Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 5, pp. 801–810, Warszawa 1973

Dislocation effects in elastic structured solids

D. R. AXELRAD, J. W. PROVAN and S. EL HELBAWI (MONTREAL)

THE INFLUENCE of mobile line dislocations, anchored to slip planes in isotropic elastic crystals on the linear elastic response of a polycrystalline solid, is investigated according to the random theory of deformation in this paper. The formulation incorporates the inactivated Frank-Read source as its fundamental mechanism and examines its effect on the response characteristics through the use of statistical correlation theory. The independent parameters used in the correlation analysis are the slip plane orientation and the direction of the dislocation line segments which differ from crystal to crystal in a random fashion. The relation between this linear theory and other theories currently being discussed in the literature are investigated.

W oparciu o losową teorię odkształcenia przeanalizowano w pracy wpływ ruchomych linii dyslokacji zakotwionych w płaszczyznach poślizgu sprężystych kryształów izotropowych na liniową charakterystykę ciała polikrystalicznego. Sformułowanie teorii opiera się na podstawowym mechanizmie źródeł Franka-Reada i analizuje ich wpływ na charakterystykę reakcji materiału za pomocą statystycznej teorii korelacji. Za niezależne parametry analizy korelacyjnej przyjęto orientację płaszczyzn poślizgu oraz kierunki odcinków linii dyslokacji zmieniające się w sposób losowy od kryształu do kryształu. Zbadano związek tej teorii liniowej z innymi teoriami aktualnie rozważanymi w literaturze tego zagadnienia.

На основе вероятностной теории деформаций исследовано влияние подвижных линий дислокаций, закрепленных в плоскостях скольжения изотропных кристаллов, на линейную характеристику поликристаллического тела. Формулировка теории исходит из основного механизма источников Франка-Рида. Анализ влияния этих источников на характеристику реакции материала выполнен при помощи статистической теории корреляции. В качестве независимых параметров в корреляционном анализе приняты: ориентация плоскостей скольжения и направления отрезков дислокационных линий, изменяющиеся случайным образом при переходе от одного кристалла к другому. Изучена связь предложенной теории с другими теориями, обсуждаемыми в последнее время в научной литературе, посвященной этим вопросам.

1. Introduction

METALS comprise of crystals separated from each other by boundaries. Each crystal is a three-dimensional network of atoms, arranged in unit cells possessing an order which tends to reduce their potential energy to a minimum. Due to reasons as yet not fully understood, the lattice of actual crystals is never perfect, being disturbed by the presence of vacancies, line imperfections such as dislocations, in addition to the grain boundaries which are themselves surface defects. Further, impurity atoms frequently penetrate the lattice structure causing further distortions. Apart from these geometrical defects, lattice vibrations cause further discrepancies between the observed mechanical response characteristics of a crystal and those theoretically predicted by assuming the crystal lattice to be perfect.

The scatter frequently encountered in experimental data is, to a great extent, inadequately described by the phenomenological laws and theories of elasticity and plasticity.

http://rcin.org.pl

These laws either neglect non-linear effects on the response of a solid or are designed to smooth them out over macroscopic regions of the material specimen. In this manner, local mechanisms and their effects on the theoretical response are not taken into account. Higher order continuum theories have recently been developed for the purpose of accounting for such anomalies in the response behaviour of metals but are frequently too vague and illdefined to be of use in the solution of problems for which they were formulated.

Since imperfections exist in crystal lattices, their effects must be included in any formulation which describes the material response. For this purpose, dislocations are considered in this paper to be the main mechanism for the description of the non-ideal response of solids. Since, however, dislocations have both long range stress fields and induce small atomic displacements compared to inter-atomic distances beyond a few lattice vectors away from a dislocation line, it is considered unnecessary to deal directly with the atomic structure of dislocations. This permits the use of linear elasticity for the description of their local effects, i.e., the individual crystal is assumed to comprise of an isotropic elastic matrix observing the laws of linear elasticity in which a three-dimensional network of dislocations is embedded. Segments of this network, lying in slip planes, are anchored to interstitial atoms. From the analysis of the deformation of these segments and that of the elastic matrix, a quasi-static linear elastic response of the polycrystalline metal is then derived. In this way, the overall mechanical response will be correlated to the response of crystals containing dislocations.

