199.

NOTE ON THE THEORY OF ELLIPTIC MOTION.

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If, as usual, r, θ denote the radius vector and longitude, and μ the central mass, then the Vis Viva and Force function are respectively

$$T = \tfrac{1}{2}(r'^2 + r^2\theta'^2),$$

$$U = \frac{\mu}{r};$$

and writing

$$\frac{dT}{dr'} = r' = p,$$

$$\frac{dT}{d\theta'} = r^2\theta' = q,$$

we have $r' = p$, $\theta' = \frac{q}{r^2}$, and $T = \frac{1}{2}\left(p^2 + \frac{q^2}{r^2}\right)$, whence, putting $H = T - U$, the value of H is

$$H = \tfrac{1}{2}\left(p^2 + \frac{q^2}{r^2}\right) - \frac{\mu}{r};$$

and by Sir W. R. Hamilton's theory, the equations of motion are

$$\frac{dr}{dt} = \frac{dH}{dp}, \quad \frac{dp}{dt} = -\frac{dH}{dr},$$

$$\frac{d\theta}{dt} = \frac{dH}{dq}, \quad \frac{dq}{dt} = -\frac{dH}{d\theta}:$$

or substituting for H its value, the equations of motion are

$$\frac{dr}{dt}=p,$$

$$\frac{d\theta}{dt}=\frac{q}{r^2},$$

$$\frac{dp}{dt}=\frac{q^2}{r^3}+\frac{\mu}{r^2},$$

$$\frac{dq}{dt}=0.$$

Putting, as usual, $\mu=n^2a^3$, and introducing the eccentric anomaly u, which is given as a function of t by means of the equation

$$nt+c=u-e\sin u,$$

$\left(\text{so that } \dfrac{du}{dt}=\dfrac{n}{1-e\cos u}\right)$, the integral equations are

$$q=na^2\sqrt{1-e^2},$$

$$p=\frac{nae\sin u}{1-e\cos u},$$

$$r=a\,(1-e\cos u),$$

$$\theta-\varpi=\tan^{-1}\left(\frac{\sqrt{1-e^2}\sin u}{\cos u-e}\right);$$

where the constants of integration a, e, c, ϖ denote as usual the mean distance, the eccentricity, the mean anomaly at epoch, and the longitude of pericentre.

Suppose that q_0, p_0, r_0, θ_0, u_0 correspond to the time t_0 (q is constant, so that $q_0=q$), and write

$$V=na^2\,(u-u_0+e\sin u-e\sin u_0);$$

joining to this the equations

$$r=a\,(1-e\cos u),\quad r_0=a\,(1-e\cos u_0),$$

$$\theta-\theta_0=\tan^{-1}\left(\frac{\sqrt{1-e^2}\sin u}{\cos u-e}\right)-\tan^{-1}\left(\frac{\sqrt{1-e^2}\sin u_0}{\cos u-e}\right),$$

u, u_0, e will be functions of a, r, r_0, θ, θ_0, and consequently (n being throughout considered as a function of a) V will be a function of a, r, r_0, θ, θ_0. The function V so expressed as a function of a, r, r_0, θ, θ_0 is, in fact, the characteristic function of Sir W. R. Hamilton, and according to his theory we ought to have

$$dV=\tfrac{1}{2}n^2a\,(t-t_0)\,da+pdr+qd\theta-p_0dr_0-q_0d\theta_0.$$

To verify this, I form the equation

$$\begin{aligned} dV = & \tfrac{1}{2}na\,(u - u_0 + e\sin u - e\sin u_0)\,da \\ & + na^2\,[(1+e\cos u)\,du - (1+e\cos u_0)\,du_0] \\ & + na^2\,(\sin u - \sin u_0)\,de \\ & + \frac{nae\sin u}{1-e\cos u}\{dr - (1-e\cos u)\,da - ae\sin u\,du + a\cos u\,de\} \\ & - \frac{nae\sin u_0}{1-e\cos u_0}\{dr_0 - (1-e\cos u_0)\,da - ae\sin u_0\,du_0 + a\cos u_0\,de\} \\ & + na^2\sqrt{1-e^2}\left\{ d\theta - \frac{1}{\sqrt{1-e^2}\,(1-e\cos u)}[(1-e^2)\,du + \sin u\,de] \right. \\ & \qquad \left. - d\theta_0 + \frac{1}{\sqrt{1-e^2}\,(1-e\cos u_0)}[(1-e^2)\,du_0 + \sin u_0\,de]\right\}; \end{aligned}$$

the coefficient of du on the right-hand side is

$$na^2(1+e\cos u) - \frac{na^2e^2\sin^2 u}{1-e\cos u} - \frac{na^2(1-e^2)}{1-e\cos u}$$

$$= na^2\left(1 + e\cos u - \frac{1-e^2+e^2\sin^2 u}{1-e\cos u}\right),$$

which vanishes, and similarly the coefficient of du_0 also vanishes: the coefficient of de is the difference of two parts, the first of which is

$$na^2\sin u + \frac{na^2e\sin u\cos u}{1-e\cos u} - \frac{na^2\sin u}{1-e\cos u}$$

$$= na^2\sin u\left(1 - \frac{1-e\cos u}{1-e\cos u}\right),$$

which vanishes, and the second part in like manner also vanishes; the coefficient of da is the difference of two parts, the first of which is

$$\tfrac{1}{2}na\,(u + e\sin u) - nae\sin u = \tfrac{1}{2}na\,(u - e\sin u),$$

and the second is the like function of u_0; the entire coefficient therefore is

$$\tfrac{1}{2}\,na\,(u - u_0 - e\sin u + e\sin u_0).$$

We have therefore

$$\begin{aligned} dV = & \tfrac{1}{2}na\,(u - u_0 - e\sin u + e\sin u_0)\,da \\ & + \frac{nae\sin u}{1-e\cos u}\,dr + na^2\sqrt{1-e^2}\,d\theta \\ & - \frac{nae\sin u_0}{1-e\cos u_0}\,dr_0 - na^2\sqrt{1-e^2}\,d\theta_0\,; \end{aligned}$$

or what is the same thing,

$$dV = \tfrac{1}{2}n^2a\,(t - t_0)\,da + p\,dr + q\,d\theta - p_0\,dr_0 - q_0\,d\theta_0,$$

the equation which was to be verified.

2, *Stone Buildings, March* 28, 1856.