186.

ON THE DETERMINATION OF THE VALUE OF A CERTAIN DETERMINANT.

CONSIDERING the determinant

$$\begin{vmatrix} \theta, & 1, & . & . & . & \cdots \\ n, & \theta, & 2 \\ . & n-1, & \theta, & 3 \\ & & n-2, & \theta, & 4 \\ \vdots \end{vmatrix}$$

let the successive diagonal minors be $U_0, U_1, U_2, \ldots U_x \ldots$, it is easy to find

$$U_0 = 1,$$
$$U_1 = \theta,$$
$$U_2 = (\theta^2 - 1) - (n - 1),$$
$$U_3 = \theta (\theta^2 - 4) - 3 (n - 2) \theta,$$
$$U_4 = (\theta^2 - 1)(\theta^2 - 9) - 6 (n - 3)(\theta^2 - 1) + 3 (n - 3)(n - 1),$$

which in fact suggests the law, viz.

$$U_x = (\theta + x - 1)(\theta + x - 3)(\theta + x - 5)\ldots(\theta - x + 5)(\theta - x + 3)(\theta - x + 1)$$

$$- \frac{x(x-1)}{2} (n - x + 1)(\theta + x - 3)(\theta + x - 5)\ldots(\theta - x + 5)(\theta - x + 3)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{2 \cdot 4} (n - x + 1)(n - x + 3)(\theta + x - 5)\ldots(\theta - x + 5)$$

$$- \&c.$$

$$\vdots$$

$$+ (-)^s \frac{x(x-1)\ldots(x - 2s + 1)}{2 \cdot 4 \ldots 2s} (n - x + 1)(n - x + 3)\ldots(n - x + 2s - 1)$$

$$(\theta + x - 2s - 1)(\theta + x - 2s - 3)\ldots(\theta - x + 2s + 1)$$

$$\vdots$$

to $s = \frac{1}{2}x$ or $\frac{1}{2}(x - 1)$, as x is even or odd.

And of course if x denote the number of lines or columns of the determinant, then U_x is the value of the determinant. This theorem, or a particular case of it, is due to Prof. Sylvester: I have not been able to find an easier demonstration than the following one, which, it must be admitted, is somewhat complicated. I observe that U_x satisfies the equation

$$U_x - \theta U_{x-1} + (x-1)(n-x+2) U_{x-2} = 0.$$

Hence writing $x-1$ and $x-2$ for x, we have the system

$$U_x - \theta U_{x-1} + (x-1)(n-x+2) U_{x-2} \qquad\qquad = 0,$$
$$U_{x-1} - \qquad\qquad \theta U_{x-2} + (x-2)(n-x+3) U_{x-3} \qquad = 0,$$
$$U_{x-2} - \qquad\qquad \theta U_{x-3} + (x-3)(n-x+4) U_{x-4} = 0,$$

or, eliminating U_{x-1} and U_{x-3},

$$U_x + \{(x-1)(n-x+2) + (x-2)(n-x+3) - \theta^2\} U_{x-2}$$
$$+ (x-2)(x-3)(n-x+3)(n-x+4) U_{x-4} = 0.$$

Suppose, for shortness,

$$(\theta + x - 1)(\theta + x - 3)(\theta + x - 5)\ldots(\theta - x + 5)(\theta - x + 3)(\theta - x + 1) = H_x,$$

and assume

$$U_x = A_{x,0} H_x - A_{x,1} H_{x-2} \ldots + (-)^s A_{x,s} H_{x-2s} \ldots,$$

where $A_{x,s}$ is independent of θ, then

$$U_x \quad \text{contains the term } (-)^s A_{x,s} \quad H_{x-2s},$$
$$U_{x-2} \text{ contains the term } (-)^s A_{x-2,s} H_{x-2s-2},$$

which is to be multiplied by

$$(x-1)(n-x+2) + (x-2)(n-x+3) - \theta^2.$$

This multiplier may be written under the form

$$(x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s-1)^2 - \{\theta^2 - (x-2s-1)^2\}$$
$$= M_{x,s} - \{\theta^2 - (x-2s-1)^2\},$$

if, for shortness,

$$M_{x,s} = (x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s-1)^2.$$

Now

$$M_{x,s} - \{\theta^2 - (x-2s-1)^2\}$$

multiplied into

$$(-)^s A_{x-2,s} H_{x-2s-2}$$

gives rise to the terms

$$(-)^s M_{x,s} A_{x-2,s} H_{x-2s-2} - (-)^s A_{x-2,s} H_{x-2s},$$

(since $\{\theta^2 - (x - 2s - 1)^2\} H_{x-2s-2} = H_{x-2s}$), or, what is the same thing,

$$-(-)^s M_{x,s-1} A_{x-2,s-1} H_{x-2s} - (-)^s A_{x-2,s} H_{x-2s}$$

$$= -(-)^s \{M_{x,s-1} A_{x-2,s-1} + A_{x-2,s}\} H_{x-2s},$$

and moreover

$$U_{x-4} \text{ contains the term } (-)^s A_{x-4,s} H_{x-2s-4},$$

or, what is the same thing, $(-)^s A_{x-4,s-2} H_{x-2s}$.