2. Dislocation mechanism

The dislocation mechanism is well understood and several excellent texts (see HIRTH and LOTHE [1], for example) are available. Some standard results required in the subsequent analysis have been extracted from these texts without their full derivation being repeated in this presentation.

Furthermore, a clear understanding of the three measuring scales introduced in earlier work [2, 3], should be noted (see also Ref. [4]). Similar to that work, an ensemble of micro-elements, $\alpha = 1, \ldots, N$, where α designates a particular crystal in a mesodomain M, $M = 1, \ldots, P$ will be used.

In polycrystalline solids, dislocations form three-dimensional networks randomly distributed within each crystal. Segments of each network are anchored to impurity atoms by the Cottrell force. Upon the application of an external stress field to a crystal containing such a three-dimensional network, a dislocation of length l lying in a slip plane will deform according to the sequence 1, 2, 3, 4, 5 (indicated in Fig. 1) with an increase in the applied stress. This topology was first proposed by FRANK and READ [5] and is now known as the Frank-Read source. In the atomic theory of plasticity, this mechanism plays a significant role, since it acts as a source from which an immense number of dislocations may be generated. In the first stage of deformation, a dislocation segment behaves like a string fixed at two ends. Approximating this curve in the initial bow-out by two straight segments pulled at the midspan to an angle ψ measured from the base line, it is

http://rcin.org.pl

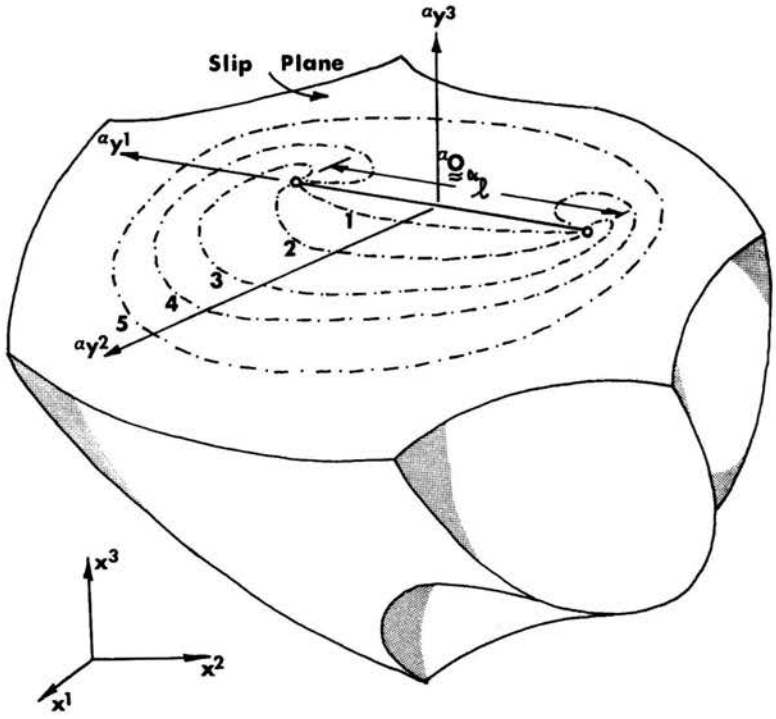

FIG. 1.

seen that the line tension T is, to a first approximation, equal to the self-energy ζ per unit length of the dislocation. Since, at this stage of the deformation, the line dislocation is to a large extent an edge dislocation whose self-energy is:

$$\zeta = Gb^2/4\pi(1-\nu)\ln r/r_0, \tag{2.1}$$

the line tension is also given by this expression. In (2.1), G is the crystalline shear modulus, $b = |\mathbf{b}|$ is the Burgers vector, r is the radial distance from the edge dislocation and r_0 is the cut-off radius. Substituting $\bar{r} = 10^5\, b$, corresponding to the radius of a typical crystal, and choosing the cut-off radius $r_0 = 5b$, the line tension becomes:

$$T = \pi G b^2/4(1-\nu), \tag{2.2}$$

where $\ln 2 \times 10^4$ has been approximated by π^2.

