179.

ON CERTAIN FORMS OF THE EQUATION OF A CONIC.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 44—48.]

To find the general equation of a conic which passes through two given points and touches a given line.

Let the coordinates of the given points be (α, β, γ), $(\alpha', \beta', \gamma')$, and the equation of the given line be $\lambda x+\mu y+\nu z=0$. Then writing

$$u=\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix}, \quad v=s\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}, \quad w=\begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix},$$

where a, b, c, s are arbitrary coefficients, the general equation of a conic passing through the two given points will be

$$uw-v^2=0.$$

We have identically

$$s\begin{vmatrix} \lambda x+\mu y+\nu z, & x, & y, & z \\ \lambda\alpha+\mu\beta+\nu\gamma, & \alpha, & \beta, & \gamma \\ \lambda\alpha'+\mu\beta'+\nu\gamma', & \alpha', & \beta', & \gamma' \\ \lambda a+\mu b+\nu c, & a, & b, & c \end{vmatrix}=0;$$

and hence putting

$$\nabla=\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix},$$

$$A = \quad (\lambda\alpha' + \mu\beta' + \nu\gamma')\, s,$$
$$B = -(\lambda a + \mu b + \nu c\ )\, s,$$
$$C = -(\lambda\alpha + \mu\beta + \nu\gamma\ )\, s,$$

we have

$$(\lambda x + \mu y + \nu z)\, s\nabla + Au + Bv + Cw = 0,$$

and consequently the equation $\lambda x + \mu y + \nu z = 0$ is equivalent to

$$Au + Bv + Cw = 0,$$

and we have only to express that the line represented by this equation touches the conic $uw - v^2 = 0$.

Combining the two equations, we find $Au + Cw + B\sqrt{(uw)} = 0$, that is

$$(Au + Cw)^2 - B^3 uw = 0,$$

an equation which must have equal roots; and the condition for this is obviously $4AC - B^2 = 0$. Or putting the condition under the form $-B + 2\sqrt{(AC)} = 0$ and substituting for A, B, C their values, the condition becomes $\{i = \sqrt{(-1)}$ as usual$\}$

$$\lambda a + \mu b + \nu c + 2is\sqrt{\{(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma')\}} = 0.$$

We have consequently

$$s^2 = -\frac{(\lambda a + \mu b + \nu c)^2}{4(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma')},$$

and the equation of the conic is

$$4(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma') \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix} \begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix} + (\lambda a + \mu b + \nu c)^2 \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}^2 = 0.$$

But the equation of the conic may be obtained in a different form as follows: we may first write $B^2v^2 = 4ACuw$, and then substituting for $-Bv$ the value

$$(\lambda x + \mu y + \nu z)\, s\nabla + Au + Cw,$$

the equation becomes

$$\{Au + Cw + (\lambda x + \mu y + \nu z)\, s\nabla\}^2 = 4ACuw,$$

or, extracting the root of each side and transposing,

$$\{\sqrt{(Au)} + \sqrt{(Cw)}\}^2 + (\lambda x + \mu y + \nu z)\, s\nabla = 0,$$

and thence

$$\sqrt{(Au)} + \sqrt{(Cw)} + i\sqrt{(s\nabla)}\sqrt{(\lambda x + \mu y + \nu z)} = 0,$$

or substituting the values of A, C, ∇, u, w, and omitting the common factor $\sqrt{(s)}$ the equation becomes

$$\sqrt{(\lambda\alpha' + \mu\beta + \nu\gamma')}\sqrt{\left(\begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix}\right)} + i\sqrt{(\lambda\alpha + \mu\beta + \nu\gamma)}\sqrt{\left(\begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix}\right)}$$

$$+ i\sqrt{(\lambda x + \mu y + \nu z)}\sqrt{\left(\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix}\right)} = 0,$$

a form symmetrically related to the three lines

$$\lambda x + \mu y + \nu z = 0, \quad \begin{vmatrix} x, & y, & z \\ \alpha, & \beta, & \gamma \\ a, & b, & c \end{vmatrix} = 0, \quad \begin{vmatrix} x, & y, & z \\ \alpha', & \beta', & \gamma' \\ a, & b, & c \end{vmatrix} = 0.$$

