Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 5, pp. 753–763, Warszawa 1973

Influence of random perturbations on self-excited vibrations of a system with one degree of freedom

K. PISZCZEK (KRAKÓW)

THE PAPER presents an approximate method of analysis of the influence of random perturbations on the amplitude and the angular vibration frequency of self-excited vibrations. The essential feature of the method consists in the method of derivation of the differential Eq. (1.11) describing the stochastic process. The theoretical analysis is illustrated by two examples concerning the Van der Pol equation with "white noise" perturbations, and the Rayleigh-type equation describing the lathe tool vibrations.

Podano przybliżoną metodę analizy wpływu zaburzeń przypadkowych na amplitudę i częstość kątową drgań samowzbudnych. Istotą metody jest sposób otrzymywania równania różniczkowego (1.11), opisującego proces stochastyczny. Część teoretyczną zilustrowano dwoma przykładami dotyczącymi równania Van der Pola przy zaburzeniu w postaci "białego szumu" oraz równania typu Rayleigha, opisującego drgania noża tokarki.

Дается метод приближенного анализа влияния случайных возмущений на амплитуду и угловую частоту автоколебаний. Существо метода состоит в построении дифференциального уравнения (1.11), описывающего случайный процесс. Теоретическая часть иллюстрируется двумя подробно вычисленными примерами, относящимися к (а) уравнению Ван дер Поля при возмущениях типа "белого шума" и к (б) уравнению типа Релея, описывающего колебания токарного резца.

Introduction

THEORY of self-excited vibrations constitutes one of the most important sections of non-linear dynamics. In the deterministic approach, the amplitude of stationary vibrations and the angular frequency corresponding to stable limit cycles are determined in an approximate manner. Analysis concerning the Van der Pol equation [4] or the Rayleigh equation [2] may serve as a good example of these types of problems.

This paper is aimed at the determination of the influence of random external perturbations, considered as a stationary stochastic process, on the approximate value of the square of amplitude of self-excited vibrations, and on the mean square value of angular frequency of these vibrations. One of the papers dealing with the subject is [1] which in a very particular manner concerns the Van der Pol equation. The linearization-correlation method was applied in that paper; it consisted of an approximate evaluation of the self-correlation function and of the spectral density corresponding to the non-linear term of the differential equation, the latter being then divided into two linear equations: one of them corresponds to the deterministic self-excited vibrations, the other represents the random perturbation.

The method presented in this paper is general and may be applied to the entire group of problems connected with the influence of external perturbations of random character upon the characteristics of non-linear vibrations.

1. Essential features of the method

Let the differential equation

$$L[\ddot{x}, \dot{x}, x; p] \equiv \ddot{x}+f_1(x, \dot{x})+\omega_0^2 x-p(t) = 0 \tag{1.1}$$

describe the vibrations of a system with one degree of freedom, the function $f_1(x, \dot{x})$ having the property that under vanishing external excitations $p(t) \equiv 0$ stable self-excited vibrations appear in the system. There exist numerous criteria which make it possible to establish the existence of limit cycles of self-excited vibrations on the sole basis of properties of coefficients of the corresponding differential equation of motion. These are, for instance, the Lienard and Bendixon criteria (cf. [6]). Function $p(t)$ is assumed to constitute a small perturbation in the form of a stationary, normal stochastic process with a zero mean value.

Let us denote by ξ_1 the random vibrations produced by external excitations $p(t)$, and by ξ_2 — the stationary self-excited vibrations which, in the general case, may be assumed in the form of a Fourier series ([5], p. 49),

$$\xi_2(t) = \xi_{02}+A\sin\omega t+a_2\sin(2\omega t+\varphi_2)+ \ldots \tag{1.2}$$

Here, ξ_{02} = const represents the value of displacement of the vibration center, A is the first approximation of "amplitude" of self-excited vibrations, and ω — the corresponding "angular frequency". The remaining terms in Eq. (1.2) represent higher order harmonic components. The values of $A, \omega, a_2, \varphi_2 \ldots$ are considered as constants.

