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THE FIRST PART OF A MEMOIR ON THE DEVELOPMENT OF THE DISTURBING FUNCTION IN THE LUNAR AND PLANETARY THEORIES.

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THE development, as is well known, depends upon that of the reciprocal of the distance of the two planets: and Hansen's Memoir "Entwickelung der negativen und ungeraden Potenzen der Quadratwurzel der Function $r^2 + r'^2 - 2rr'(\cos U \cos U' + \sin U \sin U' \cos J)$," *Abh. der K. Sächs. Ges. zu Leipzig,* t. II., pp. 286—376 (1854), contains a formula which is truly fundamental, viz. the expression of the coefficient of the general term

$$\frac{r^n}{r'^{n+1}} \cos (jU + j'U')$$

of the development of the reciprocal of the distance as expressed in the above-mentioned form, where r, r' are the radius vectors of the inferior and superior planets respectively, and U, U' are the angular distances from the mutual node. In the lunar theory, where the higher powers of $\frac{r}{r'}$ are neglected, we have in this manner a small number of terms each of which is to be separately developed in multiple cosines of the mean anomalies. This can be effected as in the *Fundamenta Nova,* and my "Memoir on the Development of the Disturbing Function in the Lunar Theory," *R. Ast. Soc. Mem.,* t. XXVII., 1859, [213], which is a mere completion of Hansen's process(1). In fact if f, f' are the true anomalies, and ϖ, ϖ' the distances

1 I take the opportunity of mentioning the memoir of Hansen's which immediately precedes that above referred to, viz. "Entwickelung des Products einer Potenz des Radius Vectors mit dem Sinus oder Cosinus eines Vielfachen der wahren Anomalie in Reihen die nach den Sinussen oder Cosinussen der Vielfachen der wahren der excentrischen oder mittleren Anomalie fortschreiten," t. II., pp. 183—281 (1853).

of the pericentres from the mutual node, then we have $U = f + \varpi$, $U' = f' + \varpi'$, and the general term is

$$\frac{r^n}{r'^{n+1}} \cos(jf + j'f' + j\varpi + j'\varpi')$$

where r, f are given functions of the mean anomaly g, and r', f' are the like functions of the mean anomaly g'. And the development depends upon those of

$$r^n \frac{\cos}{\sin} jf, \qquad r'^{-n-1} \frac{\cos}{\sin} j'f',$$

which (if we consider as well negative as positive values of the index n) are each of the form

$$r^n \frac{\cos}{\sin} jf,$$

and when the developments of these expressions are known, we obtain at once by the mere addition and subtraction of the coefficients of the cosines and sines of the different multiples of g and g', the development of

$$\frac{r^n}{r'^{n+1}} \cos(jf + j'f' + j\varpi + j'\varpi')$$

in the tabular form employed in my memoir just referred to. In the planetary theory we must unite together the terms containing the different powers of $\frac{r}{r'}$, so as to form the entire coefficient $D(j, j')$ of $\cos(jU + j'U')$; if then we write

$$r = a\,(1 + x),\ r' = a'\,(1 + x'),$$

and develope the coefficient in powers of x, x' we have the general term

$$x^a x'^a \cos(jU + j'U')$$

which admits of development in multiple cosines of the mean anomalies, in the same manner precisely as the before-mentioned general term

$$\frac{r^n}{r'^{n+1}} \cos(jU + j'U')$$

in the lunar theory. It is proper to remark that this method is really identical with that commonly made use of in the planetary theory: the only difference is, that by Hansen's fundamental formula, we have the complete expression of the coefficient $D(j, j')$ developed in powers of sin or of $\tan \frac{1}{2}\phi$, instead of (as in the ordinary methods) the first two or three terms of this development.

The required development of

$$x^{\alpha} x'^{\alpha'} \cos(jU + j'U')$$

or, what is the same thing,

$$x^{\alpha} x'^{\alpha'} \cos(jf + j'f' + j\varpi + j'\varpi')$$

depends on the developments of

$$x^{\alpha} \frac{\cos}{\sin} jf, \quad x'^{\alpha'} \frac{\cos}{\sin} j'f'.$$

These are functions of the same form, and we may consider only

$$x^{\alpha} \frac{\cos}{\sin} jf.$$

The value of x is $\left(\frac{r}{a} - 1\right)$ and we could of course calculate

$$\left(\frac{r}{a} - 1\right)^{\alpha} \frac{\cos}{\sin} jf$$

by the methods of the *Fundamenta Nova* or the memoir of Hansen's referred to in the foot-note. But if we write $f = g + y$ (y is the equation of the centre), then the required expressions depend on

$$x^{\alpha} \frac{\cos}{\sin} jy$$

which are actually calculated as far as e^7 for $\alpha = 0, 1, 2 \ldots 7$, and j an undetermined symbol, by Leverrier in the *Annales de l'Observatoire de Paris*, t. I. pp. 346—348 (1855). Hence, by the mere substitution, in Leverrier's formula, of the numerical values of j and j', and by the addition and subtraction of the coefficients of the cosines or sines of the different multiples of g and g', we may obtain the development of

$$x^{\alpha} x'^{\alpha'} \cos(jf + j'f' + j\varpi + j'\varpi')$$

in a tabular form similar to that employed in my memoir already referred to.

I have thought it desirable to put together the various results above referred to, and to investigate by a different process the expression for Hansen's coefficient, which in his memoir is obtained by means of a long series of transformations which it is not very easy to follow, and is not exhibited in quite the most simple form. And this is what I have done in the present first part of a memoir on the development of the disturbing function. My object has been to exhibit, in as complete a form as possible, the preliminary development in multiple cosines of the true anomalies; and to indicate the process of the ulterior development in multiple cosines of the mean anomalies.

I.

Let 𝔐 be the inferior, 𝔐′ the superior of the two planets (in the lunar theory 𝔐 is the moon, 𝔐′ the sun) and let the quantities relating to the two bodies be, for 𝔐,

r, the radius-vector,
U, the distance from node,
$\mho$, the distance of pericentre from node,
f, the true anomaly,
g, the mean anomaly,
a, the mean distance,
e, the eccentricity,

and for 𝔐′ the accented letters, r', &c., in the like significations.

The node referred to is the ascending node of the orbit of 𝔐 upon that of 𝔐′, and I write also

Φ, the inclination of the orbit of 𝔐 to that of 𝔐′,
$\eta, = \sin \frac{1}{2}\Phi$.

The disturbing function is in the first instance given as a function of r, r', H, where

$$\cos H = \cos U \cos U' + \sin U \sin U' \cos \Phi,$$

that is, as a function of r, r', U, U', Φ, or, what is the same thing, of r, r', U, U', η. And the preliminary development is a development in multiple cosines of U, U'. We have then

$$U = f + \mho,$$
$$U' = f' + \mho',$$

and finally

$$r = a \operatorname{elqr}(e, g),$$
$$f = \operatorname{elta}(e, g),$$
$$r' = a' \operatorname{elqr}(e', g'),$$
$$f' = \operatorname{elta}(e', g'),$$

and the ulterior development is a development in multiple cosines of g, g', $\mho$, $\mho'$, the coefficients involving, as before, η, and also a, e, a', e'. But as usual it is not attempted to carry the development further, by introducing in the coefficients in place of the relative quantities $\mho$, $\mho'$, η, the remaining elements of the two orbits, which, if it were necessary to use them, would be

for 𝔐,

θ, the longitude of node,
σ, the departure of node,
ϕ, the inclination,
ϖ, the departure of pericentre,

and for 𝔐′, the like accented quantities, the orbit of reference being any fixed or moveable orbit whatever.

II.

The expression for the disturbing function on $\mathfrak{M}$ (that is, when the superior planet disturbs the inferior) is

$$\mathfrak{M}'\left\{-\frac{r\cos H}{r'^2}+\frac{1}{\sqrt{r^2+r'^2-2rr'\cos H}}\right\}$$

and that for the disturbing function on $\mathfrak{M}'$ (that is, when the inferior planet disturbs the superior) is

$$\mathfrak{M}\left\{-\frac{r'\cos H}{r^2}+\frac{1}{\sqrt{r^2+r'^2-2rr'\cos H}}\right\}$$

where the disturbing function is taken with Lagrange's sign ($=-R$, if R be the disturbing function of the *Mécanique Céleste*).

