# 189.

## NOTE ON A FORMULA IN FINITE DIFFERENCES.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 198—201.]

In Jacobi's Memoir "De usu legitimo formulæ summatoriæ Maclaurinianæ," *Crelle*, t. XII. [1834], pp. 263—273 (1834), expressions are given for the sums of the odd powers of the natural numbers 1, 2, 3...$x$ in terms of the quantity.

$$u = x(x+1),$$

viz. putting for shortness

$$Sx^r = 1^r + 2^r + \ldots + x^r,$$

the expressions in question are

$$\begin{aligned}
Sx^3 &= \tfrac{1}{4}u^2,\\
Sx^5 &= \tfrac{1}{6}u^2\left(u - \tfrac{1}{2}\right),\\
Sx^7 &= \tfrac{1}{8}u^2\left(u^2 - \tfrac{4}{3}u + \tfrac{2}{3}\right),\\
Sx^9 &= \tfrac{1}{10}u^2\left(u^3 - \tfrac{5}{2}u^2 + 3u - \tfrac{3}{2}\right),\\
Sx^{11} &= \tfrac{1}{12}u^2\left(u^4 - 4u^3 + \tfrac{17}{2}u^2 - 10u + 5\right),\\
Sx^{13} &= \tfrac{1}{14}u^2\left(u^5 - \tfrac{35}{6}u^4 + \tfrac{287}{15}u^3 - \tfrac{118}{3}u^2 + \tfrac{691}{15}u - \tfrac{691}{30}\right),\\
&\text{\&c.,}
\end{aligned}$$

which, especially as regards the lower powers, are more simple than the ordinary expressions in terms of $x$.

The expressions are continued by means of a recurring formula, viz. if

$$Sx^{2p-3} = \frac{1}{2p-2}\{u^{p-1} - a_1u^{p-2} \ldots + (-)^{p-1}a_{p-3}u^2\},$$

$$Sx^{2p-1} = \frac{1}{2p}\{u^p - b_1u^{p-2} \ldots + (-)^p b_{p-3}u^2\},$$

then

$$\begin{aligned}
2p\,(2p-1)\,a_1 &= (2p-2)\,(2p-3)\,b_1 - p\,(p-1),\\
2p\,(2p-1)\,a_2 &= (2p-4)\,(2p-5)\,b_2 - (p-1)\,(p-2)\,b_1,\\
2p\,(2p-1)\,a_3 &= (2p-6)\,(2p-7)\,b_3 - (p-2)\,(p-3)\,b_2,\\
&\vdots\\
2p\,(2p-1)\,a_{p-3} &= 5\,.\,6\quad b_{p-3} - 3\,.\,4\quad b_{p-4},\\
0 &= 3\,.\,4\quad b_{p-2} - 2\,.\,3\quad b_{p-3},
\end{aligned}$$

by means of which the coefficients $b$ can be determined when the coefficients $a$ are known.

Jacobi remarks also that the expressions for the sums of the even powers may be obtained from those for the odd powers by means of the formula

$$Sx^{2p} = \frac{1}{2p+1}\,\partial_x Sx^{2p+1},$$

which shows that any such sum will be of the form $(2x+1)\,u$ into a rational and integral function of $u$: thus in particular

$$Sx^2 = \tfrac{1}{6}\,(2x+1)\,u.$$

To show *à priori* that $Sx^{2p+1}$ can be expressed as a rational and integral function of $u$, it may be remarked that $Sx^{2p+1} = \phi_1 x$ where $\phi_1 x$ denotes the summatory integral $\Sigma\,(x+1)^{2p+1}$, taken so as to vanish for $x=0$: $\phi_1 x$ is a rational and integral function of $x$ of the degree $2p+2$, and which, as is well known, contains $x^2$ as a factor. Suppose that $y$ is any positive or negative integer less than $x$, we have

$$\phi_1 x - \phi_1 y = (y+1)^{2p+1} + (y+2)^{2p+1} \ldots + x^{2p+1},$$

and in particular putting $y = -1-x$,

$$\phi_1 x - \phi_1\,(-1-x) = (-x)^{2p+1} + (1-x)^{2p+1} \ldots + x^{2p+1}, = 0,$$

since the terms destroy each other in pairs; we have therefore $\phi_1 x = \phi_1\,(-1-x)$. Now $u = x^2 + x$, or writing this equation under the form $x^2 = -x + u$, we see that any rational and integral function of $x$ may be reduced to the form $Px + Q$, where $P$ and $Q$ are rational and integral functions of $u$. Write therefore $\phi_1 x = Px + Q$: the substitution of $-1-x$ in the place of $x$ leaves $u$ unaltered, and the equation $\phi_1 x = \phi_1\,(-1-x)$ thus shows that $P = 0$; we have therefore $\phi_1 x = Q$, a rational and integral function of $u$. Moreover $\phi_1 x$ as containing the factor $x^2$, must clearly contain the factor $u^2$, and the expressions for $Sx^{2p+1}$ are thus shown to be of the form given by Jacobi.