Similarly to the deterministic approach to the problem of influence of external excitations on self-excited vibrations (cf. [5], Eqs. (41.1), (42.13)), the solution of (1.1) is assumed in the form

$$x = \xi_1+\xi_2. \tag{1.3}$$

This assumption demonstrates the influence of random excitations ξ_1 on the final form of the solution x which will be found to be close to the solution ξ_2 under small random perturbations. Substituting the relation (1.3) into Eq. (1.1), the form of solution (1.2) being taken into account, and representing the resulting expression in the form of a Fourier series, we obtain

$$\begin{aligned}\psi_0(\ddot{\xi}_1, \dot{\xi}_1, \xi_1; A, \omega, a_2, \varphi_2, \ldots)+\psi_1(\dot{\xi}_1, \xi_1; A, \omega, a_2, \varphi_2 \ldots)\cos\omega t \\ +\chi_1(\dot{\xi}_1, \xi_1; A, \omega, a_2, \varphi_2, \ldots)\sin\omega t+\psi_2(\dot{\xi}_1, \xi_1; A, \omega, a_2, \varphi_2, \ldots)\cos 2\omega t \\ +\chi_2(\dot{\xi}_1, \xi_1; A, \omega, a_2, \varphi_2, \ldots)\sin 2\omega t+ \ldots = 0.\end{aligned} \tag{1.4}$$

In view of the requirement that the above equality should be satisfied identically with respect to time t, all the coefficients of expansion (1.4) must vanish. Confining ourselves to the first approximation, i.e. preserving in Eq. (1.2) only the first two terms or the first three terms in the Fourier expansion (1.4), we obtain

$$\begin{aligned}\psi_0(\ddot{\xi}_1, \dot{\xi}_1, \xi_1; A, \omega) = 0, \\ \psi_1(\dot{\xi}_1, \xi_1; A, \omega) = 0, \\ \chi_1(\dot{\xi}_1, \xi_1; A, \omega) = 0.\end{aligned} \tag{1.5}$$

Applying the formulae for the Fourier series coefficients, Eqs. (1.5) are written in the form

$$\int_0^T L[\ddot{\tilde{x}}, \dot{\tilde{x}}, \tilde{x}; p]\,dt = 0,$$

(1.6)

$$\int_0^T L[\ddot{\tilde{x}}, \dot{\tilde{x}}, \tilde{x}; p]\cos\omega t\,dt = 0, \qquad \int_0^T L[\ddot{\tilde{x}}, \dot{\tilde{x}}, \tilde{x}; p]\sin\omega t\,dt = 0.$$

Here $T = 2\pi/\omega$, and

(1.7) $$\tilde{x} = \xi_1 + \xi_{02} + A\sin\omega t.$$

In integrating the expressions (1.6), the magnitudes $\xi_1, \dot{\xi}_1, \ddot{\xi}_1, p$ should be considered as constants.

Expression (1.7) is now substituted in (1.6) to yield, the notation of (1.1) being taken into account, the equations

$$\ddot{\xi}_1 + \omega_0^2\xi_1 + \omega_0^2\xi_{02} + \frac{1}{T}\int_0^T f_1[\xi_1 + \xi_{02} + A\sin\omega t, \dot{\xi}_1 + A\omega\cos\omega t]\,dt - p = 0,$$

(1.8) $$\int_0^T f_1[\xi_1 + \xi_{02} + A\sin\omega t, \dot{\xi}_1 + A\omega\cos\omega t]\cos\omega t\,dt = 0,$$

$$\int_0^T f_1[\xi_1 + \xi_{02} + A\sin\omega t, \dot{\xi}_1 + A\omega\cos\omega t]\sin\omega t\,dt = 0.$$

A and ω may be calculated from the last two equations and expressed as functions of the variables $\xi_1, \dot{\xi}_1, \xi_{02}$. Explicit form of these functions depends on the type of the function $f_1(x, \dot{x})$ appearing in the original differential equation (1.1). In certain cases, what is evident on the basis of a detailed analysis of two particular examples, the two last equations of (1.8) may be written as follows:

(1.9)
$$\omega^2A^2 = \Phi_1^0 + \Phi_1(\xi_1, \dot{\xi}_1),$$
$$\omega^2 = \Phi_2^0 + \Phi_2(\xi_1, \dot{\xi}_1).$$

Here

$$\Phi_1(0, 0) = 0, \quad \Phi_2(0, 0) = 0, \text{ and}$$
$$\Phi_1^0 = \overline{\omega}^2\overline{A}^2, \quad \Phi_2^0 = \overline{\omega}^2.$$

$\overline{\omega}$ and $\overline{A}$ denote the angular frequency and the amplitude of self-excited vibrations, respectively, influence of random perturbations being disregarded.