From this line tension, the minimum value of the applied stress required to activate a Frank-Read source can be obtained by considering the equilibrium of forces acting upon a semi-circular dislocation loop. Hence:

$$|^{\alpha}\mathbf{F}|^{\alpha}l = {}^{\alpha}\xi_{32}\, b^{\alpha}l = 2T, \tag{2.3}$$

where $^{\alpha}F = |'F|$ is the force per unit length acting on a dislocation, $^{\gamma}\xi_{32}$ is the component of the microstress tensor $^{\gamma}\boldsymbol{\xi}$ acting in the $^{\alpha}y^2$ direction on a slip-plane whose normal is in the $^{\alpha}y^3$ direction (see Fig. 1). Substituting for T from expression (2.2), it is observed

http://rcin.org.pl

that for a linear elastic response to be maintained, no stresses within the polycrystalline solid may exceed the value:

$$\xi_{32} = \pi Gb/2(1-\nu)^{\gamma}l. \tag{2.4}$$

Configuration 2 shown in Fig. 1 corresponds to this stress.

The displacement of the string in the ${}^{\alpha}y^2$ direction measured from the equilibrium position can be estimated by solving the differential equation expressing the balance of forces between the line tension and the force exerted by the external stress, neglecting the Peierls stress. The resulting equilibrium equation can be written in the form:

$$T\frac{d^2\,{}^{\gamma}y^2}{d({}^{\alpha}y^1)^2} = -\xi_{32}b = -Gb\,{}^{\gamma}e_{32}, \tag{2.5}$$

where ${}^{\alpha}y^2$ is the displacement of the string in the ${}^{\alpha}y^2$ direction and ${}^{\gamma}e_{32}$ is the component of the linear strain tensor ${}^{\alpha}\mathbf{e}$ acting in the ${}^{\gamma}y^2$ direction on a face whose normal is in the ${}^{\alpha}y^3$ direction. Solving for ${}^{\alpha}y^2$ from (2.5), the average displacement expressed in terms of the strain component is as follows:

$${}^{\alpha}\overline{y}^2 = \frac{2}{3\pi}\{(1-\nu)^{\gamma}l^2/b\}{}^{\alpha}e_{32}, \tag{2.6}$$

where use has been made of (2.2).

The effect of activating the Frank-Read sources interior to a crystal and of the damping induced by the retardation forces encountered by moving dislocations are beyond the scope of the present study. In what follows, the linear relation between the displacement and the applied shear strain given by Eq. (2.6) will be used in the equilibrium equations describing the quasi-static linear elastic response of polycrystalline metals.

3. Proposed analytic model

The mechanical response of a polycrystalline aggregate is different from, although dependent upon, the response of its individual crystals. It is, therefore, essential to correlate the mechanical response of the aggregate to the response of individual crystals, so that the important modes of deformation of the crystalline level can be identified. An approach of this kind can lead either to a refinement of the phenomenological laws describing the macroscopic response of the aggregate, hence leading to more reliable predictions of its physical characteristics, or to the elimination of undesirable modes through the design of new materials. However, without certain assumptions, it is inconceivable that these modes can be identified due to the seemingly incoherent structure of real polycrystalline solids. The following assumptions are made concerning each microdomain:

(i) The crystal matrix whether free from or containing imperfections, responds to an applied stress according to the constitutive equations of a linear isotropic elastic material. These relations contain the two independent material coefficients G and ν.

(ii) The density of line dislocations, ${}^{\alpha}\varrho_d$ unit lengths per unit volume in the α^{th} crystal, is inversely proportional to its average radius ${}^{\alpha}\overline{r}$. The reasoning behind this is that since

http://rcin.org.pl

a crystal has a preference to grow in the form of a close-packed structure tending to reduce its surface potential energy to a minimum, crystals tend to grow spherically around nucleation points in the melt. Hence, in the growth stage of a crystal from a radius r to a radius $r+\Delta r$, the misfit area $\delta\pi r\Delta r$ corresponds to dislocation lines. From such a geometrical consideration, the total misfit area ${}^{\alpha}\mathscr{S}$ of an individual microscopic region of average radius ${}^{\alpha}\bar{r}$ will be, to a first approximation, $4\pi{}^{\alpha}\bar{r}^2$. The density of dislocations ${}^{\alpha}\varrho_d$ is then the total length $2\,\mathscr{S}/b$ of the dislocation line divided by the crystal's volume, i.e.:

$$\varrho_d = \frac{6}{b\,{}^{\alpha}\bar{r}}. \tag{3.1}$$

iii) The dislocation density does not vary from crystal to crystal, i.e., ${}^{\alpha}\varrho_d = \varrho_d$. This implies, in conjunction with (ii), that there is no variation in the average radius and volume of the microscopic regions, i.e., ${}^{\alpha}\bar{r} = \bar{r}$; ${}^{\alpha}\mathring{v} = \mathring{v}$; ${}^{\alpha}l = l$. By introducing this assumption, the effect of fluctuations in the line dislocation density on the mechanical response of a solid is neglected.

