Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 5, pp. 765–780, Warszawa 1973

Inhomogeneities in second-grade fluid bodies and isotropic solid bodies

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A GLOBAL theory for materially uniform, smooth, second-grade fluid bodies and isotropic solid bodies is developed in this paper. The main results include: isotropy groups and representations for the response function relative to distorted references in general, representations for the response field in arbitrary configuration, and geometric characterization for local homogeneity, local pseudo-homogeneity, and local inhomogeneity in those second-grade bodies.

Przedstawiono globalną teorię materialnie jednorodnych, gładkich ciał ciekłych rzędu drugiego jak również izotropowych ciał stałych. Uzyskano następujące wyniki podstawowe: grupy izotropii i reprezentacje funkcji reakcji materiału przy odkształceniu układu współrzędnych, reprezentacje pola reakcji dla dowolnej konfiguracji, charakterystykę geometryczną lokalnej jednorodności, lokalnej pseudojednorodności i lokalnej niejednorodności dla rozważanych ciał drugiego rzędu.

Изложена глобальная теория материально однородных, гладких жидких тел второго порядка, а также изотропных твердых тел. Получены следующие основные результаты: группы изотропии и представления функций реакции материала при деформировании, представление поля реакций в произвольной конфигурации, геометрические характеристики локальной однородности, локальной псевдооднородности и локальной неоднородности исследуемых тел второго порядка.

1. Introduction

As EXPLAINED in the preceding paper [1], a *material of second-grade* is defined by a constitutive equation of the form

$$\tilde{R} = R(\boldsymbol{\varphi}_{*p}, \boldsymbol{\varphi}_{**p}, p), \tag{1.1}$$

where $(\boldsymbol{\varphi}_{*p}, \boldsymbol{\varphi}_{**p})$ denotes the second-grade class induced by a configuration $\boldsymbol{\varphi}$ of a body $\mathscr{A}$ at the point $p \in \mathscr{A}$. R is called the *response function* whose value $\tilde{R}$ is a state function of p in the configuration $\boldsymbol{\varphi}$ such as the stress tensor, the stored energy, *etc.* For the purpose of this paper, the physical significance of $\tilde{R}$ is not important and therefore not specified.

Since R depends only on the second-grade class $(\boldsymbol{\varphi}_{*p}, \boldsymbol{\varphi}_{**p})$, it can be represented by a *relative response function* $R_\varkappa$ with respect to a reference configuration $\boldsymbol{\varkappa}$, *wiz.*,

$$R(\boldsymbol{\varphi}_{*p}, \boldsymbol{\varphi}_{**p}, p) = R_\varkappa(\mathbf{F}, \mathsf{G}, p), \tag{1.2}$$

where $(\mathbf{F}, \mathsf{G})$ denotes the second-grade deformation of $\boldsymbol{\varphi}$ relative to $\boldsymbol{\varkappa}$. In the notation of [1], we write

$$(\boldsymbol{\varphi}_{*p}, \boldsymbol{\varphi}_{**p}) = (\mathbf{F}, \mathsf{G}) \circ (\boldsymbol{\varkappa}_{*p}, \boldsymbol{\varkappa}_{**p}). \tag{1.3}$$

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From (1.2) and (1.3), the relative response function obeys the following transformation rule:

$$R_{\mu}(\mathbf{F}, \mathsf{G}, p) = R_{\varkappa}((\mathbf{F}, \mathsf{G}) \circ (\mathbf{H}, \mathsf{K}), p)), \tag{1.4}$$

where $(\mathbf{H}, \mathsf{K})$ denotes the second-grade deformation from $\varkappa$ to μ, *viz.*,

$$(\mu_{*p}, \mu_{**p}) = (\mathbf{H}, \mathsf{K}) \circ (\varkappa_{*p}, \varkappa_{**p}). \tag{1.5}$$

From chain rule, the second-grade deformation obeys the composition formula:

$$(\mathbf{F}, \mathsf{G}) \circ (\mathbf{H}, \mathsf{K}) = (\mathbf{FH}, \mathbf{F}\mathsf{K} + \mathsf{G}[\mathbf{H}, \mathbf{H}]) \tag{1.6}$$

cf. [1, (3.4)].

In accordance with Noll's general definition, the isotropy group relative to $\varkappa$ is the group $g_{\varkappa}(p)$ consisting in all second-grade deformations $(\mathbf{H}, \mathsf{K})$ such that

$$R_{\varkappa}((\mathbf{F}, \mathsf{G}) \circ (\mathbf{H}, \mathsf{K}), p = R_{\varkappa}(\mathbf{F}, \mathsf{G}, p), \tag{1.7}$$

for all deformations $(\mathbf{F}, \mathsf{G})$. Comparing (1.7) with (1.4), we see that $(\mathbf{H}, \mathsf{K})$ belongs to $g_{\varkappa}(p)$ if, and only if, the relative response function is invariant under the change of local reference configuration from $(\varkappa_{*p}, \varkappa_{p*p})$ to $(\mathbf{H}, \mathsf{K}) \circ (\varkappa_{*p}, \varkappa_{**p})$.

In [1], it is proposed that a *(second-grade) fluid point p* be characterized by the following condition: Let there be a local reference configuration $(\varkappa_{*p}, \varkappa_{**p})$ such that the isotropy group $g_{\varkappa}(p)$ is formed by all second-grade deformations $(\mathbf{H}, \mathsf{K})$ which verify the conditions:

$$\text{i)} \qquad |\det \mathbf{H}| = 1,$$
$$\text{ii)} \qquad \operatorname{tr}(\mathbf{H}^{-1}\mathsf{K}) = \mathbf{0},$$

where

$$\operatorname{tr}(\mathbf{A}\mathsf{B})_i \equiv A^a{}_b B^b{}_{ai}. \tag{1.8}$$

The local reference configuration $(\varkappa_{*p}, \varkappa_{**p})$, which is distinguished by Conditions i) and ii), is called an *undistorted* local configuration. It is proved in [1] that the response function relative to $(\varkappa_{*p}, \varkappa_{**p})$ has the following representation:

$$R_{\varkappa}(\mathbf{F}, \mathsf{G}, p) = \hat{R}_{\varkappa}(|\det \mathbf{F}|, \operatorname{grad}|\det \mathbf{F}|, p), \tag{1.9}$$

where $\operatorname{grad}|\det \mathbf{F}|$ is the spatial gradient of $|\det \mathbf{F}|$ at the point p and is given by the formula

$$\operatorname{grad}|\det \mathbf{F}| = |\det \mathbf{F}|(\mathbf{F}^{-1})^T \operatorname{tr}(\mathbf{F}^{-1}\mathsf{G}). \tag{1.10}$$

The representation (1.9) means that:

$$(\mathbf{F}, \mathsf{G})^{-1} \circ (\overline{\mathbf{F}}, \overline{\mathsf{G}}) \in g_{\varkappa}(p) \Leftrightarrow |\det \mathbf{F}| = |\det \overline{\mathbf{F}}|, \operatorname{grad}|\det \mathbf{F}| = \operatorname{grad}|\det \overline{\mathbf{F}}|. \tag{1.11}$$

In particular, since $(\mathbf{1}, \mathsf{0}) \in g_{\varkappa}(p)$, we have

$$(\mathbf{H}, \mathsf{K}) \in g_{\varkappa}(p) \Leftrightarrow |\det \mathbf{H}| = 1, \quad \operatorname{grad}|\det \mathbf{H}| = \mathbf{0}. \tag{1.12}$$

In [1] it is proposed also that a (second-grade) solid point p be characterized by the following condition: Let there be a local reference configuration $(\varkappa_{**p}, \varkappa_{*p})$ such that the isotropy group $g_{\varkappa}(p)$ is formed by some second-grade deformations $(\mathbf{H}, \mathsf{K})$ which verify the conditions:

$$\text{iii)} \qquad \mathbf{H}^T = \mathbf{H}^{-1},$$
$$\text{iv)} \qquad \mathsf{K} = \mathsf{0}.$$