The expressions for A and ω thus derived are now substituted into the Eq. $(1.8)_1$ to yield

$$\ddot{\xi}_1 + F(\xi_1, \dot{\xi}_1; \xi_{02}) = p(t),$$

where F denotes a certain function of the arguments indicated in the formula. Consequently, this function is represented in the form of a sum

$$F(\xi_1, \dot{\xi}_1; \xi_{02}) = F_1(\xi_1, \dot{\xi}_1; \xi_{02}) + g(\xi_{02}),$$

the function F_1 being of such a type that

$$F(0, 0; \xi_{02}) = F_1(0, 0; \xi_{02}) = 0.$$

The equation

$$g(\xi_{02}) = 0 \tag{1.10}$$

serves to determine ξ_{02}. Let us denote its roots by $\bar{\xi}_{02}$. With this notation taken into account in the expression for F written as a sum of F_1 and g, we obtain

$$F(\xi_1, \dot{\xi}_1; \bar{\xi}_{02}) = F_1(\xi_1, \dot{\xi}_1; \bar{\xi}_{02}) = f(\xi_1, \dot{\xi}_1)$$

and Eq. (1.1) takes the form

$$\ddot{\xi}_1 + f(\xi_1, \dot{\xi}_1) = p(t). \tag{1.11}$$

The functions F, F_1 and g cannot be defined more precisely in the general case since they depend on the type of non-linearity appearing in the original Eq. (1.1). The method of separation of F into functions F_1 and g, as presented above, is general. Detailed expressions are obtained in each particular case separately.

2. Evaluation of the mean square value of displacements produced by random vibrations

The differential Eq. (1.11) is approximately solved by means of the method of equivalent statistic linearization [7].

The solution of Eq. (1.11) is sought in the form

$$\xi_1 = \langle \xi_1 \rangle + x_1 = m + x_1, \tag{2.1}$$

where $\langle \xi_1 \rangle = m$ represents the mean value of the random function ξ_1, and x_1 is a centred value (with a zero mean value). From the assumption, $\langle p \rangle = 0$. Equation (1.11) is then represented in an equivalent form

$$\ddot{x}_1 + k\dot{x}_1 + \Omega^2 x_1 = p(t) \tag{2.2}$$

in which the constants k and Ω^2 are expressed by the formulae

$$\begin{aligned} k &= \frac{\langle f(m+x_1, \dot{x}_1)\dot{x}_1 \rangle}{\langle \dot{x}_1^2 \rangle}, \\ \Omega^2 &= \frac{\langle f(m+x_1, \dot{x}_1)x_1 \rangle}{\langle x_1^2 \rangle}. \end{aligned} \tag{2.3}$$

In view of $\langle \ddot{\xi}_1 \rangle = 0$, we obtain

$$\langle f(m+x_1, \dot{x}_1) \rangle = 0. \tag{2.4}$$

The solution x_1 is assumed in the form

$$x_1 = a\sin(\Omega t + \Phi) = a\sin\theta. \tag{2.5}$$

Here a and Φ are random functions which vary slowly in time, therefore we may assume

$$\dot{x}_1 = a\Omega\cos\theta, \tag{2.6}$$

The expressions (2.3), (2.4) are now written as

$$(2.7)\qquad k = \frac{1}{2\pi\langle \dot{x}_1^2\rangle}\int_0^{2\pi}\int_0^{\infty} f(m+a\sin\theta,\, a\Omega\cos\theta)\, a\Omega\cos\theta\, w(a)\, da\, d\theta,$$

$$\Omega^2 = \frac{1}{2\pi\langle x_1^2\rangle}\int_0^{2\pi}\int_0^{\infty} f(m+a\sin\theta,\, a\Omega\cos\theta)\, a\sin\theta\, w(a)\, da\, d\theta$$

and

$$(2.8)\qquad \int_0^{2\pi}\int_0^{\infty} f(m+a\sin\theta,\, a\Omega\cos\theta)\, w(a)\, da\, d\theta = 0.$$