(iv) A certain ratio σ of the line dislocations within a crystal are mobile. A typical value of σ for a representative crystal of average radius $\bar{r} = 10^5 b$ may be established as follows. From expression (3.1), ϱ_d is of the order of 10^{11} cm/cm^3. From the internal friction experiments of GRANATO [6], the density of mobile dislocations is approximately 10^7 cm/cm^3. Thus a representative value for the ratio σ may be taken as 10^{-4}.

(v) Out of the numerous possible slip systems within the α^{th} microscopic region, there is only one slip direction ${}^{\alpha}O_{\beta j}$, $\beta = 2, 3$ for the motion of mobile dislocations (see Fig. 1). The ${}^{\alpha}O_{\beta j}$ represent the direction cosines between the ${}^{\alpha}y^{\beta}$ and X^j axes. ${}^{\alpha}y^2$ denotes the direction of bow-out of the inactivated Frank-Read source, while ${}^{\alpha}y^3$ designates the normal to the slip plane.

(vi) The direction cosines $\mathbf{O}$ have a random distribution over the crystals in a mesoscopic region.

(vii) Dislocations only interact with externally applied stress fields and not among themselves.

(viii) The deformation is completely reversible.

In summary, the model proposed for the study of the linear response characteristics of a polycrystalline aggregate consists of a large number of close-packed crystals of equal volume. Each microscopic element, or crystal, in turn contains the same density of anchored semi-mobile dislocation segments which can bow-out in one preferred slip directon on one slip plane both of which vary from crystal to crystal in a random fashion. Contributions due to the interaction between dislocation segments are neglected.

4. The effect of dislocations on the linear elastic response of polycrystalline metals

Within the framework of the assumptions made in the previous section, the total free energy ${}^{M}\mathring{U}$ stored in a mesoscopic region of the solid due to an applied external load is equal to the sum of the strain energy stored in the elastic matrix and the increase in the self-energy of the anchored dislocation segments in each crystal summed over the number N

http://rcin.org.pl

of microscopic elements within the mesoscopic region. This may be expressed as follows:

$$^{M}\mathring{U} = \frac{1}{2}\sum_{\alpha=1}^{N}\int_{v}(\underline{\underline{E}}+\eta\,{}^{\alpha}\underline{\underline{\Gamma}})\,{}^{\alpha}\mathbf{e}\,{}^{\alpha}\mathbf{e}\,dv, \tag{4.1}$$

where:

$$\underline{\underline{E}} = \frac{2Gv}{1-2v}\delta_{ij}\delta_{kl}+2G\delta_{ik}\delta_{jl} = E_{ijkl} \tag{4.2}$$

is the fourth-order isotropic tensor describing the material properties of the matrix and $\boldsymbol{\delta}$ is the Kronecker delta. The double bar under a symbol indicates a fourth order quantity. The terms:

$$\eta = \frac{2}{3\pi}G(1-\nu)l^2\sigma\varrho_d, \tag{4.3}$$

$$^{\alpha}\underline{\underline{\Gamma}} = {}^{\alpha}O_{3i}\,{}^{\alpha}O_{2j}\,{}^{\alpha}O_{3k}\,{}^{\alpha}O_{2l} = {}^{\alpha}\Gamma_{ijkl}, \tag{4.4}$$

describe the influence of the deformation of the anchored dislocations on the work done by an external force field. η is the product of the work done by the force $|^{\alpha}F| = Gb^{\alpha}e_{32}$ given by combining (2.3) and (2.5), acting on a dislocation segment of length l to displace it a distance $^{\alpha}\bar{y}^2$, defined by equation (3.1), times the number of mobile dislocation segments σ in the crystal. The transformation of the local strain components $^{\alpha}\mathbf{e}$ from the $^{\alpha}y^j$ to X^i axes introduces the fourth-order tensor $^{\alpha}\underline{\underline{\Gamma}}$ of Eq. (4.4). Relation (4.1) is that used in Ref. [4]. Upon differentiating (4.1) with respect to $^{\alpha}\mathbf{e}v$, the microstresses in the α^{th} crystal are obtained:

$$^{\alpha}\boldsymbol{\xi} = (\underline{\underline{E}}+\eta\,{}^{\alpha}\underline{\underline{\Gamma}})\,{}^{\alpha}\mathbf{e}. \tag{4.5}$$