But we may in the first instance consider the development of the reciprocal of the distance of the two planets

$$=\frac{1}{\sqrt{r^2+r'^2-2rr'\cos H}}.$$

The preceding expression for $\cos H$ may be written

$$\cos H=\cos U\cos U'+\sin U\sin U'(1-2\eta^2),$$

or in either of the two forms

$$\cos H=\cos(U-U')-2\eta^2\sin U\sin U',$$

$$\cos H=(1-\eta^2)\cos(U-U')+\eta^2\cos(U+U').$$

Now imagine the function developed in ascending powers of $\frac{r}{r'}$, the coefficient of $\frac{r^n}{r'^{n+1}}$ will contain $\cos^n H$, $\cos^{n-2}H$,... to $\cos H$ or 1 according as n is even or odd; and if we then substitute for $\cos H$ the last given expression, and express the different powers of $\cos H$ in multiple cosines of $U-U'$ and $U+U'$, and make the final expression contain the cosines of opposite arguments each with the same coefficient, it is easy to see that the form of the general term is

$$\frac{r^n}{r'^{n+1}}C_n(j,j')\cos(jU+j'U'),$$

where j, j', each of them extend through the values n, $n-2$,... $-n$, and where

$$C_n(-j,-j')=C_n(j,j').$$

Thus in particular for $n=0$, 1, 2, 3, the combinations (j, j') belonging to the several arguments are,

$n=0$	$n=1$		$n=2$			$n=3$			
0, 0	1, 1	−1, 1	2, 2	0, 2	−2, 2	3, 3	1, 3	−1, 3	−3, 3
	1, −1	−1, −1	2, 0	0, 0	−2, 0	3, 1	1, 1	−1, 1	−3, 1
			2, −2	0, −2	−2, −2	3, −1	1, −1	−1, −1	−3, −1
						3, −3	1, −3	−1, −3	−3, −3

But as the coefficients C satisfy the condition $C(-j, -j') = C(j, j')$, the two terms $C(j, j')\cos(jU+j'U')$ and $C(-j, -j')\cos(-jU-j'U')$ are equal to each other, and they may be combined together into a single term. The general term may consequently be written

$$\frac{r^n}{r'^{n+1}}\, 2C_n(j, j')\cos(jU+j'U'),$$

where j has only the values $n, n-2, \ldots 1$ or 0, and j' has as before the values $n, n-2, \ldots -n$; except (which occurs only when n is even) for $j=0$, when j' has only the values $n, n-2, \ldots 0$: and in the particular case $j=j'=0$, the last-mentioned expression for the general term must be multiplied by $\frac{1}{2}$, or, what is the same thing, the factor 2 must be omitted. In particular for $n=0$, 1, 2, 3, 4, the combinations (j, j') belonging to the several arguments are

$n=0$	$n=1$		$n=2$			$n=3$				$n=4$				
0, 0	1, 1		2, 2	0, 2		3, 3	1, 3			4, 4	2, 4	0, 4		
	1, −1		2, 0	0, 0		3, 1	1, 1			4, 2	2, 2	0, 2		
			2, −2			3, −1	1, −1			4, 0	2, 0	0, 0		
						3, −3	1, −3			4, −2	2, −2			
										4, −4	2, −4			

I remark that in a series $K(j, j') \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, jU+j'U'$, where each argument occurs positively and negatively, and $K(-j, -j') = \pm K(j, j')$ according as the series is one of cosines or of sines, we may say that the *discrete* general term is $K(j,\ j') \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (jU+j'U')$, or that $K(j,\ j')$ is the discrete coefficient of the cosine or sine, but if we unite together the terms with opposite arguments so as to form the general term $2K(j,\ j') \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (jU+j'U')$, then this may be called the *concrete* general term, and $2K(j,\ j')$ may be said to be the concrete coefficient of the cosine or sine. In a sine series the term corresponding to the argument zero vanishes; in a cosine series this is not in general the case, and the concrete term corresponding to the argument zero must be multiplied by $\frac{1}{2}$.

Returning to the question in hand, from the symmetry of the expression to be developed, we have

$$C_n(j,\ j') = C_n(j',\ j),$$

and it follows that the only coefficients which need be calculated are those for which j is not negative, and not less in absolute magnitude than j'; the remaining coefficients are respectively equal to coefficients which satisfy these conditions. Thus for $n=0,\ 1,\ 2,\ 3,\ 4$, the combinations $(j,\ j')$ corresponding to the coefficients in question are,

0, 0

,

1, 1	
1, −1	

,

2, 2		
2, 0	0, 0	
2, −2		

,

3, 3			
3, 1	1, 1		
3, −1	1, −1		
3, −3			

,

4, 4				
4, 2	2, 2			
4, 0	2, 0	0, 0		
4, −2	2, −2			
4, −4	2, −4			

;

and we have, for instance, $C(2,\ 4) = C(4,\ 2)$, $C(-2,\ -4) = C(2,\ 4) = C(4,\ 2)$, &c. Under the preceding restriction, viz. j not negative, and not less in absolute magnitude than j', the expression for $C_n(j,\ j')$ (deduced from the formulæ of Hansen's Memoir) is as follows; viz. putting as usual $\Pi x = 1.2.3\ldots x$, and also $\Pi_1(x-\frac{1}{2}) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\ldots(x-\frac{1}{2})$, and representing the hypergeometric series

$$1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha.\alpha+1.\beta.\beta+1}{1.\ \ 2.\ \ \gamma.\gamma+1}x^2 + \&\text{c.}$$

by $F(\alpha,\ \beta,\ \gamma,\ x)$, we have

$$C(j,\ j') = \frac{2^{j+j'}\,\Pi_1(\frac{1}{2}(n+j)-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+j')-\frac{1}{2})}{\Pi\,\frac{1}{2}(n-j)\,\Pi\,\frac{1}{2}(n-j')\,\Pi\,(j+j')}$$

$$\times\,\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2)$$

which, it is to be noticed, is a rational and integral function of η, the highest power being η^{2n}, and the lowest power or order of the coefficient being $\eta^{j+j'}$. I reserve the demonstration of this formula for a separate paragraph.

Let the discrete general term involving $\cos(jU+j'U')$ be represented by

$$D(j, j')\cos(jU+j'U');$$

we have, in like manner as for the coefficients C, $D(-j, -j') = D(j, j')$, $D(j', j) = D(j, j')$, and consequently $D(j, j')$ will be known, if we know its value when the before-mentioned conditions are satisfied; viz., if j be not negative and not less in absolute magnitude than j'; and collecting together the different terms which involve the cosine in question, we find at once, the conditions being satisfied,

$$D(j, j') = \frac{r^j}{r'^{j+1}} C_j(j, j') + \frac{r^{j+2}}{r'^{j+3}} C_{j+2}(j, j') + \&c....$$

so that the value of $D(j, j')$ is known. A transformation of this expression will be given in the sequel.

III.

The *concrete* general term is

$$2D(j, j')\cos(jU+j'U');$$

in particular the concrete terms involving the arguments $U-U'$, $U+U'$, are

$$2D(1, -1)\cos(U-U'),$$
$$2D(1, \quad 1)\cos(U+U'),$$

or, substituting for the D coefficients their values, the terms are

$$2\left\{\frac{r}{r'^2} C_1(1, -1) + \frac{r^3}{r'^4} C_3(1, -1) + \ldots\right\}\cos(U-U'),$$

$$2\left\{\frac{r}{r'^2} C_1(1, \quad 1) + \frac{r^3}{r'^4} C_3(1, \quad 1) + \ldots\right\}\cos(U+U').$$

So far I have considered the reciprocal of the radius vector; but if we consider, instead, the disturbing functions, the only difference will be that we must add the term

$$-\frac{r}{r'^2}\cos H,$$

or

$$-\frac{r}{r^2}\cos H,$$

according as the superior planet disturbs the inferior one, or the inferior planet the superior one. The value of $\cos H$ is

$$= \quad (1-\eta^2)\cos(U-U') + \quad \eta^2 \quad \cos(U+U'),$$

$$= 2\,C_1(1,-1)\cos(U-U') + 2\,C_1(1,\ 1)\cos(U+U'),$$

observing that

$$C_1(1,-1) = \tfrac{1}{2}(1-\eta^2), \qquad C_1(1,\ 1) = \tfrac{1}{2}\eta^2\,;$$

and the terms to be added are consequently, when the superior planet disturbs the inferior,

$$-2\frac{r}{r'^2}C_1(1,-1)\cos(U-U')$$

$$-2\frac{r}{r'^2}C_1(1,\quad 1)\cos(U+U'),$$

the effect of which is simply to destroy the same terms contained with the opposite sign in the reciprocal of the distance;

and when the inferior planet disturbs the superior one, the terms to be added are

$$-2\frac{r'}{r^2}C_1(1,-1)\cos(U-U')$$

$$-2\frac{r'}{r^2}C_1(1,\quad 1)\cos(U+U'),$$

which are not equal to any terms in the reciprocal of the distance. We may write as follows:

Reciprocal of Distance is,

Disturbing function ÷ Mass of Disturbing Planet is,

		U	$+U'$
$D(0, 0)$	cos	0	0
$+2D(1, -1)$		1	-1
[1] $\left[-2\frac{r}{r'^2}C(1, -1)\right]$		1	-1
[2] $\left[-2\frac{r'}{r^2}C(1, -1)\right]$		1	-1
$+2D(1, 1)$		1	1
[1] $\left[-2\frac{r}{r'^2}C(1, 1)\right]$		1	1
[2] $\left[-2\frac{r'}{r^2}C(1, 1)\right]$		1	1
$+2D(2, -2)$		2	-2
$+2D(2, 0)$		2	0
$+2D(0, 2)$		0	2
$+2D(2, 2)$		2	2
$+2D(3, -3)$		3	-3
$+2D(3, -1)$		3	-1
$+2D(1, -3)$		1	-3
$+2D(3, 1)$		3	1
$+2D(1, 3)$		1	3
$+2D(3, 3)$		3	3
$+2D(4, -4)$		4	-4
$+2D(4, -2)$		4	-2
$+2D(2, -4)$		2	-4
$+2D(4, 0)$		4	0
$+2D(0, 4)$		0	4
$+2D(4, 2)$		4	2
$+2D(2, 4)$		2	4
$+2D(4, 4)$		4	4
+ &c., &c.			

