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APROPOS OF PARTITIONS.

[From the *Quarterly Mathematical Journal*, vol. I. (1857), pp. 183—184.]

LET $\Pi(1-x^a) = (1-x^a)\ (1-x^b)\ldots(\kappa \text{ factors})$ and assume that $\left[\dfrac{1}{\Pi(1-x^a)}\right]_{1-x}$ is the part of $\dfrac{1}{\Pi(1-x^a)}$ which involves negative powers of $1-x$, then

$$\text{Coefficient } x^q \text{ in } \left[\frac{1}{\Pi(1-x^a)}\right]_{1-x} = \text{coefficient } z^{\kappa-1} \text{ in } \left(\frac{1}{(1-z)^{q+1}}\,\Pi\,\frac{z}{1-(1-z)^a}\right),$$

which suggests the question of the expansion in powers of z, of the function

$$\frac{1}{(1-z)^{q+1}}\,\Pi\,\frac{z}{1-(1-z)^a}.$$

Now by the definition of Bernoulli's numbers

$$\frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + B_1\frac{t}{1\,.\,2} - B_2\frac{t^3}{1\,.\,2\,.\,3\,.\,4} + B_3\frac{t^5}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6} - \&c.$$

from which it is easy to deduce

$$\frac{t}{1-e^{-t}} = e^{\frac{1}{2}t - B_1\frac{t^2}{1\,.\,2^2} + B_2\frac{t^4}{1\,.\,2\,.\,3\,.\,4^2} - B_3\frac{t^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6^2} + \&c.};$$

and, writing in this formula $t = -a\log(1-z)$, we have

$$\frac{-a\log(1-z)}{1-(1-z)^a} = e^{-\frac{1}{2}\log(1-z)\,a - B_1\frac{\log^2(1-z)}{1\,.\,2^2}a^2 + B_2\frac{\log^4(1-z)}{1\,.\,2\,.\,3\,.\,4^2}a^4 + \&c.},$$

i.e.

$$\frac{z}{1-(1-z)^a} = \frac{1}{a}\left(\frac{-z}{\log(1-z)}\right) e^{-\frac{1}{2}\log(1-z)\,a - \&c.} = \frac{1}{a} e^{\log\left(\frac{-z}{\log(1-z)}\right) - \frac{1}{2}\log(1-z)\,a - \&c.},$$

and putting p_κ for $abc\ldots$ and S_1, $S_2\ldots$ for the sums of the powers, we have, taking the product

$$\Pi \frac{z}{1-(1-z)^a} = \frac{1}{p_\kappa} e^{\kappa\log\left(\frac{-z}{\log(1-z)}\right) - \frac{1}{2}\log(1-z)\,S_1 - B_1\frac{\log^2(1-z)}{1\,.\,2^2}S_2 + B_2\frac{\log_4(1-z)}{1\,.\,2\,.\,3\,.\,4^2}S_4 - \&c.},$$

whence also

$$\frac{1}{(1-z)^{q+1}}\Pi \frac{z}{1-(1-z)^a} = \frac{1}{p_\kappa} e^{\kappa\log\left(\frac{-z}{\log(1-z)}\right) - (q+1+\frac{1}{2}S_1)\log(1-z) - B_1\frac{\log^2(1-z)}{1\,.\,2^2}S_2 + B_2\frac{\log^4(1-z)}{1\,.\,2\,.\,3\,.\,4^2}S_4 - \&c.}$$

from which the development may be found.

The index of e is

$$\begin{aligned} &(q+1-\tfrac{1}{2}\kappa+\tfrac{1}{2}S_1)\,z \\ &+(\tfrac{1}{2}q+\tfrac{1}{2}-\tfrac{5}{24}\kappa+\tfrac{1}{4}S_1-\tfrac{1}{24}S_2)\,z^2 \\ &+(\tfrac{1}{3}q+\tfrac{1}{3}-\tfrac{1}{8}\kappa+\tfrac{1}{6}S_1-\tfrac{1}{24}S_2)\,z^3 \\ &+\&c. \end{aligned}$$

and developing the exponential,

1°. The coefficient of z is

$$q+\tfrac{1}{2}S_1-\tfrac{1}{2}(\kappa-2).$$

2°. The coefficient of z^2 is

$$\tfrac{1}{2}q^2+q\,\{\tfrac{1}{2}S_1-\tfrac{1}{2}(\kappa-3)\}+\tfrac{1}{8}S_1^2-\tfrac{1}{24}S_2-\tfrac{1}{4}(\kappa-3)\,S_1+(\kappa-3)\,(\tfrac{1}{8}\kappa-\tfrac{1}{3}),$$

and so on.

The peculiarity is the appearance of the factors $\kappa-2$, $\kappa-3$, &c. If we neglect these terms, and consider as well q as $a, b, c\ldots$ to be each of them of the dimension unity, the coefficients will be homogeneous.