In the present formulation, the strains $^{\alpha}\mathbf{e}$ and the microstresses $^{\alpha}\boldsymbol{\xi}$ pertaining to the α^{th} microscopic region are random variables dependent upon the orientation tensor $\mathbf{O}$. These random quantities can be expressed in terms of an average, denoted by $\langle\cdot\rangle_N$, taken over the crystals in the mesoscopic region, and a fluctuating component, denoted by $\overset{*}{\cdot}$, where by definition $\langle\overset{*}{\cdot}\rangle_N \equiv \mathbf{O}$. Accordingly, when:

$$^{\alpha} = \underline{\underline{\Gamma}}\langle\underline{\underline{\Gamma}}\rangle_N+{}^{\alpha}\dot{\underline{\underline{\Gamma}}}, \quad {}^{\alpha}\mathbf{e} = \langle\mathbf{e}\rangle_N+{}^{\alpha}\dot{\mathbf{e}}, \quad {}^{\alpha}\boldsymbol{\xi} = \langle\boldsymbol{\xi}\rangle_N+{}^{\alpha}\dot{\boldsymbol{\xi}} \tag{4.6}$$

are substituted into Eq. (4.5), the expressions for the average and fluctuating components of the microstress tensor $^{\alpha}\boldsymbol{\xi}$ are given by:

$$\langle\boldsymbol{\xi}\rangle_N = \underline{\underline{E}}\langle\mathbf{e}\rangle_N+\eta\langle\underline{\underline{\Gamma}}\rangle_N\langle\mathbf{e}\rangle_N+\eta\langle\dot{\underline{\underline{\Gamma}}}\dot{\mathbf{e}}\rangle_N, \tag{4.7}$$

$$^{\alpha}\dot{\boldsymbol{\xi}} = \underline{\underline{E}}\,{}^{\alpha}\dot{\mathbf{e}}+\eta\{\langle\underline{\underline{\Gamma}}\rangle_N\,{}^{\alpha}\dot{\mathbf{e}}+{}^{\alpha}\dot{\underline{\underline{\Gamma}}}\langle\mathbf{e}\rangle_N+{}^{\alpha}\dot{\underline{\underline{\Gamma}}}\,{}^{\alpha}\dot{\mathbf{e}}-\langle\dot{\underline{\underline{\Gamma}}}\dot{\mathbf{e}}\rangle_N\}. \tag{4.8}$$

It is important to note the dependence of $^{\alpha}\boldsymbol{\xi}$ on the dislocation density. Indeed, when $\eta = 0$ the stresses and strains are only related through the isotropic material coefficient tensor $\underline{\underline{E}}$ given in Eq. (4.2). On the other hand, when the fluctuating components of $^{\alpha}\underline{\underline{\Gamma}}$ vanish, the average stresses depend, in addition, on the preferred macroscopic direction of slip. The third term in (4.7) shows that, due to the assumptions made concerning the nature

http://rcin.org.pl

of the polycrystalline material, there exists a correlation between the possible orientations and the strains defined as follows:

$$\langle \underline{\underline{\dot{\Gamma}}}\,\dot{\mathbf{e}} \rangle_N = \frac{1}{N} \sum_{\alpha=1}^{N} {}^{\alpha}\underline{\underline{\dot{\Gamma}}}\,{}^{\alpha}\dot{\mathbf{e}}. \tag{4.9}$$

Similar remarks can be made concerning the fluctuating components ${}^{\alpha}\dot{\boldsymbol{\xi}}$, given by Eq. (4.8).

At this stage of the analysis, probabilistic concepts concerning the existence of a stress correlation function, i.e. a second-order statistical moment of the stress distribution, are introduced. The correlation theory hypothesizes that for the description of a random process, its mean and correlation function have to be estimated (see YAGLOM [7]). However, for a random function to be uniquely specified by its first and second moments, it is necessary and sufficient that its distribution be Gaussian. Thus, due to lack of evidence to the contrary, and since random processes occurring in nature are often observed to be Gaussian, the independent variables ${}^{\alpha}\mathbf{O}$, as well as the dependent variables $\mathbf{e}$ and ${}^{\alpha}\boldsymbol{\xi}$, will be assumed normally distributed.

