# 188.

## ON THE SIMULTANEOUS TRANSFORMATION OF TWO HOMOGENEOUS FUNCTIONS OF THE SECOND ORDER.

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In a former paper with this title, *Cambridge and Dublin Math. Journal*, t. IV. [1849], pp. 47—50 [74], I gave (founded on the methods of Jacobi and Prof. Boole) a simple solution of the problem, but the solution may I think be presented in an improved form as follows, where as before I consider for greater convenience the case of three variables only.

Suppose that by the linear transformation[1]

$$(x,\ y,\ z) = \left(\begin{array}{ccc} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{array}\right)\!\!\!\left(x_1,\ y_1,\ z_1\right),$$

we have identically

$$(a,\ b,\ c,\ f,\ g,\ h \,\big)\!\!\big( x,\ y,\ z)^2 = (a_1,\ b_1,\ c_1,\ f_1,\ g_1,\ h_1 \,\big)\!\!\big( x_1,\ y_1,\ z_1)^2,$$

$$(A,\ B,\ C,\ F,\ G,\ H \,\big)\!\!\big( x,\ y,\ z)^2 = (A_1,\ B_1,\ C_1,\ F_1,\ G_1,\ H_1 \,\big)\!\!\big( x_1,\ y_1,\ z_1)^2;$$

and write also

$$(\xi_1,\ \eta_1,\ \zeta_1) = \left(\begin{array}{ccc} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{array}\right)\!\!\!\left(\xi\ \eta,\ \zeta\right).$$

[1] I represent in this manner the system of equations

$$x = \alpha x_1 + \beta y_1 + \gamma z_1,\ \&c.$$

and so in all like cases.

Comparing these with the relations between $(x, y, z)$ and $(x_1, y_1, z_1)$, we see that

$$(\xi, \eta, \zeta \between x, y, z) = (\xi_1, \eta_1, \zeta_1 \between x_1, y_1, z_1),$$

and multiplying the first of the relations between two quadrics by an indeterminate quantity $\lambda$, and adding it to the second, we have

$$(\lambda a + A, \ldots \between x, y, z)^2 = (\lambda a_1 + A_1, \ldots \between x_1, y_1, z_1)^2.$$

We have thus a linear function and a quadric transformed into functions of the same form by means of the linear substitutions, and any invariant of the system will remain unaltered to a factor près, such factor being a power of the determinant of substitution. The invariants are, 1° the discriminant of the quadric; 2° the reciprocant, considered not as a contravariant of the quadric, but as an invariant of the system. And if we write

$$K = \text{Disc. } (\lambda a + A, \ldots \between x, y, z)^2,$$

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between \xi, \eta, \zeta)^2 = \text{Recip. } (\lambda a + A, \ldots \between x, y, z)^2,$$

then $K_1$, &c. being the analogous expressions for the transformed functions, and the determinant of substitution being represented by $\Pi$, we have

$$K_1 = \Pi^2 K,$$

$$(\mathfrak{A}_1, \ldots \between \xi_1, \eta_1, \zeta_1)^2 = \Pi^2 (\mathfrak{A}, \ldots \between \xi, \eta, \zeta)^2,$$

and substituting for $\xi_1, \eta_1, \zeta_1$ their values in terms of $\xi, \eta, \zeta$, the last equation breaks up into six equations, and we have

$$\begin{aligned} K_1 &= \Pi^2 K, \\ (\mathfrak{A}_1, \ldots \between \alpha, \alpha', \alpha'')^2 &= \Pi^2 \mathfrak{A}, \\ &\vdots \\ (\mathfrak{A}_1, \ldots \between \beta, \beta', \beta'')(\gamma, \gamma', \gamma'') &= \Pi^2 \mathfrak{F}, \\ &\vdots \end{aligned}$$

which is the system obtained in a somewhat different manner in my former paper. Putting $f_1 = g_1 = h_1 = F_1 = G_1 = H_1 = 0$, and writing also (which is no additional loss of generality) $a_1 = b_1 = c_1 = 1$, the formulæ become

$$(a, b, c, f, g, h \between x, y, z)^2 = (1, 1, 1 \between x_1^2, y_1^2, z_1^2),$$

$$(A, B, C, F, G, H \between x, y, z)^2 = (A_1, B_1, C_1 \between x_1^2, y_1^2, z_1^2),$$

viz. there are two given quadrics which are to be by the same linear substitution transformed, one of them into the form $x_1^2 + y_1^2 + z_1^2$ and the other into the form $A_1 x_1^2 + B_1 y_1^2 + C_1 z_1^2$, where $A_1, B_1, C_1$ have to be determined. The solution is contained in the following system of formulæ, viz.

$$(A_1 + \lambda)(B_1 + \lambda)(C_1 + \lambda) = \Pi^2 \text{ Disc. } (\lambda a + A, \ldots),$$

which gives $A_1$, $B_1$, $C_1$ as the roots of a cubic equation, and gives also

$$1 = \Pi^2 \text{Disc.}\, (a, \ldots) = \Pi^2\kappa, \text{ or } \Pi^2 = \frac{1}{\kappa} \text{ suppose,}$$

and we have then, writing for shortness, $(*\between X, Y, Z)$ for

$$((B_1+\lambda)(C_1+\lambda),\ (C_1+\lambda)(A_1+\lambda),\ (A_1+\lambda)(B_1+\lambda)\between X,\ Y,\ Z),$$

$$(*\between \alpha^2,\quad \alpha'^2,\quad \alpha''^2) = \frac{1}{\kappa}\mathfrak{A},$$

$$(*\between \beta^2,\quad \beta'^2,\quad \beta''^2) = \frac{1}{\kappa}\mathfrak{B},$$

$$(*\between \gamma^2,\quad \gamma'^2,\quad \gamma''^2) = \frac{1}{\kappa}\mathfrak{C},$$

$$(*\between \beta\gamma,\ \beta'\gamma',\ \beta''\gamma'') = \frac{1}{\kappa}\mathfrak{F},$$

$$(*\between \gamma\alpha,\ \gamma'\alpha',\ \gamma''\alpha'') = \frac{1}{\kappa}\mathfrak{G},$$

$$(*\between \alpha\beta,\ \alpha'\beta',\ \alpha''\beta'') = \frac{1}{\kappa}\mathfrak{H},$$

where $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})$ are the coefficients of the reciprocant of $(\lambda a + A, \ldots\between x, y, z)^2$. Writing $\lambda = -A_1$, $-B_1$, or $-C_1$ the quadric functions on the left-hand side become mere monomials, and we have the actual values of the squares and products $\alpha^2$, $\beta\gamma$, &c. of the coefficients of the linear substitutions: thus $\alpha^2$, $\beta^2$, $\gamma^2$, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ are respectively equal to $\mathfrak{A}_0$, $\mathfrak{B}_0$, $\mathfrak{C}_0$, $\mathfrak{F}_0$, $\mathfrak{G}_0$, $\mathfrak{H}_0$ each into the common factor

$$\frac{1}{\kappa}\,(B_1 - A_1)\,(C_1 - A_1),$$

the suffix denoting that we are to write in the expressions for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ the value $-A_1$ for $\lambda$; and similarly for the sets $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$.

2, *Stone Buildings*, *27th March*, 1857.