187.

ON THE SUMS OF CERTAIN SERIES ARISING FROM THE EQUATION $x = u + tfx$.

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LAGRANGE has given the following formula for the sum of the inverse n^{th} powers of the roots of the equation $x = u + tfx$,

$$\Sigma (z^{-n}) = u^{-n} + (-nu^{-n-1} fu)\frac{t}{1} + (-nu^{-n-1} f^2u)' \frac{t^2}{1.2} + \&c. \qquad (1)$$

where n is a positive integer and the series on the second side of the equation is to be continued as long as the exponent of u remains negative (*Théorie des Équations Numériques*, p. 225). Applying this to the equation $x = 1 + tx^s$, we have

$$\Sigma (z^{-n}) = 1^{-n} - \frac{n}{1} t . 1^{-n+s-1} + \frac{n(n-2s+1)}{1.2} t^2 . 1^{-n+2s-2} \ldots$$

$$+ (-)^q \frac{n(n-qs+q-1)\ldots(n-qs+1)}{1.2\ldots q} t^q . 1^{-n+qs-q} - \&c. \qquad (2)$$

to be continued while the exponent of 1 remains negative.

Let $n = \mu s + \rho$, ρ being not greater than $s-1$, the series may always be continued up to $q = \mu$, and no further. In fact writing the above value for n and putting $q = \mu + \theta$, the general term is

$$(-)^{\mu+\theta} \frac{t^{\mu+\theta}}{1.2\ldots(\mu+\theta)} (\mu s + \rho)(\rho - \theta s + \mu + \theta - 1)\ldots(\rho - \theta s + 1)\, 1^{-(\rho - \theta s + \mu + \theta)}.$$

Now if $\rho + \mu - \theta(s-1)$ is negative or zero, the term is to be rejected on account of the index of 1 not being negative, and if this quantity be positive, then since

$\rho-\theta s+1$ is necessarily negative for any value of θ greater than zero, the factorial $(\rho-\theta s+\mu+\theta-1)\ldots(\rho-\theta s+1)$ begins with a positive and ends with a negative factor, and since the successive factors diminish by unity, one of them is necessarily equal to zero, or the term vanishes; hence the series is always to be continued up to $q=\mu$.

Hence

$$\Sigma(z^{-\mu s-\rho}) = 1 - \frac{\mu s+\rho}{1}t + \frac{(\mu s+\rho)\{(\mu-2)s+\rho+1\}}{1.2}t^2 \ldots$$
$$+(-)^q \frac{(\mu s+\rho)\{(\mu-q)s+\rho+q-1\}\ldots\{(\mu-q)s+\rho+1\}}{1.2\ldots q}t^q$$
$$-\&c. \qquad (3)$$

continued to $q=\mu$.

By taking the terms in a reverse order, it is easy to derive

$$(-)^\mu t^{-\mu}\,\Sigma(z^{-\mu s-\rho}) = (\mu s+\rho)\left\{\begin{array}{l}\dfrac{(\mu+\rho-1)\ldots(\mu+1)}{2.3\ldots\rho} - \dfrac{(\mu+\rho+s-2)\ldots\mu}{2.3\ldots\rho+s}t^{-1} \\ +(-)^q\dfrac{(\mu+\rho+qs-q-1)\ldots(\mu+1-q)}{2.3\ldots(\rho+q)s}t^{-q} \\ -\&c.\end{array}\right. \qquad (4)$$

continued to $q=\mu$.

Suppose in particular $s=2$, and $t=-\dfrac{\alpha+1}{\alpha^2}$, so that the equation in x becomes $\dfrac{x-1}{x^2}=-\dfrac{\alpha+1}{\alpha^2}$, whence $x=-\alpha$ or $x=\dfrac{\alpha}{\alpha+1}$, or substituting in (2), we find

$$\frac{(\alpha+1)^n}{\alpha^n}+\frac{(-)^n}{\alpha^n} = 1+\frac{n}{1}\frac{\alpha+1}{\alpha^2}+\frac{n(n-3)}{2}\left(\frac{\alpha+1}{\alpha^2}\right)^2+\&c. \qquad (5)$$

continued to the term involving $\left(\dfrac{\alpha+1}{\alpha^2}\right)^{\frac{1}{2}n}$ or $\left(\dfrac{\alpha+1}{\alpha^2}\right)^{\frac{1}{2}(n-1)}$.