Let it be required to find the conic passing through the two points (α, β, γ), $(\alpha', \beta', \gamma')$, and touching the three lines

$$\lambda_1 x + \mu_1 y + \nu_1 z = 0, \quad \lambda_2 x + \mu_2 y + \nu_2 z = 0, \quad \lambda_3 x + \mu_3 y + \nu_3 z = 0.$$

The constants a, b, c have to be determined in such manner that the equations obtained from the preceding, by writing successively $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_2, \nu_3)$ for (λ, μ, ν) may represent one and the same equation; the three equations so obtained will therefore subsist simultaneously, and we may from the equations in question eliminate a, b, c; the resulting equation

$$\begin{vmatrix} \sqrt{(\lambda_1 x + \mu_1 y + \nu_1 z)}, & \sqrt{(\lambda_2 x + \mu_2 y + \nu_2 z)}, & \sqrt{(\lambda_3 x + \mu_3 y + \nu_3 z)} \\ \sqrt{(\lambda_1\alpha + \mu_1\beta + \nu_1\gamma)}, & \sqrt{(\lambda_2\alpha + \mu_2\beta + \nu_2\gamma)}, & \sqrt{(\lambda_3\alpha + \mu_3\beta + \nu_3\gamma)} \\ \sqrt{(\lambda_1\alpha' + \mu_1\beta' + \nu_1\gamma')}, & \sqrt{(\lambda_2\alpha' + \mu_2\beta' + \nu_2\gamma')}, & \sqrt{(\lambda_3\alpha' + \mu_3\beta' + \nu_3\gamma')} \end{vmatrix} = 0$$

is the equation of the conic in question; this is in fact evident from other considerations.

To find the condition in order that a conic passing through the points (α, β, γ), $(\alpha', \beta', \gamma')$ may touch the four lines

$$\lambda_1 x + \mu_1 y + \nu_1 z = 0, \quad \lambda_2 x + \mu_2 y + \nu_2 z = 0, \quad \lambda_3 x + \mu_3 y + \nu_3 z = 0, \quad \lambda_4 x + \mu_4 y + \nu_4 z = 0.$$

The relation first obtained between a, b, c, s gives four equations from which these quantities may be eliminated, the resulting equation

$$\begin{vmatrix} \lambda_1, & \mu_1, & \nu_1, & \surd\{(\lambda_1\alpha+\mu_1\beta+\nu_1\gamma)\ (\lambda_1\alpha'+\mu_1\beta'+\nu_1\gamma')\} \\ \lambda_2, & \mu_2, & \nu_2, & \surd\{(\lambda_2\alpha+\mu_2\beta+\nu_2\gamma)\ (\lambda_2\alpha'+\mu_2\beta'+\nu_2\gamma')\} \\ \lambda_3, & \mu_3, & \nu_3, & \surd\{(\lambda_3\alpha+\mu_3\beta+\nu_3\gamma)\ (\lambda_3\alpha'+\mu_3\beta'+\nu_3\gamma')\} \\ \lambda_4, & \mu_4, & \nu_4, & \surd\{(\lambda_4\alpha+\mu_4\beta+\nu_4\gamma)\ (\lambda_4\alpha'+\mu_4\beta'+\nu_4\gamma')\} \end{vmatrix}=0$$

is the required relation.

