210.

ON THE CUBIC TRANSFORMATION OF AN ELLIPTIC FUNCTION.

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LET

$$z = \frac{(a', b', c', d' \between x, 1)^3}{(a, b, c, d \between x, 1)^3}$$

be any cubic fraction whatever of x, then it is always possible to find quartic functions of z, x respectively, such that

$$\frac{dz}{\sqrt{(\mathrm{a, b, c, d, e} \between z, 1)^4}} = \frac{dx}{\sqrt{(A, B, C, D, E \between x, 1)^4}}.$$

This depends upon the following theorem, viz. putting for shortness,

$$U = (a, b, c, d \between x, y)^3,$$
$$U' = (a', b', c', d' \between x, y)^3,$$

and representing by the notation

$$\text{disct.}\ (aU' - a'U,\ \ bU' - b'U,\ \ cU' - c'U,\ \ dU' - d'U);$$

or more shortly by

$$\text{disct.}\ (aU' - a'U, \ldots),$$

the discriminant in regard to the facients (λ, μ) of the cubic function

$$(aU' - a'U,\ \ bU' - b'U,\ \ cU' - c'U,\ \ dU' - d'U \between \lambda, \mu)^3;$$

or what is the same thing, the cubic function

$$(a, b, c, d \between \lambda, \mu)^3 . (a', b', c', d' \between x, y)^3$$
$$-(a', b', c', d' \between \lambda, \mu)^3 . (a, b, c, d \between x, y)^3;$$

and by $J(U, U')$ the functional determinant, or Jacobian, of the two cubics U, U'; the theorem is that the discriminant contains as a factor the square of the Jacobian, or that we have

$$\text{disct.}\ (aU' - a'U, \ldots) = \{J(U, U')\}^2 . (A, B, C, D, E \between x, y)^4.$$

For assuming this to be the case, then (disregarding a mere numerical factor) we have

$$UdU' - U'dU = J(U, U')(ydx - xdy),$$

and the two equations give

$$\frac{UdU' - U'dU}{\sqrt{\text{disct.}(aU' - a'U, \ldots)}} = \frac{ydx - xdy}{\sqrt{(A, B, C, D, E \between x, y)^4}},$$

whence writing z for $U' \div U$, and putting y equal to unity, we have

$$\frac{dz}{\sqrt{\text{disct.}(az - a', \ldots)}} = \frac{dx}{\sqrt{(A, B, C, D, E \between x, 1)^4}};$$

where disct. $(az - a', \ldots)$, or at full length,

$$\text{disct.}\ (az - a',\ bz - b',\ cz - c',\ dz - d'),$$

is a given quartic function of z,

$$= (\mathrm{a, b, c, d, e} \between z, 1)^4$$

suppose; and this proves the theorem of transformation.

The assumed subsidiary theorem may be thus proved: suppose that the parameter θ is determined so that the cubic

$$U + \theta U'$$

may have a square factor, the cubic may be written

$$(a + \theta a',\ b + \theta b',\ c + \theta c',\ d + \theta d' \between x, y)^3,$$

and the requisite condition is

$$\text{disct.}\ (a + \theta a', \ldots) = 0;$$

there are consequently four roots; and calling these $\theta_1, \theta_2, \theta_3, \theta_4$, we have identically

$$\text{disct.}\ (a + \theta a', \ldots) = K(\theta - \theta_1)(\theta - \theta_2)(\theta - \theta_3)(\theta - \theta_4),$$

or what is the same thing,

$$\text{disct.}\ (aU' - a'U, \ldots) = K(U + \theta_1 U')(U + \theta_2 U')(U + \theta_3 U')(U + \theta_4 U').$$

Now any double factor of U or U' (that is the linear factor which enters twice into U or U') is a simple factor of $J(U, U')$, and we have $J(U, U') = J(U, U + \theta U')$, and consequently

$$J(U, U') = J(U, U + \theta_1 U') = \&c.;$$

hence the double factors of each of the expressions $U + \theta_1 U'$, $U + \theta_2 U'$, $U + \theta_3 U'$, $U + \theta_4 U'$ are simple factors of $J(U, U')$, or what is the same thing, $J(U, U')$ is the product of four linear factors, which are respectively double factors of the product

$$(U + \theta_1 U')(U + \theta_2 U')(U + \theta_3 U')(U + \theta_4 U'),$$

or this product contains the factor $\{J(U, U')\}^2$, which proves the theorem.

2, *Stone Buildings, W.C., March* 5, 1858.