VIII.

NOTE ON THE APPLICATION OF THE METHOD OF IMAGES TO PROBLEMS OF VIBRATIONS

« Proc. of the London Math. Soc. », Ser. 2, vol. 2, 1904, pp. 327–331.

Lord KELVIN'S method of images has yielded the solution of a large number of problems in electrostatics, in the theory of steady electric currents, in hydrodynamics (STOKES, HICKS). It has also been applied with success to problems of elastic equilibrium by BETTI, CERRUTI, SOMIGLIANA, and others. In all these cases the partial differential equations that are involved are of *elliptic* type. Applications of the method have been made also by BETTI to problems of the conduction of heat involving differential equations of *parabolic* type. I propose now to explain an application of the method to a question concerning the vibrations of elastic bodies in which, naturally, the differential equations are of *hyperbolic* type [1].

The result to which I wish to call especial attention is that in this latter case the application of the method of successive images leads to solutions containing a finite number of terms—not to infinite series—even in the case where the number of images is infinite. This result may seem somewhat paradoxical, but it will be shown to depend upon the fact that it is not necessary to use all the images of the infinite train, but only, in each case, a finite number of them. The reason for this simplification is to be sought in the fact that the characteristic surfaces of the partial differential equations of hyperbolic type are real, and thus the property in question is bound up with the hyperbolic character of the equations, and is, consequently, capable of a very wide application.

In my paper in the « Acta Mathematica », t. XVIII [*], there was given the solution of the following problem among others:

In the differential equation of vibrating membranes, viz.,

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2},$$

(1) M. HADAMARD in a very interesting paper published in the « Bulletin de la Société Mathématique de France », has also utilised the method of images in a problem of vibrations, but he has not called attention to the point which is chiefly emphasised in the present note.

(*) In queste « Opere »: vol. secondo, III, pp. 19–73 [N. d. R.].

consider x, y, t as Cartesian coordinates of a point in a space of three dimensions, and let the values of w and its normal derivative $\partial w/\partial n$ be given on a surface σ in this space. It is required to determine, whenever possible, the value of w.

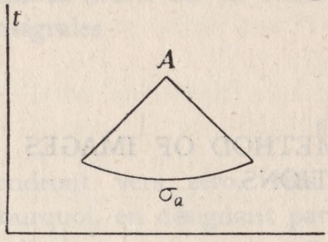

Fig. 1.

The solution is as follows:–With any point $A(x_1, y_1, t_1)$ of the space as vertex draw a right circular cone of vertical angle 90° with its axis parallel to the axis of t. Let that sheet of the cone – the "inferior" sheet, say – in which $t < t_1$ cut out on the surface σ a portion σ_a, as in fig. 1. Let a positive quantity r be determined by the equation

$$r^2 = (x - x_1)^2 + (y - y_1)^2.$$

Then w is expressed by the formula

$$\text{(A)}\quad w(x_1, y_1, t_1) = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\left\{\cos(n, t) - \frac{t_1 - t}{r}\cos(n, r)\right\} w\, d\sigma_a$$

$$+ \frac{1}{2\pi}\int_{\sigma_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\left[\frac{\partial w}{\partial t}\cos(n, t) - \left\{\frac{\partial w}{\partial x}\cos(n, x) + \frac{\partial w}{\partial y}\cos(n, y)\right\}\right] d\sigma_a\,.$$

If the surface σ is the plane of (x, y), this formula reduces to that of Poisson.

Now suppose that the surface σ is formed by that portion of the plane of (x, y) which is defined by $x > 0$ and that portion of the plane of (t, y) which is defined by $t > 0$. Tehn σ_a may be bounded by a circle in the plane of (x, y), or it may be bounded partly by an arc of a circle in the plane of (x, y) and partly by an arc of an hyperbola in the plane of (t, y). The latter case is represented in Fig. 2, which shows the trace of the cone and planes upon the plane of (x, t). The surface σ consists of a portion σ'_a bounded by a circle, and a portion σ''_a bounded by an hyperbola. The formula (A) becomes

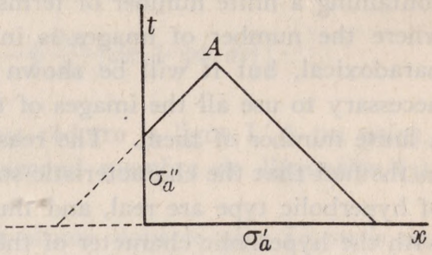

Fig. 2.

$$\text{(B)}\quad w(x_1, y_1, t_1) = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma'_a}\frac{1}{\sqrt{(t_1^2 - r^2)}}\, w\, d\sigma'_a + \frac{1}{2\pi}\int_{\sigma'_a}\frac{1}{\sqrt{(t_1^2 - r^2)}}\frac{\partial w}{\partial t}\, d\sigma'_a$$

$$- \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma''_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\frac{t_1 - t}{r}(\cos n, r)\, w\, d\sigma''_a - \frac{1}{2\pi}\int_{\sigma''_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\frac{\partial w}{\partial x}\, d\sigma''_a\,.$$

The values of w and $\partial w/\partial t$ on σ'_a are arbitrary, but those of w and $\partial w/\partial x$ on σ''_a cannot be given arbitrarily; if the values of either w or $\partial w/\partial x$ on σ''_a

are given, those of the other are determined thereby. We must therefore seek to eliminate from the formula (B) either the values of w or those of $\partial w / \partial x$ on the surface σ_a'', as is done in analogous cases where the methods of GREEN are employed. For this purpose we may have recourse to the method of images. Suppose, in fact, that, as shown in Fig. 3, A′ is the optical image of the point A in the plane of (y, t).