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The particular local reference configuration $(\varkappa_{*p}, \varkappa_{**p})$ is also called an *undistorted* local configuration of p. In the case, when $g_\varkappa(p)$ is formed by *all* pairs $(\mathbf{H}, \mathsf{K})$ satisfying Conditions iii) and iv), p is called an *isotropic* (second-grade) solid point. In [1] it is proved that relative to $(\varkappa_{*p}, \varkappa_{**p})$ the response function admits the following representation:

$$R_\varkappa(\mathbf{F}, \mathsf{G}, p) = \check{R}_\varkappa(\mathbf{B}, \operatorname{grad}\mathbf{B}, p), \tag{1.13}$$

where $\mathbf{B}$ denotes the left Cauchy-Green tensor of $\mathbf{F}$ as usual, cf. [6, (23.5)],

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \tag{1.14}$$

and $\operatorname{grad}\mathbf{B}$ denotes the spatial gradient of $\mathbf{B}$ and is given by the component form

$$(\operatorname{grad}\mathbf{B})^{ij}{}_k = G^i{}_{AB}F^j{}_A(F^{-1})^B{}_k + G^j{}_{AB}F^i{}_A(F^{-1})^B{}_k. \tag{1.15}$$

For the proof of (1.9) and (1.13), see Ref. [1]; for a general theory of representation for constitutive relations, see Ref. [2]. It should be noted that Conditions i), ii) and iii), iv), and hence the representations (1.9) and (1.13), are valid only when the local reference configurations are undistorted. In the next section, we shall consider the isotropy groups of fluids and solids relative to distorted reference configurations, and we shall derive corresponding representations for the response functions.

We shall develop a global theory for homogeneous or pseudo-homogeneous fluid bodies and isotropic solid bodies in § 3. Then, in § 4, we present a global theory for materially uniform fluid bodies and isotropic solid bodies in general. The main result of the global theory is a representation for the response field $\tilde{R}$ in terms of the deformation relative to a global reference configuration. Finally, in § 5, we prove some geometric theorems which characterize local homogeneity, local pseudo-homogeneity, and local inhomogeneity in materially uniform, smooth, second-grade fluid bodies and isotropic solid bodies.

2. Isotropy groups and representations for the response function relative to distorted references

Let $g_\varkappa(p)$ and $g_\mu(p)$ denote the isotropy groups relative to $\varkappa$ and μ, respectively, and suppose that the second-grade deformation from $\varkappa$ to μ at p is $(\mathbf{L}, \mathsf{M})$. Then it follows from Noll's general rule that

$$g_\mu(p) = (\mathbf{L}, \mathsf{M}) \circ g_\varkappa(p) \circ (\mathbf{L}, \mathsf{M})^{-1}, \tag{2.1}$$

cf. [1, (3.7)]. In the preceding section, we have defined the isotropy groups for a fluid and for an isotropic solid relative to an undistorted reference, say $\varkappa$. The formula (2.1) can now be used to determine the isotropy groups relative to an arbitrary configuration μ.

We consider first the isotropy group $g_\mu(p)$ of a fluid point p. We have the following result: *A second-grade deformation* $(\mathbf{H}, \mathsf{K})$ *belongs to the isotropy group* $g_\mu(p)$ *if, and only if, it verifies the conditions*:

$$\text{i)} \qquad |\det\mathbf{H}| = 1,$$

$$\text{ii)}' \qquad \operatorname{tr}\big(\mathbf{L}^{-1}\mathbf{H}^{-1}\mathbf{L}\operatorname{Grad}(\mathbf{L}^{-1}\mathbf{H}\mathbf{L})\big) = \mathbf{0},$$

where the left-hand side of ii) *is given by*

$$\begin{aligned}\operatorname{tr}\big(\mathbf{L}^{-1}\mathbf{H}^{-1}\mathbf{L}\operatorname{Grad}(\mathbf{L}^{-1}\mathbf{H}\mathbf{L})\big) &= -(\mathbf{L}^{-1}\mathbf{H}\mathbf{L})^T\operatorname{tr}(\mathbf{L}^{-1}\mathsf{M}) \\ &+\mathbf{L}^T\operatorname{tr}(\mathbf{H}^{-1}\mathsf{K})+\operatorname{tr}(\mathbf{L}^{-1}\mathsf{M}) = (\mathbf{1}-\mathbf{L}^{-1}\mathbf{H}\mathbf{L})^T\operatorname{tr}(\mathbf{L}^{-1}\mathsf{M})+\mathbf{L}^T\operatorname{tr}(\mathbf{H}^{-1}\mathsf{K})\end{aligned} \tag{2.2}$$

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To prove that i) and ii)′ characterize $g_{\varkappa}(p)$, we observe first the formula

$$(2.3)\qquad (\mathbf{L}, \mathsf{M})^{-1} = (L^{-1}, -\mathbf{L}^{-1}\mathsf{M}[\mathbf{L}^{-1}, \mathbf{L}^{-1}]),$$

cf. [1, (3.5)]. Now, from (2.1), we have

$$(2.4)\qquad (\mathbf{H}, \mathsf{K}) \in g_{\mu}(p) \Leftrightarrow (\mathbf{L}, \mathsf{M})^{-1} \circ (\mathbf{H}, \mathsf{K}) \circ (\mathbf{L}, \mathsf{M}) \in g_{\varkappa}(p).$$

From (2.3) and (1.6) the deformation gradient on the right-hand side of (2.4) is given by

$$(2.5)\qquad (\mathbf{L}, \mathsf{M})^{-1} \circ (\mathbf{H}, \mathsf{K}) \circ (\mathbf{L}, \mathsf{M}) = \big(\mathbf{L}^{-1}\mathbf{HL}, \mathbf{L}^{-1}\mathbf{H}\mathsf{M} + \mathbf{L}^{-1}\mathsf{K}[\mathbf{L}, \mathbf{L}] - (\mathbf{L}^{-1}\mathsf{M}(\mathbf{L}, \mathbf{L}))[\mathbf{HL}, \mathbf{HL}]\big).$$

Hence, from (2.4), $(\mathbf{H}, \mathsf{K}) \in g_{\mu}(p)$, if and only if,

$$(2.6)\qquad 1 = |\det(\mathbf{L}^{-1}\mathbf{HL})| = |\det \mathbf{H}|,$$

and

$$(2.7)\qquad \mathbf{0} = \operatorname{tr}(\mathbf{L}^{-1}\mathbf{H}^{-1}\mathbf{L}\mathbf{L}^{-1}\mathbf{H}\mathsf{M}) + \operatorname{tr}\big(\mathbf{L}^{-1}\mathbf{HL}\mathbf{L}^{-1}\mathsf{K}[\mathbf{L}, \mathbf{L}]\big) - \operatorname{tr}\big(\mathbf{L}^{-1}\mathbf{HL}\mathbf{L}^{-1}\mathsf{M}(\mathbf{L}^{-1}, \mathbf{L}^{-1})[\mathbf{HL}, \mathbf{HL}]\big) = \operatorname{tr}(\mathbf{L}^{-1}\mathsf{M}) + \mathbf{L}^{T}\operatorname{tr}(\mathbf{H}^{-1}\mathsf{K}) - (\mathbf{L}^{-1}\mathbf{HL})^{T}\operatorname{tr}(\mathbf{L}^{-1}\mathsf{M}) = (\mathbf{1} - \mathbf{L}^{-1}\mathbf{HL})^{T}\operatorname{tr}(\mathbf{L}^{-1}\mathsf{M}) + \mathbf{L}^{T}\operatorname{tr}(\mathbf{H}^{-1}\mathsf{K}).$$

Comparing (2.7) with (2.2), we have proved that Conditions i) and ii)′ characterize $g_{\mu}(p)$.

From (2.7) we see that (μ_{*p}, μ_{**p}) is undistorted, if and only if, its second-grade deformation $(\mathbf{L}, \mathsf{M})$ relative to an undistorted reference $(\varkappa_{*p}, \varkappa_{**p})$ obeys the condition

$$(2.8)\qquad \operatorname{tr}(\mathbf{L}^{-1}\mathsf{M}) = \mathbf{0}$$

or, equivalently, the condition

$$(2.9)\qquad \operatorname{grad}|\det \mathbf{L}| = \mathbf{0},$$

cf. (1.10).