In these relations $w(a)$ denotes the function of probability distribution density of the amplitude and is expressed by the Rayleigh function

$$(2.9)\qquad w(a) = \frac{a}{\langle x_1^2\rangle}\exp\left[-\frac{a^2}{2\langle x_1^2\rangle}\right],$$

while

$$(2.10)\qquad \langle \dot{x}_1^2\rangle = \Omega^2\langle x_1^2\rangle.$$

From Eq. (2.8) is calculated the mean value of m which, substituted in Eqs. (2.7), may serve, together with Eq. (2.10), to express k and Ω^2 in terms of $\langle x_1^2\rangle$ and the vibrating system parameters contained in the form of $f(\xi_1, \dot{\xi}_1)$.

From the linear theory it is known that in the case of a vibrating system with one degree of freedom we may state, in connection with Eq. (2.2), the following equality:

$$(2.11)\qquad \langle x_1^2\rangle = \int_{-\infty}^{\infty} |H(j\omega)|^2 S_p(\omega)\, d\omega.$$

Here

$$(2.12)\qquad H(j\omega) = \frac{1}{\Omega^2-\omega^2+jk\omega}$$

represents the frequency characteristics of the system, and $S_p(\omega)$ is the spectral density of excitation p. In the case of "white noise", we obtain $S_p(\omega) = S_0 = \text{const}$ and Eq. (2.11) assumes after calculations the form

$$(2.13)\qquad \langle x_1^2\rangle = \frac{\pi S_0}{k\Omega^2}.$$

In the general case, once the form of $S_p(\omega)$ is known, we are able to calculate the integral appearing in Eq. (2.11). The resulting parameters k and Ω are replaced with the values from Eqs. (2.7) to yield the algebraic equation necessary for the final determination of $\langle x_1^2\rangle$. This value and Eqs. (2.10), (2.1) make it possible to express, by means of Eq. (1.9), the following mean square values of the amplitude and the self-excited vibration frequency, the random perturbations being accounted for,

$$\langle A^2\rangle = [\Phi_1^0+\langle\Phi_1(\xi_1, \dot{\xi}_1)\rangle]\frac{1}{\langle\omega^2\rangle}, \tag{2.14}$$
$$\langle\omega^2\rangle = \Phi_2^0+\langle\Phi_2(\xi_1, \dot{\xi}_1)\rangle.$$

3. Example 1. Van der Pol equation

As the first example of the theory let us analyze the influence of small random perturbations on the amplitude of self-excited vibrations described by the Van der Pol equation

$$\ddot{x}-\varepsilon(1-\gamma x^2)\dot{x}+\omega_0^2 x = p(t), \tag{3.1}$$

where ε and γ are positive constants. From Eq. (1.1) it follows that the function

$$f_1(x, \dot{x}) \equiv -\varepsilon(1-\gamma x^2)\dot{x}. \tag{3.2}$$

Expressions (1.5) take the respective forms

$$\ddot{\xi}_1-\varepsilon\dot{\xi}_1+\omega_0^2\xi_1+\varepsilon\gamma\left(\xi_1^2+\frac{1}{2}A^2\right)\dot{\xi}_1-p(t) = 0,$$
$$-\varepsilon A\omega+\varepsilon\gamma A\omega\xi_1^2+\frac{1}{4}\varepsilon\gamma A^3\omega = 0, \tag{3.3}$$
$$2\varepsilon\gamma A\xi_1\dot{\xi}_1-A\omega^2+A\omega_0^2 = 0.$$