[1] Only to be inserted for the disturbing function when the superior planet disturbs the inferior, and having the effect of destroying portions of the coefficients of the next preceding terms.

[2] Only to be inserted for the disturbing function when the inferior planet disturbs the superior one.

For the lunar theory, to the extent to which it is necessary to carry the development, and to which it is carried in Hansen's *Fundamenta Nova*, and my memoir before referred to, we might simply write,

Disturbing function ÷ Sun's Mass is

			U	$+U'$
	$\frac{1}{r'}C_0(0,\ 0)$	cos	0	0
(1)	$+\ \frac{r^2}{r'^3}C_2(0,\ 0)$		0	0
(6)	$+\ 2\frac{r^3}{r'^4}C_3(1,\ -1)$		1	-1
(8)	$+\ 2\frac{r^3}{r'^4}C_3(1,\ 1)$		1	1
(2)	$+\ 2\frac{r^2}{r'^3}C_2(2,\ -2)$		2	-2
(3)	$+\ 2\frac{r^2}{r'^3}C_2(2,\ 0)$		2	0
(4)	$+\ 2\frac{r^2}{r'^3}C_2(0,\ 2)$		0	2
(5)	$+\ 2\frac{r^2}{r'^3}C_2(2,\ 2)$		2	2
(7)	$+\ 2\frac{r^3}{r'^4}C_3(3,\ 3)$		3	-3
(9)	$+\ 2\frac{r^3}{r'^4}C_3(3,\ -1)$		3	-1
(10)	$+\ 2\frac{r^3}{r'^4}C_3(1,\ -3)$		1	-3
	$+\ 2\frac{r^3}{r'^4}C_3(3,\ 1)$		3	1
	$+\ 2\frac{r^3}{r'^4}C_3(1,\ 3)$		1	3

where the prefixed numbers are those of Hansen's ten parts $\Omega_1, \Omega_2 \ldots \Omega_{10}$; or omitting the first term which depends only on the sun's radius vector, and the last two terms which are ultimately neglected, and substituting for the coefficients C their values we have, Disturbing function ÷ Sun's Mass is

				U		+U′
(1)	$\frac{1}{4} - \frac{3}{2}\eta^2 + \frac{3}{2}\eta^4$	$\frac{r^2}{r'^3}$	cos	0		0
(2)	$\frac{3}{4} - \frac{3}{2}\eta^2 + \frac{3}{4}\eta^4$	$\frac{r^2}{r'^3}$		2	−	2
(3)	$\frac{3}{2}\eta^2 - \frac{3}{2}\eta^4$	$\frac{r^2}{r'^3}$		2		0
(4)	$\frac{3}{2}\eta^2 - \frac{3}{2}\eta^4$	$\frac{r^2}{r'^3}$		0		2
(5)	$\frac{3}{4}\eta^4$	$\frac{r^3}{r'^4}$		2		2
(6)	$\frac{3}{8} - \frac{33}{8}\eta^2 + \frac{75}{8}\eta^4$	$\frac{r^3}{r'^4}$		1	−	1
(7)	$\frac{5}{8} - \frac{15}{8}\eta^2 + \frac{15}{8}\eta^4$	$\frac{r^3}{r'^4}$		3	−	3
(8)	$\frac{9}{4}\eta^2 - \frac{15}{2}\eta^4$	$\frac{r^3}{r'^4}$		1		1
(9)	$\frac{15}{8}\eta^2 - \frac{15}{4}\eta^4$	$\frac{r^3}{r'^4}$		3	−	1
(10)	$\frac{15}{8}\eta^2 - \frac{15}{4}\eta^4$	$\frac{r^3}{r'^4}$		1	−	3

If the same form be adopted for the planetary theory, the expressions for the leading coefficients will be,

$C_0(0,\ 0)$, $\qquad D(0,\ 0) = \frac{1}{r'} \quad 1$

2 $\qquad + \frac{r^2}{r'^3} \quad \frac{1}{4}(1 - 6\eta^2 + 6\eta^4)$

4 $\qquad + \frac{r^4}{r'^5} \quad \frac{9}{64}(1 - 20\eta^2 + 90\eta^4 - 140\eta^6 + 70\eta^8)$

6 $\qquad + \frac{r^6}{r'^7} \quad \frac{25}{256}(1 - 42\eta^2 + 420\eta^4 - 1680\eta^6 + 3150\eta^8 \ldots)$

8 $\qquad + \frac{r^8}{r'^9} \quad \frac{1225}{16384}(1 - 72\eta^2 + 1260\eta^4 - 9240\eta^6 - 34650\eta^8 \ldots)$

⋮

2p $\qquad + \frac{r^{2p}}{r'^{2p+1}} \quad \frac{\Pi_1(p-\frac{1}{2})\,\Pi_1(p-\frac{1}{2})}{\Pi p\,\Pi p} F(-2p,\ 2p+1,\ 1,\ \eta^2);$

$C_1(1,\,-1),\qquad D(1,\,-1)=\frac{r}{r'^2}\quad\tfrac{1}{2}(1-\eta^2)$

3 $\qquad +\frac{r^3}{r'^4}\ \tfrac{3}{16}(1-\eta^2)(1-10\,\eta^2+15\,\eta^4)$

5 $\qquad +\frac{r^5}{r'^6}\ \tfrac{15}{128}(1-\eta^2)(1-28\,\eta^2+168\,\eta^4-336\,\eta^6+210\,\eta^8)$

7 $\qquad +\frac{r^7}{r'^8}\ \tfrac{175}{2048}(1-\eta^2)(1-54\,\eta^2+675\,\eta^4-3100\,\eta^6+7425\,\eta^8)$

⋮

$2p+1$ $\qquad +\frac{r^{2p+1}}{r'^{2p+2}}\ \frac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p-\frac{1}{2})}{\Pi(p+1)\,\Pi p}(1-\eta^2)\,F(-2p,\,2p+3,\,1,\,\eta^2);$

$C_1(1,\,1),\qquad D(1,\,1)=\frac{r}{r'^2}\quad\tfrac{1}{2}\,\eta^2$

3 $\qquad +\frac{r^3}{r'^4}\ \tfrac{9}{8}\,\eta^2(1-\tfrac{10}{3}\,\eta^2+\tfrac{5}{3}\,\eta^4)$

5 $\qquad +\frac{r^5}{r'^6}\ \tfrac{225}{128}\,\eta^2(1-\tfrac{28}{3}\,\eta^2+28\,\eta^4-\tfrac{168}{5}\,\eta^6)$

7 $\qquad +\frac{r^7}{r'^8}\ \tfrac{1225}{512}\,\eta^2(1-18\,\eta^2+\tfrac{225}{2}\,\eta^4-330\,\eta^6)$

⋮

$2p+1$ $\qquad +\frac{r^{2p+1}}{r'^{2p+2}}\ 2\,\frac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p+\frac{1}{2})}{\Pi p\,\Pi p}\,\eta^2\,F(-2p,\,2p+3,\,3,\,\eta^2);$

$C_2(2,\,-2),\qquad D(2,\,-2)=\frac{r^2}{r'^3}\quad\tfrac{3}{8}(1-\eta^2)^2$

4 $\qquad +\frac{r^4}{r'^5}\ \tfrac{5}{32}(1-\eta^2)^2(1-14\,\eta^2+28\,\eta^4)$

6 $\qquad +\frac{r^6}{r'^7}\ \tfrac{105}{1024}(1-\eta^2)^2(1-36\,\eta^2+270\,\eta^4-660\,\eta^6+495\,\eta^8)$

⋮

$2p$ $\qquad +\frac{r^{2p}}{r'^{2p+1}}\ \frac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p-\frac{3}{2})}{\Pi(p+1)\,\Pi(p-1)}(1-\eta^2)^2\,F(-2p+2,\,2p+3,\,1,\,\eta^2);$

$C_2(2,\,0),\qquad D(2,\,0)=\frac{r^2}{r'^3}\quad\tfrac{3}{4}\,\eta^2(1-\eta^2)$

4 $\qquad +\frac{r^4}{r'^5}\ \tfrac{45}{32}\,\eta^2(1-\eta^2)(1-\tfrac{14}{3}\,\eta^2+\tfrac{14}{3}\,\eta^4)$

6 $\qquad +\frac{r^6}{r'^7}\ \tfrac{525}{256}\,\eta^2(1-\eta^2)(1-12\,\eta^2+45\,\eta^4-66\,\eta^6\ldots)$