In general (μ_{*p}, μ_{**p}) and $(\bar{\mu}_{*p}, \bar{\mu}_{**p})$ are called *materially isomorphic* if R_{μ} and $R_{\bar{\mu}}$ are identical, *viz.*,

$$(2.10)\qquad R_{\mu}(\mathbf{F}, \mathsf{G}, p) = R_{\bar{\mu}}(\mathbf{F}, \mathsf{G}, p) \quad \forall (\mathbf{F}, \mathsf{G}).$$

Comparing this condition with (1.7), we see that (μ_{*p}, μ_{**p}) and $(\bar{\mu}_{*p}, \bar{\mu}_{**p})$ are materially isomorphic if, and only if, the deformation $(\mathbf{H}, \mathsf{K})$ connecting them, say by

$$(2.11)\qquad (\bar{\mu}_{*p}, \bar{\mu}_{**p}) = (\mathbf{H}, \mathsf{K}) \circ (\mu_{*p}, \mu_{**p}),$$

belongs to $g_{\mu}(p)$. Let $(\mathbf{L}, \mathsf{M})$ and $(\bar{\mathbf{L}}, \bar{\mathsf{M}})$ be the second-grade deformations of (μ_{*p}, μ_{**p}) and $(\bar{\mu}_{*p}, \bar{\mu}_{**p})$, respectively, relative to an arbitrary reference configuration $(\varkappa_{*p}, \varkappa_{**p})$. Then, as usual, we have

$$(2.12)\qquad (\bar{\mathbf{L}}, \bar{\mathsf{M}}) = (\mathbf{H}, \mathsf{K}) \circ (\mathbf{L}, \mathsf{M}).$$

But from (2.1), we have

$$(2.13)\qquad (\bar{\mathbf{L}}, \bar{\mathsf{M}}) \circ (\mathbf{L}, \mathsf{M})^{-1} \in g_{\mu}(p) \Leftrightarrow (\mathbf{L}, \mathsf{M})^{-1} \circ (\bar{\mathbf{L}}, \bar{\mathsf{M}}) \in g_{\varkappa}(p).$$

So, if $(\varkappa_{*p}, \varkappa_{**p})$ is undistorted, then by combining (2.13) and (1.11), we obtain

$$(2.14)\qquad (\bar{\mathbf{L}}, \bar{\mathsf{M}}) \circ (\mathbf{L}, \mathsf{M})^{-1} \in g_{\mu}(p) \Leftrightarrow |\det \mathbf{L}| = |\det \bar{\mathbf{L}}|,\ \operatorname{grad}|\det \mathbf{L}| = \operatorname{grad}|\det \bar{\mathbf{L}}|,$$

which is another way to characterize $g_{\mu}(p)$ for an arbitrary reference μ.

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Having considered the isotropy group $g_\mu(p)$ of a fluid point relative to an arbitrary reference μ, we turn next to representation for R_μ. As before, let $(\mathbf{L}, \mathsf{M})$ be the second-grade deformation of μ relative to an undistorted reference $(\varkappa_{*p}, \varkappa_{**p})$. Then, from (2.14), we know that R_μ can be characterized by $|\det \mathbf{L}|$ and $\operatorname{grad}|\det \mathbf{L}|$. Indeed, if $(\mathbf{F}, \mathsf{G})$ is an arbitrary deformation relative to (μ_{*p}, μ_{**p}), then the corresponding deformation relative to $(\varkappa_{*p}, \varkappa_{**p})$ is given by

$$(2.15) \qquad (\mathbf{F}, \mathsf{G}) \circ (\mathbf{L}, \mathsf{M}) = \big(\mathbf{FL}, \mathbf{F}\mathsf{M} + \mathsf{G}[\mathbf{L}, \mathbf{L}]\big).$$

Now, since $(\varkappa_{*p}, \varkappa_{**p})$ is undistorted, $R_\varkappa$ has the representation (1.9). Substituting (2.15) into (1.9) and using (1.4), we obtain

$$(2.16) \qquad R_\mu(\mathbf{F}, \mathsf{G}, p) = R_\varkappa\big((\mathbf{F}, \mathsf{G}) \circ (\mathbf{L}, \mathsf{M}), p\big) = \hat{R}_\varkappa(|\det \mathbf{FL}), \operatorname{grad}|\det \mathbf{FL}|, p),$$

where, from (2.15) and (1.10), the arguments of $\hat{R}_\varkappa$ are given by

$$(2.17) \qquad |\det(\mathbf{FL})| = |\det \mathbf{F}||\det \mathbf{L}|,$$

and

$$(2.18) \qquad \begin{aligned} \operatorname{grad}|\det(\mathbf{FL})| &= |\det(\mathbf{FL})|\big((\mathbf{FL})^{-1})^T \operatorname{tr}\big(\mathbf{L}^{-1}\mathbf{F}^{-1}(\mathbf{F}\mathsf{M} + \mathsf{G}[\mathbf{L}, \mathbf{L}])\big) \\ &= |\det \mathbf{F}|(\mathbf{F}^{-1})^T \operatorname{grad}|\det \mathbf{L}| + |\det \mathbf{L}| \operatorname{grad}|\det \mathbf{F}|. \end{aligned}$$

Hence we have the following representation for R_μ:

$$(2.19) \qquad R_\mu(\mathbf{F}, \mathsf{G}, p) = \hat{R}_\varkappa\big(|\det \mathbf{F}||\det \mathbf{L}|, |\det \mathbf{F}|(\mathbf{F}^{-1})^T \operatorname{grad}|\det \mathbf{L}| + |\det \mathbf{L}| \operatorname{grad}|\det \mathbf{F}|, p),$$

where $\varkappa$ is undistorted and is related to μ by

$$(2.20) \qquad (\mu_{*p}, \mu_{**p}) = (\mathbf{L}, \mathsf{M}) \circ (\varkappa_{*p}, \varkappa_{**p}).$$

Next, we consider the isotropy group and the representation of the response function of an isotropic solid point p relative to an arbitrary local reference (μ_{*p}, μ_{**p}). As before, we choose an undistorted local reference $(\varkappa_{*p}, \varkappa_{**p})$ and denote the second-grade deformation of (μ_{*p}, μ_{**p}) relative to $(\varkappa_{*p}, \varkappa_{**p})$ by $(\mathbf{L}, \mathsf{M})$.

We have the following result: *A deformation* $(\mathbf{H}, \mathsf{K})$ *belongs to* $g_\mu(p)$ *if, and only if, it verifies the conditions*

$$\text{iii)}' \qquad (\mathbf{L}^{-1}\mathbf{HL})^T = (\mathbf{L}^{-1}\mathbf{HL})^{-1},$$

$$\text{iv)}' \qquad \operatorname{Grad}(\mathbf{L}^{-1}\mathbf{HL}) = \mathbf{0},$$

where the left-hand side of iv)′ *is given by*

$$(2.21) \qquad \operatorname{Grad}(\mathbf{L}^{-1}\mathbf{HL}) = -\mathbf{L}^{-1}\mathsf{M}[\mathbf{L}^{-1}\mathbf{HL}, \mathbf{L}^{-1}\mathbf{HL}] + \mathbf{L}^{-1}\mathsf{K}[\mathbf{L}, \mathbf{L}] + \mathbf{L}^{-1}\mathbf{H}\mathsf{M}.$$

The proof is essentially the same as before. Conditions iii)′ and iv)′ characterize $g_\mu(p)$ for an arbitrary reference μ. By definition, (μ_{*p}, μ_{**p}) is undistorted if, an only if, iii)′ and iv)′ reduce to iii) and iv). We claim that such is the case for μ if, and only if,

$$(2.22) \qquad \mathbf{LL}^T = \mathbf{L}^T\mathbf{L} = c\mathbf{I}, \quad c > 0,$$

and

$$(2.23) \qquad \mathsf{M} = \mathsf{O}.$$

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Sufficiency is obvious. Conversely, if every $(\mathbf{H}, \mathsf{K}) \in g_\mu(p)$ obeys iii) and iv), then iii) implies

$$(\mathbf{LL}^T)\mathbf{H} = \mathbf{H}(\mathbf{LL}^T) \tag{2.24}$$

for all orthogonal $\mathbf{H}$; it is well-known that (2.22) gives the general solution to (2.24). Similarly, iii), iv), and iv)′ imply

$$\mathbf{H}\mathsf{M} = \mathsf{M}[\mathbf{L}^{-1}\mathbf{HL}, \mathbf{L}^{-1}\mathbf{HL}] \tag{2.25}$$

for all orthogonal tensors $\mathbf{H}$. We can rewrite this equation as

$$\mathbf{L}^{-1}\mathsf{M} = \mathbf{L}^{-1}\mathbf{H}^{-1}\mathbf{L}(\mathbf{L}^{-1}\mathsf{M})[\mathbf{L}^{-1}\mathbf{HL}, \mathbf{L}^{-1}\mathbf{HL}] \tag{2.26}$$

for all orthogonal tensors $\mathbf{H}$; it is well-known that the general solution of (2.26) is

$$\mathbf{L}^{-1}\mathsf{M} = \mathsf{O}, \tag{2.27}$$

which is equivalent to (2.23). Thus the proof is complete.

