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NOTE ON JACOBI'S CANONICAL FORMULÆ FOR DISTURBED MOTION IN AN ELLIPTIC ORBIT.

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CONSIDER a body (afterwards called the disturbed body) revolving about a central body under the influence of their mutual attraction and of any disturbing forces. Then referring the disturbed body to axes through the central body and parallel to fixed lines in space, write

x, y, z, the coordinates of the disturbed body,

r, , the radius vector, $= \sqrt{(x^2 + y^2 + z^2)}$,

M , the mass of the disturbed body,

M'' , the mass of the central body.

Write also

R, the disturbing function, taken negatively, i.e. the sign of R is taken as in the Mécanique Céleste.

The equations of motion then are

$$\frac{d^2x}{dt^2} = -\frac{(M + M'')\,x}{r^3} - \frac{dR}{dx},$$

$$\frac{d^2y}{dt^2} = -\frac{(M + M'')\,y}{r^3} - \frac{dR}{dy},$$

$$\frac{d^2z}{dt^2} = -\frac{(M + M'')\,z}{r^3} - \frac{dR}{dz},$$

and the motion may be represented by supposing that the body moves in an ellipse with variable elements, and such that the direction and velocity of motion are always the same as in the actual orbit.

We may, to fix the ideas, take the plane of xy to be the plane of the ecliptic and the axis of x to be the line through the first point of Aries, or origin of longitude.

Jacobi's canonical elements may be taken to be,

First, the constant of vis viva or invariable part of the half-square of the velocity, which is equal to the sum of the masses divided by twice the mean distance and with the sign minus.

Secondly, the constant of areas, which is equal to the square root of the sum of the masses into the half of the latus rectum.

Thirdly, the constant of the reduced area (i.e. of the area described on the plane of the ecliptic), which is equal to the square root of the sum of the masses into the half of the latus rectum into the cosine of the inclination.

Fourthly, the constant attached by addition to the time t, or what is the same thing, the epoch or time of pericentric passage, taken with the sign minus.

Fifthly, the angular distance from node (or argument of latitude) of the pericentre.

Sixthly, the longitude of the node.

(The first and fourth elements are taken by Jacobi with the contrary sign, but this difference is not material.)

Representing the preceding system of canonical elements by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$, and observing that Jacobi's disturbing function Ω is the same as the disturbing function R of the Mécanique Céleste, except that the sign is reversed (i.e. $\Omega = -R$), the expressions for the variations of the canonical elements are

$$\frac{d\mathfrak{A}}{dt} = -\frac{dR}{d\mathfrak{F}}, \quad \frac{d\mathfrak{B}}{dt} = -\frac{dR}{d\mathfrak{G}}, \quad \frac{d\mathfrak{C}}{dt} = -\frac{dR}{d\mathfrak{H}},$$

$$\frac{d\mathfrak{F}}{dt} = +\frac{dR}{d\mathfrak{A}}, \quad \frac{d\mathfrak{G}}{dt} = +\frac{dR}{d\mathfrak{B}}, \quad \frac{d\mathfrak{H}}{dt} = +\frac{dR}{d\mathfrak{C}}.$$

In the ordinary case in which the disturbing force is the attraction of a third body, write

x', y', z', the coordinates of the disturbing body,

r' , the radius vector, $= \sqrt{(x'^2 + y'^2 + r'^2)}$,

M' , the mass of the disturbing body.

Then the expression for the disturbing function is

$$R = M' \left\{ \frac{xx' + yy' + zz'}{r'^3} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right\}.$$

The preceding formulæ form a convenient standard of reference for the various systems of elements which have been made use of by writers upon Physical Astronomy; any such system may be without difficulty derived from the canonical system by expressing the elements adopted in terms of the above-mentioned canonical elements.