184.

A THEOREM RELATING TO SURFACES OF THE SECOND ORDER.

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GIVEN a surface of the second order

$$(a, b, c, d, f, g, h, l, m, n)\,(x, y, z, w)^2 = 0,$$

and a fixed plane

$$\alpha x + \beta y + \gamma z + \delta w = 0,$$

imagine a variable plane

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

subjected to the condition that it always touches a surface of the second order, or what is the same thing such that the parameters ξ, η, ζ, ω satisfy a condition

$$(\mathrm{a, b, c, d, f, g, h, l, m, n})\,(\xi, \eta, \zeta, \omega)^2 = 0.$$

The given surface of the second order, and the variable plane meet in a conic, and the fixed plane and the variable plane meet in a line, it is required to find the locus of the pole of the line with respect to the conic.

The pole in question is the point in which the variable plane is intersected by the polar of the line with respect to the surface of the second order: this polar is the line joining the pole of the fixed plane with respect to the surface of the second order, and the pole of the variable plane with respect to the surface of the second order. Let α_1, β_1, γ_1, δ_1, be given linear functions of α, β, γ, δ, and ξ, η, ζ, ω, be given linear functions of ξ, η, ζ, ω, viz., if

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}),$$

are the inverse system to $(a, b, c, d, f, g, h, l, m)$, then let

$$\begin{aligned}
\alpha_1 &= \mathfrak{A}\alpha + \mathfrak{H}\beta + \mathfrak{G}\gamma + \mathfrak{L}\delta,\\
\beta_1 &= \mathfrak{H}\alpha + \mathfrak{B}\beta + \mathfrak{F}\gamma + \mathfrak{M}\delta,\\
\gamma_1 &= \mathfrak{G}\alpha + \mathfrak{F}\beta + \mathfrak{C}\gamma + \mathfrak{N}\delta,\\
\delta_1 &= \mathfrak{L}\alpha + \mathfrak{M}\beta + \mathfrak{N}\gamma + \mathfrak{D}\delta,
\end{aligned}$$

and in like manner,

$$\begin{aligned}
\xi_1 &= \mathfrak{A}\xi + \mathfrak{H}\eta + \mathfrak{G}\zeta + \mathfrak{L}\omega,\\
\eta_1 &= \mathfrak{H}\xi + \mathfrak{B}\eta + \mathfrak{F}\zeta + \mathfrak{M}\omega,\\
\zeta_1 &= \mathfrak{G}\xi + \mathfrak{F}\eta + \mathfrak{C}\zeta + \mathfrak{N}\omega,\\
\omega_1 &= \mathfrak{L}\xi + \mathfrak{M}\eta + \mathfrak{N}\zeta + \mathfrak{D}\omega,
\end{aligned}$$

then the coordinates of the pole of the fixed plane are as

$$\alpha_1 : \beta_1 : \gamma_1 : \delta_1,$$

and the coordinates of the pole of the variable plane are as

$$\xi_1 : \eta_1 : \zeta_1 : \delta_1,$$

whence the equations of the polar are

$$\left\|\begin{matrix} x, & y, & z, & w \\ \alpha_1, & \beta_1, & \gamma_1, & \delta_1 \\ \xi_1, & \eta_1, & \zeta_1, & \omega_1 \end{matrix}\right\| = 0,$$

a system of equations which may be thus represented

$$\begin{aligned}
\xi_1 &= \mathrm{K}\lambda x + \mu\alpha_1,\\
\eta_1 &= \mathrm{K}\lambda y + \mu\beta_1,\\
\zeta_1 &= \mathrm{K}\lambda z + \mu\gamma_1,\\
\omega_1 &= \mathrm{K}\lambda w + \mu\delta_1,
\end{aligned}$$

where K is the discriminant of the system

$$(a, b, c, d, e, f, g, h, l, m, n).$$

Write

$$\begin{aligned}
\mathrm{x} &= ax + hy + gz + lw,\\
\mathrm{y} &= hx + by + fz + mw,\\
\mathrm{z} &= gx + fy + cz + nw,\\
\mathrm{w} &= lx + my + nz + dw,
\end{aligned}$$

the last preceding system of equations may be written

$$\begin{aligned}\xi &= \lambda \mathrm{x} + \mu\alpha,\\ \eta &= \lambda \mathrm{y} + \mu\beta,\\ \zeta &= \lambda \mathrm{z} + \mu\gamma,\\ \omega &= \lambda \mathrm{w} + \mu\delta,\end{aligned}$$

equations in which λ, μ are indeterminate, and where x, y, z, w may be considered as current coordinates, and this system represents the polar above referred to. Combining the equations in question with the equation

$$\xi x + \eta y + \zeta z + \omega w = 0,$$

of the variable plane, we have

$$\lambda (x\mathrm{x} + y\mathrm{y} + z\mathrm{z} + w\mathrm{w}) + \mu (\alpha x + \beta y + \gamma z + \delta w) = 0,$$

i.e.

$$\lambda (a,\ldots) (x,\ y,\ z,\ w)^2 + \mu (\alpha x + \beta y + \gamma z + \delta w) = 0,$$

or what is the same thing

$$\lambda : \mu = \alpha x + \beta y + \gamma z + \delta w : (a, \ldots)(x,\ y,\ z,\ w)^2,$$

and substituting these values in the expressions for ξ, η, ζ, ω we have ξ, η, ζ, ω in terms of the coordinates x, y, z, w of the pole above referred to, i.e., if for shortness,

$$\begin{aligned}U &= (a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n)\ (x,\ y,\ z,\ w)^2,\\ P &= \alpha x + \beta y + \gamma z + \delta w,\end{aligned}$$

then

$$\begin{aligned}\xi &= \tfrac{1}{2} P d_x U - \alpha U,\\ \eta &= \tfrac{1}{2} P d_y U - \beta U,\\ \zeta &= \tfrac{1}{2} P d_z U - \gamma U,\\ \omega &= \tfrac{1}{2} P d_w U - \delta U,\end{aligned}$$

and combining with these equations the equation

$$(a,\ldots)\ (\xi,\ \eta,\ \zeta,\ \omega)^2 = 0,$$

we have

$$(\mathrm{a}\ldots)\ (\tfrac{1}{2} P d_x U - \alpha U,\ \tfrac{1}{2} P d_y U - \beta U,\ \tfrac{1}{2} P d_z U - \gamma U,\ \tfrac{1}{2} P d_w U - \delta U)^2 = 0,$$

for the required locus of the pole of the line of intersection of the variable plane and the fixed plane, with the conic of intersection of the given surface of the second order and the variable plane. The locus in question is a surface of the fourth order; and it may be remarked that this surface touches the given surface of the second order along the conic of intersection with the fixed plane.

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