From (1.13), we can also express $g_\mu(p)$ in the following way: Let $(\bar{\mathbf{L}}, \bar{\mathsf{M}})$ be the second-grade deformation of $(\bar{\mu}_{*p}, \bar{\mu}_{**p})$ relative to $(\varkappa_{*p}, \varkappa_{**p})$. Then $(\bar{\mu}_{*p}, \bar{\mu}_{**p})$ and (μ_{*p}, μ_{**p}) are materially isomorphic if, and only if,

$$\mathbf{LL}^T = \bar{\mathbf{L}}\bar{\mathbf{L}}^T, \quad \operatorname{grad}(\mathbf{LL}^T) = \operatorname{grad}(\bar{\mathbf{L}}\bar{\mathbf{L}}^T), \tag{2.28}$$

where from (1.15) we have

$$(\operatorname{grad}\mathbf{LL}^T)^{ij}{}_k = M^i{}_{AB}L^j{}_A(L^{-1})^B{}_k + M^j{}_{AB}L^i{}_A(L^{-1})^B{}_k. \tag{2.29}$$

Having characterized $g_\mu(p)$ of an isotropic solid relative to an arbitrary μ, we consider next the response function R_μ. By the same argument as before, we know that R_μ depends on μ through $\mathbf{LL}^T$ and $\operatorname{grad}(\mathbf{LL}^T)$ only, where $(\mathbf{L}, \mathsf{M})$ denotes the deformation of μ relative to an undistorted reference $\varkappa$. If $(\mathbf{F}, \mathsf{G})$ is an arbitrary deformation relative to (μ_{*p}, μ_{**p}) as before, then the corresponding deformation relative to $(\varkappa_{*p}, \varkappa_{**p})$ is given by (2.15). Substituting (2.15) into (1.13), we obtain

$$R_\mu(\mathbf{F}, \mathsf{G}, p) = \check{R}_\varkappa(\mathbf{FLL}^T\mathbf{F}^T, \operatorname{grad}(\mathbf{FLL}^T\mathbf{F}^T), p), \tag{2.30}$$

where from (1.15) the second variable of $\check{R}_\varkappa$ is given by

$$\begin{aligned}(\operatorname{grad}(\mathbf{FLL}^T\mathbf{F}^T))^{ij}{}_k = {} & (\operatorname{grad}(\mathbf{LL}^T))^{AB}{}_C F^i{}_A F^j{}_B (F^{-1})^C{}_k \\ & + (G^i{}_{AB}F^j{}_C + G^j{}_{AB}F^i{}_C)(F^{-1})^B{}_k(\mathbf{LL}^T)^{AC}.\end{aligned} \tag{2.31}$$

3. Homogeneous and pseudo-homogeneous fluid bodies and isotropic solid bodies of second-grade

In the preceding sections, we have presented a local theory for a second-grade fluid point and a second-grade solid point. As usual, a local theory corresponds to a global theory in the special case when the body is homogeneous. We shall develop this special case first in this section.

Naturally, we call $\mathscr{A}$ a *(second-grade) fluid body* or a *(second-grade) isotropic solid body* if each point $p \in \mathscr{A}$ is a (second-grade) fluid point or a (second-grade) isotropic

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solid point, respectively. For a reason which will become apparent later, we call $\mathscr{A}$ a *homogeneous body* if there exists a global reference $\boldsymbol{\varkappa}$ which verifies the following conditions:

v) The field of response functions relative to $\boldsymbol{\varkappa}$, namely $R_{\varkappa}(\mathbf{F}, \mathsf{G}, p)$, $p \in \mathscr{A}$, is independent of p.

vi) The induced field of local configurations of $\boldsymbol{\varkappa}$, namely $(\boldsymbol{\varkappa}_{*p}, \boldsymbol{\varkappa}_{**p})$, $p \in \mathscr{A}$, is undistorted at each p.

Naturally, we call the configuration $\boldsymbol{\varkappa}$, which is distinguished by Conditions v) and vi), a *homogeneous configuration* of $\mathscr{A}$. It should be pointed that Condition v) generally does not imply Condition vi). However, if Condition v) holds, and if $(\boldsymbol{\varkappa}_{*p}, \boldsymbol{\varkappa}_{**p})$ is undistorted at any one point p, then Condition vi) also holds. For definiteness, we call a body $\mathscr{A}$ that satisfies Condition v) but not Condition vi) a *pseudo-homogeneous body*. We shall develop a global theory for homogeneous bodies first.

The structure of a homogeneous body is very simple. Relative to a homogeneous configuration $\boldsymbol{\varkappa}$, the response function is given by

$$\tilde{R} = R_{\varkappa}(\mathbf{F}, \mathsf{G}). \tag{3.1}$$

This equation means that in any deformed configuration $\boldsymbol{\varphi}$ of $\mathscr{A}$ the response $\tilde{R}$ on $\boldsymbol{\varphi}(\mathscr{A})$ is given by $R_{\varkappa}(\mathbf{F}, \mathsf{G})$, where $(\mathbf{F}, \mathsf{G})$ denotes the second-grade deformations from $\boldsymbol{\varkappa}$ to $\boldsymbol{\varphi}$.

Since $\boldsymbol{\varkappa}$ is undistorted, $R_{\varkappa}$ admits a representation as shown in the preceding sections. In particular, when $\mathscr{A}$ is a fluid body, (3.1) reduces to

$$\tilde{R} = \hat{R}_{\varkappa}(|\det\mathbf{F}|, \operatorname{grad}|\det\mathbf{F}|), \tag{3.2}$$

and when **A** is an isotropic solid body, the representation is

$$\tilde{R} = \check{R}_{\varkappa}(\mathbf{B}, \operatorname{grad}\mathbf{B}). \tag{3.3}$$

Further, in the argument of $\hat{R}_{\varkappa}$ and $\check{R}_{\varkappa}$ the gradient operation reduces to the ordinary spatial gradient, *i.e.*, in coordinate form

$$(\operatorname{grad} f)_i = \frac{\partial f}{\partial x^i} \tag{3.4}$$

for any quantity f.

Next, we consider the structure of a pseudo-homogeneous body $\mathscr{A}$. By definition, there exists a configuration $\boldsymbol{\mu}$, called a *pseudo-homogeneous configuration*, relative to which the field of response functions $R_{\mu}(\mathbf{F}, \mathsf{G}, p)$, $p \in \mathscr{A}$, is independent of p, but $\boldsymbol{\mu}$ is not an undistorted configuration. In a deformed configuration $\boldsymbol{\varphi}$, the field of response $\tilde{R}$ is still given by

$$\tilde{R} = R_{\mu}(\mathbf{F}, \mathsf{G}). \tag{3.5}$$

However, since $\boldsymbol{\mu}$ is not undistorted, the representation (3.2) or (3.3) are no longer applicable.