In a manner similar to the definition of the correlation given by Eq. (4.9), the stress correlation may be approximated, since the number N of crystals is finite in a mesoscopic region, as follows:

$$B_{\xi^2}(\boldsymbol{\lambda}) = \langle \dot{\boldsymbol{\xi}}[{}^{\alpha}\boldsymbol{X}]\dot{\boldsymbol{\xi}}[{}^{\alpha}\boldsymbol{X}+\boldsymbol{\lambda}] \rangle_N = \frac{1}{N} \sum_{\alpha=1}^{N} \dot{\boldsymbol{\xi}}({}^{\iota}\boldsymbol{X})\dot{\boldsymbol{\xi}}({}^{\alpha}\boldsymbol{X}+\boldsymbol{\lambda}), \tag{4.10}$$

where $\boldsymbol{\lambda}$ is the vectorial distance between the centers of mass of the microscopic regions at ${}^{\alpha}\boldsymbol{R}$ and ${}^{\alpha}\boldsymbol{R}+\boldsymbol{\lambda}$. Analogously, the correlation functions between the fluctuating components of the strain and orientation tensors are defined as:

$$\begin{aligned}
B_{e^2}(\boldsymbol{\lambda}) &= \langle \dot{\mathbf{e}}[{}^{\alpha}\boldsymbol{X}]\dot{\mathbf{e}}[\boldsymbol{X}+\boldsymbol{\lambda}] \rangle_N, \\
B_{\Gamma^2}(\boldsymbol{\lambda}) &= \langle \underline{\underline{\dot{\Gamma}}}[{}^{\alpha}\boldsymbol{X}]\underline{\underline{\dot{\Gamma}}}[\boldsymbol{X}+\boldsymbol{\lambda}] \rangle_N, \\
B_{e\Gamma}(\boldsymbol{\lambda}) &= \langle \dot{\mathbf{e}}[{}^{\alpha}\boldsymbol{X}]\underline{\underline{\dot{\Gamma}}}[{}^{\iota}\boldsymbol{X}+\boldsymbol{\lambda}] \rangle_N, \\
B_{e^2\Gamma}(\boldsymbol{\lambda}) &= \langle \dot{\mathbf{e}}[{}^{\alpha}\boldsymbol{X}]\dot{\mathbf{e}}[\boldsymbol{X}+\boldsymbol{\lambda}]\underline{\underline{\dot{\Gamma}}}[{}^{\alpha}\boldsymbol{X}] \rangle_N, \\
B_{e\Gamma^2}(\boldsymbol{\lambda}) &= \langle \dot{\mathbf{e}}[{}^{\alpha}\boldsymbol{X}]\underline{\underline{\dot{\Gamma}}}[{}^{\alpha}\boldsymbol{X}]\underline{\underline{\dot{\Gamma}}}[{}^{\alpha}\boldsymbol{X}+\boldsymbol{\lambda}] \rangle_N, \\
B_{e^2\Gamma^2}(\boldsymbol{\lambda}) &= \langle \dot{\mathbf{e}}[{}^{\alpha}\boldsymbol{X}]\dot{\mathbf{e}}[{}^{\alpha}\boldsymbol{X}+\boldsymbol{\lambda}]\underline{\underline{\dot{\Gamma}}}[{}^{\alpha}\boldsymbol{X}]\underline{\underline{\dot{\Gamma}}}[{}^{\alpha}\boldsymbol{X}+\boldsymbol{\lambda}] \rangle_N.
\end{aligned} \tag{4.11}$$

