191.

NOTE ON THE EXPANSION OF THE TRUE ANOMALY.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 229—232.]

If the true anomaly and the mean anomaly are respectively denoted by u, m, and if e be the eccentricity, then as usual $u - e \sin u = m$; and if we write

$$\lambda = \frac{1 - \surd(1 - e^2)}{e}$$

and take c to denote the base of the hyperbolic system of logarithms, we have

$$u = m + 2\Sigma_1^\infty A_r \frac{\sin rm}{r},$$

and

$$A_r = \lambda^r c^{-\frac{1}{2}re(\lambda - \lambda^{-1})} + \lambda^{-r} c^{\frac{1}{2}re(\lambda - \lambda^{-1})},$$

where, after expanding the exponentials, the negative powers of λ are to be rejected and the term independent of λ is to be multiplied by $\frac{1}{2}$ (see *Camb. Math. Journal*, t. I. [1839] p. 228 and t. III. [1843] p. 165, [4]).

It is easily seen that e^r is the lowest power of e which enters into the value of A_r and the question arises to find the numerical coefficient of the term in question; this is readily obtained from the formula; in fact considering first a term of the form

$$\lambda^{-r} e^s (\lambda - \lambda^{-1})^s,$$

since λ is itself of the order e, when the negative powers of λ are rejected this is at least of the order e^s and it is consequently to be neglected if $s > r$. But if $s < r$ all the powers of λ are negative and the term is to be rejected. The only case to be considered is therefore that of $s=r$, in which case there is a term containing e^r. We thus obtain from $\lambda^{-r} c^{\frac{1}{2}re(\lambda-\lambda^{-1})}$ the term

$$\tfrac{1}{2}\frac{r^r e^r}{2^r.1.2.3...r}.$$

In the next place a term of the form $\lambda^r e^s (\lambda-\lambda^{-1})^s$ is at least of the order e^s if $s>r$, or the terms to be considered are those for which $s=$ or $<r$. But in such term the only part of the order e^r is

$$(-)^s \lambda^{r-s} e^s,$$

or, since neglecting higher powers of e we have $\lambda=\frac{1}{2}e$, this is

$$(-)^s 2^{-r+s} e^r,$$

and the set of terms arising from

$$\lambda^r c^{-\frac{1}{2}re(\lambda-\lambda^{-1})},$$

is

$$\frac{e^r}{2^r}\left\{1+\frac{r}{1}+\frac{r^2}{1.2}...+\frac{r^{r-1}}{1.2...(r-1)}+\tfrac{1}{2}\frac{r^r}{1.2...r}\right\},$$

the last term being divided by 2 because arising from a term independent of λ. Hence the first term of A_r is

$$\frac{e^r}{2^r}\left\{1+\frac{r}{1}+\frac{r^2}{1.2}...+\frac{r^r}{1.2...r}\right\},$$

a result which it may be remarked is contained in the general formula given in Hansen's Memoir "Entwickelung des Products u. s. w.," *Leipzig Trans.*, t. II. p. 277 (1853).

The preceding expression is

$$=\frac{e^r c^r}{2^r}\frac{1}{\Gamma(r+1)}\int_r^\infty x^r c^{-x}\,dx,$$

and to find its value when r is large, we have

$$\int_r^\infty x^r c^{-x}\,dx=\int_0^\infty (y+r)^r e^{-y-r}\,dy=r^r c^{-r}\int_0^\infty\left(1+\frac{y}{r}\right)^r e^{-y}\,dy$$

$$=r^r c^{-r}\int_0^\infty c^{-y+r\log\left(1+\frac{y}{r}\right)}\,dy$$

$$=r^r c^{-r}\int_0^\infty c^{-\frac{y^2}{2r}+\frac{y^3}{3r^2}-\&c.}\,dy$$

$$=r^r c^{-r}\int_0^\infty\left(1+\frac{y^3}{3r^2}+...\right)e^{-\frac{y^2}{2r}}\,dy$$