⋮

$2p$ $\qquad +\frac{r^{2p}}{r'^{2p+1}}\ 2\,\frac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p-\frac{1}{2})}{\Pi p\,\Pi(p-1)}\,\eta^2(1-\eta^2)\,F(-2p+2,\,2p+3,\,3,\,\eta^2);$

$C_n(0,\ 2) = C_n(2,\ 0),\qquad D(0,\ 2) = D(2,\ 0);$

$C_2(2,\ 2),\qquad D(2,\ 2) = \frac{r^2}{r'^3}\ \tfrac{3}{8}\eta^4$

4 $\quad + \frac{r^4}{r'^5}\ \tfrac{75}{32}\eta^4(1 - \tfrac{14}{5}\eta^2 + \tfrac{28}{15}\eta^4)$

6 $\quad + \frac{r^5}{r'^6}\ \tfrac{3675}{512}\eta^4(1 - \tfrac{36}{5}\eta^2 + 18\eta^4)$

⋮

$2p \quad + \frac{r^{2p}}{r'^{2p+1}}\ \tfrac{2}{3}\frac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p+\frac{1}{2})}{\Pi(p-1)\,\Pi(p-1)}\eta^4 F(-2p+2,\ 2p+3,\ 5,\ \eta^2);$

⋮

$C_3(3,\ -3),\qquad D(3,\ -3) = \frac{r^3}{r'^4}\ \tfrac{5}{16}(1-\eta^2)^3$

5 $\quad + \frac{r^5}{r'^6}\ \tfrac{35}{256}(1-\eta^2)^3(1 - 18\eta^2 + 45\eta^4)$

⋮

$2p+1 \quad + \frac{r^{2p+1}}{r'^{2p+2}}\ \frac{\Pi_1(p+\frac{3}{2})\,\Pi_1(p-\frac{3}{2})}{\Pi(p+2)\,\Pi(p-1)}(1-\eta^2)^3 F(-2p+2,\ 2p+5,\ 1,\ \eta^2);$

$C_3(3,\ -1),\qquad D(3,\ -1) = \frac{r^3}{r'^4}\ \tfrac{15}{16}\eta^2(1-\eta^2)^2$

5 $\quad + \frac{r^5}{r'^6}\ \tfrac{105}{64}\eta^2(1-\eta^2)^2(1 - 6\eta^2 + \tfrac{15}{2}\eta^4)$

⋮

$2p+1 \quad + \frac{r^{2p+1}}{r'^{2p+2}}\ 2\frac{\Pi_1(p+\frac{1}{2})\,\Pi_1(p-\frac{1}{2})}{\Pi(p+1)\,\Pi(p-1)}\eta^2(1-\eta^2)^2 F(-2p+2,\ 2p+5,\ 3,\ \eta^2);$

$C_n(1,\ -3) = C_n(3,\ -1),\qquad D(1,\ -3) = D(3,\ -1);$

$C_3(3,\ 1),\qquad D(3,\ 1) = \frac{r^3}{r'^4}\ \tfrac{15}{16}\eta^4(1-\eta^2)$

5 $\quad + \frac{r^5}{r'^6}\ \tfrac{525}{128}\eta^4(1-\eta^2)(1 - \tfrac{18}{5}\eta^2 + 3\eta^4)$

⋮

$2p+1 \quad - \frac{r^{2p+1}}{r'^{2p+2}}\ \tfrac{2}{3}\frac{\Pi_1(p+\frac{3}{2})\,\Pi_1(p+\frac{1}{2})}{\Pi p\ \Pi(p-1)}\eta^4(1-\eta^2) F(-2p+2,\ 2p+5,\ 5,\ \eta^2);$

$C_n(1,\ 3) = C_n(3,\ 1),\qquad D(1,\ 3) = D(3,\ 1);$

$C_3(3,\ 3),\qquad D(3,\ 3) = \frac{r^3}{r'^4}\ \tfrac{5}{16}\eta^6$

5 $\quad + \frac{r^5}{r'^6}\ \tfrac{245}{64}\eta^6(1 + \tfrac{18}{7}\eta^2 \ldots)$

⋮

$2p+1 \quad + \frac{r^{2p+1}}{r'^{2p+2}}\ \tfrac{4}{45}\frac{\Pi_1(p+\frac{3}{2})\,\Pi_1(p+\frac{3}{2})}{\Pi(p-1)\,\Pi(p-1)}\eta^6 F(-2p+2,\ 2p+5,\ 7,\ \eta^2);$ &c.

IV.

But for the planetary theory it is more suitable to arrange the expression of $D(j, j')$ according to the powers of η. We have, under the before-mentioned conditions, j not negative and not less in absolute magnitude than j',

$$D(j, j') = \Sigma \frac{r^{j+2\lambda}}{r'^{j+2\lambda+1}} C_{j+2\lambda}(j, j')$$

$$= \frac{r^j}{r'^{j+1}} \Sigma \left(\frac{r}{r'}\right)^{2\lambda} C_{j+2\lambda}(j, j')$$

where λ extends from 0 to ∞; and, writing in the expression of $C_n(j, j')$, $j+2\lambda$ for n, we find

$$C_{j+2\lambda}(j, j') = \frac{2^{j+j'}\,\Pi(j+\lambda-\frac{1}{2})\,\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})}{\Pi\lambda\,\Pi(\frac{1}{2}(j-j')+\lambda)\,\Pi(j+j')} F(-2\lambda,\ 2j+2\lambda+1, j+j'+1, \eta^2),$$

or, substituting for the hypergeometric series its value,

$$C_{j+2\lambda}(j, j') = \Sigma_\theta \frac{2^{j+j'}\,\Pi_1(j+\lambda-\frac{1}{2})\,\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})}{\Pi\lambda\,\Pi(\frac{1}{2}(j-j')+\lambda)\,\Pi(j+j')} \frac{(-)^\theta[2\lambda]^\theta[2j+2\lambda+\theta]^\theta}{[\theta]^\theta[j+j'+\theta]^\theta}\eta^{2\theta}$$

$$= \Sigma_\theta \frac{(-)^\theta\,2^{j+j'}\,\Pi_1(j+\lambda-\frac{1}{2})\,\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})\,\Pi\,2\lambda\,\Pi(2j+2\lambda+\theta)}{\Pi\,\lambda\,\Pi(\frac{1}{2}(j-j')+\lambda)\,\Pi(j+j'+\theta)\,\Pi\,\theta\,\Pi(2\lambda-\theta)\,\Pi(2j+2\lambda)}\eta^{2\theta}$$

from $\theta=0$ to $\theta=\infty$. Substituting in the numerator for $\Pi\,2\lambda$, $2^{2\lambda}\,\Pi\,\lambda\,\Pi_1(\lambda-\frac{1}{2})$, and in the denominator for $\Pi(2j+2\lambda)$, $2^{2j+2\lambda}\,\Pi(j+\lambda)\,\Pi_1(j+\lambda-\frac{1}{2})$, and reducing, this becomes

$$C_{j+2\lambda}(j, j') = \Sigma_\theta \frac{(-)^\theta\,\eta^{2\theta}}{\Pi(j+j'+\theta)\,\Pi\theta}\,\frac{1}{2^{j-j'}}\,\frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})\,\Pi(2j+2\lambda+\theta)\,\Pi_1(\lambda-\frac{1}{2})}{\Pi(\frac{1}{2}(j-j')+\lambda)\,\Pi(j+\lambda)\,\Pi(2\lambda-\theta)},$$

and substituting this expression in $D(j, j')$, we find

$$D(j, j') = \frac{1}{2^{j-j'}}\,\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\Sigma_\theta \frac{(-)^\theta\,\eta^{2\theta}}{\Pi(j+j'+\theta)\,\Pi\theta}$$

$$\Sigma_\lambda\left(\frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})\,\Pi(2j+2\lambda+\theta)\,\Pi_1(\lambda-\frac{1}{2})}{\Pi(\frac{1}{2}(j-j')+\lambda)\,\Pi(j+\lambda)\,\Pi(2\lambda-\theta)}\left(\frac{r}{r'}\right)^{2\lambda}\right)$$

from $\lambda=0$ to $\lambda=\infty$ and $\theta=0$ to $\theta=\infty$; which is the required expression. It is to be remarked that the series in $\left(\frac{r}{r'}\right)$ which multiplies $\eta^{2\theta}$ is not in general expressible as a hypergeometric series. If, however, we attend only to the leading term in η, or write $\theta=0$, we find for $D(j, j')$ the value

$$\frac{1}{2^{j-j'}}\,\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\frac{1}{\Pi(j+j')}$$

$$\Sigma_\lambda\left(\frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})\,\Pi(2j+2\lambda)\,\Pi_1(\lambda-\frac{1}{2})}{\Pi(\frac{1}{2}(j-j')+\lambda)\,\Pi(j+\lambda)\,\Pi\,2\lambda}\left(\frac{r}{r'}\right)^{2\lambda}\right)$$

