g' = \tfrac{1}{2} \;,,\; 4\tau + g' + \tfrac{1}{2} \;,,\; g') \\
-\;6\, e' .-\;2\; m^2 \;(\;,,\; 2\tau \;,,\; 2\tau - g' = \tfrac{1}{2} \;,,\; 4\tau - g' + \tfrac{1}{2} \;,,\; g') \\
+\;6\, e' .-\;2\; m^2 \;(\;,,\; 2\tau \;,,\; 2\tau + g' = \tfrac{1}{2} \;,,\; 4\tau + g' + \tfrac{1}{2} \;,,\; g')
\end{array}$$

so that the required term is

$$(\tfrac{99}{8} + \tfrac{99}{8} + 15 + 3 + 6 - 6 =) + \tfrac{171}{4}\, m^2e' \cos g',$$

and annexing this to the terms found Annex 2, the part of δQ which contains δv is

$$\left.\begin{array}{lll}
\tfrac{171}{4}\, m^2e' & \cos & g' \\
-\;3 & ,, & 2\tau \\
-\;\tfrac{21}{2}\quad e' & ,, & 2\tau - g' \\
+\;\tfrac{3}{2}\quad e' & ,, & 2\tau + g'
\end{array}\right|\, \delta v.$$

Annex 18.

Calculation of term of $m^2n^2\,\delta P$.

The part of δP involving $\delta v'$ and $\delta\rho'$ is

$$\delta P = \rho\left[(\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v')\,\delta\,\frac{1}{\rho'^3} + (3\sin 2v - 2v')\,\frac{\delta v'}{\rho'^3}\right];$$

$$\rho = \left.\begin{array}{l}
\\
1 - \tfrac{1}{6}\, m^2 \\
+\tfrac{3}{2}\, m^2e' \quad \cos\; g' \\
-\quad m^2 \qquad ,,\;\; 2\tau \\
-\tfrac{7}{2}\, m^2e' \quad ,,\;\; 2\tau - g' \\
+\tfrac{1}{2}\, m^2e' \quad ,,\;\; 2\tau + g'
\end{array}\right| \quad
\tfrac{1}{2} + \tfrac{3}{2}\cos 2v - 2v' = \left.\begin{array}{l}
\\
\tfrac{1}{2} - \tfrac{33}{16}\, m^2 \\
\quad -\tfrac{99}{16}\, m^2e' \quad \cos\; g' \\
+\tfrac{3}{2} \qquad\qquad ,,\;\; 2\tau \\
+(3 + \tfrac{9}{2}m)\, e' \quad ,,\;\; 2\tau - g' \\
-(3 + \tfrac{9}{2}m)\, e' \quad ,,\;\; 2\tau + g' \\
+\;\&c.
\end{array}\right| \quad
\delta\,\frac{1}{\rho'^3} = \begin{array}{l}
\\
3\, e' \quad t \\
+\,3 \qquad t\cos 2g' \\
+\,9\, e' \quad ,,\;\; 2g'
\end{array}$$

where, in the second factor, the terms belonging to the arguments which contain 4τ (i. e. the arguments 4τ, $4\tau-g'$, $4\tau+g'$) are omitted. In fact, the terms in question would, in the product of the second and third factors, give rise to terms with arguments containing 4τ, and as there are no such terms in the first factor, there is no resulting secular term.

The product of the second and third factors is

$$\begin{array}{llll}
(\frac{3}{2}-\frac{99}{16}m^2)\,e' & t & & \\
+(\frac{3}{2}-\frac{99}{16}m^2) & t\cos g' & & \\
+\frac{9}{2}\,e' & \quad ,,\;\; 2\tau & & \\
-\frac{297}{16}m^2\,e' & (\frac{1}{2}t & +\frac{1}{2}t\cos 2g' &) \\
+\frac{9}{2} & (\frac{1}{2}t\cos 2\tau-g' & +\frac{1}{2}\;\;,,\;\; 2\tau+g' &) \\
+(9+\frac{27}{2}m)\,e' & (\frac{1}{2}t\;\;,,\;\; 2\tau-2g' & +\frac{1}{2}\;\;,,\;\; 2\tau &) \\
-(9+\frac{27}{2}m)\,e' & (\frac{1}{2}\;\;,,\;\; 2\tau+2g' & +\frac{1}{2}\;\;,,\;\; 2\tau &) \\
+(\frac{9}{2}-\frac{297}{16}m^2)\,e' & t\;\;,,\;\; 2g' & & \\
+\frac{27}{2}\,e' & (\frac{1}{2}t\;\;,,\;\; 2\tau-2g' & +\frac{1}{2}t\;\;,,\;\; 2\tau+2g'), &
\end{array}$$