Put $\alpha=-\dfrac{a+b}{a}$; and therefore

$$\alpha+1=-\frac{b}{a},\quad \frac{\alpha+1}{\alpha}=\frac{b}{a+b},\quad \frac{\alpha+1}{\alpha^2}=\frac{ab}{(a+b)^2};$$

we obtain

$$\frac{a^n+b^n}{(a+b)^n} = 1-\frac{n}{1}\frac{ab}{(a+b)^2}+\frac{n(n-3)}{1.2}\frac{a^2b^2}{(a+b)^4}-\&c. \qquad (6),$$

or

$$\frac{(a+b)^n-a^n-b^n}{nab(a+b)} = (a+b)^{n-3}-\frac{n-3}{2}(a+b)^{n-5}ab$$
$$+\frac{(n-4)(n-5)}{2.3}(a+b)^{n-7}a^2b^2-\&c. \qquad (7),$$

to be continued as long as the exponent of $(a+b)$ on the second side is negative.

This formula, which is easily deducible from that for the expansion of $\cos n\theta$ in powers of $\cos\theta$, is employed by M. Stern, *Crelle*, t. xx. [1840], in proving the following theorem: If

$$S = 1 - \frac{n-3}{2} + \frac{(n-4)(n-5)}{2\,.\,3} - \&\text{c.} \tag{8}$$

continued to the first term that vanishes, then according as n is of the form $6k+3$, $6k \pm 1$, $6k$ or $6k \pm 2$,

$$S = \frac{3}{n}, \quad S = 0, \quad S = -\frac{1}{n}, \quad S = \frac{2}{n}, \tag{9}$$

which is in fact immediately deduced from it by writing $b = \omega a$, ω being one of the impossible cube roots of unity. Substituting the above values of x in the equation (4),

$$(1+\alpha)^{p+1} - (1+\alpha)^{-p} = (2p+1)\,\alpha\left\{1 + \frac{(p+1)\,p}{2\,.\,3}\,\frac{\alpha^2}{\alpha+1} + \frac{(p+2)(p+1)\,p\,(p-1)}{2\,.\,3\,.\,4\,.\,5}\,\frac{\alpha^4}{(\alpha+1)^2} + \dots\right\}, \tag{10}$$

$$(1+\alpha)^{p} + (1+\alpha)^{-p} = 2p\left\{\frac{1}{p} + \frac{p}{2}\,\frac{\alpha^2}{\alpha+1} + \frac{(p+1)\,p\,(p-1)}{2\,.\,3\,.\,4}\,\frac{\alpha^4}{(\alpha+1)^2} + \dots\right\}, \tag{11}$$

whence

$$(1+\alpha)^{p+1} + (1+\alpha)^{p} = (2p+1)\,\alpha\left\{1 + \frac{(p+1)\,p}{2\,.\,3}\,\frac{\alpha^2}{\alpha+1} + \dots\right\}$$

$$+\, 2p\left\{\frac{1}{p} + \frac{p}{2}\,\frac{\alpha^2}{\alpha+1} + \dots\right\}, = U \text{ suppose,} \tag{12}$$

i.e

$$\Delta\,(-)^p\,(1+\alpha)^p = (-)^{p+1}\,U \text{ or } (1+\alpha)^p = (-)^p\,\Sigma\,(-)^{p+1}\,U,$$

where Δ and Σ refer to the variable p. The summation is readily effected by means of the formulæ

$$\Sigma\,(-)^{p+1}\,(2p+1)\,(p+s+1)\dots(p-s) = (-)^p\,(p+s+1)\dots(p-s-1),$$

$$\Sigma\,(-)^{p+1}\,(p+s)\dots(p-s)\,2p = (-)^p\,(p+s)\dots(p-s-1),$$

and we thence find

$$(1+\alpha)^p = \left\{1 + \frac{p\,(p-1)}{1\,.\,2}\,\frac{\alpha^2}{1+\alpha} + \frac{(p+1)\,p\,(p-1)(p-2)}{1\,.\,2\,.\,3\,.\,4}\,\frac{\alpha^4}{(1+\alpha)^2} + \dots\right\}$$

$$+\,\alpha\left\{\frac{p}{1} + \frac{(p+1)\,p\,(p-1)}{1\,.\,2\,.\,3}\,\frac{\alpha^2}{1+\alpha} + \dots\right\}, \tag{13}$$

a formula of Euler's (*Pet. Trans.* 1811) demonstrated likewise by M. Catalan (*Liouville*, t. IX. [1844], pp. 161—174) by induction. It may be expressed also in the slightly different form

$$(1+\alpha)^p = \left\{1+\frac{(p+1)p}{1.2}\frac{\alpha^2}{1+\alpha}+\frac{(p+2)(p+1)p(p-1)}{1.2.3.4}\frac{\alpha^4}{(1+\alpha)^2}+\dots\right\}$$

$$+\frac{\alpha}{1+\alpha}\left\{\frac{p}{1}+\frac{(p+1)p(p-1)}{1.2.3}\frac{\alpha^2}{1+\alpha}+\dots\right\}. \qquad (14)$$