which may be simplified by putting in the numerator for $\Pi(2j+2\lambda)$,

$$2^{2j+2\lambda}\,\Pi(j+\lambda)\,\Pi_1(j+\lambda-\tfrac{1}{2}),$$

and in the denominator for $\Pi(j+j')$,

$$2^{j+j'}\,\Pi\,\tfrac{1}{2}(j+j')\,\Pi_1(\tfrac{1}{2}(j+j')-\tfrac{1}{2}),$$

and for $\Pi\,2\lambda$, $2^{2\lambda}\,\Pi\lambda\,\Pi_1(\lambda-\tfrac{1}{2})$. We thus obtain the value

$$\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\frac{1}{\Pi\,\frac{1}{2}(j+j')\,\Pi\,(\frac{1}{2}(j+j')-\frac{1}{2})}$$

$$\Sigma_\lambda\,\frac{\Pi_1(\frac{1}{2}(j+j')+\lambda-\frac{1}{2})\,\Pi_1(j+\lambda-\frac{1}{2})}{\Pi(\frac{1}{2}(j-j')+\lambda)\,\Pi\lambda}\left(\frac{r}{r'}\right)^{2\lambda},$$

which is equal to

$$\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\frac{\Pi_1(j-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')\,\Pi\,\frac{1}{2}(j-j')}$$

$$\Sigma_\lambda\,\frac{[\frac{1}{2}(j+j')+\lambda-\frac{1}{2}]^\lambda\,[j+\lambda-\frac{1}{2}]^\lambda}{[\frac{1}{2}(j-j')+\lambda]^\lambda\,[\lambda]^\lambda}\left(\frac{r}{r'}\right)^{2\lambda},$$

or finally we have, as regards the leading term in η,

$$D(j,j')=\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{r^j}{r'^{j+1}}\,\frac{\Pi_1(j-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')\,\Pi\,\frac{1}{2}(j-j')}$$

$$F\left(\tfrac{1}{2}(j+j')+\tfrac{1}{2},\ j+\tfrac{1}{2},\ \tfrac{1}{2}(j-j')+1,\ \frac{r^2}{r'^2}\right).$$

I remark that if in general

$$r^x r'^x\,(r^2+r'^2-2rr'\cos\vartheta)^{-x-\frac{1}{2}}=\Sigma R_x{}^i\cos i\vartheta,\quad R_x{}^{-i}=R_x{}^i$$

then, writing $\frac{1}{2}(j-j')$ for i and $\frac{1}{2}(j+j')$ for x, we have

$$R^{\frac{1}{2}(j-j')}_{\frac{1}{2}(j+j')}=\frac{r^j}{r'^{j+1}}\,\frac{\Pi_1(j-\frac{1}{2})}{\Pi_1(\frac{1}{2}(j+j')-\frac{1}{2})\,\Pi\,\frac{1}{2}(j-j')}\,F\left(\tfrac{1}{2}(j+j')+\tfrac{1}{2},\ j+\tfrac{1}{2},\ \tfrac{1}{2}(j-j')+1,\ \frac{r^2}{r'^2}\right)$$

and the last-mentioned expression for $D(j,j')$, as regards the leading term in η, becomes therefore

$$D(j,j')=\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\,\frac{\Pi_1(\frac{1}{2}(j+j')-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')}\,R^{\frac{1}{2}(j-j')}_{\frac{1}{2}(j+j')},$$

and we have in general

$$D(j,j')=\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\left\{\frac{\Pi_1(\frac{1}{2}(j+j')-\frac{1}{2})}{\Pi\,\frac{1}{2}(j+j')}\,R^{\frac{1}{2}(j-j')}_{\frac{1}{2}(j+j')}+\text{terms in }\eta\right\}.$$

As a verification, it may be noticed that for $j+j'=0$ we have $D(j,-j')=(1-\eta^2)^j$ $(R_0{}^j+\text{terms in }\eta)$, and for $\eta=0$, $D(j,-j)=R_0{}^j$, which is right, since the two sides each denote the coefficient of $\cos j\vartheta$ in $(r^2+r'^2-2rr'\cos\vartheta)^{-\frac{1}{2}}$. There is reason to believe that the expression for $D(j,j')$ might be further reduced so as to obtain in a convenient form the coefficients of the successive powers of η; but I have not yet accomplished this.

V.

Considering now $D(j, j')$ as a given function of r and r', we must in the planetary theory write $r = a(1+x)$, $r' = a'(1+x')$, and develope in powers of x and x'. The general term is, of course,

$$\frac{a^\alpha a'^{\alpha'}}{\Pi\alpha\,\Pi\alpha'} d_a{}^\alpha d_{a'}{}^{\alpha'} \overline{D(j, j')}\, x^\alpha x'^{\alpha'}$$

where $\overline{D(j, j')}$ is what $D(j, j')$ becomes when a, a' are substituted for r, r'; and, writing $f + \varpi$, $f' + \varpi'$, for U, U', we see that in the planetary theory the terms to be developed are of the form

$$x^\alpha x'^{\alpha'} \cos(jf + j'f' + j\varpi + j'\varpi'),$$

while, from what has preceded, the form for the lunar theory is

$$r^n r'^{n'} \cos(jf + j'f' + j\varpi + j'\varpi'),$$

$$\left(n' \text{ is always } = -n-1, \text{ except in the terms } \frac{r'}{r^2}\cos(U \pm U')\right)$$

the values which have to be substituted being

$$\begin{aligned} r &= a \operatorname{elqr}(e, g), & x &= \operatorname{elqr}(e, g) - 1, \\ f &= \operatorname{elta}(e, g), & & \\ r' &= a' \operatorname{elqr}(e', g'), & x' &= \operatorname{elqr}(e', g') - 1, \\ f' &= \operatorname{elta}(e', g'). & & \end{aligned}$$

Suppose that in the former case the developments of

$$x^\alpha \cos jf, \quad x^\alpha \sin jf, \quad x'^{\alpha'} \cos j'f', \quad x'^{\alpha'} \sin j'f'$$

and in the latter case the developments of

$$\left(\frac{r}{a}\right)^n \cos jf, \quad \left(\frac{r}{a}\right)^n \sin jf, \quad \left(\frac{r'}{a'}\right)^{n'} \cos j'f', \quad \left(\frac{r'}{a'}\right)^{n'} \sin j'f'$$

are

$$\Sigma\,[\cos]^i \cos ig, \quad \Sigma\,[\sin]^i \sin ig, \quad \Sigma\,[\cos]^{i'} \cos i'g', \quad \Sigma\,[\sin]^{i'} \sin i'g',$$

where the summations extend from i or $i' = -\infty$ to ∞, and where the coefficients $[\cos]^i$, $[\sin]^i$ satisfy

$$[\cos]^{-i} = [\cos]^i, \quad [\sin]^{-i} = -[\sin]^i,$$

and in like manner the coefficients $[\cos]^{i'}$, $[\sin]^{i'}$ satisfy

$$[\cos]^{-i'} = [\cos]^{i'}, \quad [\sin]^{-i'} = -[\sin]^{i'}.$$

It is to be observed that $[\cos]^i$, $[\sin]^i$ are functions of e, and $[\cos]^{i'}$, $[\sin]^{i'}$

functions of e'; the accents to the indices i and i' are sufficient to indicate this. Hence observing that

$$\Sigma\,[\cos]^i \cos ig \,.\, \Sigma\,[\cos]^{i'} \cos i'g' = \Sigma\Sigma\,[\cos]^i\,[\cos]^{i'} \cos(ig+i'g'),$$

$$\Sigma\,[\cos]^i \cos ig \,.\, \Sigma\,[\sin]^{i'} \sin i'g' = \Sigma\Sigma\,[\cos]^i\,[\sin]^{i'} \sin(ig+i'g'),$$

$$\Sigma\,[\sin]^i \sin ig \,.\, \Sigma\,[\cos]^{i'} \cos i'g' = \Sigma\Sigma\,[\sin]^i\,[\cos]^{i'} \sin(ig+i'g'),$$

$$\Sigma\,[\sin]^i \sin ig \,.\, \Sigma\,[\sin]^{i'} \sin i'g' = -\Sigma\Sigma\,[\sin]^i\,[\sin]^{i'} \cos(ig+i'g'),$$

we have for the products of $x^a x'^{a'}$, or as the case may be $\left(\frac{r}{a}\right)^m \left(\frac{r'}{a'}\right)^n$ into

$$\cos(jf+j'f'), \text{ the values } \Sigma\Sigma\,([\cos]^i\,[\cos]^{i'} + [\sin]^i\,[\sin]^{i'}) \cos(ig+i'g'),$$

$$\sin(jf+j'f'), \quad \text{,,} \quad \Sigma\Sigma\,([\sin]^i\,[\cos]^{i'} + [\cos]^i\,[\sin]^{i'}) \sin(ig+i'g'),$$

and thence observing that

$$\cos(j\varpi + j'\varpi')\,\Sigma\Sigma\, P^{i,i'} \cos(ig+i'g') = \Sigma\Sigma\, P^{i,i'} \cos(ig+i'g'+j\varpi+j'\varpi'),$$

$$-\sin(j\varpi + j'\varpi')\,\Sigma\Sigma\, Q^{i,i'} \sin(ig+i'g') = \Sigma\Sigma\, Q^{i,i'} \cos(ig+i'g'+j\varpi+j'\varpi'),$$

provided only that $P^{-i,-i'} = P^{i,-i'}$, $Q^{-i,-i'} = -Q^{i,i'}$, we find for the product of $x^a x'^{a'}$ or as the case may be $\left(\frac{r}{a}\right)^n \left(\frac{r'}{a'}\right)^{n'}$ into $\cos(jf+j'f'+j\varpi+j'\varpi')$ the expression

$$\Sigma\Sigma\,([\cos]^i\,[\cos]^{i'} + [\sin]^i\,[\sin]^{i'} + [\sin]^i\,[\cos]^{i'} + [\cos]^i\,[\sin]^{i'}) \cos(ig+i'g'+j\varpi+j'\varpi')$$

or, finally, the expression

$$\Sigma\Sigma\,([\cos]^i + [\sin]^i)\,([\cos]^{i'} + [\sin]^{i'}) \cos(ig+i'g'+j\varpi+j'\varpi')$$

which is the required development of the general term in multiple cosines of the mean anomalies.