In order to make use of material symmetry in this case, we need the representations (2.19) and (2.30). We consider first the case when $\mathscr{A}$ is a fluid body. As before, let $(\mathbf{L}, \mathsf{M})$ denote the second-grade deformation of $(\boldsymbol{\mu}_{*p}, \boldsymbol{\mu}_{**p})$ for a particular point $p \in \mathscr{A}$ relative to an undistorted local reference configuration $(\boldsymbol{\varkappa}_{*p}, \boldsymbol{\varkappa}_{**p})$. Since $\boldsymbol{\mu}$ is pseudo-homogeneous, the field $\boldsymbol{\zeta}$ given by

$$\boldsymbol{\zeta}(q) \equiv (\mathbf{L}, \mathsf{M})^{-1} \circ (\boldsymbol{\mu}_{*q}, \boldsymbol{\mu}_{**q}), \quad q \in \mathscr{A} \tag{3.6}$$

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is a field of undistorted local configurations on $\mathscr{A}$. As we shall prove later, such a field $\boldsymbol{\zeta}$ cannot be an induced field of a configuration of $\mathscr{A}$. However, since from (3.6) at the point p we have

$$(3.7) \qquad \boldsymbol{\zeta}(p) = (\mathbf{L}, \mathsf{M})^{-1} \circ (\boldsymbol{\mu}_{*p}, \boldsymbol{\mu}_{**p}) = (\boldsymbol{\varkappa}_{*p}, \boldsymbol{\varkappa}_{**p}),$$

the field of response function relative to $\boldsymbol{\zeta}$ is given by

$$(3.8) \qquad R_{\zeta}(\mathbf{F}, \mathsf{G}, q) = R_{\varkappa}(\mathbf{F}, \mathsf{G}, p), \quad \forall\, q \in \mathscr{A}.$$

Here, we have used the fact that $R_{\zeta}(\mathbf{F}, \mathsf{G}, q)$ is independent of q, an obvious consequence of (3.6). As explained in the preceding section, the distorted configuration $\boldsymbol{\mu}$ can be characterized by two quantities: a positive scalar $|\det \mathbf{L}|$ and a vector $\operatorname{grad}|\det \mathbf{L}|$. For brevity, we put

$$(3.9) \qquad l = |\det \mathbf{L}|, \quad \mathbf{m} = \operatorname{grad}|\det \mathbf{L}|.$$

In view of the fact that $\boldsymbol{\mu}$ is pseudo-homogeneous, $(\mathbf{L}, \mathsf{M})$ remains constant on $\boldsymbol{\mu}(A)$, and hence $(l, \mathbf{m})$. Now, let $(\mathbf{F}, \mathsf{G})$ be the second-grade deformation of $\boldsymbol{\varphi}$ relative to $\boldsymbol{\mu}$ as before. Then, from (2.19), the field of response has the representation

$$(3.10) \qquad \tilde{R} = \check{R}_{\zeta}(l|\det \mathbf{F}|, |\det \mathbf{F}|(\mathbf{F}^{-1})^{T}\mathbf{m} + l \operatorname{grad}|\det \mathbf{F}|),$$

where $\operatorname{grad}|\det \mathbf{F}|$ is given by (3.6).

Having obtained the desired representation (3.10) for $\tilde{R}$, we now show that $\mathscr{A}$ does not have any homogeneous configuration. Indeed, suppose that there exists a homogeneous configuration $\boldsymbol{\varphi}$ for $\mathscr{A}$. Let $(\mathbf{F}, \mathsf{G})$ be the second-grade deformation of $\boldsymbol{\varphi}$ relative to $\boldsymbol{\mu}$. Then, from (3.6), the second-grade deformation of $\boldsymbol{\varphi}$ relative to $\boldsymbol{\zeta}$ is given by

$$(3.11) \qquad (\boldsymbol{\varphi}_{*q}, \boldsymbol{\varphi}_{**q}) \circ \boldsymbol{\zeta}^{-1}(q) = \big(\mathbf{F}(q), \mathsf{G}(q)\big) \circ (\mathbf{L}, \mathsf{M}),\ q \in \mathscr{A},$$

where the composite deformation on the right-hand side of (3.11) corresponds to the argument in (3.10).

Now, since $\boldsymbol{\varphi}$ is required to be homogeneous, Condition v) implies that

$$(3.12) \qquad l|\det \mathbf{F}| = \text{a constant independent of } q,$$

and Condition vi) implies that

$$(3.13) \qquad |\det \mathbf{F}|(\mathbf{F}^{-1})^{T}\mathbf{m} + l \operatorname{grad}|\det \mathbf{F}| = \mathbf{o},$$

since $\mathbf{m} \neq \mathbf{o}$; otherwise, $\boldsymbol{\mu}$ would be homogeneous not pseudo-homogeneous; (3.12) and (3.13) are clearly contradictory. Thus we have completed the proof.

Since R_{μ} is independent of the position in $\boldsymbol{\mu}(\mathscr{A})$, the local configurations induced by $\boldsymbol{\mu}$ are *materially isomorphic* in a sense to be defined in general in the next section. The local configurations of $\boldsymbol{\mu}$ are not undistorted, however, since their deformation $(\mathbf{L}, \mathsf{M})$ from an undistorted reference $\boldsymbol{\zeta}$ does not satisfy (2.8). Of course, it is possible to deform $\boldsymbol{\mu}(\mathscr{A})$ in such a way that the local configuration of some point $p \in \mathscr{A}$ becomes undistorted. It is not possible, however, to deform the whole $\boldsymbol{\mu}(\mathscr{A})$ into an undistorted configuration as we have shown.

Having explained the structure of a pseudo-homogeneous fluid body in detail, we can treat that of a pseudo-homogeneous isotropic solid body in a similar way. Let $\mathscr{A}$ now denote a pseudo-homogeneous isotropic solid body, and suppose that $\boldsymbol{\mu}$ is one of its

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pseudo-homogeneous configurations. As before, let $(\mathbf{L}, \mathsf{M})$ be the second-grade deformation of $\boldsymbol{\mu}$ relative to $\boldsymbol{\zeta}$. Then we can characterize $\boldsymbol{\mu}$ by

$$\mathbf{E} \equiv \mathbf{L}\mathbf{L}^T, \quad \mathsf{D} \equiv \operatorname{grad}(\mathbf{L}\mathbf{L}^T), \tag{3.14}$$

where $\mathbf{E}$ is a metric and D is given by (2.29). From (2.30), the response field $\tilde{R}$ in any deformed configuration $\boldsymbol{\varphi}$ is given by

$$\tilde{R} = \check{\tilde{R}}_{\zeta}(\mathbf{F}\mathbf{E}\mathbf{F}^T, \mathsf{W}(\mathbf{F}, \mathsf{G}, \mathbf{E}, \mathsf{D})), \tag{3.15}$$

where W is given by

$$\left(\mathsf{W}(\mathbf{F}, \mathsf{G}, \mathbf{E}, \mathsf{D})\right)^{ij}{}_k = D^{AB}{}_C F^j{}_B F^i{}_A (F^{-1})^C{}_k + (G^i{}_{AB} F^j{}_C + G^j{}_{AB} F^i{}_C)(F^{-1})^B{}_k E^{AC}, \tag{3.16}$$

cf. (2.31). The expression (3.16) is the desired representation for $\tilde{R}$ in terms of the deformation of $\boldsymbol{\varphi}$ relative to $\boldsymbol{\mu}$.

By essentially the same argument as before, we can show that a pseudo-homogeneous isotropic solid body does not have any homogeneous configuration either.

Further, as in the case of a fluid body, the local configurations induced by $\boldsymbol{\mu}$ are materially isomorphic to one another, but those local configurations are not undistorted. Hence $\tilde{R}$ must be determined through (3.16) not through (3.3).

4. Materially uniform smooth inhomogeneous fluid bodies and isotropic solid bodies of second-grade

The global theory for homogeneous or pseudo-homogeneous fluids and isotropic solids developed in the preceding section can be generalized in an obvious way to a global theory for materially uniform but inhomogeneous bodies. Recall first that material isomorphism between two points p and q in a body $\mathscr{A}$ is defined by the condition that there exist local references $\boldsymbol{\zeta}(p)$ and $\boldsymbol{\zeta}(q)$ such that

$$R_{\zeta}(\mathbf{F}, \mathsf{G}, p) = R_{\zeta}(\mathbf{F}, \mathsf{G}, q) \tag{4.1}$$

for all second-grade deformations $(\mathbf{F}, \mathsf{G})$, *cf.* [1]. Naturally, we say that $\boldsymbol{\zeta}(p)$ and $\boldsymbol{\zeta}(q)$ are *materially isomorphic* to each other, and we call p and q *materially isomorphic* points. A body is said to be *materially uniform* if all of its points are materially isomorphic to one another.

As usual, we define *smoothness* of a materially uniform body by the condition that there be smooth local fields of materially isomorphic references in a neighborhood of any point in the body. Generally, smooth global fields of materially isomorphic references need not exist; this situation is explained in detail in the theory of smooth materially uniform simple bodies, *cf.* [3]. While it is, of course, possible to develop a general global theory for second-grade bodies with arbitrary material symmetry, this is not the purpose of this paper. Here, we are interested in second-grade fluid bodies and isotropic solid bodies only. For these special cases, the material structure can be characterized by some tensor fields, as we shall now see.

