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ON GAUSS' METHOD FOR THE ATTRACTION OF ELLIPSOIDS.

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THE following is the method employed in Gauss' Memoir "Theoria Attractionis Corporum Sphæroidicorum ellipticorum homogeneorum methodo novo tractata," 1813. *Comm. Gott. recent.*, t. II. [and *Werke* t. VI. pp. 1—22]. I have somewhat developed the geometrical considerations upon which the method depends.

The attraction of the ellipsoid is found by means of the following theorems, which apply generally to the case of a homogeneous solid bounded by a closed surface:— M denotes the attracted point, P a point of the surface, PQ is the normal (lying outside the surface) at the point P, dS is the element of the surface at this point, MQ, QX, and MX denote angles at the point P, viz. MQ the $\angle MPQ$, and QX and MX the inclinations of QP and MP respectively to a line PX drawn in a direction assumed as that of the axis of X, $\overline{MP}$ denotes the distance between the points M and P. And X is the attraction in the direction opposite to that of the axis of x; the integrations extend over the entire surface.

THEOREM. The integral

$$\iint \frac{dS \cos MQ}{MP^2}$$

has for its value

$$0, \quad -2\pi, \text{ or } -4\pi,$$

according as M is exterior to, upon, or interior to, the surface.

This is obviously a purely geometrical theorem.

THEOREM. The attraction is given by the formula

$$X = \iint \frac{dS \cos QX}{MP}.$$

THEOREM. The attraction is also given by the formula

$$X = -\iint \frac{dS \cos MQ \cos MX}{\overline{MP}}.$$

Consider now an ellipsoid, the semi-axes of which are A, B, C, and putting a, b, c for the coordinates of the attracted point M, and x, y, z for the coordinates of P, assume

$$x = A\xi,$$
$$y = B\eta,$$
$$z = C\zeta,$$

so that ξ, η, ζ are the coordinates of a point P' on a sphere, radius unity, corresponding in a definite manner to the point P on the ellipsoid; and let $d\sigma$ be the corresponding element of the spherical surface, we have

$$dS = \frac{ABC}{p}\, d\sigma,$$

where p denotes the perpendicular let fall from the centre of the ellipsoid upon the tangent plane at P.

Moreover,

$$\cos QX = \frac{p\xi}{A}, \quad \cos MX = \frac{a-x}{\overline{MP}} = \frac{a - A\xi}{\overline{MP}};$$

and therefore

$$dS \cos QX = \frac{BC\xi d\sigma}{\overline{MP}}.$$

The second theorem gives therefore

$$A\ \frac{X}{ABC} = \iint \frac{\xi d\sigma}{\overline{MP}},$$

where the integration is extended over the surface of the sphere, and the third theorem gives

$$\frac{X}{ABC} = -\iint \frac{(a - A\xi) \cos MQdS}{\overline{MP}^2},$$

where the integration is extended over the surface of the ellipsoid.

Suppose now a confocal ellipsoid, the semi-axes of which are $A + \delta A$, $B + \delta B$, $C + \delta C$, and let $P_{\prime}$ be the point on this new ellipsoid which *corresponds* to the point P on the original ellipsoid, i.e. let $P_{\prime}$ be the point whose coordinates are $(A + \delta A)\,\xi$, $(B + \delta B)\,\eta$, $(C + \delta C)\,\zeta$; the decrement of MP will be equal to the normal distance δN between the two ellipsoids at the point P, multiplied into the cosine of the angle MQ, and we have, by a property of confocal ellipsoids, $A\delta A = B\delta B = C\delta C = p\delta N$; we have therefore

$$\delta\, \overline{MP} = -\frac{A\delta A \cos MQ}{p},$$

which gives

$$-d\sigma\,.\,\delta\overline{MP} = \frac{A\delta A}{ABC}\, dS \cos MQ.$$

Now from the equation

$$A\,\frac{X}{ABC} = \iint \frac{\xi d\sigma}{\overline{MP}},$$

we find

$$A\,\delta\,\frac{X}{ABC} + \frac{X}{ABC}\,\delta A = \iint \frac{\xi d\sigma\delta\overline{MP}}{\overline{MP}^2}$$

$$= \frac{\delta A}{ABC}\iint \frac{A\xi\cos MQdS}{\overline{MP}^2}.$$

But

$$-\frac{X}{ABC}\,\delta A = \frac{\delta A}{ABC}\iint \frac{(a - A\xi)\cos MQdS}{\overline{MP}^2},$$

and consequently

$$A\,\delta\,.\,\frac{X}{ABC} = \frac{a\delta A}{ABC}\iint \frac{\cos MQdS}{\overline{MP}^2}.$$

Hence, by the first theorem:

In the case of an exterior point, we have

$$\delta\,.\,\frac{X}{ABC} = 0,$$

i.e. the attractions, in the directions of the axes, of confocal ellipsoids vary as the masses; which is Maclaurin's theorem for the attractions of ellipsoids upon an exterior point.