Hence we must have

$$A_{x,s} - (A_{x-2,s} + M_{x,s-1} A_{x-2,s-1}) + (x-2)(x-3)(n-x+3)(n-x+4) A_{x-4,s-2} = 0,$$

where

$$M_{x,s-1} = (x-1)(n-x+2) + (x-2)(n-x+3) - (x-2s+1)^2.$$

This may be satisfied by assuming

$$A_{x,s} = B_{x,s}(n-x+1)(n-x+3)\ldots(n-x+2s-1);$$

for then

$$A_{x-2,s} = B_{x-2,s}(n-x+3)\ldots(n-x+2s-1)(n-x+2s+1),$$

$$A_{x-2,s-1} = B_{x-2,s-1}(n-x+3)\ldots(n-x+2s-1),$$

$$(n-x+3)(n-x+4) A_{x-4,x-2} = B_{x-4,s-2}(n-x+4)(n-x+3)\ldots(n-x+2s-1),$$

and consequently

$$B_{x,s}(n-x+1)$$

$$- B_{x-2,s}(n-x+2s+1)$$

$$- B_{x-2,s-1} M_{s,x-1}$$

$$+ B_{x-4,s-2}(x-2)(x-3)(n-x+4) = 0;$$

and if this equation be satisfied independently of n, we must have

$$B_{x,s} - B_{x-2,s} - (2x-3) B_{x-2,s-1} + (x-2)(x-3) B_{x-4,s-2} = 0,$$

$$B_{x,s} - (2s+1) B_{x-2,s} - \{5x - 8 - (x-2s+1)^2\} B_{x-2,s-1} + 4(x-2)(x-3) B_{x-4,s-2} = 0,$$

and these are both satisfied by

$$B_{x,s} = \frac{x \cdot x-1 \ldots x-2s+1}{2^s \cdot 1 . 2 . 3 \ldots s},$$

in fact, substituting this value and omitting the factor

$$\frac{(x-2)(x-3)\ldots(x-2s+1)}{2^s \cdot 1 . 2 . 3 \ldots s},$$

the first equation becomes

$$x(x-1) - \quad (x-2s)(x-2s-1) - \quad (2x-3)2s + 4s(s-1) = 0,$$

and the second equation becomes

$$x(x-1) - (2s+1)(x-2s)(x-2s-1) - \{5x-8-(x-2s+1)^2\}2s + 16s(s-1) = 0,$$

which are each of them an identical equation, the first being

$$x^2 - \qquad x$$
$$- x^2 + (4s+1)x - 2s(2s+1)$$
$$- 4sx + 6s$$
$$+ 4s(s-1) = 0,$$

and the second being

$$x^2 - \qquad x$$
$$- (2s+1)\{x^2 - (4s+1)x + 2s(2s+1)\}$$
$$+ 2s\{x^2 - (4s+3)x + (2s-1)^2 + 8\}$$
$$+ 16s(s-1) = 0,$$

as may be easily verified.

Hence writing for $B_{x,s}$ its value and recapitulating, the equation

$$U_x + \{(x-1)(n-x+2) + (x-2)(n-x+3) - \theta^2\} U_{x-2}$$
$$+ (x-2)(x-3)(n-x+3)(n-x+4) U_{x-4} = 0$$

is satisfied by

$$U_x = A_{x,0} H_x - A_{x,1} H_{x-2} \ldots + (-)^s A_{x,s} H_{x-2s} \ldots \text{to } s = \tfrac{1}{2}x \text{ or } \tfrac{1}{2}(x-1), \text{ as } x \text{ is even or odd,}$$

where

$$H_x = (\theta + x - 1)(\theta + x - 3)(\theta + x - 5) \ldots (\theta - x + 5)(\theta - x + 3)(\theta - x + 1),$$

$$A_{x,s} = \frac{x(x-1) \ldots (x-2s+1)}{2^s . 1.2.3 \ldots s}(n-x+1)(n-x+3) \ldots (n-x+2s-1),$$

and since for $x = 0, 1, 2, 3$ the values of the expression U_x coincide with those of the first four diagonal minors, the expression gives in general the value of the diagonal minor, or when x denotes the number of lines or columns of the determinant, then the value of the determinant.

2, *Stone Buildings*, *1st April*, 1857.