Assuming that $A \neq 0$, $\omega \neq 0$, the two latter equations take the form

$$A^2 = \frac{4}{\gamma}(1-\gamma\xi_1^2), \quad \omega^2 = \omega_0^2+2\varepsilon\gamma\xi_1\dot{\xi}_1. \tag{3.4}$$

Substituting this in Eq. $(3.3)_1$, we have

$$\ddot{\xi}_1+\varepsilon\dot{\xi}_1+\omega_0^2\xi_1-\varepsilon\gamma\xi_1^2\dot{\xi}_1 = p(t). \tag{3.5}$$

In the case considered, the expressions appearing in Eq. (1.9) are

$$\Phi_1^0 = \frac{4}{\gamma}\omega^2, \quad \Phi_1 = -4\xi_1^2\omega^2, \tag{3.6}$$
$$\Phi_2^0 = \omega_0^2, \quad \Phi_2 = 2\varepsilon\gamma\xi_1\dot{\xi}_1.$$

Equation (3.5) therefore assumes the form of Eq. (1.11),

$$f(\xi_1, \dot{\xi}_1) = \varepsilon\dot{\xi}_1+\omega_0^2\xi_1-\varepsilon\gamma\xi_1^2\dot{\xi}_1 \tag{3.7}$$

and

$$\xi_{02} = 0, \tag{3.8}$$

and thus we assume

$$g(\xi_{02}) = 0. \tag{3.9}$$

From Eq. (2.8), we obtain

$$m = 0 \tag{3.10}$$

and from Eq. (2.7), we determine

$$k = \varepsilon(1-\gamma\langle x_1^2\rangle), \quad \Omega^2 = \omega_0^2. \tag{3.11}$$

Confining the considerations to excitations of the "white noise" form, we calculate from Eqs. (2.13) and (3.11)

$$\langle x_1^2\rangle = \frac{1}{2\gamma}\left(1-\sqrt{1-\frac{4\pi S_0\gamma}{\omega_0^2\varepsilon}}\right), \tag{3.12}$$

where the minus sign is selected owing to the fact that for $S_0 \to 0$ also $\langle x_1^2\rangle \to 0$.

If it is assumed that

$$\frac{S_0\gamma}{\omega_0^2\varepsilon} \ll 1 \tag{3.13}$$

then Eq. (3.12) yields

$$\langle x_1^2\rangle \approx \frac{\pi S_0}{\varepsilon\omega_0^2}, \tag{3.14}$$

that is a value which is independent of the factor connected with non-linearity determined by the expression containing the coefficient γ. In the case considered, taking into account the relations (1.1), (3.6), (3.10) and (3.12), the expressions (2.14) are written in the form

$$\langle A^2\rangle = \frac{2}{\gamma}\left[1+\sqrt{1-\frac{4\pi S_0\gamma}{\omega_0^2\varepsilon}}\right], \quad \langle\omega^2\rangle = \omega_0^2. \tag{3.15}$$

This formula is identical with the result derived in [1] by an entirely different, probably more complicated, method. From Eq. (3.15) it is seen that increasing noise intensity leads to decreasing amplitudes of self-excited vibrations which may be reduced even by one half when compared to the amplitudes evaluated without the noises,

$$A_0^2 = \frac{4}{\gamma}, \tag{3.16}$$

provided

$$\frac{4\pi S_0\gamma}{\omega_0^2\varepsilon} = 1. \tag{3.17}$$

Equation (3.12) implies that increasing noise intensity increases the mean square value of amplitude of the vibration components produced exclusively by random noises. In the case of small intensity of random perturbations, the relation (3.14) remains valid, and thus $\langle x_1^2\rangle$ is independent of A_0^2.

The mean square value of angular frequency of the self-excited vibrations is derived from the second of Eqs. (2.14). Using Eqs. (3.6), we obtain

$$\omega^2 = \omega_0^2, \tag{3.18}$$

since in a stationary process $\langle\xi_1\dot{\xi}_1\rangle = 0$. Random perturbations have thus no influence on the mean square value of angular frequency.