The two series (13), (14) are each of them supposed to contain $p+1$ terms, p being an integer; but since the terms after these all of them vanish, the series may be continued indefinitely. Suppose the two sides expanded in powers of p, the coefficients will be separately equal, and thus the identity of the two sides will be independent of the particular values of p, or the equations (13), (14), and similarly, (10), (11), (12) are true for any values of p whatever. It is to be observed that the series for negative values of p do not differ essentially from those for the corresponding positive values; as may be seen immediately by writing $-p$ for p, and $\frac{-\alpha}{1+\alpha}$ for α.

Suppose next $s=3$, or that the equation in x is $x=1+tx^3$; to rationalise the roots of this, assume $t=\frac{4(\beta^2-1)^2}{(\beta^2+3)^3}$, then values of x are

$$x=\frac{\beta^2+3}{2(\beta+1)},\ x=-\frac{\beta^2+3}{2(\beta-1)},\ x=\frac{\beta^2+3}{\beta^2-1},$$

and hence

$$\frac{2^n\{(\beta+1)^n+(-)^n(\beta-1)^n\}+(\beta^2-1)^n}{(\beta^2+3)^r}=$$

$$1-\frac{n}{1}t+\frac{n(n-5)}{1.2}t^2-\frac{n(n-7)(n-8)}{1.2.3}t^3\dots+(-)^r\frac{n(n-2r-1)\dots(n-3r+1)}{1.2\dots r}t^r+\&\text{c}. \qquad (15)$$

where $t=\frac{4(\beta^2-1)^2}{(\beta^2+3)^3}$, and the series is to be continued up to the term involving $t^{\frac{1}{3}n}$, $t^{\frac{1}{3}(n-1)}$ or $t^{\frac{1}{3}(n-2)}$.

Again, from the formula (4) we deduce the three following forms,

$$(-)^\mu\frac{2^3\{(\beta+1)^{3\mu}+(-)^\mu(\beta-1)^{3\mu}\}+(\beta^2-1)^{3\mu}}{2^{2\mu}(\beta^2-1)^{2\mu}}=$$

$$3\mu\left\{\frac{1}{\mu}-\frac{(\mu+1)\mu}{2.3}t^{-1}+\frac{(\mu+3)(\mu+2)(\mu+1)\mu(\mu-1)}{2.3.4.5.6}t^{-2}+\dots(-)^q\frac{(\mu+2q-1)\dots(\mu-q+1)}{2.3\dots3q}t^{-q}\dots\right\}, \qquad (16)$$

$$(-)^{\mu}\,\frac{2^{3\mu+1}\{(\beta+1)^{3\mu+1}-(-)^{\mu}(\beta-1)^{3\mu+1}\}+(\beta^2-1)^{3\mu+1}}{2^{2\mu}(\beta^2-1)^{2\mu}(\beta^2+3)}=$$

$$(3\mu+1)\left\{1-\frac{(\mu+2)(\mu+1)\mu}{2\,.\,3\,.\,4}t^{-1}\ldots+(-)^q\frac{(\mu+2q)\ldots(\mu-q+1)}{2\,.\,3\ldots3q+1}t^{-q}\ldots\right\},\ (17)$$

$$(-)^{\mu}\,\frac{2^{3\mu+2}\{(+1)^{3\mu+2}+(-)^{\mu}(\beta-1)^{3\mu+2}\}+(\beta^2-1)^{3\mu+2}}{2^{2\mu}(\beta^2-1)^2(\beta^2+3)^2}=$$

$$(3\mu+2)\left\{\frac{\mu+1}{2}-\frac{(\mu+3)(\mu+2)(\mu+1)\mu}{2\,.\,3\,.\,4\,.\,5}t^{-1}\ldots+(-)^q\frac{(\mu+2q+1)\ldots(\mu-q+1)}{2\,.\,3\ldots(3q+2)}t^{-q}\ldots\right\},\ (18)$$

all of them continued up to $q=\mu$.

2, *Stone Buildings*, 1*st April*, 1857.