and we may in this product omit the terms with arguments containing $2g'$, since the first factor does not contain any such term. The product then is

$$\begin{array}{rlll}
(-\frac{99}{16}-\frac{297}{32}=-\frac{495}{32}) & (\frac{3}{2}-\frac{495}{32}m^2)\,e' & & \cdot\, t \\
 & +\,(\frac{3}{2}-\frac{99}{16}m^2) & & t\cos g' \\
((\frac{9}{2}+\frac{9}{2}-\frac{9}{2})\,e'+(\frac{27}{4}-\frac{27}{4})\,me'=) & +\,\frac{9}{2}\,e' & ,, & 2\tau \\
 & +\,\frac{9}{4} & ,, & 2\tau-g' \\
 & +\,\frac{9}{4} & ,, & 2\tau+g',
\end{array}$$

which is to be multiplied by the first factor, ρ, and the whole by the factor m^2n^2.

The term in the product is

$$\begin{array}{lll}
m^2n^2e't & (\frac{3}{2}-\frac{495}{32}m^2)\,(1-\frac{1}{6}m^2) & \\
 & +\,\frac{1}{2}\cdot\frac{3}{2} & \cdot\;\;\frac{3}{2}m^2 \\
 & +\,\frac{1}{2}\cdot\frac{9}{2} & \cdot\,-\;\;m^2 \\
 & +\,\frac{1}{2}\cdot\frac{9}{4} & \cdot\,-\,\frac{7}{2}m^2 \\
 & +\,\frac{1}{2}\cdot\frac{9}{4} & \cdot\;\;\frac{1}{2}m^2,
\end{array}$$

giving the term $\frac{3}{2}m^2n^2e't$, which was found above, Annex 9, and the new terms

$$(-\tfrac{495}{32}-\tfrac{1}{4}+\tfrac{9}{8}-\tfrac{9}{4}-\tfrac{63}{16}+\tfrac{9}{16}=)-\tfrac{647}{32}\,m^4n^2\,e'\,t.$$

The term of the part containing $\delta v'$ is found to be $=0$; in fact we have

$\rho=$			$3\sin 2v-2v'=$			$\dfrac{\delta v'}{\rho'^3}=$		
$1-\frac{1}{6}m^2$			3	sin	2τ	2	$t\sin$	g'
$+\frac{3}{2}m^2e'$	cos	g'	$+(6+9m)e'$	„	$2\tau-g'$	$+\frac{11}{2}e'$	„	$2g'$
$-\ m^2$	„	2τ	$-(6+9m)e'$	„	$2\tau+g'$			
$-\frac{7}{2}m^2e'$	„	$2\tau-g'$	&c.					
$+\frac{1}{2}m^2e'$	„	$2\tau+g'$						

where in the second factor the terms with arguments containing 4τ are for the before-mentioned reason omitted.

The product of the second and third factors is

$$\begin{aligned}
&6\,(\tfrac{1}{2}\,t\cos 2\tau-g' \quad -\tfrac{1}{2}\,t\cos 2\tau+g'\ )\\
+(12+18m)\,e'&(\tfrac{1}{2}\quad ,,\ 2\tau-2g' \quad -\tfrac{1}{2}\quad ,,\ 2\tau\quad)\\
-(12+18m)\,e'&(\tfrac{1}{2}\quad ,,\ 2\tau \quad -\tfrac{1}{2}\quad ,,\ 2\tau+2g')\\
+\tfrac{33}{2}\quad e'&(\tfrac{1}{2}\quad ,,\ 2\tau-2g' \quad -\tfrac{1}{2}\quad ,,\ 2\tau+2g')
\end{aligned}$$