VI.

Investigation of the coefficient $C_n(j, j')$.

It is possible that there might be some advantage in developing in the first instance according to the powers of $\cos H$, a process which, as it has been seen, leads very readily to the form of the general term; but the mode which I have adopted is to develope in the first instance according to the powers of η. I put therefore

$$\cos H = \cos(U-U') - 2\eta^2 \sin U \sin U',$$

and we have then for the reciprocal of the distance,

$$\{r^2 + r'^2 - 2rr' \cos(U-U') - rr'(-4 \sin U \sin U')\,\eta^2\}^{-\frac{1}{2}}$$

which is to be developed in ascending powers of $\frac{r}{r'}$, and we have for the discrete general term of the development

$$\frac{r^n}{r'^{n+1}} C_n(j, j') \cos(jU + j'U');$$

which is the definition of $C_n(j, j')$, the coefficient the value of which is sought for. It has been seen that j, j', are each of them of the same parity with n (even or odd according as n is even or odd), and it has been seen also that it is sufficient to consider the case where j is not negative and not inferior in absolute magnitude to j'.

Expanding in powers of η, and putting as before

$$\Pi x = 1.2.3 \ldots x,\ \Pi_1(x - \tfrac{1}{2}) = \tfrac{1}{2}.\tfrac{3}{2}.\tfrac{5}{2} \ldots (x - \tfrac{1}{2})$$

the general term is

$$\frac{\Pi_1(x-\frac{1}{2})}{\Pi x} r^x r'^x \{(r^2 + r'^2 - 2rr' \cos(U - U')\}^{-x-\frac{1}{2}} (-4 \sin U \sin U')^x \eta^{2x}$$

where x extends from 0 to ∞.

The factor $(-4 \sin U \sin U')^x$ consists of a series of multiple cosines, and as usual it is assumed that the cosines to opposite arguments are made to occur with equal coefficients. The form of the general term is $\cos(\lambda U + \lambda' U')$, where λ, λ' have each of them the values $x, x-2, x-4, \ldots -x$, that is, λ, λ' are each of them of the same parity with x. Hence j, j' being as before of the same parity with each other, and ϑ being even or odd according as j, j' and x are of the same parity or of opposite parities, the development of $(-4 \sin U \sin U')^x$ will contain a series of terms $\cos[(j+\vartheta) U + (j' - \vartheta) U']$, which (since the other factor contains only multiple cosines of $U - U'$) are the only terms which give rise to a term $\cos(jU + j'U')$. I represent the discrete term of $(-4 \sin U \sin U')^x$ which contains the before-mentioned argument by

$$M_x{}^{\vartheta} \cos[(j+\vartheta) U + (j - \vartheta) U'].$$

On the before-mentioned assumption, j not negative and not less in absolute magnitude than j', we have $j + j'$ and $j - j'$, each of them not negative. We must have $j + \vartheta \not> x$ and $j' - \vartheta \not> x$, that is $\vartheta \not> (x - j)$ and $\not< -(x - j')$, consequently $x - j \not< -(x - j')$ or $2x \not< j + j'$. And this relation being assumed, ϑ extends from the inferior limit $-(x - j')$ to the superior limit $(x - j)$ by steps of two units, the extreme terms being

$$\vartheta = -(x - j'),\quad M_x{}^{\vartheta} \cos[(j + j' - x) U + xU'],$$

$$\vartheta = \quad (x - j),\quad M_x{}^{\vartheta} \cos[xU + (j + j' - x) U'],$$

the coefficient of U increasing from $j + j' - x$ to x, and that of U' diminishing from x to $j + j' - x$ by steps of two units.

Now expanding in ascending powers of $\frac{r}{r'}$, write

$$r^x r'^x \{(r^2 + r'^2 - 2rr' \cos(U - U')\}^{-x-\frac{1}{2}} = \Sigma R_x^i \cos i (U - U')$$

where i extends from $-\infty$ to $+\infty$ and $R_x^{-i} = R_x^i$; so that $R_x^i \cos i (U - U')$ is the discrete general term of the development. The term $R_x^{\vartheta} \cos \vartheta (U - U')$ in combination with the term $M_x^{\vartheta} \cos [(j + \vartheta) U + (j' - \vartheta) U']$ gives rise to $M_x^{\vartheta} R_x^{\vartheta} \cos (jU + j'U)$, and restoring the multiplier which has been omitted, and giving to ϑ and x the different admissible values, we find for the discrete general term containing $\cos (jU + j'U')$ the value

$$\Sigma \frac{\Pi_1 (x - \frac{1}{2})}{\Pi x} \eta^{2x} \Sigma (M_x^{\vartheta} R_x^{\vartheta}) \cos (jU + j'U')$$

where ϑ extends from the inferior limit $\vartheta = -(x - j')$ to the superior limit $\vartheta = x - j$, by steps of two units, and x extends from $x = \frac{1}{2}(j + j')$ to $x = \infty$. The portion of this containing $\frac{r^n}{r'^{n+1}} \cos (jU + j'U')$ is

$$\frac{r^n}{r'^{n+1}} C_n (j, j') \cos (jU + j'U')$$

and we have therefore

$$C_n (j, j') = \text{coeff. } \frac{r^n}{r'^{n+1}} \text{ in } \Sigma \frac{\Pi_1 (x - \frac{1}{2})}{\Pi x} \eta^{2x} \Sigma (M_x^{\vartheta} R_x^{\vartheta}).$$

There is some speciality in the case $j + j' = 0$, but the result just obtained subsists without variation. To find M_x^{ϑ} we have

$$M_x^{\vartheta} = \text{Discrete coeff. } \cos [(j + \vartheta) U + (j' - \vartheta) U'] \text{ in } (-4 \sin U \sin U')^x,$$

and putting $\sin U = \frac{1}{2i} \left(v - \frac{1}{v}\right)$, $\sin U' = \frac{1}{2i} \left(v' - \frac{1}{v'}\right)$ where $i = \sqrt{-1}$ we have

$$(-4 \sin U \sin U')^x = \left(v - \frac{1}{v}\right)^x \left(v' - \frac{1}{v'}\right)^x,$$

and the function on the right hand contains the term

$$(-)^{f+f'} \frac{\Pi x}{\Pi f \Pi (x - f)} \frac{\Pi x}{\Pi f' \Pi (x - f')} v^{x+2f} v'^{x-2f'},$$

or putting

$$x - 2f = j + \vartheta, \quad x - 2f' = j' - \vartheta,$$

and therefore

$$f = \tfrac{1}{2}(x - j - \vartheta), \quad f' = \tfrac{1}{2}(x - j' + \vartheta),$$

which give integer values for f, f', since ϑ is even or odd according as j, j', and x are of the same parity or of opposite parities, and replacing $v^{x-2f} \; v'^{x-2f'}$ by the *half* of its value $2 \cos [(j + \vartheta) U + (j' - \vartheta) U']$, the term is

$$(-)^{x+\frac{1}{2}(j+j')} \frac{\Pi x}{\Pi \frac{1}{2}(x - j - \vartheta) \Pi \frac{1}{2}(x + j + \vartheta)} \frac{\Pi x}{\Pi \frac{1}{2}(x - j' + \vartheta) \Pi \frac{1}{2}(x + j' - \vartheta)} \times \cos [(j + \vartheta) U + (j' - \vartheta) U'],$$

and consequently,

$$M_x^{\vartheta} = (-)^{x-\frac{1}{2}(j+j')} \frac{\Pi x}{\Pi\frac{1}{2}(x-j-\vartheta)\,\Pi\frac{1}{2}(x+j+\vartheta)} \frac{\Pi x}{\Pi\frac{1}{2}(x-j'+\vartheta)\,\Pi\frac{1}{2}(x+j'-\vartheta)},$$

which is the expression for M_x^{ϑ}.