We may obtain a finite expression for $Sx^n$ in terms of the differences of $0^n$ as follows: we have

$$Sx^n = 1^n + 2^n \ldots + x^n = \{(1+\Delta) + (1+\Delta)^2 \ldots + (1+\Delta)^x\}\,0^n = \frac{1+\Delta}{\Delta}\,\{(1+\Delta)^x - 1\}0^n,$$

and putting $(1+\Delta)^x = e^{x\log(1+\Delta)}$ and observing that the term independent of $x$ vanishes, and that the terms containing powers higher than $x^{n+1}$ also vanish, we have

$$Sx^n = S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^n \cdot \frac{x^k}{\Pi k},$$

where the summation with respect to $k$, extends from $k=1$ to $k=n+1$, or what is the same thing (since the term corresponding to $k=1$ in fact vanishes) from $k=2$ to $k=n+1$.

The equation $x^2 = -x+u$ gives

$$x^k = P_k x + Q_k,$$

and it is easy to see that writing for shortness

$$M_k = 1 + \frac{k-3}{1}u + \frac{k-4\,.\,k-5}{1\,.\,2}u^2 + \frac{k-5\,.\,k-6\,.\,k-7}{1\,.\,2\,.\,3}u^3 + \ldots,$$

where the series is to be continued to the term $u^{\frac{1}{2}(k-2)}$ or $u^{\frac{1}{2}(k-3)}$ according as $k$ is even or odd, we have

$$P_k = (-)^{k+1}M_{k+1},\quad Q_k = (-)^k u M_k,$$

we have consequently

$$Sx^n = xS_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^n \cdot \frac{(-)^{k+1}M_{k+1}}{\Pi k}$$
$$+ S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^n \cdot \frac{(-)^k u M_k}{\Pi k}.$$

If $n$ is odd, $=2p+1$, then (by what precedes) the first term vanishes, or we have

$$S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p+1} \cdot \frac{(-)^{k+1}M_{k+1}}{\Pi k} = 0,\ (k=1 \text{ to } k=2p+2),$$

and the formula becomes

$$Sx^{2p+1} = S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p+1} \cdot \frac{(-)^k u M_k}{\Pi k},\quad (k=1 \text{ to } k=2p+2),$$

which it may be noticed puts in evidence the factor $u$ but not the factor $u^2$.

If $n$ is even, $=2p$, then (by what precedes) the coefficient of $x$ is to the constant term in the ratio $2:1$, or we have

$$S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p} \cdot \frac{(-)^{k+1}(M_{k+1}-2uM_k)}{\Pi k} = 0,\ (k=1 \text{ to } k=2p+1),$$

and the formula becomes

$$Sx^{2p} = (2x+1)\,S_k\left\{\frac{1+\Delta}{\Delta}\log^k(1+\Delta)\right\}0^{2p} \cdot \frac{(-)^k u M_k}{\Pi k},\ (k=1 \text{ to } k=2p+1).$$

The values of the functions $M$ are as follows:

$$\begin{aligned}
&M_1 = 0,\\
&M_2 = 1,\\
&M_3 = 1,\\
&M_4 = 1 + u,\\
&M_5 = 1 + 2u,\\
&M_6 = 1 + 3u + u^2,\\
&M_7 = 1 + 4u + 3u^2,\\
&\&c.
\end{aligned}$$

As a simple example of the formulæ, we have

$$\begin{aligned}
Sx^3 = &\left\{\frac{1+\Delta}{\Delta}\log^2(1+\Delta)\right\} 0^3 \,.\, \tfrac{1}{2}u\\
&+ \left\{\frac{1+\Delta}{\Delta}\log^3(1+\Delta)\right\} 0^3 \,.\, -\tfrac{1}{6}u\\
&+ \left\{\frac{1+\Delta}{\Delta}\log^4(1+\Delta)\right\} 0^3 \,.\, \tfrac{1}{24}(u+u^2),
\end{aligned}$$

and the coefficients are

$$\begin{aligned}
(\Delta - \tfrac{1}{12}\Delta^3)\, 0^3 &= 1 - \tfrac{1}{12}6 = \tfrac{1}{2},\\
(\Delta^2 - \tfrac{1}{2}\Delta^3)\, 0^3 &= 6 - \tfrac{1}{2}6 = 3,\\
\Delta^3\ 0^3 &= \quad\quad\ \ 6,
\end{aligned}$$

and therefore

$$Sx^3 = \tfrac{1}{4}u - \tfrac{1}{2}u + \tfrac{1}{4}(u+u^2) = \tfrac{1}{4}u^2,$$

which is right; the example shows however that the calculation for the higher powers would be effected more readily by means of Jacobi's recurring formula.

2, *Stone Buildings*, 27*th Oct*., 1857.