which, omitting the terms with arguments containing $2g'$, is

$$\begin{aligned}
-(6+9m+6+9m)=-(12+18m)\,e'\quad &t\cos 2\tau\\
+\ 3\qquad\qquad &,,\ 2\tau-g'\\
-\ 3\qquad\qquad &,,\ 2\tau+g'
\end{aligned}$$

which is to be multiplied by the first factor, ρ, and the whole by m^2n^2. The term is

$$\begin{aligned}
m^4n^2e'\,t\,.\quad &\tfrac{1}{2}\,.-12\,.-1\\
+&\tfrac{1}{2}\,.\quad 3\ .-\tfrac{7}{2}\\
+&\tfrac{1}{2}\,.-3\ .\quad \tfrac{1}{2}
\end{aligned}$$

which is

$$(6-\tfrac{21}{4}-\tfrac{3}{4}=)\ 0\ \ m^4n^2\,e'\,t.$$

Hence the entire term in question is the before-mentioned value

$$-\tfrac{647}{32}\,m^4n^2\,e'\,t.$$

Annex 19.

Calculation of term in $m^2n^2\dfrac{2}{\rho}\dfrac{dv}{dt}\left(C+\int\delta Q\,dt\right)$.

We have

$$\frac{2}{\rho}\frac{dv}{dt} = \qquad\qquad\qquad\qquad m^2n^2\left(C + \int \delta Q\, dt\right) =$$

$$\begin{array}{lll|lll} n\times & & & n\times & & \\ 2 + \frac{1}{3}\ m^2 & & & -\frac{1455}{32}\ m^4e' & t & \\ -\ 9\ \ m^2e' & \cos & g' & 0 & t\cos & g' \\ +\ \frac{15}{2}\ \ m^2 & ,, & 2\tau & -\ \frac{15}{4}\ \ m^2 & ,, & 2\tau \\ +\ \frac{105}{4}\ \ m^2e' & ,, & 2\tau - g' & +\ \frac{21}{8}\ \ m^2 & ,, & 2\tau - g' \\ -\ \frac{15}{4}\ \ m^2e' & ,, & 2\tau + g' & \frac{3}{8}\ \ m^2 & ,, & 2\tau - g' \end{array}$$

See *post* Annex 20. | For the term $-\frac{1455}{32} m^4e'\, t$ see *post* Annex 21.

The term in the product therefore is

$$\begin{array}{llll} m^4n^2e'\, t\ . & 2 & . & -\frac{1455}{32} \\ & +\frac{1}{2}\ . & \frac{15}{2} & . -\ \frac{15}{4} \\ & +\frac{1}{2}\ . & \frac{105}{4} & . -\ \frac{21}{8} \\ & +\frac{1}{2}\ . -& \frac{15}{4} & .\ \ \ \frac{3}{8} \end{array}$$

which is

$$= \left(2\ .\ -\frac{1455}{32} - \frac{225}{16} + \frac{2205}{64} + \frac{45}{64}\right) m^4n^2e'\, t$$

$$= \left(2\ .\ -\frac{1455}{32} + \frac{675}{32} = \right)\quad -\frac{2235}{32}\, m^4n^2e'\, t.$$

Annex 20.

Calculation of $\frac{2}{\rho}\frac{dv}{dt}$.

We have

$$\frac{dv}{dt} = \qquad\qquad\qquad\qquad \frac{2}{\rho} =$$

$$\begin{array}{lll|lll} n\times & & & & & \\ 1 & & & 2 + \frac{1}{3}\ m^2 & & \\ -\ 3\ \ m^2e' & \cos & g' & -\ 3\ m^2e' & \cos & g' \\ +\ \frac{11}{4}\ \ m^2e' & ,, & 2\tau & +\ 2\ m^2 & ,, & 2\tau \\ +\ \frac{77}{8}\ \ m^2e' & ,, & 2\tau - g' & +\ 7\ m^2e' & ,, & 2\tau - g' \\ -\ \frac{11}{8}\ \ m^2e' & ,, & 2\tau + g' & -\ 1\ m^2e' & ,, & 2\tau + g' \end{array}$$