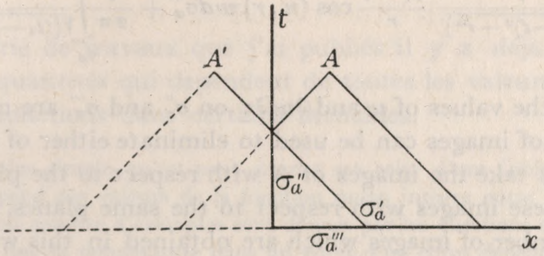

Fig. 3.

The inferior sheet of a cone drawn from A′ in the same way as the former cone was drawn from A will cut out on the plane of (h, t) an area σ_a'' bounded by an arc of an hyperbola, and it it will cut out on the plane of (x, y) an area σ_a''' bounded by an arc of a circle, and we shall have the formula

$$\text{(C)} \qquad 0 = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a'''}\frac{1}{\sqrt{(t_1^2 - r'^2)}}\, w\, d\sigma_a''' + \frac{1}{2\pi}\int_{\sigma_a'''}\frac{1}{\sqrt{(t_1^2 - r'^2)}}\frac{\partial w}{\partial t}\, d\sigma_a'''$$

$$+ \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a''}\frac{1}{\sqrt{\{(t_1 - t)^2 - r'^2\}}}\,\frac{t_1 - t}{r}\cos(n, r)\, w\, d\sigma_a'' - \frac{1}{2\pi}\int_{\sigma_a''}\frac{1}{\sqrt{\{(t_1 - t)^2 - r'^2\}}}\frac{\partial w}{\partial x}\, d\sigma_a'',$$

where

$$r'^2 = (x + x_1)^2 + (y - y_1)^2.$$

By adding the formulæ (B) and (C) we eliminate w; by subtracting (C) from (B) we eliminate $\partial w / \partial x$. Thus the method of images leads easily to the desired result.

We proceed to consider successive images. Suppose that the surface σ is formed by the strip σ' of the plane of (x, y) defined by $a > x > 0$, and by the two half planes $(x = 0, t > 0)$ and $(x = a, t > 0)$, denoted respectively by σ'' and σ'''. It is clear that,

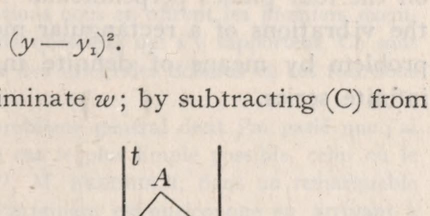

Fig. 4.

if the cone with its vertex at A cuts out on σ' the area σ_a, and on σ'' and σ''' the areas σ_a'' and σ_a''' bounded by arcs of hyperbolas, the formula (A) becomes

$$w(x_1, y_1, t_1) = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma'_a}\frac{1}{\sqrt{(t_1^2 - r^2)}} w \, d\sigma_a + \frac{1}{2\pi}\int_{\sigma'_a}\frac{1}{\sqrt{(t_1^2 - r^2)}}\frac{\partial w}{\partial t} d\sigma'_a$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma''_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\frac{t_1 - t}{r}\cos(n, r)\, w \, d\sigma''_a - \frac{1}{2\pi}\int_{\sigma''_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\frac{\partial w}{\partial x} d\sigma''_a$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma'''_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\frac{t_1 - t}{r}\cos(n, r)\, w \, d\sigma'''_a + \frac{1}{2\pi}\int_{\sigma'''_a}\frac{1}{\sqrt{\{(t_1 - t)^2 - r^2\}}}\frac{\partial w}{\partial x} d\sigma'''_a.$$

In this formula the values of w and $\partial w/\partial x$ on σ''_a and σ'''_a are not independent, and the method of images can be used to eliminate either of them. For this purpose we must take the images of A with respect to the planes σ'' and σ''', the images of these images with respect to the same planes, and so on indefinitely. The number of images which are obtained in this way is, of course, infinite, but it is important to observe that it is not necessary to use them all. A finite number of them suffices for the solution of the problem.

With each of the images as vertex draw a cone of vertical angle 90° and with its axis parallel to the axis of t, and take the inferior sheets of these cones. If the vertex of any one is at a distance from σ'' or σ''' which exceeds t_1, that one does not cut these half planes, and it may be omitted. It is clear that the values of w or of $\partial w/\partial x$ on σ''_a and σ'''_a can be eliminated from the preceding formulæ by taking account of those images only which are at a less distance than t_1 from σ'' and σ'''.

We conclude from this argument that the method of images does not in this case lead to infinite series, but to solutions with a finite number of terms. In place of the three planes σ'', σ' and σ''' we might have five, of which four are perpendicular to σ and cut it in a rectangle. If w vanishes on the four planes perpendicular to σ, we have the well known problem of the vibrations of a rectangular membrane, and we have a solution of this problem by means of definite integrals without having to introduce any infinite series.