First, we consider the structure of a smooth, materially uniform, second-grade fluid body. Denoting the body by $\mathscr{A}$ as before, we know that $\mathscr{A}$ can be covered by local smooth

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fields of materially isomorphic references. Since these fields are not required to be induced fields, without loss of generality we may choose them to be fields of materially isomorphic undistorted references. We denote a typical one of these fields by $\boldsymbol{\zeta}$.

Now, let $\boldsymbol{\mu}$ be a global configuration of $\mathscr{A}$. Then, as explained in the preceding section, we can characterize the local configurations induced by $\boldsymbol{\mu}$ by l and $\mathbf{m}$. While in general $\boldsymbol{\zeta}$ need not be defined globally, the fields l and $\mathbf{m}$ are global and are smooth in $\boldsymbol{\mu}(\mathscr{A})$. In the special case when $\mathscr{A}$ is homogeneous, l is constant and $\mathbf{m}$ vanishes throughout $\boldsymbol{\mu}(\mathscr{A})$, while in the case when $\mathscr{A}$ is pseudo-homogeneous, both l and $\mathbf{m}$ are constant and non-vanishing. In general, of course, l and $\mathbf{m}$ need not be constant on $\boldsymbol{\mu}(\mathscr{A})$. In that case, we call $\boldsymbol{\mu}$ an *inhomogeneous configuration* of $\mathscr{A}$. Naturally, we call $\mathscr{A}$ an *inhomogeneous body* if all configurations of $\mathscr{A}$ are inhomogeneous.

It should be noted that a homogeneous body or a pseudo-homogeneous body possesses homogeneous or pseudo-homogeneous configurations as well as inhomogeneous configurations. Hence the mere fact that a configuration $\boldsymbol{\mu}$ is inhomogeneous for $\mathscr{A}$ does not imply inhomogeneity of the body. In order to show that $\mathscr{A}$ is inhomogeneous, we must verify that *all* configurations of $\mathscr{A}$ are inhomogeneous. Therefore it is important to know the transformation rule for l and $\mathbf{m}$ under a change of configuration.

In the preceding section we have given the transformation rule for l and $\mathbf{m}$ relative to a homogeneous or a pseudo-homogeneous configuration. Now, let $\boldsymbol{\mu}$ and $\overline{\boldsymbol{\mu}}$ be configuration of $\mathscr{A}$ in general. We denote the second-grade deformation field from $\boldsymbol{\mu}$ to $\overline{\boldsymbol{\mu}}$ by $(\mathbf{F}, \mathsf{G})$. Then l, $\mathbf{m}$ and $\bar{l}$, $\overline{\mathbf{m}}$ on $\boldsymbol{\mu}(A)$ and $\overline{\boldsymbol{\mu}}(A)$, respectively, are related by

$$\bar{l} = l|\det\mathbf{F}|, \quad \overline{\mathbf{m}} = |\det\mathbf{F}|(\mathbf{F}^{-1})^T\mathbf{m} + l\,\mathrm{grad}|\det\mathbf{F}|. \tag{4.2}$$

The proof is the same as (2.17) and (2.18), since (4.2) is a pointwise result.

The transformation rule $(4.2)_2$ shows that $\mathbf{m}$ does *not* correspond to a material[1] vector field in a deformation from $\boldsymbol{\mu}$ to $\overline{\boldsymbol{\mu}}$ in general. However, if the deformation is isochoric, then $\mathbf{m}$ transforms as a material covector field, since in this case, (4.2) reduces to

$$\bar{l} = l, \quad \overline{\mathbf{m}} = (\mathbf{F}^{-1})^T\mathbf{m}. \tag{4.3}$$

We can use this special case to describe the inhomogeneity in a particular fluid body: We consider a fluid body that is equipped with a global configuration $\boldsymbol{\mu}$ in which l is constant but $\mathbf{m}$ is not constant, say $\mathbf{m}$ vanishes at some points but does not vanish at some other points. Then, from (4.3), such a fluid body is clearly inhomogeneous.

Knowing that a fluid body in general may be inhomogeneous, we now seek a representation for $\tilde{R}$ when the reference configuration $\boldsymbol{\mu}$ is arbitrary. This representation can be read off from (4.2) and (2.19), namely,

$$\tilde{R} = \hat{R}_{\zeta}(l|\det\mathbf{F}|, |\det\mathbf{F}|(\mathbf{F}^{-1})^T\mathbf{m} + l\,\mathrm{grad}|\det\mathbf{F}|), \tag{4.4}$$

which has the same form as (3.10), except that here l and $\mathbf{m}$ need not be constant, so that $\tilde{R}_{\zeta}$ depends implicitly on the position in $\boldsymbol{\mu}(\mathscr{A})$ through l and $\mathbf{m}$.

The representation (4.4) expresses $\tilde{R}$ in terms of the field of second-grade deformation $(\mathbf{F}, \mathsf{G})$ relative to $\boldsymbol{\mu}$.

Next, we consider the structure of a smooth, materially uniform, second-grade, iso-

[1] A *material* vector field or tensor field is a vector field or tensor field on the body manifold, *cf.* [3].

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tropic, solid body. Again, we denote the body by $\mathscr{A}$, and we choose a set of local fields of materially isomorphic undistorted references $\boldsymbol{\zeta}$ to cover $\mathscr{A}$. Then, as explained in the preceding section, we can characterize the local configurations induced by a global configuration $\boldsymbol{\mu}$ by $\mathbf{E}$ and D. In the special case when $\mathscr{A}$ is homogeneous, $\mathbf{E}$ is constant and D vanishes, while in the case when $\mathscr{A}$ is pseudo-homogeneous, both $\mathbf{E}$ and D are constant and non-vanishing. In general, of course, $\mathbf{E}$ and D need not be constant on $\boldsymbol{\mu}(\mathscr{A})$.

Examples of inhomogeneous, second-grade, isotropic, solid bodies are easy to construct. Indeed, any inhomogeneous simple (*i.e.*, first-grade) isotropic solid body corresponds to an inhomogeneous second-grade body. Further, even if $\mathbf{E}$ is constant and coincides with the Euclidean metric on $\boldsymbol{\mu}(\mathscr{A})$, $\mathscr{A}$ may still be inhomogeneous provided that D be non-vanishing.

The transformation rules for $\mathbf{E}$ and D under a change of configuration from $\boldsymbol{\mu}$ to $\bar{\boldsymbol{\mu}}$ are

$$\bar{\mathbf{E}} = \mathbf{F}\mathbf{E}\mathbf{F}^T \tag{4.5}$$

and

$$\overline{D}^{ij}{}_k = D^{AB}{}_C F^i{}_A F^j{}_B (F^{-1})^C{}_k + (G^i{}_{AB} F^j{}_C + G^j{}_{AB} F^i{}_C)(F^{-1})^B{}_k E^{AC}, \tag{4.6}$$

where $(\mathbf{F}, \mathsf{G})$ denote the second-grade deformation from $\boldsymbol{\mu}$ to $\bar{\boldsymbol{\mu}}$, *cf.* (2.30) and (2.31). From (4.4), when $\mathbf{E}$ is the identity (metric) tensor, then so is $\bar{\mathbf{E}}$ if, and only if, $\mathbf{F}$ is a constant rotation. In this case G vanishes, and (4.6) reduces to

$$\overline{D}^{ij}{}_k = D^{AB}{}_c F^i{}_A F^j{}_B (F^{-1})^c{}_k, \tag{4.7}$$

which shows that the field D transforms as a third-order material tensor in such a special deformation.