so that the product is

$$\begin{array}{lllll} & \multicolumn{2}{c}{\overbrace{2+\tfrac{1}{3}\quad m^2}^{n\,\times}} & & \\ (-6-3=) & -9 & m^2e' & \cos & g' \\ (+\tfrac{11}{2}+1=) & +\tfrac{15}{2} & m^2 & ,, & 2\tau \\ (+\tfrac{77}{4}+7=) & +\tfrac{105}{4} & m^2e' & ,, & 2\tau-g' \\ (-\tfrac{11}{4}\quad 1=) & -\tfrac{15}{8} & m^2e' & ,, & 2\tau+g'. \end{array}$$

Annex 21.

Calculation of a term in $m^2n^2\left(C+\int \delta Q\,dt\right)$.

The part $\dfrac{\rho}{\rho'^3}(-3\sin 2v-2v')\,\delta\rho$ of δQ gives

$$\frac{\rho}{\rho'^3}(-3\sin 2v-2v') = \qquad\qquad \delta\rho =$$

$$\begin{array}{llll|lllll} & & & & \multicolumn{5}{c}{\overbrace{3\quad m^3\qquad\quad \sin\ g'}^{n^{-1}\,\times}} \\ -3 & & \sin & 2\tau & -\tfrac{203}{12} & m^2e' & & ,, & 2\tau \\ & \tfrac{21}{2}e' & ,, & 2\tau\quad g & +\tfrac{64}{24} & m^2 & & ,, & 2\tau-g' \\ + & \tfrac{3}{2}e' & ,, & 2\tau+g' & -\tfrac{91}{24} & m^2 & & ,, & 2\tau+g' \end{array}$$

and we have thence in $m^2n^2\,\delta Q$ the term

$$\begin{array}{llll} m^4\,n\,e' & \tfrac{1}{2}\,. & -3 & .-\tfrac{203}{12} \\ & +\tfrac{1}{2}\,. & -\tfrac{21}{2}\,. & \tfrac{61}{24} \\ & +\tfrac{1}{2}\,. & \tfrac{3}{2} & .-\tfrac{91}{24}\,; \end{array}$$

that is

$$(\tfrac{203}{8}-\tfrac{427}{32}-\tfrac{91}{32}=)+\tfrac{147}{16}\,m^4\,n\,e'.$$

The part $\dfrac{\rho^2}{\rho'^3}(3\cos 2v-2v')\,\delta v$ of δQ gives

$$\frac{\rho^2}{\rho'^3}(-3\cos 2v-2v') = \qquad\qquad \delta v =$$

$$\begin{array}{llll|lllll} & & & & \multicolumn{5}{c}{\overbrace{-\ 3\qquad\qquad\quad \cos\ g'}^{n^{-1}}} \\ \tfrac{171}{4} & m^2e' & \cos & g' & -\tfrac{74}{3} & m^2e' & & ,, & 2\tau \\ -\ 3 & & ,, & 2\tau & +\tfrac{215}{48} & m^2 & & ,, & 2\tau-g' \\ -\ \tfrac{21}{2} & e' & ,, & 2\tau-g' & -\tfrac{257}{48} & m^2 & & ,, & 2\tau+g' \\ +\ \tfrac{3}{2} & e' & ,, & 2\tau+g' & & & & & \end{array}$$

See *ante*, Annex 17.

and we have thence in $m^2n^2\,\delta Q$ the term

$$\begin{array}{lllll} m^4n\,e' & \tfrac12 . & \tfrac{171}{4} . & -\ 3 \\ & +\ \tfrac12 . & -\ 3 . & -\ \tfrac{74}{3} \\ & +\ \tfrac12 . & -\ \tfrac{21}{2} . & \tfrac{215}{48} \\ & +\ \tfrac12 . & \tfrac32 . & -\ \tfrac{257}{48} \end{array}$$

that is

$$(-\tfrac{513}{8}+37-\tfrac{1505}{64}-\tfrac{257}{64}=)-\tfrac{1749}{32}\,m^4n\,e';$$

and, combining this with the other term just obtained, the two together are

$$(\tfrac{147}{16}-\tfrac{1749}{32}=)\ \tfrac{1455}{32}\,m^4n\,e';$$

and this term in $m^2n^2\,\delta Q$ gives in $m^2n^2\left(C+\int\delta Q\,dt\right)$ the term

$$-\tfrac{1455}{32}\,m^4n\,e'\,t.$$

Annex 22.