4. Example 2. Lathe tool vibrations

The problem of self-excited vibrations of a lathe tool during metal cutting is discussed in the book [3]. The corresponging differential equation has the form [at $p(t) \equiv 0$]

$$\ddot{x}-\left(a_1^2+\frac{1}{2}\beta^2\dot{x}-\frac{1}{3}\gamma^2\dot{x}^2\right)\dot{x}+\omega_0^2x = p(t) \tag{4.1}$$

in which the non-linear characteristics of the damping force is seen to be asymmetric, provided $\beta^2 \neq 0$.

Let us analyze the influence of random perturbations upon the self-excited vibration amplitude for two types of perturbations, namely in the cases when their correlation functions have the form

$$R_p(\tau) = 2\pi S_0\delta(\tau), \tag{4.2}$$

or

$$R_p(\tau) = R_0e^{-\alpha|\tau|}. \tag{4.3}$$

Here S_0, R_0, α are positive constants, and $\delta(\tau)$ is the Dirac function.

Performing the integrations indicated by Eq. $(1.6)_{2,3}$, we transform the relations (1.9)

$$A^2 = \frac{4a_1^2}{\gamma^2\omega_0^2}+\frac{4}{\gamma^2\omega_0^2}(\beta^2\dot{\xi}_1-\gamma^2\dot{\xi}_1^2), \quad \omega^2 = \omega_0^2. \tag{4.4}$$

The first of Eqs. (1.6), after rearrangements and application of Eqs. (4.4), separates into two equations (1.10), (1.11) which now assume the form

$$\begin{aligned} &\ddot{\xi}_1+a_2^2\dot{\xi}_1+\frac{5}{2}\beta^2\dot{\xi}_1^2-\frac{5}{3}\gamma^2\dot{\xi}_1^3+\omega_0^2\xi_1 = p(t),\\ &\xi_{02}-\frac{\beta^2a_1^2}{\gamma^2\omega_0^2} = 0, \end{aligned} \tag{4.5}$$

with the notation

$$a_2^2 = a_1^2\left(1-\frac{\beta^4}{a_1^2\gamma^2}\right). \tag{4.6}$$

It will be assumed that

$$\frac{\beta^4}{a_1^2\gamma^2} < 1. \tag{4.7}$$

In the case considered, we have therefore

$$f(\xi_1, \dot{\xi}_1) \equiv a_2^2\dot{\xi}_1+\frac{5}{2}\beta^2\dot{\xi}_1^2-\frac{5}{3}\gamma^2\dot{\xi}_1^3+\omega_0^2\xi_1. \tag{4.8}$$

Taking into account Eqs. (2.1), (2.5), (2.6), we derive from Eq. (2.4)

$$m = \langle\xi_1\rangle = -\frac{5}{2}\beta^2\langle x_1^2\rangle, \tag{4.9}$$

while expressions (2.3) as the linearization coefficients are equal

$$k = a_2^2-5\gamma^2\omega_0^2\langle x_1^2\rangle, \quad \Omega^2 = \omega_0^2. \tag{4.10}$$

Let us consider the two types of excitation: (4.2) and (4.3).

1. Similarly as in the preceding example, we obtain in the case of excitation in the "white noise" form

(4.11)
$$\langle x_1^2 \rangle = \frac{a_2^2}{10\gamma^2\omega_0^2}\left(1-\sqrt{1-20\frac{\pi S_0\gamma^2}{a_2^4}}\right)$$

which, under small random perturbation conditions is reduced to

(4.12)
$$\langle x_1^2 \rangle \approx \frac{\pi S_0}{\omega_0^2 a_2^2}.$$

Inserting (4.11) into (4.4), we obtain the mean square value of the self-excited vibration amplitude accounting for the random perturbations. It is given by the formula

(4.13)
$$\langle A^2 \rangle = \frac{4a_1^2}{\gamma^2\omega_0^2}\left[1-0.1\frac{a_2^2}{a_1^2}\left(1-\sqrt{1-20\frac{\pi S_0\gamma^2}{a_2^4}}\right)\right].$$

If the expression appearing at the left-hand side of Eq. (4.7) is much smaller than unity (or of $\beta^2 = 0$), then Eq. (4.13) assumes the form

