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NOTE ON MR SALMON'S EQUATION OF THE ORTHOTOMIC CIRCLE.

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LET $U_1=0$, $U_2=0$, $U_3=0$ be the equations of three circles, and let V be the functional determinant of U_1, U_2, U_3, the functions being in the first instance made homogeneous by the introduction of a variable z, which is ultimately replaced by unity; then the equation of the circle cutting at right angles the three given circles, or, as it may be called, the orthotomic circle, is $V=0$. This elegant theorem of Mr Salmon's is connected with the theory developed by Hesse in the memoir, "Ueber die Wendepuncte der Curven dritter Ordnung," *Crelle*, t. XXVIII. (1844), p. 97).

In fact, let $U_1=0$, $U_2=0$, $U_3=0$ be the equations of three conics, the locus of a point such that its polars with respect to each of these conics, or indeed with respect to any conic having for its equation $\lambda U_1+\mu U_2+\nu U_3=0$ (where λ, μ, ν are arbitrary), pass through the same point, is a curve of the third order $V=0$, where V is the functional determinant of U_1, U_2, U_3.

Conversely, if the curve of the third order $V=0$ be given, and U be a function of the third order, such that the functional determinant of $\frac{dU}{dx}$, $\frac{dU}{dy}$, $\frac{dU}{dz}$, or, what is the same thing, the "Hessian" of the function U is equal to V, a condition which may be written $V=H(U)$, then we may take for the conics any three conics the equations of which are of the form $\lambda\frac{dU}{dx}+\mu\frac{dU}{dy}+\nu\frac{dU}{dz}=0$. The equation $V=H(U)$ affords the means of determining U; in fact, we shall have $U=aV+bH(V)$, where a and b are constants to be determined. This gives $H(U)=H(aV+bH(V))=AV+BH(V)$, where A and B are given functions of a, b (a practical method of determining these functions was first given in Aronhold's memoir, "Zur Theorie der homogenen Functionen dritten Grades von zwei Variabeln," *Crelle*, t. XXXIX. (1850), pp. 140—159); and we have therefore

$V = AV + BH(V)$, i.e. $A=1$, $B=0$: the latter equation determines, what is alone important, the ratio $b : a$; the equation is of the third order, so that there are in general three distinct solutions $U = aV + BH(V) = 0$.

In the particular case in which the curves of the third order $V=0$ is made up of a line $P=0$ and a conic $W=0$, i.e. where $V = PW = 0$, the curve $H(V)=0$ is made up of the same line $P=0$ and of a conic having double contact with the conic $W=0$ at the point of intersection with the line $P=0$, i.e. $H(PW) = P(lW + mP^2)$. And $U = aPW + bH(PW)$ is consequently a function of the same form, i.e. the cubic $U=0$ is made up of the line $P=0$ and of a conic having double contact with the conic $W=0$ at its points of intersection with the line $P=0$. We may therefore write $U = P(fW + gP^2)$, and forming with this value the equation $PW = P(FW + GP^2)$, it may be noticed that, owing to the occurrence of a special factor which may be rejected, the resulting equation $G=0$ gives only a single value for the ratio $f : g$. Forming from the value $U = P(fW + gP^2)$, the equation $\lambda \frac{dU}{dx} + \mu \frac{dU}{dy} + \nu \frac{dU}{dz} = 0$, the equation thus obtained will be of the form $W + PQ = 0$, which is the equation of a conic passing through the points of intersection of the line and conic $P=0$, $W=0$, and besides intersecting the conic $W=0$ in two other points. And it may also be shown that the four points of intersection, (i.e. the points given by the equations $W=0$, $W+PQ=0$), the pole of the line $P=0$ with respect to the conic $W=0$, and the pole of this same line with respect to the conic $W+PQ=0$, lie all six in the same conic. We see, therefore, that, given a curve of the third order, the aggregate of a line $P=0$ and a conic $W=0$, as the locus of the point such that its polars, with respect to three several conics (or a system depending on three conics), meet in a point, each conic of the system is a conic passing through the points of intersection of the line and conic $P=0$, $W=0$, and, moreover, such that the four points of intersection with the conic $W=0$ and the poles of the line $P=0$, with respect to the conic of the system, and with respect to the conic $W=0$, lie all six in the same conic. In the particular case in which the line and conic $P=0$, $W=0$ are the line at ∞ and a circle, each conic of the system is a circle such that its points of intersection with the circle $W=0$ and the centres of the two circles lie in a circle, i.e. the conics are circles cutting at right angles the circle $W=0$, which agrees with Mr Salmon's theorem.

To verify the assumed theorems in the case of the curve of the third order $V = PW = 0$, we may take

$$P = \alpha x + \beta y + \gamma z = 0$$

for the equation of the line, and

$$W = x^2 + y^2 + z^2 = 0$$

for the equation of the conic. I write, for greater convenience, $U = P\left(\tfrac{1}{2} fW + \tfrac{1}{6} gP^2\right)$; the Hessian of this is

$$\begin{vmatrix} f(P + 2\alpha x) + g\alpha^2 P, & f(\beta x + \alpha y) + g\alpha\beta P, & f(\alpha z + \gamma x) + g\gamma\alpha P \\ f(\beta x + \alpha y) + g\alpha\beta P, & f(P + 2\beta y) + g\beta^2 P, & f(\gamma y + \beta z) + g\beta\gamma P \\ f(\alpha z + \gamma x) + g\gamma\alpha P, & f(\gamma y + \beta z) + g\beta\gamma P, & f(P + 2\gamma z) + g\gamma^2 P \end{vmatrix},$$

which is equal to

$$f^3P\left(4P^2-(\alpha^2+\beta^2+\gamma^2)\,W\right)+f^2gP\,(\alpha^2+\beta^2+\gamma^2)\,P^2,$$

i.e. we must have $4f+g\,(\alpha^2+\beta^2+\gamma^2)=0$, or putting $g=-24$ and $\therefore f=6\,(\alpha^2+\beta^2+\gamma^2)$, we have

$$U=P\left(3\,(\alpha^2+\beta^2+\gamma^2)\,W-4P^2\right).$$

Forming the function $\lambda\dfrac{dU}{dx}+\mu\dfrac{dU}{dy}+\nu\dfrac{dU}{dz}$, and dividing by the constant factor $3\,(\alpha^2+\beta^2+\gamma^2)(\lambda\alpha+\mu\beta+\nu\gamma)$, we have for the equation of any one of the conics

$$W+P\left\{\frac{2\,(\lambda x+\mu y+\nu z)}{\lambda\alpha+\mu\beta+\nu\gamma}-\frac{4P}{\alpha^2+\beta^2+\gamma^2}\right\}=0,$$

which may be written under the form $W+2PQ=0$, where

$$Q=ax+by+cz=\frac{\lambda x+\mu y+\nu z}{\lambda\alpha+\mu\beta+\nu\gamma}-\frac{2\,(\alpha x+\beta y+\gamma z)}{\alpha^2+\beta^2+\gamma^2}.$$

We have therefore $a\alpha+b\beta+c\gamma=1-2=-1$, i.e. $a\alpha+b\beta+c\gamma+1=0$. And there is no difficulty in showing that, given the two conics

$$x^2+y^2+z^2=0,$$

$$x^2+y^2+z^2+2\,(\alpha x+\beta y+\gamma z)(ax+by+cz)=0,$$

the condition in order that the four points of intersection and the poles, with respect to each conic, of the line $\alpha x+\beta y+\gamma z=0$, may lie in a conic is precisely this equation $a\alpha+b\beta+c\gamma+1=0$.