The preceding investigations apply directly to the circle, which is a conic passing through two given points. Thus the equation of a circle touching the three lines

$$Ax+By+C=0,$$
$$A'x+B'y+C'=0,$$
$$A''x+B''y+C''=0,$$

is

$$\begin{vmatrix} \surd(Ax+By+C), & \surd(A'x+B'y+C'), & \surd(A''x+B''y+C'') \\ \surd(A+Bi), & \surd(A'+B'i), & \surd(A''+B''i) \\ \surd(A-Bi), & \surd(A'-B'i), & \surd(A''+B''i) \end{vmatrix}=0.$$

Hence also the equation of a circle touching the three lines

$$x\cos\alpha+y\sin\alpha-p=0,$$
$$x\cos\beta+y\sin\beta-q=0,$$
$$x\cos\gamma+y\sin\gamma-r=0,$$

is

$$\sin\tfrac{1}{2}(\beta-\gamma)\surd(x\cos\alpha+y\sin\alpha-p)+\sin\tfrac{1}{2}(\gamma-\alpha)\surd(x\cos\beta+y\sin\beta-q)$$
$$+\sin\tfrac{1}{2}(\alpha-\beta)\surd(x\cos\gamma+y\sin\gamma-r)=0.$$

To rationalise the equation, I remark that an equation $\surd(A)+\surd(B)+\surd(C)=0$ gives in general

$$(1,\ 1,\ 1,\ \bar{1},\ \bar{1},\ \bar{1})\ (A,\ B,\ C)^2=0,$$

and that

$$(1,\ 1,\ 1,\ \bar{1},\ \bar{1},\ \bar{1}\between 2p\sin^2\tfrac{1}{2}(\beta-\gamma),\quad 2q\sin^2\tfrac{1}{2}(\gamma-\alpha),\ 2r\sin^2\tfrac{1}{2}(\alpha-\beta)^2)$$

or as it may also be written

$$(1,\ 1,\ 1,\ \bar{1},\ \bar{1},\ \bar{1}\between p\{1-\cos(\beta-\gamma)\},\ q\{1-\cos(\gamma-\alpha)\},\ r\{1-\cos(\alpha-\beta)\})^2,$$

is identically equal to

$$\{p(\sin\beta-\sin\gamma)+q(\sin\gamma-\sin\alpha)+r(\sin\alpha-\sin\beta)\}^2$$
$$+\{p(\cos\beta-\cos\gamma)+q(\cos\gamma-\cos\alpha)+r(\cos\alpha-\cos\beta)\}^2$$
$$-\{p\sin(\beta-\gamma)+q\sin(\gamma-\alpha)+r\sin(\alpha-\beta)\}^2.$$

Hence if we replace p, q, r by

$$x\cos\alpha + y\sin\alpha - p,\quad x\cos\beta + y\sin\beta - q,\quad x\cos\gamma + y\sin\beta - r,$$

the last-mentioned expression equated to zero will give the equation of the circle, and we obtain

$$\begin{aligned}&\{\nabla x + p(\sin\beta - \sin\gamma) + q(\sin\gamma - \sin\alpha) + r(\sin\alpha - \sin\beta)\}^2\\ &+ \{\nabla y - p(\cos\beta - \cos\gamma) - q(\cos\gamma - \cos\alpha) - r(\cos\alpha - \cos\beta)\}^2\\ &- \{p\sin(\beta - \gamma) + q\sin(\gamma - \alpha) + r\sin(\alpha - \beta)\}^2 = 0,\end{aligned}$$

where

$$\nabla = \sin(\beta - \gamma) + \sin(\gamma - \alpha) + \sin(\alpha - \beta),$$

and we have thus the equation of the circle in the usual form with the coordinates of the centre and the radius put in evidence.

The condition that there may be a circle touching the four lines

$$\begin{aligned}Ax + By + C &= 0,\\ A'x + B'y + C' &= 0,\\ A''x + B''y + C'' &= 0,\\ A'''x + B'''y + C''' &= 0,\end{aligned}$$

is by the general formula shown to be

$$\begin{vmatrix} A, & B, & C, & \sqrt{(A^2 + B^2)} \\ A', & B', & C', & \sqrt{(A'^2 + B'^2)} \\ A'', & B'', & C'', & \sqrt{(A''^2 + B''^2)} \\ A''', & B''', & C''', & \sqrt{(A'''^2 + B'''^2)} \end{vmatrix} = 0,$$

which is in fact obvious from other considerations.