(4.14)
$$\langle A^2 \rangle = \frac{2a_1^2}{5\omega_0^2\gamma^2}\left(9+\sqrt{1-20\frac{\pi S_0\gamma^2}{a_1^4}}\right).$$

Both relations (4.13), (4.14) are expressed, at very small perturbations, in the following manner:

(4.15)
$$\langle A^2 \rangle \approx \frac{4a_1^2}{\gamma^2\omega_0^2}\left[1-\frac{\pi S_0\gamma^2}{a_1^2 a_2^2}\right],$$
$$\langle A^2 \rangle \approx \frac{4a_1^2}{\gamma^2\omega_1^2}\left[1-\frac{\pi S_0\gamma^2}{a_1^4}\right].$$

These results imply the following conclusions.

(a) Asymmetry of the damping characteristic considerably influences the amplitude of perturbed self-excited vibrations. From both Eqs. (4.13) and $(4.15)_1$ it follows that with increasing β^2 the perturbed self-excited vibration amplitude decreases. It may be demonstrated that the expression (4.14) is always greater than (4.13).

(b) Random perturbations reduce the amplitude of self-excited vibrations when compared to the case without perturbations; in the case considered here, the amplitude is equal to

(4.16)
$$A_0^2 = \frac{4a_1^2}{\gamma^2\omega_0^2}$$

and is independent of the factor responsible for the asymmetry of the characteristics.

(c) If $\beta^2 = 0$, increasing of the non-linearity coefficient reduces the value of vibration amplitude. With $\beta^2 \neq 0$, the variation of amplitude depends on the remaining parameters of the system.

2. In the case of perturbation given by Eq. (4.3), spectral density of this process has the form

(4.17)
$$S_p(\omega) = \frac{R_0}{\pi}\frac{\alpha}{\alpha^2+\omega^2}.$$

As a result of the known transformations applied to the linear Eq. (2.2), we obtain

$$\langle x_1^2\rangle = \frac{R_0(\alpha+R)}{\omega_0^2 k(\alpha^2+k\alpha+\omega_0^2)}. \tag{4.18}$$

Taking account of the first relation (4.10) and substituting

$$k = a_2^2\xi, \tag{4.19}$$

we obtain from the Eq. (4.18) an algebraic equation of third order in the variable ξ,

$$\xi^3+\xi^2(u-1)+\xi(v-u)+w = 0, \tag{4.20}$$

where the notations

$$u = \frac{\alpha^2+\omega_0^2}{\alpha a_2^2}, \quad v = \frac{5R_0\gamma^2}{\alpha a_2^4}, \quad w = \frac{5R_0\gamma^2}{a_2^6}, \tag{4.21}$$

are introduced. The real root of Eq. (4.20) contained in the interval $0 < \xi < 1$ [what follows from Eq. (4.10)] and such that $\xi \to 1$ when $R_0 \to 0$ yields the value of k which will be used to express $\langle x_1^2\rangle$. The root of Eq. (4.20) is then found by the method of consecutive approximations. If we assume $\xi = 1+\varepsilon_0 w$ and confine ourselves to $\varepsilon_0 w$ in the first power (for small random excitations), the approximate value of the root of Eq. (4.20) will be equal to

$$\xi = 1-\frac{5R_0\gamma^2(\alpha+a_2^2)}{a_2^4(\alpha^2+\alpha a_2^2+\omega_0^2)}. \tag{4.22}$$

From the Eqs. (4.10) and (4.19), we determine

$$\langle x_1^2\rangle = \frac{R_0(\alpha+a_2^2)}{\omega_0^2 a_2^2(\alpha^2+\alpha a_2^2+\omega_0^2)}, \tag{4.23}$$

and on the basis of Eq. (4.4), we obtain

$$\langle A^2\rangle = \frac{4a_1^2}{\gamma^2\omega_0^2}\left[1-\frac{R_0\gamma^2(\alpha+a_2^2)}{a_1^2 a_2^2(\alpha^2+\alpha a_2^2+\omega_0^2)}\right]. \tag{4.24}$$

This result enables us to determine the influence of parameter α on the amplitude of self-excited vibrations. It is easily established that, in the approximation considered, if

$$\alpha > \alpha_1 = \omega_0-a_2^2 > 0, \tag{4.25}$$

then the second term in Eq. (4.24) decreases with increasing values of α, while the value of the entire expression for A^2 increases. For $\alpha < \alpha_1$, the situation is reversed. With $\omega_0-a_2^2 < 0$, A^2 is always an increasing function of α.