Using, for instance, Eq. (4.8), the stress correlation tensor given by (4.10) may be expressed in terms of the orientation and strain correlations and their product moments, given by Eq. (4.11), in the following way:

$$\begin{aligned}
B_{\xi^2}(\boldsymbol{\lambda}) = {} & \underline{\underline{\underline{\mathbf{A}}}}\,B_{e^2}(\boldsymbol{\lambda}) + \underline{\underline{\mathbf{B}}}\,B_{\Gamma^2}(\boldsymbol{\lambda}) + \underline{\underline{\mathbf{C}}}\,B_{e\Gamma}(\boldsymbol{\lambda}) - \eta^2 B_{e\Gamma}(\boldsymbol{\lambda})B_{e\Gamma}(\boldsymbol{\lambda}) + \underline{\underline{\mathbf{D}}}\,B_{e^2\Gamma}(\boldsymbol{\lambda}) \\
& + \mathbf{F}B_{e\Gamma^2}(\boldsymbol{\lambda}) + \eta^2 B_{e^2\Gamma^2}(\boldsymbol{\lambda}),
\end{aligned} \tag{4.12}$$

http://rcin.org.pl

where:

$$\begin{aligned}
\underline{\underline{\underline{\underline{\mathbf{A}}}}} &= (\underline{\underline{E}}+\eta\langle\underline{\underline{\Gamma}}\rangle_N)^2,\\
\underline{\underline{B}} &= \eta^2\langle\mathbf{e}\rangle_N\langle\mathbf{e}\rangle_N,\\
\underline{\underline{\underline{C}}} &= 2\eta(\underline{\underline{E}}+\eta\langle\underline{\underline{\Gamma}}\rangle_N)\langle\mathbf{e}\rangle_N,\\
\underline{\underline{D}} &= 2\eta\underline{\underline{E}},\\
\mathbf{F} &= 2\eta^2\langle\mathbf{e}\rangle_N.
\end{aligned} \tag{4.13}$$

Implicit in the above derivation is the assumption of statistical homogeneity, i.e., independence of the specific position ${}^{\alpha}X$.

Thus Eq. (4.12) represents the general expression for the stress correlations for the case where the stresses and strains are dependent on the orientations. In this context, it is important to note that once the strain and orientation correlation tensors, as well as their product moments, have been experimentally determined, the stress correlation tensor can be estimated from expression (4.12). This stress correlation function along with the mean stress determined from (4.7) will uniquely specify the probability distribution of the stresses in the polycrystalline aggregate. An experimental determination of the random displacement of crystals in a model of a two-phase structure has been shown to be possible using holographic interferometry by AXELRAD and KALOUSEK [8].

In discussing the general expression for the stress correlations given in (4.12), four cases which demonstrate the implications of some of the underlying assumptions concerning the linear response characteristics of solids may be discerned.

(i) If the slip-plane orientations have a uniform random distribution, i.e., $\boldsymbol{\lambda}$ is replaced by $|\boldsymbol{\lambda}|$ in the correlation function, then statistical isotropy is implied in addition to statistical homogeneity. This case has been partially dealt with by BARENBLATT and GORODSTOV [9] in their discussion of the random field of microstresses in the steady plastic flow of polycrystalline solids. However, these authors did not include in their formulation the dependence of the stresses on either the orientation of the crystals or the dislocation density. In the present analysis, statistical isotropy implies that $|\boldsymbol{\lambda}|$ replaces the vector $\boldsymbol{\lambda}$ and $m \ll N$ replaces N in the correlations and moments defined in Eqs. (4.10) and (4.11). Thus λ may be visualized as representing the radius of a sphere centered at ${}^{\alpha}X$ on the surface of which lie the centers of mass of m microscopic regions. Since the influence of the neighbouring crystals on the state of stress of the α^{th} crystal decreases with increasing λ, it may be further conjectured that m is the number of crystals in direct contact with the αth microscopic region. The assumption of isotropic homogeneity is frequently made in the analysis of turbulence in fluids [10].

(ii) If the strains are independent of the orientations, the terms on the right-hand side of Eq. (4.12) which contain second and third-order product moments will vanish, PUGACHEV [11]. Moreover, the last term in this expression will become the product of the orientation and strain correlation tensors. Hence the stress correlations reduce to:

$$B_{\xi^2}(\lambda) = \underline{\underline{\underline{\underline{\mathbf{A}}}}} B_{e^2}(\lambda)+\underline{\underline{B}}B_{\Gamma^2}(\lambda)+\eta^2 B_{e^2}(\lambda)B_{\Gamma^2}(\lambda) \tag{4.14}$$

which specifies that the only quantities required to be experimentally determined for the unique specification of the stress state are the strain and orientation correlations.