Now, let $\boldsymbol{\mu}$ be an arbitrary reference configuration of $\mathscr{A}$, homogeneous or inhomogeneous. We define the global fields $\mathbf{E}$ and D on $\boldsymbol{\mu}(\mathscr{A})$ relative to an undistorted reference $\boldsymbol{\zeta}$. Then the response field $\tilde{R}$ in any deformed configuration $\boldsymbol{\varphi}$ can be determined by

$$\tilde{R} = \check{R}_{\zeta}(\mathbf{F}\mathbf{E}\mathbf{F}^T, \mathsf{W}(\mathbf{F}, \mathsf{G}, \mathbf{E}, \mathsf{D}), \tag{4.8}$$

where W is given by (3.16). The representation (4.8) has the same form as (3.15) except that here $\mathbf{E}$ and D need not be constant, so that $\hat{R}_\zeta$ depends implicitly on the position in $\boldsymbol{\mu}(\mathscr{A})$ through $\mathbf{E}$ and D. The representation (4.8) expresses $\tilde{R}$ in terms of the deformation fields $\mathbf{F}$ and G relative to $\boldsymbol{\mu}$.

5. Local homogeneity, local pseudo-homogeneity, and local inhomogeneity in fluid bodies and isotropic solid bodies

In the theory of simple bodies it is known that every materially uniform, smooth, simple fluid body is locally homogeneous, and a materially uniform, smooth, isotropic solid body is locally homogeneous if, and only if, its characteristic Riemannian metric is flat. A locally homogeneous simple body need not be globally homogeneous; for more details about inhomogeneities in simple bodies see refs. [3, 4 and 5].

In the preceding section we have shown that, unlike a simple fluid body, a second-grade fluid body, generally, need not be homogeneous. Indeed, there are three types of

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materially uniform smooth second-grade fluid bodies: homogeneous, pseudo-homogeneous or inhomogeneous fluid bodies. In the first part of this section we shall characterize these three types of fluid bodies explicitly by some geometric conditions. However, since we shall use local differential geometry only, those conditions are local, *i.e.*, they characterize(²) *local homogeneity*, *local pseudo-homogeneity*, and *local inhomogeneity* only.

Our first result is

THEOREM 5.1. *A second-grade fluid body $\mathscr{A}$ is locally homogeneous near $p \in \mathscr{A}$ if, and only if, in any configuration $\boldsymbol{\mu}$ the characteristic fields l and $\mathbf{m}$ are related by*

$$\mathbf{m} = \mathrm{Grad}\, l \tag{5.1}$$

near p.

Proof. Sufficiency. From the theory of simple fluid bodies we know that, locally, there exist deformations relative to $\boldsymbol{\mu}$ such that in the deformed configuration $\bar{\boldsymbol{\mu}}$ the field l is constant near p. From $(4.2)_1$, we then have

$$\mathbf{o} = \mathrm{grad}(l|\det\mathbf{F}|). \tag{5.2}$$

Using the chain rule and the formula for the derivative of the determinant, we can rewrite (5.2) as

$$\mathbf{o} = |\det\mathbf{F}|(\mathbf{F}^{-1})^T\mathrm{Grad}\, l + l\,\mathrm{grad}|\det\mathbf{F}|. \tag{5.3}$$

Substituting (5.1) into (5.3) and comparing the result with $(4.2)_2$, we see that

$$\bar{\mathbf{m}} = \mathbf{o} \tag{5.4}$$

in the neighborhood of p, where (5.2) holds. Thus $\bar{\boldsymbol{\mu}}$ is locally homogeneous near p, and hence $\mathscr{A}$.

Necessity. Reversing the preceding argument, we assume that $\boldsymbol{\mu}$ is locally homogeneous near p. Then, locally, l is constant, and $\mathbf{m}$ vanishes. Now, let $\bar{\boldsymbol{\mu}}$ be an arbitrary configuration of $\mathscr{A}$. From (4.2), we obtain

$$\bar{l} = l|\det\mathbf{F}|, \quad \bar{\mathbf{m}} = l\,\mathrm{grad}|\det\mathbf{F}| = \mathrm{grad}\,\bar{l}. \tag{5.5}$$

Hence (5.1) holds locally in $\bar{\boldsymbol{\mu}}$ near the position of p. But since $\bar{\boldsymbol{\mu}}$ is arbitrary, necessity is proved.

The preceding theorem suggests that we define a vector field $\mathbf{m}_0$ on any configuration $\boldsymbol{\mu}$ of $\mathscr{A}$ by

$$\mathbf{m}_0 \equiv \frac{1}{l}(\mathbf{m} - \mathrm{Grad}\, l). \tag{5.6}$$

Then we have the following

LEMMA 5.2. *The vector field $\mathbf{m}_0$ defined by* (5.6) *transforms as a material covector in any deformation of $\mathscr{A}$, i.e., in any configuration $\bar{\boldsymbol{\mu}}$ with deformation gradient $\mathbf{F}$ relative to $\boldsymbol{\mu}$, we have*

$$\bar{\mathbf{m}}_0 = (\mathbf{F}^{-1})^T\mathbf{m}_0. \tag{5.7}$$

The proof of this lemma is obvious.

(²) As usual, we say that $\mathscr{A}$ is *locally homogeneous* near p if a neighborhood of p is homogeneous, *cf.* [3]; *local pseudo-homogeneity* and *local inhomogeneity* are defined similarly.

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In view of the preceding lemma, we see that Theorem 5.1 can be restated as

THEOREM 5.1′. *A second-grade fluid body $\mathscr{A}$ is locally homogeneous near $p \in \mathscr{A}$ if, and only if, the characteristic field* $\mathbf{m}_0$ *vanishes locally in any configuration* $\boldsymbol{\mu}$.

Having characterized the locally homogeneous case, we consider next local pseudo-homogeneity in a second-grade fluid body.

THEOREM 5.3. *A second-grade fluid body $\mathscr{A}$ is locally pseudo-homogeneous near p if, and only if, in any configuration $\boldsymbol{\mu}$ the characteristic field* $\mathbf{m}_0$ *does not vanish but satisfies locally the condition*

$$\operatorname{Curl}\mathbf{m}_0 = \mathbf{o}. \tag{5.8}$$

Proof. Necessity. Suppose that $\mathscr{A}$ is locally pseudo-homogeneous near p. Then there exists a reference configuration $\boldsymbol{\mu}$ in which l and $\mathbf{m}_0$ are constant and non-vanishing near p. The field $\overline{\mathbf{m}}_0$ in any configuration $\bar{\boldsymbol{\mu}}$ is given by (5.7), *viz.*,

$$\overline{\mathbf{m}}_0 = (\mathbf{F}^{-1})^T\mathbf{m}_0 \neq \mathbf{o}, \tag{5.9}$$

which clearly satisfies the condition

$$\operatorname{curl}\overline{\mathbf{m}}_0 = \mathbf{o}. \tag{5.10}$$

Sufficiency. Suppose that $\mathbf{m}_0 \neq \mathbf{o}$ and satisfies (5.10) locally in $\boldsymbol{\mu}$. Then in a neighborhood of p we can find a function f such that

$$\mathbf{m}_0 = \operatorname{Grad} f \neq \mathbf{o}. \tag{5.11}$$

Denoting the coordinate system corresponding to $\boldsymbol{\mu}$ by (X^A), we define the deformation functions x^i near p by

$$x^1 = f(X^1, X^2, X^2), \quad x^2 = x^2(X^1, X^2, X^3), \quad x^3 = x^3(X^1, X^2, X^3), \tag{5.12}$$

where x^2 and x^3 are chosen in such a way that

$$|\det\mathbf{F}| = \frac{1}{l}. \tag{5.13}$$

In $\bar{\boldsymbol{\mu}}$ we then have, locally,

$$l = 1, \quad \overline{\mathbf{m}} = \overline{\mathbf{m}}_0 = (\mathbf{F}^{-1})^T\operatorname{Grad} f = \operatorname{grad} x^1. \tag{5.14}$$

Consequently, $\bar{\boldsymbol{\mu}}$ is locally pseudo-homogeneous near p, and hence $\mathscr{A}$.

