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ON SOME FORMULÆ RELATING TO THE VARIATION OF THE PLANE OF A PLANET'S ORBIT.

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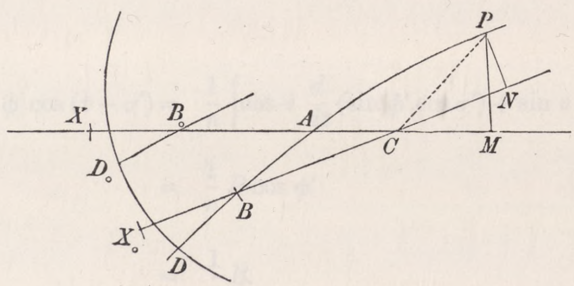
In Hansen's Memoir, "Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten," *Abhand. der K. Sächs. Gesell.* t. V. (1856), are contained, § 8, some very elegant formulæ for taking account of the variation of the plane of the orbit. These, in fact, depend upon the following geometrical theorem, viz., if (in the figure) ABC is a spherical triangle; P, a point on the side AB; and PM, PN, the perpendiculars let fall from P on the other two sides AC, CB; then we have



$$\cos PM \sin (BC + CM) = \cos PN \sin BN - \tan \tfrac{1}{2} C \cos BC (\sin PM + \sin PN),$$
$$\cos PM \cos (BC + CM) = \cos PN \cos BN + \tan \tfrac{1}{2} C \sin BC (\sin PM + \sin PN).$$

These equations, in fact, give

$$\cos PM \sin CM = \cos PN \sin CN - \tan \tfrac{1}{2} C (\sin PM + \sin PN),$$
$$\cos PM \cos CM = \cos PN \cos CN;$$

the latter of which is at once seen to be true, since joining the points C and P, the two sides are respectively equal to $\cos CP$. To verify the former one, write $\angle PCM = C_1$, $\angle PCN = C_2$, so that $C = C_1 - C_2$. Then, since $\cos CP = \cos PM \cos CM = \cos PN \cos CN$, $\sin PM = \sin CP \sin C_1$, $\sin PN = \sin CP \sin C_2$, the equation becomes $\cos CP (\tan CM - \tan CN) = -\tan \tfrac{1}{2} C \sin CP (\sin C_1 + \sin C_2)$, or since $\tan CM = \tan CP \cos C_1$, $\tan CN = \tan CP \cos C_2$, this is

$$\cos C_1 - \cos C_2 = -\tan \tfrac{1}{2} C (\sin C_1 + \sin C_2),$$

which is identically true, in virtue of the equation $C = C_1 - C_2$; and, conversely, we have the original two equations.

Suppose that XM is the ecliptic, X being the origin of longitudes, DP the instantaneous orbit, D the departure-point therein, and P the planet, DD_0 the orthogonal trajectory of the successive positions of the orbit; and writing

þ, the departure of the planet,

v, the longitude of ditto,

y, the latitude of ditto,

θ, the longitude of node,

σ, the departure of ditto,

ϕ, the inclination;

then, in the figure, $DP =$ þ, $XM = v$, $PM = y$, $XA = \theta$, $DA = \sigma$, $\angle A = \phi$.

The quantities θ_0, σ_0, ϕ_0, might be considered as altogether arbitrary; but to fix the ideas it is better to assume at once that they denote

θ_0, the longitude of node,

σ_0, the departure,

ϕ_0, the inclination,

for the initial position of the orbit, viz., in the figure $XB_0 = \theta_0$, $D_0B_0 = \sigma_0$, $\angle B_0 = \phi_0$.

Take $DB = \sigma_0$, $\angle B = \phi_0$, $BX_0 = \theta_0$, this determines a travelling orbit of reference X_0N, and origin of longitudes X_0 therein; such that, with respect to this travelling orbit, the position of the planet's orbit is determined by

θ_0, the longitude of node,

σ_0, the departure of node,

ϕ_0, the inclination.