$$=r^r c^{-r}\sqrt{2r}\int_0^\infty\left(1+\frac{2\sqrt{2}}{3\sqrt{r}}z^3+...\right)e^{-z^2}\,dz,$$

or neglecting all the terms except the first, this is

$$= r^r c^{-r} \sqrt{2r} \int_0^\infty e^{-z^2}\, dz$$

$$= \sqrt{2\pi r}\, r^r c^{-r}.$$

Hence multiplying by $\dfrac{1}{2^r}\, e^r c^r\, \dfrac{1}{\Gamma(r+1)}$ and observing that when r is large, we have, by a well-known formula,

$$\Gamma(r+1) = \sqrt{2\pi r}\, r^r c^{-r},$$

we obtain finally the result that when r is large the first term of A_r is approximately

$$= \left(\frac{ec}{2}\right)^r.$$

I take the opportunity of mentioning the following somewhat singular theorem, which seems to belong to a more general theory: viz. if $u - e \sin u = m$, then we have

$$\log(1 - e\cos u) = \frac{1}{\alpha}\log(1 - \alpha e \cos\phi),$$

where

$$\phi - \frac{1}{\alpha}\tan\phi = m,$$

provided that the negative powers of α are rejected, and α is then put equal to unity.

To show this, we have by Lagrange's theorem, observing that

$$\frac{d}{dm} F(1 - e\cos m) = e \sin m\, F'(1 - e\cos m),$$

$$F(1 - e\cos u) = F(1 - e\cos m) + \frac{e^2}{1}\sin^2 m\, F'(1 - e\cos m)$$

$$+ \frac{e^3}{1\,.\,2}\, \frac{d}{dm}\sin^3 m\, F'(1 - e\cos m) + \text{\&c.},$$

and the coefficient of e^r in $F(1 - e\cos u)$ is

$$\frac{(-)^r}{1\,.\,2\ldots(r-1)}\left\{\frac{1}{r}\, F_r \cos^r m + \frac{r-1}{1}\, F_{r-1} \cos^{r-2} m \sin^2 m\right.$$

$$\left. - \frac{(r-1)(r-2)}{1\,.\,2}\, F_{r-2}\, \frac{d}{dm}(\cos^{r-3} m \sin^3 m) + \text{\&c.}\right\},$$

where $F_r = F^r(1)$.

Hence in particular when $Fx = \log x$, $F_r = (-)^{r-1}\, 1\,.\,2 \ldots (r-1)$ and thence the coefficient of e^r in $\log(1 - e\cos u)$ is

$$-\left\{\frac{1}{r}\cos^r m - \frac{1}{1}\cos^{r-2} m \sin^2 m - \frac{1}{1\,.\,2}\frac{d}{dm}(\cos^{r-3} m \sin^2 m) - \&\text{c.}\right\},$$

continued as long as the exponent of $\cos m$ is not negative. Now in the expansion of $\frac{1}{\alpha}\log(1 - \alpha e\cos\phi)$, where $\phi - \frac{1}{\alpha}\tan\phi = m$, the coefficient of e^r is $-\frac{1}{r}\alpha^{r-1}\cos^r\phi$, and this (by Lagrange's theorem) is equal to

$$-\frac{1}{r}\alpha^{r-1}\left\{\cos^r m - \frac{1}{1\,.\,\alpha}\, r\cos^{r-1} m \sin m \tan m - \frac{1}{1\,.\,2\,.\,\alpha^2}\frac{d}{dm}(r\cos^{r-1} m \sin m \tan^2 m) - \&\text{c.}\right\}$$

$$= -\left\{\frac{1}{r}\alpha^{r-1}\cos^r m - \frac{1}{1}\alpha^{r-2}\cos^{r-2} m \sin^2 m - \frac{1}{1\,.\,2}\alpha^{r-3}\cos^{r-3} m \sin^3 m - \&\text{c.}\right\},$$

where the series is continued indefinitely; but if we reject the negative powers of α and then put α equal to unity this is precisely equal to the former expression for the coefficient of e^r, and the formula is thus shown to be true.

2, *Stone Buildings, W.C.,* 17*th Nov.,* 1857.