If we now exclude the locally homogeneous case and the locally pseudo-homogeneous case, then the locally inhomogeneous case can be characterized by

THEOREM 5.4. *A second-grade fluid body $\mathscr{A}$ is locally inhomogeneous near p if, and only if, in any configuration $\boldsymbol{\mu}$ the characteristic field* $\mathbf{m}_0$ *satisfies locally one of the following two conditions*:

vii) $\mathbf{m}_0$ *and* Curl $\mathbf{m}_0$ *are non-vanishing near p.*

viii) $\mathbf{m}_0$ *vanishes at p but it does not vanish identically near p.*

The preceding theorems show that local homogeneity, local pseudo-homogeneity, or local inhomogeneity in a materially uniform, smooth, second-grade, fluid body can be characterized by l and $\mathbf{m}_0$ in any configuration $\boldsymbol{\mu}$. When we change the configuration $\boldsymbol{\mu}$ by a deformation, l and $\mathbf{m}_0$ transform as a material relative scalar field and a material covector field, respectively. In abstract differential geometry, this situation corresponds

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to a volume tensor field $\boldsymbol{\nu}$ and a differential 1-form $\boldsymbol{\theta}$ on the body manifold $\mathscr{A}$. Local homogeneity or inhomogeneity in $\mathscr{A}$ can now be expressed directly in terms of $\boldsymbol{\nu}$ and $\boldsymbol{\theta}$, independent of any configuration.

THEOREM 5.5. *The structure of a second-grade fluid body $\mathscr{A}$ may be characterized by a volume tensor field $\boldsymbol{\nu}$ and a differential 1-form $\boldsymbol{\theta}$ on the body manifold $\mathscr{A}$ such that*

I) *$\mathscr{A}$ is locally homogeneous if, and only if, $\boldsymbol{\theta}$ vanishes.*

II) *$\mathscr{A}$ is locally pseudo-homogeneous if, and only if, $\boldsymbol{\theta}$ is non-vanishing but is closed, i.e., the exterior derivative $d\,\boldsymbol{\theta}$ of $\boldsymbol{\theta}$ vanishes.*

III) *$\mathscr{A}$ is locally inhomogeneous if, and only if, $\boldsymbol{\theta}$ is not closed or $\boldsymbol{\theta}$ vanishes at some points but does not vanish at some other points in $\mathscr{A}$.*

Having considered local homogeneity, local pseudo-homogeneity, and local inhomogeneity in a fluid body, we consider next the corresponding situations in an isotropic solid body.

THEOREM 5.6. *A second-grade isotropic solid body $\mathscr{A}$ is locally homogeneous near p if, and only if, in any configuration $\boldsymbol{\mu}$, the characteristic fields $\mathbf{E}$ and D verify the following two conditions:*

ix) *The curvature tensor of $\mathbf{E}$ vanishes near p.*

x) *The tensor field D is given locally by*

$$\mathsf{D} = \operatorname{Grad}\mathbf{E}. \tag{5.15}$$

The proof of this theorem is essentially the same as that of Theorem 5.1.

The preceding theorem suggests that we define a tensor field D_0 on any configuration $\boldsymbol{\mu}$ by

$$\mathsf{D}_0 \equiv \mathsf{D} - \operatorname{Grad}\mathbf{E}. \tag{5.16}$$

As before, we have the following

LEMMA 5.7. *The tensor field D_0 transforms as a material tensor in any deformation of $\mathscr{A}$, i.e., in any configuration $\bar{\boldsymbol{\mu}}$ with deformation gradient $\mathbf{F}$ relative to $\boldsymbol{\mu}$, we have*

$$\bar{\mathsf{D}}_0 = \mathsf{D}_0[\mathbf{F}, \mathbf{F}, \mathbf{F}^{-1}]. \tag{5.17}$$

As before, we can restate Theorem 5.6 as

THEOREM 5.6′. *A second-grade isotropic solid body $\mathscr{A}$ is locally homogeneous near p if, and only if, the characteristic tensor D_0 and the curvature tensor of the tensor $\mathbf{E}$ both vanish near p in any configuration $\boldsymbol{\mu}$.*

Next, we characterize local pseudo-homogeneity. Our result is

THEOREM 5.8. *A second-grade isotropic solid body $\mathscr{A}$ is locally pseudo-homogeneous near p if, and only if, in any configuration $\boldsymbol{\mu}$ the characteristic metric satisfies Condition* ix), *while the characteristic tensor field D_0 satisfies the condition*

x)′ *The tensor field D_0 is constant but non-vanishing with respect to the metric $\mathbf{E}$, i.e., the covariant derivative of D_0 relative to $\mathbf{E}$ vanishes near p.*

In component form, Condition x′) means that $\mathsf{D}_0 \neq \mathrm{O}$ but

$$o = D_0^{AB}{}_{C,D} = \frac{\partial D_0^{AB}{}_C}{\partial X^D} + D_0^{KB}{}_C\left\{{A \atop KD}\right\} + D_0^{AK}{}_C\left\{{B \atop KD}\right\} - D_0^{AB}{}_K\left\{{K \atop CD}\right\}, \tag{5.18}$$

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where $\{^{A}_{BC}\}$ denotes the Christoffel symbols of the metric $\mathbf{E}$. To prove Theorem 5.8, let $\boldsymbol{\mu}$ be a locally pseudo-homogeneous configuration for $\mathscr{A}$. Then Conditions ix) and x') hold in $\boldsymbol{\mu}$. But since $\mathbf{E}$ and D_0 are material tensors with respect to any deformation, the same must hold in all configurations. Conversely, if Conditions ix) and x') hold in $\boldsymbol{\mu}$, then there exists a configuration $\bar{\boldsymbol{\mu}}$, in which $\bar{\mathbf{E}}$ is the Euclidean metric and $\bar{\mathsf{D}}_0$ is equal to D and is constant in a neighborhood of p. Therefore $\bar{\boldsymbol{\mu}}$ is locally pseudo-homogeneous, and hence $\mathscr{A}$.

Having characterized local homogeneity and local pseudo-homogeneity in a materially uniform, smooth, second-grade, isotropic solid body, we can characterize the remaining case of local inhomogeneity by

THEOREM 5.9. *A second-grade isotropic solid body $\mathscr{A}$ is locally inhomogeneous near p if, and only if, in any configuration $\boldsymbol{\mu}$, the characteristic fields $\mathbf{E}$ and D_0 satisfy one of the following two conditions:*

xi) *The curvature tensor of $\mathbf{E}$ does not vanish near p.*

xii) *The curvature tensor of $\mathbf{E}$ vanishes near p but the tensor D_0 does not remain constant relative to $\mathbf{E}$.*

Theorems 5.6–5.9 show that local homogeneity, local pseudo-homogeneity, or local inhomogeneity in a materially uniform, smooth, second-grade, isotropic solid body can be characterized by $\mathbf{E}$ and D_0 in any configuration $\boldsymbol{\mu}$ of the body. When we change the configuration $\boldsymbol{\mu}$ by a deformation, $\mathbf{E}$ and D_0 transform as material tensor fields. Consequently, there exist a Riemannian metric $\boldsymbol{\Sigma}$ and a third-order tensor field $\boldsymbol{\Gamma}$ on the body manifold $\mathscr{A}$ corresponding to $\mathbf{E}$ and D_0, respectively. Further, local homogeneity or inhomogeneity in $\mathscr{A}$ can be expressed directly in terms of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Gamma}$, independent of any configuration.

THEOREM 5.10. *The structure of a second-grade isotropic solid body $\mathscr{A}$ may be characterized by a Riemannian metric $\boldsymbol{\Sigma}$ and a third-order tensor field $\boldsymbol{\Gamma}$ on the body manifold $\mathscr{A}$ such that*

IV) *$\mathscr{A}$ is locally homogeneous if, and only if, $\boldsymbol{\Gamma}$ and the curvature tensor of $\boldsymbol{\Sigma}$ both vanish.*

V) *$\mathscr{A}$ is locally pseudo-homogeneous if, and only if, the curvature tensor of $\boldsymbol{\Sigma}$ and the covariant derivative of $\boldsymbol{\Gamma}_0$ both vanish but $\boldsymbol{\Gamma}_0$ does not vanish.*

VI) *$\mathscr{A}$ is locally inhomogeneous if, and only if, the curvature tensor of $\boldsymbol{\Sigma}$ does not vanish, or if the curvature tensor of $\boldsymbol{\Sigma}$ vanishes, the covariant derivative of $\boldsymbol{\Gamma}_0$ does not vanish.*

Now, we have characterized local homogeneity, local pseudo-homogeneity, and local inhomogeneity geometrically in any materially uniform, smooth, second-grade fluid bodies and isotropic solid bodies, so we close.

Acknowledgement

The research reported in the paper was supported by Grant 23773 from the U. S. National Science Foundation to Rice University. I am grateful to J. CROSS who checked the preprint of this paper.

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Received September 28, 1972.

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