We have in the triangle ABC, $AB = \sigma - \sigma_0$, $\angle B = \phi_0$, $\angle A = 180^0 - \phi$; and if the other parts of the triangle are represented by

$$BC = \omega,$$

$$AC = \theta_0 - \theta + \omega + \Gamma,$$

$$\angle C = \Phi;$$

then ω, Γ, Φ, are given in terms of $\sigma - \sigma_0$, ϕ_0, ϕ; and we have, moreover, $XC = \theta + AC = \theta_0 + \omega + \Gamma$, $X_0C = \sigma_0 + \omega$; that is, the position of the travelling orbit X_0N, and origin of longitudes X_0 therein, are determined by

$$\begin{aligned} &\theta_0 + \omega + \Gamma, \text{ the longitude of node,} \\ &\sigma_0 + \omega \quad\;\;, \text{ the departure of node,} \\ &\Phi \qquad\quad\;\;, \text{ the inclination.} \end{aligned}$$

Suppose that in reference to this travelling orbit and origin of longitudes therein, we have

$$\begin{aligned} &v', \text{ the longitude of planet,} \\ &y', \text{ the latitude of ditto,} \end{aligned}$$

viz., in the figure $X_0N = v'$ (and therefore $BN = v' - \theta_0$), $PN = y'$.

Moreover, $BC + CM = BC + AM - AC = \omega + (v - \theta) - (\theta_0 - \theta + \omega + \Gamma) = v - \theta_0 - \Gamma$, hence the two equations are

$$\begin{aligned} \cos y \sin (v - \theta_0 - \Gamma) &= \cos y' \sin (v' - \theta_0) - \tan \tfrac{1}{2} \Phi \cos \omega (\sin y + \sin y'), \\ \cos y \cos (v - \theta_0 - \Gamma) &= \cos y' \cos (v' - \theta_0) + \tan \tfrac{1}{2} \Phi \sin \omega (\sin y + \sin y'), \end{aligned}$$

or, as they may also be written,

$$\begin{aligned} \cos y \sin (v - \theta_0 - \Gamma) &= \cos \phi_0 \sin (\text{þ} - \sigma_0) - \tan \tfrac{1}{2} \Phi \cos \omega (\sin y + \sin y'), \\ \cos y \cos (v - \theta_0 - \Gamma) &= \qquad\;\; \cos (\text{þ} - \sigma_0) + \tan \tfrac{1}{2} \Phi \sin \omega (\sin y + \sin y'), \end{aligned}$$

or, if we put $s = \sin y + \sin y'$, then observing that $\sin y = \sin \phi \sin (\text{þ} - \sigma)$, $\sin y' = \sin \phi_0 \sin (\text{þ} - \sigma_0)$, these become

$$\begin{aligned} \cos y \sin (v - \theta_0 - \Gamma) &= \cos \phi_0 \sin (\text{þ} - \sigma_0) - \tan \tfrac{1}{2} \Phi \,.\, s \cos \omega, \\ \cos y \cos (v - \theta_0 - \Gamma) &= \qquad\;\; \cos (\text{þ} - \sigma_0) + \tan \tfrac{1}{2} \Phi \,.\, s \sin \omega, \\ \sin y \,\{= \sin \phi \sin (\text{þ} - \sigma)\} &= - \sin \phi_0 \sin (\text{þ} - \sigma_0) + s, \end{aligned}$$

which are, in fact, Hansen's formulæ (16), p. 75, the letters corresponding as follows, viz.,

$$\begin{aligned} &v,\ \text{þ},\ y,\ \sigma,\ \sigma_0,\ \theta,\ \theta_0,\ \phi,\ \phi_0,\ \Phi,\ \Gamma,\ \omega \text{ (suprà) to} \\ &l,\ v,\ b,\ \sigma,\ h\ ,\ \theta,\ h\ ,\ i\ ,\ -k\ ,\ 2\eta,\ \Gamma,\ \omega \text{ (Hansen).} \end{aligned}$$

where, of course, the correspondence ϕ_0 to $-k$, shows that these angles are measured in a contrary direction. I had from Hansen's equations expected that the above formulæ would have contained $\sin y - \sin y'$ in place of $\sin y + \sin y'$.

2, *Stone Buildings, W. C., 4th Dec*, 1860.