The expression for R_x^i is found in a similar manner, viz., by substituting for $\cos(U-U')$ its exponential expression, by which means the function

$$(r^2 + r'^2 - 2rr'\cos(U-U'))^{-x-\frac{1}{2}}$$

breaks up into a pair of factors, each of which can be separately expanded; the result, i being positive, or zero, is

$$R_x^i = \frac{r^{x+i}}{r'^{x+i+1}} \Sigma \frac{\Pi_1(x+i+m-\frac{1}{2})\,\Pi_1(x+m-\frac{1}{2})}{\Pi(i+m)\,\Pi_1(x-\frac{1}{2})\,\Pi m\,\Pi_1(x-\frac{1}{2})} \left(\frac{r}{r'}\right)^{2m}$$

from $m=0$ to $m=\infty$: this may also be written

$$R_x^i = \frac{r^{x+i}}{r'^{x+i+1}} \frac{\Pi_1(x+i-\frac{1}{2})}{\Pi i\,\Pi_1(x-\frac{1}{2})} F\left(x+i+\tfrac{1}{2},\ x+\tfrac{1}{2},\ i+1,\ \frac{r^2}{r'^2}\right);$$

writing now ϑ for i, and $x+\vartheta+2m=n$, that is $m=\frac{1}{2}(n-x-\vartheta)$ (n is of the same parity with j, j', and ϑ is even or odd according as j, j' and x are of the same parity or of opposite parities, m is therefore, as it should be, an integer), R_x^{ϑ} contains the term

$$\frac{r^n}{r'^{n+1}} \frac{\Pi_1(\frac{1}{2}(n+x+\vartheta)-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+x-\vartheta)-\frac{1}{2})}{\Pi\frac{1}{2}(n-x+\vartheta)\,\Pi_1(x-\frac{1}{2})\,\Pi\frac{1}{2}(n-x-\vartheta)\,\Pi_1(x-\frac{1}{2})},$$

$$= \frac{r^n}{r'^{n+1}} K_x^{\vartheta}$$

if for shortness

$$K_x^{\vartheta} = \frac{\Pi_1(\frac{1}{2}(n+x+\vartheta)-\frac{1}{2})\,\Pi_1(\frac{1}{2}(n+x-\vartheta)-\frac{1}{2})}{\Pi\frac{1}{2}(n-x+\vartheta)\,\Pi_1(x-\frac{1}{2})\,\Pi_1\frac{1}{2}(n-x-\vartheta)\,\Pi_1(x-\frac{1}{2})},$$

a formula which I assume to subsist as well for negative as for positive values of ϑ, so that $K_x^{-\vartheta} = K_x^{\vartheta}$. It is to be noticed that if $x > n$, then either $n-x+\vartheta$ or $n-x-\vartheta$ vanishes, and we have therefore $K_x^{\vartheta} = 0$.

Substituting for R_x^{ϑ} its value we have

$$C_n(j, j') = \Sigma \frac{\Pi_1(x-\frac{1}{2})}{\Pi x} \eta^{2x} \Sigma (M_x^{\vartheta} K_x^{\vartheta})$$

where M_x^{ϑ}, K_x^{ϑ} have the values already obtained, and as before ϑ extends to $\vartheta = -(x-j')$ to $\vartheta = (x-j)$ by steps of two units, and x (since K_x^{ϑ} vanishes for $x < n$) extends from $x = \frac{1}{2}(j+j')$ to $x = n$.

To simplify; write $x = \frac{1}{2}(j+j') + u$; $\vartheta = -(x-j') + 2s, = (x-j') - 2t$, so that $s+t=u$, s and t being any integer values (zero not excluded) which satisfy this relation, and lastly u extends from $u=0$ to $u = n - \frac{1}{2}(j+j')$. We have

$$C_n(j, j') = \eta^{j+j'} \Sigma \eta^{2u} \frac{\Pi_1(\frac{1}{2}(j+j') + u - \frac{1}{2})}{\Pi(\frac{1}{2}(j+j) + u)} \Sigma M_u^s K_u^s,$$

where

$$\overline{M_u^s} = (-)^u \frac{\Pi(\frac{1}{2}(j+j') + u)}{\Pi t \Pi(\frac{1}{2}(j+j') + s)} \frac{\Pi(\frac{1}{2}(j+j') + u)}{\Pi s \Pi(\frac{1}{2}(j+j') + t)},$$

$$\overline{K_u^s} = \frac{\Pi_1(\frac{1}{2}(n+j') + s - \frac{1}{2})}{\Pi(\frac{1}{2}(n-j) - t) \Pi_1(\frac{1}{2}(j+j') + u - \frac{1}{2})} \frac{\Pi_1(\frac{1}{2}(n+j) + t - \frac{1}{2})}{\Pi(\frac{1}{2}(n-j') - s) \Pi_1(\frac{1}{2}(j+j') + u - \frac{1}{2})},$$

and substituting these values, and observing that the result contains a factor $\frac{1}{\Pi(\frac{1}{2}(j+j') + u) \Pi_1(\frac{1}{2}(j+j') + u - \frac{1}{2})}$ which may be replaced by $\frac{2^{j+j'+2u}}{\Pi(j+j'+2u)}$ the result is

$$C_n(j, j') = 2^{j+j'} \eta^{j+j'} \Sigma_u \frac{(-)^u 2^{2u} \eta^{2u}}{\Pi(j+j'+2u)}$$

$$\times \Sigma \left\{ \frac{\Pi(\frac{1}{2}(j+j') + u)}{\Pi(\frac{1}{2}(j+j') + s)} \frac{\Pi(\frac{1}{2}(j+j') + u)}{\Pi(\frac{1}{2}(j+j') + t)} \frac{1}{\Pi s \Pi t} \frac{\Pi_1(\frac{1}{2}(n+j') + s - \frac{1}{2}) \Pi_1(\frac{1}{2}(n+j) + t - \frac{1}{2})}{\Pi(\frac{1}{2}(n-j') - s) \Pi(\frac{1}{2}(n-j) - t)} \right\}$$

which is easily transformed into

$$C_n(j, j') = j^{j+j'} \eta^{j+j'} \frac{\Pi_1(\frac{1}{2}(n+j) - \frac{1}{2}) \Pi_1(\frac{1}{2}(n+j') - \frac{1}{2})}{\Pi\frac{1}{2}(n-j) \Pi\frac{1}{2}(n-j') \Pi(j+j')} S$$

if for shortness $S =$

$$\Sigma_u \frac{(-)^u 2^{2u} \eta^{2u}}{[j+j'+2u]^{2u}} \Sigma \frac{[\frac{1}{2}(j+j')+u]^s [\frac{1}{2}(n+j')+s-\frac{1}{2}]^s [\frac{1}{2}(n-j')]^s [\frac{1}{2}(j+j')+u]^t [\frac{1}{2}(n+j)+t-\frac{1}{2}]^t [\frac{1}{2}(n-j)]}{[s]^s [t]^t}$$

$$s+t=u,\ u=0 \text{ to } u = n - \tfrac{1}{2}(j+j');$$

the value of the sum S is

$$S = (1-\eta^2)^{\frac{1}{2}(j-j')} F(-n+j, n+j+1, j+j'+1, \eta^2),$$

and we have consequently

$$C_n(j, j') = \frac{2^{j+j'} \Pi_1(\frac{1}{2}(n+j) - \frac{1}{2}) \Pi_1(\frac{1}{2}(n+j') - \frac{1}{2})}{\Pi\frac{1}{2}(n-j) \Pi\frac{1}{2}(n-j') \Pi(j+j')}$$

$$\times \eta^{j+j'} (1-\eta^2)^{\frac{1}{2}(j-j')} F(-n+j, n+j+1, j+j'+1, \eta^2),$$

the required expression for $C_n(j, j')$. It only remains to prove the formula for S.

For this purpose, observing the equation $s+t=u$, I form the equations

$$\frac{2^{2u}}{[j+j'+2u]^{2u}}=\frac{1}{[\tfrac{1}{2}(j+j')+u]^u\,[\tfrac{1}{2}(j+j')+u-\tfrac{1}{2}]^u},$$

$$[\tfrac{1}{2}(j+j')+u]^s=\frac{[\tfrac{1}{2}(j+j')+u]^u}{[\tfrac{1}{2}(j+j')+t]^t},$$

$$[\tfrac{1}{2}(j+j')+u]^t=\frac{[\tfrac{1}{2}(j+j')+u]^u}{[\tfrac{1}{2}(j+j')+s]^s},$$

and thence

$$\frac{2^{2u}}{[j+j'+2u]^{2u}}=[\tfrac{1}{2}(j+j')+u]^s\,[\tfrac{1}{2}(j+j')+u]^t$$

$$=\frac{[\tfrac{1}{2}(j+j')+u]^u}{[\tfrac{1}{2}(j+j')+u-\tfrac{1}{2}]^u}\,\frac{1}{[\tfrac{1}{2}(j+j')+s]^s\,[\tfrac{1}{2}(j+j')+t]^t},$$

