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ON A THEOREM RELATING TO HYPERGEOMETRIC SERIES.

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IN attempting to verify a formula of Hansen's relating to the development of the disturbing function in the planetary theory, I was led to a theorem in hypergeometric series: viz. writing, as usual,

$$F(\alpha,\ \beta,\ \gamma,\ x) = 1 + \frac{\alpha\,.\,\beta}{1\,.\,\gamma}\,x + \frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1}{1\,.\,2\,.\,\gamma\,.\,\gamma+1}\,x^2 + \ldots$$

then the product

$$F(\alpha,\ \beta,\ \gamma+\tfrac{1}{2},\ x)\,F(\gamma-\alpha,\ \gamma-\beta,\ \gamma+\tfrac{1}{2},\ x)$$

is connected with

$$(1-x)^{-(\gamma-\alpha-\beta)}\,F(2\alpha,\ 2\beta,\ 2\gamma,\ x)$$

by a simple relation; for if the last-mentioned expression is put equal to

$$1 + Bx + Cx^2 + Dx^3 + \ldots$$

then the product in question is equal to

$$1 + \frac{\gamma}{\gamma+\frac{1}{2}}\,Bx + \frac{\gamma\,.\,\gamma+1}{\gamma+\frac{1}{2}\cdot\gamma+\frac{3}{2}}\,Cx^2 + \frac{\gamma\,.\,\gamma+1\,.\,\gamma+2}{\gamma+\frac{1}{2}\cdot\gamma+\frac{3}{2}\cdot\gamma+\frac{5}{2}}\,Dx^3 + \text{\&c.}$$

The form of the identity thus arrived at will be best perceived by considering a particular case. Thus, comparing the coefficients of x^3, we have

$$\frac{\alpha\,.\,\alpha+1\,.\,\alpha+2\,.\,\beta\,.\,\beta+1\,.\,\beta+2}{1\,.\,2\,.\,3\,.\,\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}\,.\,\gamma+\frac{5}{2}}\,.\,1$$

$$+\frac{\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1}{1\,2\,.\,\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}}\,.\,\frac{\gamma-\alpha\,.\,\gamma-\beta}{1\,.\,\gamma+\frac{1}{2}}$$

$$+\frac{\alpha\,.\,\beta}{1\,.\,2}\,.\,\frac{\gamma-\alpha\,.\,\gamma-\alpha+1\,.\,\gamma-\beta\,.\,\gamma-\beta+1}{1\,.\,2\,.\,\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}}$$

$$+1\,.\,\frac{\gamma-\alpha\,.\,\gamma-\alpha+1\,.\,\gamma-\alpha+2\,.\,\gamma-\beta\,.\,\gamma-\beta+1\,.\,\gamma-\beta+2}{1\,.\,2\,.\,3\,.\,\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}\,.\,\gamma+\frac{5}{2}}$$

$$=\left.\begin{array}{l}\dfrac{2\alpha\,.\,2\alpha+1\,.\,2\alpha+2\,.\,2\beta\,.\,2\beta+1\,.\,2\beta+2}{1\,.\,2\,.\,3\,.\,2\gamma\,.\,2\gamma+1\,.\,2\gamma+2}\,.\,1\\[2ex] +\dfrac{2\alpha\,.\,2\alpha+1\,.\,2\beta\,\,2\beta+1}{1\,.\,2\,.\,2\gamma\,.\,2\gamma+1}\,.\,\dfrac{\gamma-\alpha-\beta}{1}\\[2ex] +\dfrac{2\alpha\,.\,2\beta}{1\,.\,2\gamma}\,\quad.\,\dfrac{\gamma-\alpha-\beta\,.\,\gamma-\alpha-\beta+1}{1\,.\,2}\\[2ex] +1\qquad.\,\dfrac{\gamma-\alpha-\beta\,.\,\gamma-\alpha-\beta+1\,.\,\gamma-\alpha-\beta+2}{1\,.\,2\,.\,3}\end{array}\right\}\frac{\gamma\,.\,\gamma+1\,.\,\gamma+2}{\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}\,.\,\gamma+\frac{5}{2}}.$$

It may be observed that the function on the right-hand side is, as regards α, a rational and integral function of the degree 3, and as such may be expanded in the form

$$\begin{array}{lll} A\,\alpha\,.\,\alpha+1\,.\,\alpha+2 & & \\ +B\,\alpha\,.\,\alpha+1 & .\,\gamma-\alpha & \\ +C\,\alpha\,. & .\,\gamma-\alpha\,.\,\gamma-\alpha+1 & \\ +D & \gamma-\alpha\,.\,\gamma-\alpha+1\,.\,\gamma-\alpha+2, & \end{array}$$

and that the last coefficient D can be obtained at once by writing $\alpha=0$; this in fact gives

$$D\gamma\,.\,\gamma+1\,.\,\gamma+2=\frac{\gamma-\beta\,.\,\gamma-\beta+1\,.\,\gamma-\beta+2}{1\,.\,2\,.\,3}\,\frac{\gamma\,.\,\gamma+1\,.\,\gamma+2}{\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}\,.\,\gamma+\frac{5}{2}},$$

and thence

$$D=\frac{\gamma-\beta\,.\,\gamma-\beta+1\,.\,\gamma-\beta+2}{1\,.\,2\,.\,3\,.\,\gamma+\frac{1}{2}\,.\,\gamma+\frac{3}{2}\,.\,\gamma+\frac{5}{2}},$$

which agrees with the left-hand side of the equation: and the value of the first coefficient A may be obtained in like manner with a little more difficulty; but I have not succeeded in obtaining a direct proof of the equation. The form of the equation shows that the left-hand side should vanish for $\gamma=-2$, which may be at once verified.

Grassmere, August 25, 1858.