Let us now assume that

$$\frac{\alpha(\alpha+a_2^2)}{\omega_0^2} \ll 1 \tag{4.26}$$

and investigate the influence of R_0 on the vibrations. Under such an assumption also the inequality

$$\frac{\alpha(\alpha+k)}{\omega_0^2} \ll 1$$

is satisfied and Eq. (4.18) may be written as

$$\langle x_1^2 \rangle \approx \frac{R_0(\alpha+k)}{k\omega_0^4}. \tag{4.27}$$

Taking here into account the first of Eqs. (4.10), we obtain

$$\langle x_1^2 \rangle = \frac{R_0}{2\omega_0^4} + \frac{a_2^2}{10\gamma^2\omega_0^2}\left[1-\sqrt{\left(1-\frac{5R_0\gamma^2}{a_2^2\omega_0^2}\right)^2 - 20\frac{R_0\gamma^2\alpha}{\omega_0^2 a_2^4}}\right]. \tag{4.28}$$

Substitution of Eq. (4.28) in Eq. (4.4) yields the mean square value of the amplitude of self-excited vibrations accounting for the influence of random perturbations

$$\langle A^2 \rangle = \frac{4a_1^2}{\gamma^2\omega_0^2}\left\{1-0.1\frac{a_2^2}{a_1^2}\left[1-\sqrt{\left(1-\frac{5R_0\gamma^2}{a_2^2\omega_0^2}\right)^2 - 20\frac{R_0\gamma^2\alpha}{\omega_0^2 a_2^4}}\right]\right\} - \frac{2R_0}{\omega_0^4}. \tag{4.29}$$

The following conclusions may be drawn from the foregoing considerations:

(a) From Eqs. (4.23) and (4.28) it follows that, under asymmetric characteristics of damping ($\beta^2 \neq 0$), the mean perturbation amplitude increases with increasing parameter β^2.

(b) As a consequence of (a), the amplitude of perturbed self-excited vibrations given by Eqs. (4.24), (4.29) decreases with increasing β^2.

(c) Increasing perturbation intensity (increasing R_0) reduces the amplitude of self-excited vibrations.

The conclusions formulated above are of a qualitative nature. Corresponding expressions referring to any particular case yield the required numerical values of vibration amplitudes, provided all parameters of the system are known.

References

1. T. K. CAUGHEY, *Response of Van der Pol's oscillator to random excitation*, J. Appl. Mech., 345–348, Sept. 1959.
2. N. MINORSKI, *Non-linear vibrations* [in Polish], PWN, Warszawa 1967.
3. Я. Г. ПАНОВКО [YA. G. PANOVKO], *Основы прикладной теории упругих колебаний* [Foundations of the applied theory of elastic vibrations, in Russian]. Машиностроение, Москва [Mashinostroienie, Moskva], 1967.
4. K. PISZCZEK, J. WALCZAK, *Vibrations in machine construction* [in Polish], PWN, Warszawa 1972.
5. К. Ф. ТЕОДОРЧИК [K. F. TEODORCHIK], *Автоколебательные системы* [Self-excited vibration systems, in Russian], Москва-Ленинград [Moskva-Leningrad], 1952.
6. И. М. БАБАКОВ [I. M. BABAKOV], *Теория колебаний* [Theory of vibrations, in Russian]. Издат. "Наука", Москва [«Nauka», Moskva], 1965.
7. И. Е. КАЗАКОВ [I. E. KAZAKOV], *Статические методы проектирования систем управления* [Statistical methods of designing of guidance systems, in Russian], Машиностроение, Москва [Mashinostroienie, Moskva], 1969.

TECHNICAL UNIVERSITY OF KRAKÓW.

Received September 22, 1972.