and we then have (putting also $(-)^u\,\eta^{2u}=(-)^{s+t}\,\eta^{2s+2t}$),

$$S=\Sigma_u\,\frac{[\tfrac{1}{2}(j+j')+u]^u}{[\tfrac{1}{2}(j+j')+u-\tfrac{1}{2}]^u}\,\Sigma\,\frac{(-)^s\,[\tfrac{1}{2}(n-j')]^s\,[\tfrac{1}{2}(n+j')+s-\tfrac{1}{2}]^s}{[s]^s\,[\tfrac{1}{2}(j+j')+s]^s}\,\eta^{2s}$$

$$\times\frac{(-)^t\,[\tfrac{1}{2}(\eta-j)]^t\,[\tfrac{1}{2}(n+j)+t-\tfrac{1}{2}]^t}{[t]^t\,[\tfrac{1}{2}(j+j')+t]^t}\,\eta^{2t},$$

and it is proper to remark that the summation as regards u may be continued indefinitely; for, if $u=s+t$ be $>n-\tfrac{1}{2}(j+j')$, then one at least of the relations $s>\tfrac{1}{2}(n-j)$, $t>\tfrac{1}{2}(n-j)$, must hold good, and at least one of the factorials $[\tfrac{1}{2}(n-j)]^s$, $[\tfrac{1}{2}(n-j)]^t$, will vanish. The two factors in the second sum are the general terms of two hypergeometric series. In fact, if we put

$$\tfrac{1}{2}(n+j)+\tfrac{1}{2}=\beta,$$

$$\tfrac{1}{2}(-n+j)\quad=\alpha,$$

$$\tfrac{1}{2}(j+j')+\tfrac{1}{2}=\epsilon,$$

and therefore

$$\tfrac{1}{2}(n+j')+\tfrac{1}{2}=\epsilon-\alpha,$$

$$\tfrac{1}{2}(-n+j')\quad=\epsilon-\beta,$$

and if, for shortness, we use $\{\alpha\}^s$ to denote the factorial $\alpha(\alpha+1)\ldots(\alpha+s-1)$ of the increment positive unity, then we have

$$S=\Sigma_u\,\frac{\{\epsilon+\tfrac{1}{2}\}^u}{\{\epsilon\}^u}\,\Sigma\,\frac{\{\alpha\}^s\,\{\beta\}^s}{\{1\}^s\,\{\epsilon+\tfrac{1}{2}\}^s}\,\eta^{2s}\,\frac{\{\epsilon-\alpha\}^s\,\{\epsilon-\beta\}^s}{\{1\}^s\,\{\epsilon+\tfrac{1}{2}\}^s}\,\eta^{2t}$$

the two hypergeometric series being consequently

$$F(\alpha,\ \beta,\ \epsilon+\tfrac{1}{2},\ \eta^2)\qquad=1+\frac{\alpha\,.\ \beta}{1\,.\,\epsilon+\tfrac{1}{2}}\,\eta^2\quad+\frac{\alpha\,.\,\alpha+1\,.\ \ \beta\ \ .\,\beta+1}{1\,.\ \ 2\ \ \ .\,\epsilon+\tfrac{1}{2}\,.\,\epsilon\ +\tfrac{3}{2}}\,\eta^4+\&\text{c.},$$

$$F(\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon+\tfrac{1}{2},\ \eta^2)=1+\frac{\epsilon-\alpha\,.\,\epsilon-\beta}{1\ \ .\,\epsilon+\tfrac{1}{2}}\,\eta^2+\frac{\epsilon-\alpha\,.\,\epsilon-\alpha+1\,.\,\epsilon-\beta\,.\,\epsilon-\beta+1}{1\,.\qquad 2\ \ .\,\epsilon+\tfrac{1}{2}\,.\,\epsilon+\tfrac{3}{2}}\,\eta^4+\&\text{c.}$$

I assume the truth of the following remarkable theorem, [see 211] viz.:

"The series formed from the product of the two hypergeometric series,

$$F(\alpha,\ \beta,\ \epsilon+\tfrac{1}{2},\ \eta^2),\ F(\epsilon-\alpha,\ \epsilon-\beta,\ \epsilon+\tfrac{1}{2},\ \eta^2),$$

by multiplying the successive terms of the product by 1, $\frac{\epsilon+\frac{1}{2}}{\epsilon}$, $\frac{\epsilon+\frac{1}{2}\,.\,\epsilon+\frac{3}{2}}{\epsilon\quad.\,\epsilon+1}$, &c. respectively, is

$$=(1-\eta^2)^{\alpha+\beta-\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2)."$$

Hence, observing that the general term of S is formed precisely in the manner in question, we have

$$S=(1-\eta^2)^{\alpha+\beta-\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2)$$

or, substituting for α, β, ϵ their values

$$S=(1-\eta^2)^{\frac{1}{2}(j-j')}\ F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2),$$

which is the required value.

It is clear that α, β are interchangeable with $\epsilon-\alpha$, $\epsilon-\beta$; that is, we have

$$\begin{aligned}S&=(1-\eta^2)^{\alpha+\beta-\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2)\\&=(1-\eta^2)^{\epsilon-\alpha-\beta}\ F(2\epsilon-2\alpha,\ 2\epsilon-2\beta,\ 2\epsilon,\ \eta^2)\end{aligned}$$

or

$$F(2\epsilon-2\alpha,\ 2\epsilon-2\beta,\ 2\epsilon,\ \eta^2)=(1-\eta^2)^{2\alpha+2\beta-2\epsilon}\ F(2\alpha,\ 2\beta,\ 2\epsilon,\ \eta^2),$$

which is a known property of hypergeometric series. The form

$$S=(1-\eta^2)^{-\frac{1}{2}(j-j')}F(-n+j',\ n+j'+1,\ j+j'+1,\ \eta^2)$$

is obviously less convenient than the one above mentioned, since the new expression is encumbered by a denominator $(1-\eta^2)^{\frac{1}{2}(j-j')}$, which really divides out, the finite hypergeometric series containing as a factor the square of such denominator. I have only noticed this for the verification which it affords by showing that j, j' may be interchanged.

VII.

The above expression of $C_n(j,\ j')$ is not, in its actual form, given in Hansen's memoir. The comparison with Hansen's formulæ is as follows:—The formula (36), p. 329, is

$$\begin{aligned}&C\,(n-2f,\ -(n-2f-2g))\\&\qquad=H\cos^{2n}\tfrac{1}{2}\,J\tan^{2g}\tfrac{1}{2}\,J\ F(-(2n-2f-2g),\ -2f,\ 2g+1,\ -\tan^2\tfrac{1}{2}\,J)\end{aligned}$$

where

$$H=\frac{\Pi\,(2n-2f)\,\Pi\,(2f+2g)}{2^{2n}\,\Pi\,(n-f)\,\Pi f\,\Pi\,(n-f-g)\,\Pi\,(f+g)\,\Pi\,2g};$$

and the formula (37), p. 330, is

$$F(-(2n-2f-2g),\ -2f,\ 2g+1,\ -\tan^2\tfrac{1}{2}J)$$
$$= \cos^{-4n+4f+4g}\tfrac{1}{2}J\ F(-(2n-2f-2g),\ 2f+2g+1,\ 2g+1,\ \sin^2\tfrac{1}{2}J).$$

Combining these, and putting $\sin\frac{1}{2}J = \eta$, we have

$$C(n-2f,\ -(n-2f-2g))$$
$$= H\eta^{2g}(1-\eta^2)^{-n+2f+g}\ F(-(2n-2f-2g),\ 2f+2g+1,\ 2g+1,\ \eta^2),$$

and then, putting

$$n-2f \qquad = j',$$
$$-n+2f+2g = j,$$

and for $C(j', j), = C(j, j')$, writing $C_n(j, j')$, we have

$$C_n(j, j') = H\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\ F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2)$$

where

$$H = \frac{\Pi(n+j')\,\Pi(n+j)}{2^{2n}\,\Pi\tfrac{1}{2}(n+j')\,\Pi\tfrac{1}{2}(n-j')\,\Pi\tfrac{1}{2}(n+j)\,\Pi\tfrac{1}{2}(n-j)\,\Pi(j+j')},$$

or, finally, since

$$\Pi(n+j) = 2^{n+j}\,\Pi\tfrac{1}{2}(n+j)\,\Pi_1(\tfrac{1}{2}(n+j)-\tfrac{1}{2}),$$
$$\Pi_1(n+j') = 2^{n+j'}\,\Pi\tfrac{1}{2}(n+j')\,\Pi_1(\tfrac{1}{2}(n+j')-\tfrac{1}{2}),$$

we find

$$C_n(j, j') = \frac{2^{j+j'}\,\Pi_1(\tfrac{1}{2}(n+j)-\tfrac{1}{2})\,\Pi_1(\tfrac{1}{2}(n+j)-\tfrac{1}{2})}{\Pi\tfrac{1}{2}(n-j)\,\Pi\tfrac{1}{2}(n-j')\,\Pi(j+j')}$$
$$\times\,\eta^{j+j'}(1-\eta^2)^{\frac{1}{2}(j-j')}\ F(-n+j,\ n+j+1,\ j+j'+1,\ \eta^2),$$

which is the formula of the present memoir.