In the case of an interior point, we have

$$\delta\,.\,\frac{X}{ABC} = -\,4\pi\,\frac{a\delta A}{A^2BC};$$

or, taking α, β, γ as the semi-axes of an ellipsoid confocal with the ellipsoid (A, B, C), but exterior to it, and supposing that (X) refers to the ellipsoid (α, β, γ), we have

$$\delta\,\frac{(X)}{\alpha\beta\gamma} = -\,4\pi a\,\frac{\delta\alpha}{\alpha^2\beta\gamma}.$$

Now introducing instead of α the new variable θ, such that $\alpha^2 = A^2 + \theta$, we have $\frac{\delta\alpha}{\alpha^2} = \frac{\delta\theta}{(A^2+\theta)^3}$, $\beta = (B^2+\theta)^{\frac{1}{2}}$, $\gamma = (C^2+\theta)^{\frac{1}{2}}$, and consequently writing d for δ,

$$d\,\frac{(X)}{\alpha\beta\gamma} = -\,4\pi a\,\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}\,(B^2+\theta)^{\frac{1}{2}}\,(C^2+\theta)^{\frac{1}{2}}},$$

and thence, effecting the integration,

$$\frac{X_{,}}{(A^2+\theta_{,})^{\frac{1}{2}}(B^2+\theta_{,})^{\frac{1}{2}}(C^2+\theta_{,})^{\frac{1}{2}}}-\frac{X}{ABC}=-4\pi a\int_0^{\theta_{,}}\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}(B^2+\theta)^{\frac{1}{2}}(C^2+\theta)^{\frac{1}{2}}},$$

where $X_{,}$ refers to the ellipsoid whose semi-axes are $(A^2+\theta_{,})^{\frac{1}{2}}$, $(B^2+\theta_{,})^{\frac{1}{2}}$, $(C^2+\theta_{,})^{\frac{1}{2}}$. In the case where $\theta_{,}=\infty$, we have

$$\frac{X_{,}}{(A^2+\theta_{,})^{\frac{1}{2}}(B^2+\theta_{,})^{\frac{1}{2}}(C^2+\theta_{,})^{\frac{1}{2}}}=0,$$

and consequently

$$X=4\pi a\,ABC\int_0^{\infty}\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}(B^2+\theta)^{\frac{1}{2}}(C^2+\theta)^{\frac{1}{2}}},$$

which is the expression for the attraction, in the direction opposite to that of the axis of x, of the ellipsoid (A, B, C) upon an interior point, the coordinates of which are (a, b, c).

In the case of an exterior point, let $A_{,}$, $B_{,}$, $C_{,}$ be the semi-axes of the confocal ellipsoid passing through the attracted point; so that putting $A_{,}=\sqrt{(A^2+\eta)}$, $B_{,}=\sqrt{(B^2+\eta)}$, $C_{,}=\sqrt{(C^2+\eta)}$, we have

$$\frac{a^2}{A^2+\eta}+\frac{b^2}{B^2+\eta}+\frac{c^2}{C^2+\eta}=1,$$

the attraction is equal to $\dfrac{ABC}{A_{,}B_{,}C_{,}}\times$ attraction of the ellipsoid which passes through the point, i.e.

$$X=4\pi a\,ABC\int_0^{\infty}\frac{d\theta}{(A_{,}^2+\theta)^{\frac{3}{2}}(B_{,}^2+\theta)^{\frac{1}{2}}(C_{,}^2+\theta)^{\frac{1}{2}}},$$

or, putting $\theta-\eta$ instead of θ,

$$X=4\pi a\,ABC\int_{\eta}^{\infty}\frac{d\theta}{(A^2+\theta)^{\frac{3}{2}}(B^2+\theta)^{\frac{1}{2}}(C^2+\theta)^{\frac{1}{2}}},$$

which is the expression for the attraction upon an exterior point. The formulæ coincide, as they ought to do, in the case of a point upon the surface.

2, *Stone Buildings*, *9th April*, 1855.