Calculation of term in $-\dfrac{2}{\rho}\dfrac{dv}{dt}\,\delta\rho$.

We have

$-\dfrac{2}{\rho}\dfrac{dv}{dt}$ (see Annex 19) =

$$\begin{array}{llll} & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{n\times} & & \\ -\ 2 & -\ \tfrac13 & m^2 & \\ & +\ 9 & m^2e' & \cos g' \\ & -\ \tfrac{15}{2} & m^2 & \ \ ,,\ \ 2\tau \\ & -\ \tfrac{105}{4} & m^2e' & \ \ ,,\ \ 2\tau-g' \\ & +\ \tfrac{15}{4} & m^2e' & \ \ ,,\ \ 2\tau+g' \end{array}$$

$\rho =$

$$\begin{array}{llll} (\tfrac32\,m^2-\tfrac{1973}{32}\,m^4)\,e' & & t & \\ +\ \tfrac32 & m^2 & t\ \cos g' & \\ +\ 5 & m^2e' & \ \ ,, & 2\tau \\ -\ \tfrac72 & m^2e' & \ \ ,, & 2\tau-g' \\ +\ \tfrac12 & m^2 & \ \ ,, & 2\tau+g' \end{array}$$

giving, besides the term $\tfrac32\,m^2n\,e'\,t$ already taken account of, the term

$$\begin{array}{lllll} m^4n\,e'\,t\ . & +\ \tfrac{1973}{16} . & -\ \tfrac13 . & \tfrac32 \\ & +\ \tfrac12 . & 9 . & \tfrac32 \\ & +\ \tfrac12 . & -\ \tfrac{15}{2} . & 5 \\ & +\ \tfrac12 . & -\ \tfrac{105}{4} . & -\ \tfrac72 \\ & +\ \tfrac12 . & \tfrac{15}{4} . & \tfrac12 \end{array}$$

which is

$$=(\tfrac{1973}{16}-\tfrac12+\tfrac{27}{4}-\tfrac{75}{4}+\tfrac{735}{16}-\tfrac{15}{16})=(\tfrac{1973}{16}+\tfrac{275}{8}=)\ \tfrac{2523}{16}\,m^4n^2e'\,t.$$

Annex 23.

Calculation of term in $\dfrac{m^2n^2}{\rho^2}\left(C+\int \delta Q\,dt\right)$.

We have

$\dfrac{1}{\rho^2}=$		$m^2n^2\left(C+\int \delta Q\,dt\right)=$	
		$n\times$	
$1+\frac{1}{3}\,m^2$		$\overbrace{-\tfrac{1455}{32}\,m^2e'\;t}$	
$-\,3\,m^2e'$	$\cos g'$		
$+\,2\,m^2$	„ 2τ	$-\,\frac{15}{4}\,m^2e'\,t$	$\cos 2\tau$
$+\,7\,m^2e'$	„ $2\tau-g'$	$+\,\frac{21}{8}\,m^2e'$	„ $2\tau-g'$
$-\,1\,m^2e'$	„ $2\tau+g$	$-\,\frac{3}{8}\,m^2e'$	„ $2\tau+g'$

giving the term

$$m^4ne'\,t \quad -\tfrac{1455}{32}$$
$$+\tfrac{1}{2}\,.\;\;2\,.\,-\tfrac{15}{4}$$
$$+\tfrac{1}{2}\,.\;\;\;7\,.\;\;\;\tfrac{21}{8}$$
$$+\tfrac{1}{2}\,.-1\,.-\;\tfrac{3}{8}$$

which is

$$=\left(-\tfrac{1455}{32}-\tfrac{15}{4}+\tfrac{147}{16}\;\tfrac{3}{16}\right)=\left(-\tfrac{1455}{32}+\tfrac{45}{8}=\right)-\tfrac{1275}{32}\,m^4ne'\,t,$$

and this completes the series of calculations.