http://rcin.org.pl

(iii) If the probability distribution of the orientation tensor takes the form of a delta-function, i.e., if all mobile dislocations in the mesoscopic region are constrained to move in one direction in slip planes having the same normal, then the stress correlation tensor has the form:

$$B_{\xi^2}(\lambda) = \underset{\equiv\equiv}{\mathbf{A}} B_{e^2}(\lambda), \tag{4.15}$$

i.e., the stress correlation is expressed in terms of the dislocation density, the average slip-plane orientation and the strain correlation tensors. This corresponds to the assumption made both in the work of KOEHLER [12] and GRANATO [6], dealing with the internal friction in solids, and in the derivation by ZORSKI [13] of the equations of motion for a compound medium comprising of an elastic matrix and a dislocation fluid.

(iv) For completeness, although not directly related to the present analysis, an assumption, frequently encountered in the literature dealing with plastic deformation of a polycrystalline metal, is that the strain in each crystal corresponds to the strain on the mascroscopic body, BISHOP and HILL [14, 15]. Assuming that this is the case, the fluctuating components of the strain will vanish and the stress correlation tensor may be expressed as follows:

$$B_{\xi^2}(\lambda) = \underset{\equiv}{\mathbf{B}} B_{\Gamma^2}(\lambda), \tag{4.16}$$

i.e., it is only dependent on the dislocation density, the average strains and the orientation correlation tensor.

Finally, a relation similar to (4.12) expressing the strain correlation tensor in terms of the stress and orientation correlation functions can be obtained in an analogous manner to the derivation of Eq. (4.12). However, such an expression would be of no practical significance due to the difficulties encountered in evaluating the stress distribution. On the other hand, Eq. (4.12) yields, at least in principle, to experimental evaluation.

References

1. J. P. HIRTH and J. LOTHE, *Theory of dislocation*, McGraw Hill Co., 1968.
2. D. R. AXELRAD, *Rheology of structured media*, Arch. Mech. Stos., **23**, 131–140, 1971.
3. D. R. AXELRAD, *Random theory of deformation of structured media*, Int. Centre Mech. Sci., Udine 1972.
4. D. R. AXELRAD and J. W. PROVAN, *Deformation theory of elastic poly-crystalline materials*, Arch. Mech. Stos. **25**, 811–821, 1973.
5. F. C. FRANK and W. T. READ, *Mechanism for the multiplication of dislocations*, Phys. Rev., **79**, 722–723, 1951.
6. A. V. GRANATO, *Internal friction studies of dislocation motion*, Dislocation dynamics, 117–157, McGraw Hill Co., N. Y., 1968.
7. A. M. YAGLOM, *Theory of stationary random functions*, Prentice Hall Inc., 1962.
8. D. R. AXELRAD and J. KALOUSEK, *Measurement of microdeformations by holographic X-ray diffractions*, Int. Symp. on Experimental Mechanics, Univ. of Waterloo, Ontario, Canada, June 12–16, 1972.
9. G. J. BARENBLATT and V. A. GORODSTOV, *On the structure of the field of microstresses of steady plastic flow*, P. M. M., 1964.
10. G. BIRKHOFF and J. KAMPE DE FERIET, *Kinematics of homogeneous turbulence*, J. Math. Mech., **1**, 663–704, 1958.

http://rcin.org.pl

11. V. S. PUGACHEV, *Theory of random functions*, Addison-Wesley Publ., Mass., 1965.
12. J. S. KOEHLER, *The influence of dislocations and impurities on the damping and the elastic constants of metal single crystals*, in: Imperfections Nearly Perfect Crystals, Shockley HOLLMAN, MAURER, SEITZ, (eds.), J. Wiley and Chapmann Hall, 1952.
13. H. ZORSKI, *Statistical theory of dislocations*, Int. J. Solids Structure, 4, 959–974, Pergamon Press, 1968.
14. J. F. W. BISHOP and R. HILL, *A theory of the plastic distorsion of a polycrystalline aggregate under combined stresses*, Phil. Mag., 414 –427, 1951a.
15. J. F. W. BISHOP and R. HILL, *A theoretical derivation of the plastic properties of a policrystalline face-centred-metal*, Phil. Mag., **42**, 1298–1302, 1951b.

MICROMECHANICS OF SOLIDS LABORATORY
McGILL UNIVERSITY, MONTREAL, CANADA.

Received October 25, 1972.

http://rcin.org.pl